

Near-optimal separators in string graphs

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Abstract

Let G be a string graph (an intersection graph of continuous arcs in the plane) with m edges. Fox and Pach proved that G has a separator consisting of $O(m^{3/4}\sqrt{\log m})$ vertices, and they conjectured that the bound of $O(\sqrt{m})$ actually holds. We obtain separators with $O(\sqrt{m} \log m)$ vertices.

Let $G = (V, E)$ be a graph with n vertices. A *separator* in G is a set $S \subseteq V$ of vertices such that there is a partition $V = V_1 \cup V_2 \cup S$ with $|V_1|, |V_2| \leq \frac{2}{3}n$ and no edges connecting V_1 to V_2 . The graph G is a *string graph* if it is an intersection graph of curves in the plane, i.e., if there is a system $(\gamma_v : v \in V)$ of curves (continuous arcs) such that $\gamma_u \cap \gamma_v \neq \emptyset$ iff $\{u, v\} \in E(G)$ or $u = v$.

Fox and Pach [FP10] proved that every string graph has a separator with $O(m^{3/4}\sqrt{\log m})$ vertices, where m is the number of edges of G .

We should mention that they actually proved the result for the weighted case, where each vertex $v \in V$ has a positive real weight, and the size of the components of $G \setminus S$ is measured by the sum of vertex weights (while the size of S is still measured as the number of vertices). Our result can also be extended to the weighted case, either by deriving it from the unweighted case along the lines of [FP10], or by using appropriate vertex-weighted versions (available in the cited sources) of the tools used in the proof. However, for simplicity, we stick to the unweighted case in this note.

Pach and Fox conjectured that string graphs actually have separators of size $O(\sqrt{m})$ (which, if true, would be asymptotically optimal in the worst case). Earlier, in [FP08], they proved some special cases of this conjecture: most notably, if every two curves γ_u, γ_v in the string representation intersect in at most k points, where k is a constant. As they kindly informed me in February 2013, they also have an (unpublished) proof of existence of separators of size $O(\sqrt{n})$ in string graphs with maximum degree bounded by a constant. Here we obtain the following result.

Theorem 1. *Every string graph G with $m \geq 2$ edges has a separator with $O(\sqrt{m} \log m)$ vertices.*

*Supported by the ERC Advanced Grant No. 267165 and by GRADR Eurogiga GIG/11/E023.

Clearly, we may assume G connected, and then the theorem immediately follows from Lemmas 2 and 3 below. Lemma 2 combines the considerations of [FP10] with those of [KM04] and adjusts them for vertex congestion instead of edge congestion. Lemma 3 replaces an approximate duality between sparsity of edge cuts and edge congestion due to Leighton and Rao [LR99] used in [KM04] with an approximate duality between sparsity of vertex cuts and vertex congestion, which is an immediate consequence of the results of Feige et al. [FHL08].

Fox and Pach [FP13] obtained several interesting applications of Theorem 1. Here we mention yet another consequence.

Crossing number versus pair-crossing number. The *crossing number* $\text{cr}(G)$ of a graph G is the minimum possible number of edge crossings in a drawing of G in the plane, while the *pair-crossing number* $\text{pcr}(G)$ is the minimum possible number of pairs of edges that cross in a drawing of G .

One of the most tantalizing questions in the theory of graph drawing is whether $\text{cr}(G) = \text{pcr}(G)$ for all graphs G [PT00], and in the absence of a solution, researchers have been trying to bound $\text{cr}(G)$ from above by a function of $\text{pcr}(G)$. The strongest result so far by Tóth [Tót12] was $\text{cr}(G) = O(p^{7/4}(\log p)^{3/2})$, where $p = \text{pcr}(G)$. It is based on the Fox–Pach separator theorem for string graphs discussed above, and by replacing their bound by Theorem 1 in Tóth’s proof, one obtains the improved estimate $\text{cr}(G) = O(p^{3/2} \log^2 p)$.

Vertex congestion in string graphs. Let \mathcal{P} denote the set of all paths in G , and for each pair $\{u, v\} \in \binom{V}{2}$ of vertices, let $\mathcal{P}_{uv} \subseteq \mathcal{P}$ be all paths from u to v . An *all-pair unit-demand multicommodity flow* in G is a mapping $\varphi: \mathcal{P} \rightarrow [0, 1]$ such that $\sum_{P \in \mathcal{P}_{uv}} \varphi(P) = 1$ for every $\{u, v\} \in \binom{V}{2}$. The *congestion* $\text{cong}(w)$ of a vertex $w \in V$ under φ is the total flow through w where, for conformity with [FHL08], we count only half of the flow through a path P if w is one of the endpoints of P . That is,

$$\text{cong}(w) = \sum_{P \in \mathcal{P}: w \text{ internal vertex of } P} \varphi(P) + \frac{1}{2} \sum_{P \in \mathcal{P}: w \text{ endpoint of } P} \varphi(P).$$

We define $\text{vcong}(G) := \min_{\varphi} \max_{w \in V} \text{cong}(w)$, where the minimum is over all all-pair unit-demand multicommodity flows.¹

Lemma 2. *If G is a connected string graph, then $\text{vcong}(G) \geq cn^2/\sqrt{m}$ (for a suitable constant $c > 0$).*

Proof. Let φ be a flow for which $\text{vcong}(G)$ is attained, and let $(\gamma_v : v \in V)$ be a string representation of G . We construct a drawing of K_V , the complete graph on the vertex set V , as follows.

