

Simulations and Bisimulations For Coalgebraic Modal Logics

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Abstract. We define a notion of Λ -simulation for coalgebraic modal logics, parametric on the choice Λ of predicate liftings for a functor T . We show this notion is adequate in several ways: i) it preserves truth of positive formulas, ii) for Λ a separating set of monotone predicate liftings, the associated notion of Λ -bisimulation corresponds to T -behavioural equivalence (moreover Λ -bisimulations correspond to T - n -behavioural equivalence), and iii) in fact, for Λ -separating and T preserving weak pullbacks, difunctional Λ -bisimulations are T -bisimulations. In essence, we arrive at a modular notion of equivalence that, when used with a separating set of monotone predicate liftings, coincides with T -behavioural equivalence regardless of whether T preserves weak pullbacks (unlike the notion of T -bisimilarity).

1 Introduction

As the basic notion of equivalence in coalgebra, T -behavioural equivalence has emerged, which declares two states to be equivalent if they are identified by some pair of coalgebra morphisms; in case the type functor T admits a final coalgebra, T -behavioural equivalence is just identification in the final T -coalgebra. As a proof principle, however, T -behavioural equivalence is comparatively unwieldy, thus motivating the search for bisimulation-type proof principles whereby two states can be shown to be behaviourally equivalent by exhibiting a *bisimulation relation* between them. The advantage of such approaches is that bisimulation relations may be comparatively small, making equivalence proofs by bisimulation more manageable than direct proofs of behavioural equivalence.

The downside is that while behavioural equivalence is a canonical notion that works for any type of coalgebras, it is rather less clear what a bisimulation is in general. In case the type functor preserves weak pullbacks, the standard notion of T -bisimulation gives a satisfactory answer: it can be uniformly defined for any T , it is always sound for T -behavioural equivalence, if T preserves weak pullbacks it is complete for T -behavioural equivalence, and it coincides with standard notions in the main examples. For functors that fail to preserve weak pullbacks, however, the search for a good generic notion of bisimilarity remains largely open.

Here, we present a modally-inspired notion of bisimulation that partly solves these problems, specifically it does so for functors that admit a separating set of *monotone* predicate liftings. Our notion of Λ -bisimilarity depends on distinguishing a modal signature Λ that we assume to consist of monotone operators. Key features of Λ -bisimilarity are

- It is related to a corresponding notion of Λ -simulation, which bears a clear relation to modal logic: all positive modal formulas over Λ are preserved by Λ -simulations.
- If Λ is separating, then Λ -bisimulation is sound and complete for behavioural equivalence.
- We have a finite-lookahead version of Λ -bisimilarity. This Λ - n -bisimilarity is sound and complete for the standard notion of n -behavioural equivalence defined via the terminal sequence.
- Λ -bisimulation allows bisimulation proofs up to difunctionality (i.e. closure under zig-zags).
- If T preserves weak pullbacks, then Λ -bisimulations are essentially the same as T -bisimulations, at least when we restrict to difunctional relations.

Related Work: Recent yet unpublished work by Enqvist [2] introduces a notion of Λ -homomorphism that is almost a special case of a Λ -simulation, and in fact shows that such Λ -homomorphisms can be induced by a relator in the sense of [5], so that the notion of Λ -simulation can itself be regarded as implicit in that work. When we say ‘almost’, we mean that the implication in the definition of Λ -homomorphism goes the other way in Enqvist’s work than it does here, so that in particular Theorem 16 would fail for his notion. The notion of Λ -homomorphism in the version that appears here has been under discussion between the authors’ group and international coauthors from late 2011.

In [6] it is shown that so-called *lax extensions of T preserving diagonals* induce notions of bisimulation that are sound and complete for behavioural equivalence, and that a finitary functor has such an extension iff it admits a separating set of finitary monotone predicate liftings. Our result, while otherwise working with similar assumptions, does not suppose finitariness of the functor.

In [5] a generic theory of coalgebraic simulation is developed using *relators*. One can show that our notion of Λ -simulation is induced by a relator and therefore subsumed by that framework. We cannot currently make out that any of our results about Λ -(bi)simulation could be obtained by instantiating the generic results, however.

2 Preliminaries

The framework of *coalgebraic modal logic* [7] covers a broad range of modalities beyond the standard relational setup, including probabilistic and game-theoretic phenomena as well as neighbourhood semantics and non-material conditionals [9]. This framework is parametric in syntax and semantics. The syntax is given by a *similarity type* Λ , i.e. a set of *modal operators* with finite arities ≥ 0 (hence possibly including propositional atoms). To simplify notation, we will pretend that all operators are unary.

Definition 1. The set $L(\Lambda)$ of Λ -formulas is given by the grammar:

$$\phi, \psi ::= \top \mid \neg\phi \mid \phi \wedge \psi \mid \heartsuit\phi \quad (\heartsuit \in \Lambda).$$

We use the standard derived Boolean operators \vee , \rightarrow , etc. We use $rank(\phi)$ to denote the maximum number of nested occurrences of $\heartsuit \in \Lambda$ in ϕ .