We draw each vertex $v \in V$ as a point $p_v \in \gamma_v$, in such a way that all the p_v are distinct.

For every edge $\{u, v\} \in \binom{V}{2}$ of the complete graph, we pick a path P_{uv} from \mathcal{P}_{uv} at random, where each $P \in \mathcal{P}_{uv}$ is chosen with probability $\varphi(P)$, the choices being independent for different $\{u, v\}$. Let us enumerate the vertices along P_{uv} as $v_0 = u, v_1, v_2, \dots, v_k = v$. Then we draw the edge $\{u, v\}$ of K_V in the following manner: We start at p_u , follow γ_u until some (arbitrarily chosen) intersection with γ_{v_1} , then we follow γ_{v_1} until some intersection with γ_{v_2} , etc., until we reach γ_v and p_v on it.

Let us estimate the expected number of pairs $\{\{u, v\}, \{u', v'\}\}$ of edges of K_V that intersect in this drawing.

¹It is well known, and easy to check by a compactness argument, that min is attained.

The drawings of $\{u, v\}$ and $\{u', v'\}$ may intersect only if there are vertices $w \in P_{uv}$ and $w' \in P_{u'v'}$ such that $\gamma_w \cap \gamma_{w'} \neq \emptyset$, i.e., $\{w, w'\} \in E(G)$ or $w = w'$. For every choice of $\{w, w'\} \in E(G)$ or $w = w' \in V$, the expected number of pairs $\{P_{uv}, P_{u'v'}\}$ with $w \in P_{uv}$ and $w' \in P_{u'v'}$ is easily seen to be bounded above by $4 \text{vcong}(G)^2$ (using linearity of expectation and independence). Thus, the total expected number of intersecting pairs of edges of K_V is at most $4(m+n) \text{vcong}(G)^2 \leq 4(2m+1) \text{vcong}(G)^2$.

At the same time, it is well known that $\text{pcr}(K_V) = \Omega(n^4)$, i.e., any drawing of K_V has $\Omega(n^4)$ intersecting pairs of edges (see, e.g., [PT00, Thm. 3]). So $m \text{vcong}(G)^2 = \Omega(n^4)$ and the lemma follows. \square

Vertex congestion and separators. Let us define $\text{vcong}^*(G) := \min\{\text{vcong}(H) : H \text{ is an induced subgraph of } G \text{ on at least } \frac{2}{3}n \text{ vertices}\}$.

Lemma 3. *Every graph G on n vertices has a vertex separator with $O((n^2 \log n)/\text{vcong}^*(G))$ vertices.*

Proof. The proof goes in the following steps, all of them contained in [FHL08] (also see [BLR10], especially Sec. 5.2 there, for a similar use of [FHL08]).

1. Let $s: V \rightarrow [0, \infty)$ be an assignment of real weights to the vertices of G , let the weight of an edge $e = \{u, v\} \in E(G)$ be $(s(u) + s(v))/2$, and let d_s be the shortest-path pseudometric in G with these edge weights. By the duality of linear programming, it is easy to derive (see [FHL08, Sec. 4])

$$\frac{1}{\text{vcong}(G)} = \min\left\{\sum_{v \in V} s(v) : \sum_{\{u, v\} \in \binom{V}{2}} d_s(u, v) = 1\right\}.$$

2. Let s^* be a vertex weighting attaining the minimum in the last formula. By using a famous result of Bourgain suitably, see [FHL08, Theorem 3.1], we get that there exists a function $f: V \rightarrow \mathbb{R}$ that is 1-Lipschitz w.r.t. s^* , i.e., $|f(u) - f(v)| \leq d_{s^*}(u, v)$ for all $u, v \in V$, and such that $\sum_{\{u, v\} \in \binom{V}{2}} |f(u) - f(v)| = \Omega((\sum_{\{u, v\} \in \binom{V}{2}} d_{s^*}(u, v))/\log n) = \Omega(1/\log n)$.
3. Let (A, B, S) be a partition of the vertex set of a graph G into three disjoint subsets with $A \neq \emptyset \neq B$ and no edges between A and B . Let the *sparsity* of (A, B, S) be $\frac{|S|}{|A \cup S| \cdot |B \cup S|}$. By [FHL08, Lemma 3.7], given a function f as above for G , there exists a partition (A, B, S) of the vertex set with sparsity $O((\sum_{v \in V} s^*(v)) \log n) = O((\log n)/\text{vcong}(G))$.
4. A standard procedure, starting with G and repeatedly finding a sparse partition until the size of all components drops below $\frac{2}{3}n$ (see, e.g., [FHL08, Sec. 6]), then finds a separator of size $O((n^2 \log n)/\text{vcong}^*(G))$ in G as claimed.

\square

Remark. Although Lemma 3 is tight for arbitrary graphs, a possible way towards proving the Fox–Pach conjecture, separators for string graphs of size $O(\sqrt{m})$, would be removing the $\log n$ factor in Lemma 3 under the assumption that G is a string graph. More concretely, the improvement might be achievable in item 2 of the proof above: indeed, if G is a planar graph or, more generally, belongs to a minor-closed class of graphs with a forbidden minor, then, in

the setting of item 2, the 1-Lipschitz f can even be made to satisfy $\sum_{\{u,v\} \in \binom{V}{2}} |f(u) - f(v)| = \Omega(1)$ [Rab08] (also see [FHL08, Thm. 3.2]). Thus, a similar improvement for string graphs is perhaps not out of reach.

Acknowledgment. I would like to thank Jacob Fox and János Pach, as well as an anonymous referee, for very useful comments.

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