Semantics are parametrized by associating a Λ -structure $\langle T, \{\llbracket \heartsuit_\lambda \rrbracket\}_{\lambda \in \Lambda} \rangle$ to a similarity type Λ . Here T is an endofunctor T on the category Set and, each $\llbracket \heartsuit_\lambda \rrbracket$ is a *predicate lifting*, that is, a natural transformation $\llbracket \heartsuit \rrbracket : \mathcal{Q} \rightarrow \mathcal{Q} \circ T^{op}$, where \mathcal{Q} is the contravariant powerset functor $\text{Set}^{op} \rightarrow \text{Set}$ (that is, $\mathcal{Q}X \mapsto 2^X$ for every set X , and given $f : X \rightarrow Y$, $\mathcal{Q}f : 2^Y \rightarrow 2^X$ is given by $\mathcal{Q}f \mapsto \lambda A. f^{-1}[A]$). For the extension of predicate liftings to the higher-arity case see [10].

Assumption 2. We can assume w.l.o.g. that T preserves injective maps [1]. For convenience of notation, we will in fact sometimes assume that subset inclusions $X \hookrightarrow Y$ are mapped to subset inclusions $TX \hookrightarrow TY$. Moreover, we assume w.l.o.g. that T is non-trivial, i.e. $TX = \emptyset \implies X = \emptyset$ (otherwise, $TX = \emptyset$ for all X).

We typically identify a similarity type Λ and its associated Λ -structure, and refer to both as Λ . Unless otherwise stated, T stands for the underlying functor of the given Λ -structure.

For a given choice of Λ , a model for $L(\Lambda)$ is just a T -coalgebra $\langle X, \xi \rangle$, i.e. a non-empty set X (the set of *states*) and *transition function* $\gamma : X \rightarrow TX$. Given $x \in X$, the truth value of $L(\Lambda)$ -formulas is defined as:

$$x \models_\gamma \top \quad \text{always} \quad (1)$$

$$x \models_\gamma \neg\phi \iff x \not\models_\gamma \phi \quad (2)$$

$$x \models_\gamma \phi \wedge \psi \iff x \models_\gamma \phi \text{ and } x \models_\gamma \psi \quad (3)$$

$$x \models_\gamma \heartsuit\phi \iff \gamma(x) \models \heartsuit\llbracket \phi \rrbracket_\gamma \quad (4)$$

where $\llbracket \phi \rrbracket_\gamma$, the extension of ϕ in γ is given by $\llbracket \phi \rrbracket_\gamma = \{z \in X \mid z \models_\gamma \phi\}$. and for $t \in TX$ and $A \subseteq X$, $t \models \heartsuit A$ is a more suggestive notation for $t \in \llbracket \heartsuit \rrbracket_X A$. When clear from context, we shall write simply $x \models \phi$ and $\llbracket \phi \rrbracket$.

Example 3. Coalgebras for the (covariant) finite powerset functor \mathcal{P}_ω are finitely branching directed graphs. For a similarity type $\Lambda = \{\square, \diamond\}$ consider the associated predicate liftings:

$$\llbracket \square \rrbracket_X(A) := \{B \mid B \subseteq A\} \quad (5)$$

$$\llbracket \diamond \rrbracket_X(A) := \{B \mid B \cap A \neq \emptyset\} \quad (6)$$

They correspond to the classical modal operators of relational modal logics, so the logic we get in this case is essentially the mono-modal version of the Hennessy-Milner logic [4]. To obtain the basic modal logic K one needs to enrich the coalgebra structure with an interpretation for propositions. So let V be a set of proposition symbols and let C_V be the constant functor that maps every set X to 2^V . For each $p \in V$, the (nullary) predicate lifting $\llbracket p \rrbracket_X := \{\pi \in 2^V \mid p \in \pi\}$ describes structures satisfying p . The Kripke functor K is then defined as $KX := C_V \times \mathcal{P}X$ and the similarity type $\Lambda = V \cup \{\diamond, \square\}$ is interpreted using the corresponding predicate liftings on the appropriate projections.

Example 4. The language of *graded modal logic* corresponds to the set $\Lambda = \{\diamond_k \mid k \in \mathbb{N}\}$ and is interpreted over the infinite multiset functor \mathcal{B}_∞ , i.e., $\mathcal{B}_\infty X \mapsto$

$\{f : X \rightarrow \mathbb{N} \cup \{\infty\} \mid f \text{ has finite support}\}$. Coalgebras for \mathcal{B}_∞ are finitely branching multigraphs (with potentially infinite cardinalities). Interpretation of the modal operators is by way of the following family of predicate liftings, for each $k \in \mathbb{N}$:

$$\llbracket \Diamond_k \rrbracket_X(A) := \{b \in \mathcal{B}_\infty X \mid b(A) > k\} \quad (7)$$

where by $b(A)$ we denote $\sum_{x \in A} b(x)$, i.e. we use $b \in \mathcal{B}_\infty X$ like measure on X .

Example 5. Probabilistic modal logics are obtained when one takes the functor \mathcal{D} that maps X to the set of finitely-supported probability distributions over X . For the language $\mathcal{L}_M = \{M_p \mid p \in [0, 1] \cap \mathbb{Q}\}$, with M_p informally read as “with probability more than p ”, the corresponding predicate liftings are defined analogously as for graded modal logics. One can instead take $\mathcal{L}_P = \{L_p \mid p \in [0, 1] \cap \mathbb{Q}\}$, with L_p read as “with probability at least p ”, and interpreted using:

$$\llbracket L_p \rrbracket_X(A) := \{\mu \in \mathcal{D}X \mid \mu(A) \geq p\} . \quad (8)$$

Example 6. As a final example, consider the subfunctor \mathcal{M} of $\mathcal{Q} \circ \mathcal{Q}$ given by $\mathcal{M}X = \{S \in \mathcal{Q}\mathcal{Q}X \mid S \text{ is upwards closed}\}$. Over this functor one can obtain the monotone neighborhood semantics of modal logic with $\mathcal{L} = \{\Box\}$ using the predicate lifting $\llbracket \Box \rrbracket_X(A) := \{S \in \mathcal{M}X \mid A \in S\}$.

A modal operator \heartsuit is called *monotone* if it satisfies the condition

$$A \subseteq B \subseteq X \text{ implies } \llbracket \heartsuit \rrbracket_X A \subseteq \llbracket \heartsuit \rrbracket_X B . \quad (9)$$

While all the examples above are monotone, it is worth stressing that the framework of coalgebraic modal logics can indeed accommodate non-monotone logics. We will however focus on the monotone case.

Assumption 7. In the following, we assume all modal operators to be monotone.

For a given endofunctor T , the choice of both the similarity type \mathcal{L} and the associated \mathcal{L} -structure over T may vary (although the number of choices is formally limited [10]), and each choice yields a potentially different logic. When the choice of predicates of liftings in \mathcal{L} is rich enough as to uniquely describe every element in TX , we call such \mathcal{L} *separating* [8]:

Definition 8. We say that \mathcal{L} is *separating* if $t \in TX$ is uniquely determined by the set $\{(\heartsuit, A) \in \mathcal{L} \times \mathcal{P}X \mid t \models \heartsuit A\}$.

It is not hard to see that, for example, $V \cup \{\Box\}$ as well as $V \cup \{\Diamond\}$ are separating over the Kripke functor K of Example 3. The reader is referred to [10] for characterizations of functors that admit separating sets of predicate liftings.

Definition 9. Given T -coalgebras C and D , we say that states x in C and y in D are *behaviourally equivalent*, and write $(C, x) \approx (D, y)$, or shortly $x \approx y$, whenever there exists a T -coalgebra E and coalgebra morphisms $f : C \rightarrow E$ and $g : D \rightarrow E$ such that $f(x) = g(y)$.

Simulations like the ones we will present in Section 3 occur frequently when dealing with logics that do not contain a Boolean basis; typically, negation is absent or only allowed on restricted positions (e.g., in front of atoms). The notion of *positive* formula is a generalization of this idea.

Definition 10. The language $L^+(\Gamma)$ of *positive Λ -formulas* is given by:

$$\phi, \psi ::= \top \mid \perp \mid \phi \wedge \psi \mid \phi \vee \psi \mid \heartsuit \phi \quad (\heartsuit \in \Lambda).$$

We can regard $L^+(\Lambda)$ as a syntactic fragment of $L(\Lambda)$ where \vee is now taken as primitive. The Boolean connectives of $L^+(\Lambda)$ allow expressing all the monotone Boolean functions, but notice that Λ may contain dual operators (e.g., $\Lambda = \{\Box, \Diamond\}$) — in fact if Λ is closed under dual operators then $L^+(\Lambda)$ is as expressive as $L(\Lambda)$. In general, of course, $L^+(\Lambda)$ is a proper fragment of $L(\Lambda)$.

3 Coalgebraic simulation

We now proceed to introduce our notion of modal simulation. We use standard notation for relations; in particular, given a binary relation $S \subseteq X \times Y$ and $A \subseteq X$, we denote by $S[A]$ the relational image $S[A] = \{y \mid \exists x \in A. xSy\}$.

Definition 11 (Λ -Simulation, Λ -Homomorphism). Let $C = (X, \xi)$ and $D = (Y, \zeta)$ be T -coalgebras. A Λ -simulation $S : C \rightarrow D$ (of D by C) is a relation $S \subseteq X \times Y$ such that whenever xSy then for all $\heartsuit \in \Lambda$ and all $A \subseteq X$

$$\xi(x) \models \heartsuit A \text{ implies } \zeta(y) \models \heartsuit S[A].$$

A function $f : X \rightarrow Y$ is a Λ -homomorphism if its graph is a Λ -simulation.

Lemma 12. Λ -simulations are stable under unions and relational composition. Moreover, equality is always a Λ -simulation.

Definition 13 (Λ -ordering). The Λ -preorder \leq_Λ on TX is defined by

$$s \leq_\Lambda t \iff \forall \heartsuit \in \Lambda, A \subseteq X. (s \models \heartsuit A \implies t \models \heartsuit A).$$

Lemma 14. Let $C = (X, \xi)$ and $D = (Y, \zeta)$ be T -coalgebras. A map $f : X \rightarrow Y$ is a Λ -homomorphism iff for all $x \in Y$,

$$Tf(\xi(x)) \leq_\Lambda \zeta(f(x)). \tag{10}$$

Proof. ‘Only if’: Let $\heartsuit \in \Lambda, A \subseteq Y$. Then

$$\begin{aligned} Tf(\xi(x)) \models \heartsuit A &\iff \xi(x) \models \heartsuit f^{-1}[A] && \text{(naturality)} \\ &\implies \zeta(f(x)) \models \heartsuit f[f^{-1}[A]] && \text{(simulation)} \\ &\implies \zeta(f(x)) \models \heartsuit A && \text{(monotony).} \end{aligned}$$

‘If’: Let $\xi(x) \models \heartsuit A$. We have to show $\zeta(f(x)) \models \heartsuit f[A]$, which will follow by (10) from $Tf(\xi(x)) \models \heartsuit f[A]$. By naturality, the latter is equivalent to $\xi(x) \models \heartsuit f^{-1}[f[A]]$. This however follows from $\xi(x) \models \heartsuit A$ by monotony. \square

Remark 15. In the notation of the above lemma, another equivalent formulation of f being a Λ -homomorphism is that $\xi(x) \models \heartsuit f^{-1}[A]$ implies $\zeta(f(x)) \models \heartsuit A$ for $\heartsuit \in \Lambda$, $A \subseteq Y$. This is an immediate consequence of the lemma by naturality of predicate liftings.

As announced, Λ -simulations preserve the truth of positive modal formulas over Λ :

Theorem 16. *If S is a simulation and xSy , then $x \models \phi$ implies $y \models \phi$ for every positive Λ -formula ϕ .*

Proof. Induction over ϕ , with trivial Boolean cases (noting that these do not include negation). For the modal case, we have

$$\begin{aligned} x \models \heartsuit \phi &\iff \xi(x) \models \heartsuit[\![\phi]\!] \\ &\implies \zeta(y) \models \heartsuit\{\{y' \mid \exists x'.(x' \models \phi \wedge x'Sy')\}\} \\ &\implies \zeta(y) \models \heartsuit[\![\phi]\!] \\ &\iff y \models \heartsuit \phi. \end{aligned}$$

□

- Example 17.** 1. When $\Lambda = \{\diamond\}$, then a Λ -simulation $S : C \rightarrow D$ is just a simulation $C \rightarrow D$ in the usual sense. (Proof: ‘only if’: if xSy and $x' \in \xi(x)$, then $\xi(x) \models \diamond\{x'\}$ and hence $\zeta(y) \models \diamond\{y' \mid x'Sy'\}$, i.e. there exists y' such that $x'Sy'$ and $y' \in \zeta(y)$. ‘If’: If $\xi(x) \models \diamond A$, then there exists $x' \in A \cap \xi(x)$ and hence we have $y' \in \zeta(y)$ such that $x'Sy'$, so that $\zeta(y) \models \diamond\{y'' \mid \exists x'' \in \xi(x). x''Sy''\}$.)
2. When $\Lambda = \{\square\}$, then a Λ -simulation $S : C \rightarrow D$ is just a simulation $D \rightarrow C$ in the usual sense. (Proof: ‘only if’: Let xSy and $y' \in \zeta(y)$. Assume that we cannot find $x' \in \xi(x)$ such that $x'Sy'$; that is, $\xi(x) \models \square\{x' \mid \neg(x'Sy')\}$. Then by the definition of Λ -simulation, $\zeta(y) \models \square A$ for an A with $y' \notin A$, contradiction. ‘If’: Let $\xi(x) \models \square A$. To show that $\zeta(y) \models \square\{y' \mid \exists x' \in A. x'Sy'\}$, let $y' \in \zeta(y)$. By the simulation property, there exists $x' \in \xi(x)$ such that $x'Sy'$, and since $\xi(x) \models \square A$, we have $x' \in A$.)
3. For probabilistic modal logic, with $\Lambda = \{L_p \mid p \in [0, 1] \cap \mathbb{Q}\}$, a relation $S \subseteq X \times Y$ between \mathcal{D} -coalgebras (X, ξ) and (Y, ζ) is a Λ -simulation iff for all xSy and all $A \subseteq X$,

$$\zeta(y)(S[A]) \geq \xi(x)(A)$$

(keep in mind that $\xi(x)$ and $\zeta(y)$ are probability measures that we can apply to subsets). The same comes out when we take $\Lambda = \{M_p \mid p \in [0, 1] \cap \mathbb{Q}\}$. Note that standardly, probabilistic bisimulations (see the next section for the definition of bisimulations) are defined only for the case where S is an equivalence relation, in which case the notion coincides with the above.

4. For graded modal logic, with $\Lambda = \{\diamond_k \mid k \in \mathbb{N}\}$, we obtain the same inequality characterizing Λ -simulations as for probabilistic logic (keeping in mind that we can see $\xi(x) \in \mathcal{B}_\infty(X)$, $\zeta(y) \in \mathcal{B}_\infty(Y)$ as discrete $\mathbb{N} \cup \{\infty\}$ -valued measures).
5. For monotone neighbourhood logic, with $\Lambda = \{\square\}$, we have that a relation $S \subseteq X \times Y$ between \mathcal{M} -coalgebras (X, ξ) and (Y, ζ) is a Λ -simulation iff for xSy , $A \in \xi(x)$ implies $S[A] \in \zeta(y)$. This is easily seen to be equivalent to the forth condition in the definition of monotone bisimulation, attributed to Pauly in [3].

For many purposes, simulations can be already too strong, e.g. when we are interested in preservation results for positive formulas up to a certain modal depth. It is therefore natural to consider n -simulations.

Definition 18 (Λ - n -simulation). Let $C = (X, \xi)$ and $D = (Y, \zeta)$ be T -coalgebras. We define the notion of Λ - n -simulation inductively as follows. Any $S_0 \subseteq X \times Y$ is a Λ -0-simulation. A relation $S_{n+1} \subseteq X \times Y$ is a Λ - $(n+1)$ -simulation if there exists a Λ - n -simulation S_n such that $S_{n+1} \subseteq S_n$ and for all x, y , $xS_{n+1}y$ implies that for all $\heartsuit \in \Lambda$, $A \subseteq X$

$$\xi(x) \models \heartsuit A \text{ implies } \zeta(y) \models \heartsuit S_n[A].$$

Theorem 19. *If S is a Λ - n -simulation and xSy , then $x \models \phi$ implies $y \models \phi$ for every positive Λ -formula ϕ of rank at most n .*

Proof. Induction on n . The base case $n = 0$ is trivial since then ϕ is equivalent to either \top or \perp . For $n > 0$, we proceed by induction on ϕ , the interesting case being:

$$\begin{aligned} x \models \heartsuit \psi &\iff \xi(x) \models \heartsuit \llbracket \psi \rrbracket \\ &\implies \zeta(y) \models \heartsuit S_{n-1} \llbracket \llbracket \psi \rrbracket \rrbracket \\ &\implies \zeta(y) \models \heartsuit \llbracket \psi \rrbracket && \text{(outer IH + monotony)} \\ &\iff y \models \heartsuit \psi. \end{aligned}$$

□

4 Bisimulations for all

The notion of Λ - (n) -simulation naturally yields a notion of bisimulation (i.e., simulations in both directions). The yardstick for any notion of bisimulation is T -behavioural equivalence (see Section 2). We say that a notion of bisimulation is *sound for T -behavioural equivalence* if any two states related by bisimulation are T -behaviourally equivalent, and *complete for T -behavioural equivalence* if any two T -behaviourally equivalent states can be related by a bisimulation.

The standard coalgebraic notion of T -bisimulation that we recall below is always sound for T -behavioural equivalence, and complete for T -behavioural equivalence if T preserves weak pullbacks. We will show that our notion of Λ -bisimilarity is always sound and complete for T -behavioural equivalence, provided that Λ is separating. Notice also that Λ -bisimulations enjoy nice closure properties, in particular under unions and composition, which for T -bisimulations is only the case, again, when T preserves weak pullbacks.

Definition 20. If S and its converse S^{-1} are Λ - n -simulations, then S is a Λ - n -bisimulation. Analogously, a Λ -bisimulation is a Λ -simulation S such that S^{-1} is a Λ -simulation as well.

Lemma 21. *If C, D are T -coalgebras and $f : C \rightarrow D$ is a coalgebra morphism, then the graph of f is a Λ -bisimulation.*

Proof. It follows from Lemma 14 that the graph of f is a Λ -simulation. To see that its converse is a Λ -simulation, let $C = (X, \xi)$, $D = (Y, \zeta)$, and let $x \in X$, $\heartsuit \in \Lambda$, $A \subseteq Y$ such that $\zeta(f(x)) \models \heartsuit A$. Now $\zeta(f(x)) = Tf(\xi(x))$ because f is a coalgebra morphism, so we obtain $\xi(x) \models \heartsuit f^{-1}[A]$ by naturality of predicate liftings, as required. \square

It is easy to see that Λ - n -bisimulations preserve and reflect the truth of formulas with up to n nested modalities. A similar notion of preservation, *n-step-equivalence* was considered in [11], obtained by projecting into the terminal sequence. We can show that n -step-equivalence coincides with Λ - n -bisimilarity when Λ is separating.

Definition 22. The *terminal sequence* of a given functor T is the sequence given by $T_0 = 1$ (some singleton set) and $T_{n+1} = TT_n$, connected by functions $p_n : T_{n+1} \rightarrow T_n$, where $p_{n+1} = Tp_n$. Every T -coalgebra $C = (X, \xi)$ defines a cone over the terminal sequence by $\xi_0 : C \rightarrow 1$ (uniquely defined) and $\xi_{n+1} = T\xi_n \circ \xi$. Given T -coalgebras (X, ξ) and (Y, ζ) and elements $x \in X$, $y \in Y$, we say that x and y are *n-step equivalent* (notation: $x \approx_n y$) whenever $\xi_n(x) = \zeta_n(y)$.

Lemma 23. Let $C = (X, \xi)$ and $D = (Y, \zeta)$ be T -coalgebras. The *n-step-equivalence relation* $\approx_n \subseteq X \times Y$ is a Λ - n -bisimulation.

Proof. Of course, it suffices to show that \approx_n is a Λ - n -simulation. We proceed by induction on n . Clearly, $\approx_0 = X \times Y$ is a Λ -0-simulation. For the inductive step, let $x \approx_{n+1} y$ and let $\heartsuit \in \Lambda$, $A \subseteq X$ such that $\xi(x) \in \heartsuit_C A$. We then have (writing \mathcal{P} and \mathcal{Q} for the covariant and contravariant powerset functors, respectively):

$$\begin{aligned}
\xi(x) \models \heartsuit A &\implies \xi(x) \in \llbracket \heartsuit \rrbracket_C \circ \mathcal{Q}\xi_n \circ \mathcal{P}\xi_n A && \text{(monotony)} \\
&\implies \xi(x) \in \mathcal{Q}(T\xi_n) \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A && \text{(naturality)} \\
&\implies x \in \mathcal{Q}\xi \circ \mathcal{Q}(T\xi_n) \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A \\
&\implies x \in \mathcal{Q}\xi_{n+1} \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A && \text{(functoriality)} \\
&\implies y \in \mathcal{Q}\zeta_{n+1} \circ \mathcal{P}\xi_{n+1} \circ \mathcal{Q}\xi_{n+1} \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A && (x \approx_{n+1} y) \\
&\implies y \in \mathcal{Q}\zeta_{n+1} \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A && ((\mathcal{P}f \circ \mathcal{Q}f)X \subseteq X) \\
&= \mathcal{Q}\zeta \circ \mathcal{Q}(T\xi_n) \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A \\
&\implies \zeta(y) \in \mathcal{Q}(T\xi_n) \circ \llbracket \heartsuit \rrbracket_{T_n} \circ \mathcal{P}\xi_n A \\
&\implies \zeta(y) \in \llbracket \heartsuit \rrbracket_D \circ \mathcal{Q}\zeta_n \circ \mathcal{P}\xi_n A && \text{(naturality)} \\
&\implies \zeta(y) \models \heartsuit \approx_n[A].
\end{aligned}$$

By the inductive hypothesis, \approx_n is a Λ - n -simulation, and, moreover, $\approx_{n+1} \subseteq \approx_n$, so \approx_{n+1} is a Λ - $(n+1)$ -simulation. \square

Of course, the converse of this lemma does not hold in general (e.g., take T to be the multiset functor and consider $\Lambda = \{\diamond_0\}$). However, we do have the following.

Theorem 24. If Λ is a separating set of predicate liftings, then $S_n \subseteq \approx_n$ for every Λ - n -bisimulation S_n .

Proof. Induction on n . Let $C = (X, \xi)$, $D = (Y, \zeta)$ be T -coalgebras, let $S_{n+1} \subseteq X \times Y$ be a Λ - $(n+1)$ -bisimulation, and let $xS_{n+1}y$. Let $S_n \supseteq S_{n+1}$ be an n -bisimulation as in the definition of Λ - $(n+1)$ -bisimilarity.

We show $\xi_{n+1}(x) = \zeta_{n+1}(y)$ using separation. Thus, let $\heartsuit \in \Lambda$, $A \subseteq T_n$. We have to show that $\xi_{n+1}(x) \models \heartsuit A$ iff $\zeta_{n+1}(y) \models \heartsuit A$; by symmetry, it suffices to prove ‘only if’. Since $\xi_{n+1} = T\xi\xi_n$, we have, by naturality, $\xi(x) \models \heartsuit \xi_n^{-1}[A]$. By simulation, it follows that $\zeta(y) \models \heartsuit S_n[\xi_n^{-1}[A]]$. By the inductive hypothesis, $S_n \subseteq \approx_n$, so that we obtain $\zeta(y) \models \heartsuit \approx_n[\xi_n^{-1}[A]]$ by monotony. Now $\approx_n[\xi_n^{-1}[A]] = \zeta_n^{-1}[A]$ by definition of \approx_n , and hence $\zeta_{n+1}(y) = T\zeta_n\zeta(y) \models \heartsuit A$ by naturality. \square

In other words, Λ - n -bisimulation is always complete for n -step equivalence, and sound if Λ is separating.

Similar results hold for Λ -bisimulations. Specifically, we have

Lemma 25. *The behavioural equivalence relation \approx between two given T -coalgebras is a Λ -bisimulation.*

In other words, Λ -bisimulation is always complete for behavioural equivalence.

Proof. Let $C = (X, \xi)$, $D = (Y, \zeta)$ be T -coalgebras; it suffices to show that behavioural equivalence \approx (as a relation between X and Y) is a Λ -simulation between C and D . Given $x \approx y$, $\heartsuit \in \Lambda$ and $A \subseteq X$ such that $\xi(x) \models \heartsuit A$, we then have to show that $\zeta(y) \models \heartsuit (\approx[A])$. So let E be a T -coalgebra and $f : C \rightarrow E$ and $g : D \rightarrow E$ be coalgebra morphisms such that $f(x) = g(y)$. By Lemma 21, and by stability of simulations under composition, the relation $g^{-1}f = \{(x', y') \mid f(x') = g(y')\}$ is a Λ -simulation. Thus, we have $\zeta(y) \models \heartsuit g^{-1}[f[A]]$; and because $g^{-1}f$ is contained in \approx we are done by monotony. \square

As in the bounded-depth setting, soundness depends, of course, on separation:

Theorem 26. *If Λ is separating, then Λ -bisimilarity is sound and complete for behavioural equivalence.*

Proof. As stated above, Lemma 25 proves completeness; it remains to show soundness. Let $C = (X, \xi)$ and $D = (Y, \zeta)$ be T -coalgebras, and let $S \subseteq X \times Y$ be a Λ -bisimulation. Let Z be the quotient of the disjoint sum $X + Y$ by the equivalence relation generated by S , and let $\kappa_1 : X \rightarrow Z$ and $\kappa_2 : Y \rightarrow Z$ denote the prolongations of the coproduct injections into the quotient. It suffices to define a coalgebra structure χ on Z that makes κ_1 and κ_2 into coalgebra morphisms. We thus have to show that putting

$$\begin{aligned}\chi(\kappa_1(x)) &= T\kappa_1(\xi(x)) \\ \chi(\kappa_2(y)) &= T\kappa_2(\zeta(y))\end{aligned}$$

yields a well-defined map $Z \rightarrow TZ$. To this end, it suffices to show that $T\kappa_1(\xi(x)) = T\kappa_2(\zeta(y))$ whenever xSy . We prove this using separation by showing that $T\kappa_1(\xi(x)) \models \heartsuit A$ iff $T\kappa_2(\zeta(y)) \models \heartsuit A$ for $\heartsuit \in \Lambda$, $A \subseteq Z$. We prove only the left-to-right implication, the converse one being symmetric. So let $T\kappa_1(\xi(x)) \models \heartsuit A$. Then $\xi(x) \models \heartsuit \kappa_1^{-1}[A]$ by naturality, and hence $\zeta(y) \models \heartsuit S[\kappa_1^{-1}[A]]$ since S is a Λ -simulation. Now clearly $S[\kappa_1^{-1}[A]] \subseteq \kappa_2^{-1}[A]$, so that $\zeta(y) \models \heartsuit \kappa_2^{-1}[A]$ by monotony. We are done by naturality. \square

In the case where T preserves weak pullbacks, it is well-known that T -bisimilarity in the sense of Aczel and Mendler is also sound and complete for behavioural equivalence, so that T -bisimilarity and Λ -bisimilarity coincide when Λ is separating. But we can do better: T -bisimulations are Λ -bisimulations (so Λ -simulations are at least as convenient a tool as T -bisimulations), and for T preserving weak pullbacks and Λ separating, difunctional Λ -bisimulations are T -bisimulations. We recall the relevant definitions:

Definition 27. A T -bisimulation between T -coalgebras (X, ξ) and (Y, ζ) is a relation $S \subseteq X \times Y$ such that there exists a coalgebra structure $\rho : S \rightarrow TS$ that makes the projections $S \rightarrow X$ and $S \rightarrow Y$ into coalgebra morphisms.

Definition 28. A binary relation $S \subseteq X \times Y$ is *difunctional* if whenever xSy , zSy , and zSw , then xSw .

Essentially, we obtain a difunctional relation if we take an equivalence relation S on the disjoint union $X + Y$ of two sets and restrict it to $X \times Y$, i.e. take $S \cap (X \times Y)$ (where originally $S \subseteq (X + Y) \times (X + Y)$).

We now prove that all T -bisimulations are Λ -bisimulations, for any Λ and T , and that the converse holds for difunctional relations if T preserves weak pullbacks. We conjecture that the assumption of difunctionality can actually be removed. Nevertheless, we note the following. To begin, every relation $S \subseteq X \times Y$ has a difunctional closure \bar{S} , where $x\bar{S}y$ iff there exists chains $x = x_0, \dots, x_n$ in X and $y_0, \dots, y_n = y$ in Y such that $x_i S y_i$ for $i = 0, \dots, n$ and $x_{i+1} S y_i$ for $i = 0, \dots, n-1$.

Definition 29. A Λ -bisimulation up to difunctionality between T -coalgebras (X, ξ) and (Y, ζ) is a relation $S \subseteq X \times Y$ such that whenever xSy and $\xi(x) \models \heartsuit A$ for $\heartsuit \in \Lambda$, $A \subseteq X$, then $\zeta(y) \models \heartsuit \bar{S}[A]$, where \bar{S} denotes the difunctional closure of S , and the analogous condition holds for S^{-1} .

Proposition 30. Let $S \subseteq X \times Y$ be a relation between T -coalgebras (X, ξ) and (Y, ζ) . Then S is a Λ -bisimulation up to difunctionality iff the difunctional closure of S is a Λ -bisimulation.

Proof. ‘If’ is trivial; we show ‘only if’. Let \bar{S} be the difunctional closure of S . Let $\heartsuit \in \Lambda$, $A \subseteq X$ such that $\xi(x) \models \heartsuit A$, and let $x\bar{S}y$, i.e. we have $x = x_0, \dots, x_n \in X$ and $y_0, \dots, y_n = y \in Y$ such that $x_i S y_i$ for $i = 0, \dots, n$ and $x_{i+1} S y_i$ for $i = 0, \dots, n-1$. We define $A_0, \dots, A_n \subseteq X$ and $B_0, \dots, B_n \subseteq Y$ inductively by $A_0 = A$, $B_i = \bar{S}[A_i]$, and $A_{i+1} = \bar{S}^{-1}[B_i]$. By induction, $\xi(x_i) \models \heartsuit A_i$ and $\zeta(y_i) \models \heartsuit B_i$ for all i . Moreover, by difunctionality of \bar{S} , $B_i = \bar{S}[A]$ for all i , so that $\zeta(y) = \zeta(y_n) \models \heartsuit \bar{S}[A]$ as required. The proof that \bar{S}^{-1} is also a Λ -simulation is completely analogous. \square

Corollary 31. Let Λ be separating. Then Λ -bisimilarity up to difunctionality is sound and complete for T -behavioural equivalence.

To complement this, we explicitly define a notion of T -bisimulation up to difunctionality:

Definition 32. A T -bisimulation up to difunctionality between T -coalgebras (X, ξ) and (Y, ζ) is a relation $S \subseteq X \times Y$ such that there exists a map $\rho : S \rightarrow T\bar{S}$, where \bar{S} denotes the difunctional closure of S , such that $T\bar{p}_1\rho = \xi p_1$ and $T\bar{p}_2\rho = \zeta p_2$. Here $p_1 : S \rightarrow X$, $p_2 : S \rightarrow Y$, $\bar{p}_1 : \bar{S} \rightarrow X$, and $\bar{p}_2 : \bar{S} \rightarrow Y$ denote the projections.

It does not seem clear in general that an analogue of Proposition 30 holds for T -bisimulations. For the case where T preserves weak pullbacks, such an analogue will follow from the identification with Λ -bisimulations.

Theorem 33. *Every T -bisimulation (up to difunctionality) is a Λ -bisimulation (up to difunctionality).*

Proof. Let (X, ξ) and (Y, ζ) be T -coalgebras. For the plain case, let $S \subseteq X \times Y$ be a T -bisimulation between them. Thus, we have $\rho : S \rightarrow TS$ such that $p_1 : S \rightarrow X$ and $p_2 : S \rightarrow Y$ are coalgebra morphisms. Now let $\heartsuit \in \Lambda$, $A \subseteq X$, and xSy such that $\xi(x) \models \heartsuit A$. We have to show $\zeta(y) \models \heartsuit S[A]$. Now $\xi(x) = Tp_1\rho(x, y)$, and hence $\rho(x, y) \models \heartsuit p_1^{-1}[A]$. Since $\zeta(y) = Tp_2\rho(x, y)$, we have to show $\rho(x, y) \models \heartsuit p_2^{-1}S[A]$. By monotonicity, it suffices to show that $p_1^{-1}[A] \subseteq p_2^{-1}S[A]$. So let $(x', y') \in S$ such that $x' \in A$; we have to show $y' \in S[A]$, which holds by definition of $S[A]$.

For the second part, let S be a T -bisimulation up to difunctionality between (X, ξ) and (Y, ζ) , and let \bar{S} denote the difunctional closure of S . Thus, we have $\rho : S \rightarrow T\bar{S}$ such that $T\bar{p}_1\rho = \xi p_1$ and $T\bar{p}_2\rho = \zeta p_2$, where $p_1 : S \rightarrow X$, $p_2 : S \rightarrow Y$, $\bar{p}_1 : \bar{S} \rightarrow X$, $\bar{p}_2 : \bar{S} \rightarrow Y$ denote the projections. Let $\heartsuit \in \Lambda$, $A \subseteq X$ such that $\xi(x) \models \heartsuit A$; we have to show $\zeta(y) \models \heartsuit \bar{S}[A]$. As above, we find that we equivalently need to show $\rho(x, y) \models \heartsuit \bar{p}_2^{-1}[\bar{S}[A]]$ from $\rho(x, y) \models \heartsuit \bar{p}_1^{-1}[A]$, which follows from $\bar{p}_1^{-1}[A] \subseteq \bar{p}_2^{-1}[\bar{S}[A]]$. \square

The announced partial converse to this is

Theorem 34. *If Λ is separating and T preserves weak pullbacks, then difunctional Λ -bisimulations are T -bisimulations, and Λ -bisimulations up to difunctionality are T -bisimulations up to difunctionality.*

Proof. For the first part, let $S \subseteq X \times Y$ be a difunctional Λ -bisimulation between T -coalgebras (X, ξ) and (Y, ζ) . Let $p_1 : S \rightarrow X$ and $p_2 : S \rightarrow Y$ denote the projections. Let

$$\begin{array}{ccc} S & \xrightarrow{p_1} & X \\ p_2 \downarrow & & \downarrow q_2 \\ Y & \xrightarrow{q_2} & Z \end{array}$$

be a pushout; since S is difunctional, this is also a pullback. Now observe that the square

$$\begin{array}{ccccc} S & \xrightarrow{p_1} & X & \xrightarrow{\xi} & TX \\ p_2 \downarrow & & & & \downarrow Tq_2 \\ Y & & & & TZ \\ \zeta \downarrow & & & & \\ TY & \xrightarrow{Tq_2} & & & TZ \end{array}$$

commutes. To show this, we use separation: let $\heartsuit \in \Lambda$, and let $A \subseteq Z$. After one application of naturality, we have to show that when xSy then $\xi(x) \models \heartsuit q_1^{-1}[A]$ iff

$\zeta(x) \models \heartsuit q_2^{-1}[A]$. We show ‘only if’: observe that Z arises from $X + Y$ by quotienting modulo the equivalence relation \sim_S generated by S . Thus $q_1^{-1}[A]$ consists of the elements of X that are \sim_S -equivalent to some element of A , similarly for $q_2^{-1}[A]$. From $\xi(x) \models \heartsuit q_1^{-1}[A]$ we conclude $\zeta(y) \models \heartsuit S[q_1^{-1}[A]]$ because S is a Λ -simulation. But $S[q_1^{-1}[A]] \subseteq q_2^{-1}[A]$ because clearly each element of $S[q_1^{-1}[A]]$ is \sim_S -equivalent to an element of $q_1^{-1}[A]$ and hence to an element of A . Therefore, $\zeta(y) \models \heartsuit q_2^{-1}[A]$. The converse implication is shown dually.

For the second part, let S be a Λ -bisimulation up to difunctionality. By Proposition 30, the difunctional closure \bar{S} of S is a Λ -bisimulation and hence, by the first part, a T -bisimulation. By composing the T -coalgebra structure $\rho : \bar{S} \rightarrow T\bar{S}$ as in the definition of T -bisimulation with the inclusion $S \hookrightarrow \bar{S}$, we see that S is a T -bisimulation up to difunctionality. \square

Corollary 35. *If T preserves weak pullbacks, then T -bisimulations up to difunctionality are sound (and complete) for T -behavioural equivalence.*

5 Conclusions

We have introduced novel notions of Λ -simulation and Λ -bisimulation that work well in a setting where the coalgebraic type functor admits a separating set Λ of monotone predicate liftings. In particular, we have shown that Λ -bisimilarity is, in this setting, always sound and complete for T -behavioural equivalence, and moreover always admits a natural notion of bisimulation up to difunctionality. We have shown that T -bisimulations are always Λ -bisimulations, similarly for versions up to difunctionality, and that the converse holds for versions up to difunctionality in case T preserves weak pullbacks. We leave the question whether the converse holds in the plain case under preservation of weak pullbacks as an open problem.

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