An inverse spectral problem related to the Geng–Xue two-component peakon equation

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Abstract

We solve a spectral and an inverse spectral problem arising in the computation of peakon solutions to the two-component PDE derived by Geng and Xue as a generalization of the Novikov and Degasperis–Procesi equations. Like the spectral problems for those equations, this one is of a 'discrete cubic string' type – a nonselfadjoint generalization of a classical inhomogeneous string – but presents some interesting novel features: there are two Lax pairs, both of which contribute to the correct complete spectral data, and the solution to the inverse problem can be expressed using quantities related to Cauchy biorthogonal polynomials with two different spectral measures. The latter extends the range of previous applications of Cauchy biorthogonal polynomials to peakons, which featured either two identical, or two closely related, measures. The method used to solve the spectral problem hinges on the hidden presence of oscillatory kernels of Gantmacher–Krein type implying that the spectrum of the boundary value problem is positive and simple. The inverse spectral problem is solved by a method which generalizes, to a nonselfadjoint case, M. G. Krein's solution of the inverse problem for the Stieltjes string.

1 Introduction

In this paper, we solve an inverse spectral problem which appears in the context of computing explicit solutions to a two-component integrable PDE in $1 + 1$ dimensions found by Geng and Xue [14]. Denoting the two unknown functions by $u(x, t)$ and $v(x, t)$, and introducing the auxiliary quantities

$$m = u - u_{xx}, \qquad n = v - v_{xx}, \tag{1.1}$$

we can write the Geng–Xue equation as

$$\begin{aligned} m_t + (m_x u + 3mu_x)v = 0, \\ n_t + (n_x v + 3nv_x)u = 0. \end{aligned} \tag{1.2}$$

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(Subscripts denote partial derivatives, as usual.) This system arises as the compatibility condition of a Lax pair with spectral parameter z,

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & zn & 1\\ 0 & 0 & zm\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{1.3a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}-v_x u & v_x z^{-1} - vunz & v_x u_x\\ uz^{-1} & v_x u - vu_x - z^{-2} & -u_x z^{-1} - vumz\\ -vu & vz^{-1} & vu_x\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{1.3b}$$

but also (because of the symmetry in (1.2)) as the compatibility condition of a different Lax pair obtained by interchanging u and v,

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & zm & 1\\ 0 & 0 & zn\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \tag{1.4a}$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}-u_x v & u_x z^{-1} - uvmz & u_x v_x\\ vz^{-1} & u_x v - uv_x - z^{-2} & -v_x z^{-1} - uvnz\\ -uv & uz^{-1} & uv_x\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}. \tag{1.4b}$$

The subject of our paper is the inverse problem of recovering m and n from spectral data obtained by imposing suitable boundary conditions on equations (1.3a) and (1.4a), in the case when m and n are both discrete measures (finite linear combinations of Dirac deltas) with disjoint supports. To explain why this is of interest, we will give a short historical background.

When $u = v$ (and consequently also $m = n$), the Lax pairs above reduce to the Lax pair found by Hone and Wang for V. Novikov's integrable PDE [28, 18]

$$m_t + m_x u^2 + 3muu_x = 0, \tag{1.5}$$

and it was by generalizing that Lax pair to (1.3) that Geng and Xue came up with their new integrable PDE (1.2). Novikov's equation, in turn, was found as a cubically nonlinear counterpart to some previously known integrable PDEs with quadratic nonlinearities, namely the Camassa–Holm equation [8]

$$m_t + m_x u + 2mu_x = 0 \tag{1.6}$$

and the Degasperis–Procesi equation [10, 9]

$$m_t + m_x u + 3mu_x = 0. \tag{1.7}$$

The equations (1.6) and (1.7) have been much studied in the literature, and the references are far too numerous to survey here. Novikov's equation (1.5) is also beginning to attract attention; see [16, 17, 19, 21, 25, 27, 30]. What these equations have in common is that they admit weak solutions called *peakons* (peaked solitons), taking the form

$$u(x,t) = \sum_{k=1}^{N} m_k(t)\, e^{-|x-x_k(t)|}, \tag{1.8}$$

where the functions $x_k(t)$ and $m_k(t)$ satisfy an integrable system of $2N$ ODEs, whose general solution can be written down explicitly in terms of elementary functions with the help of inverse spectral techniques. In the Camassa–Holm case, this involves very classical mathematics surrounding the inverse spectral theory of the vibrating string with mass density $g(y)$, whose eigenmodes are determined by the Dirichlet problem

$$\begin{aligned} -\phi''(y) = z\, g(y)\, \phi(y) \quad \text{for } -1 < y < 1, \\ \phi(-1) = 0, \qquad \phi(1) = 0. \end{aligned} \tag{1.9}$$

In particular, one considers in this context the *discrete string* consisting of point masses connected by weightless thread, so that g is not a function but a linear combination of Dirac delta distributions. Then the solution to the inverse spectral problem can be expressed in terms of orthogonal polynomials and Stieltjes continued fractions [1, 2, 3, 26]. The reason for the appearance of Dirac deltas here is that when u has the form (1.8), the first derivative u_x has a jump of size $-2m_k$ at each point $x = x_k$, and this gives deltas in u_{xx} when derivatives are taken in the sense of distributions. In each interval between these points, u is a linear combination of e^x and e^{-x}, so $u_{xx} = u$ there; thus $m = u - u_{xx} = 2\sum_{k=1}^{N} m_k\, \delta_{x_k}$ is a purely discrete distribution (or a discrete measure if one prefers). The measure $g(y)$ in (1.9) is related to the measure $m(x)$ through a so-called Liouville transformation, and g will be discrete when m is discrete.

In the case of the Degasperis–Procesi and Novikov equations (and also for the Geng–Xue equation, as we shall see), the corresponding role is instead played by variants of a third-order nonselfadjoint spectral problem called the *cubic string* [22, 23, 20, 24, 17, 5]; in its basic form it reads

$$\begin{aligned} -\phi'''(y) = z\, g(y)\, \phi(y) \quad \text{for } -1 < y < 1, \\ \phi(-1) = \phi'(-1) = 0, \qquad \phi(1) = 0. \end{aligned} \tag{1.10}$$

The study of the discrete cubic string has prompted the development of a theory of *Cauchy biorthogonal polynomials* by Bertola, Gekhtman and Szmigielski [6, 5, 4, 7]; see Appendix A. In previous applications to peakon equations, the two measures α and β in the general setup of this theory have coincided ($\alpha = \beta$), but in this paper we will actually see two different spectral measures α and β entering the picture in a very natural way.

Like the above-mentioned PDEs, the Geng–Xue equation also admits peakon solutions, but now with two components,

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{N} m_k(t)\, e^{-|x-x_k(t)|}, \\ v(x,t) &= \sum_{k=1}^{N} n_k(t)\, e^{-|x-x_k(t)|}, \end{aligned} \tag{1.11}$$

where, for each k, at most one of m_k and n_k is nonzero (i.e., $m_k n_k = 0$ for all k). In this case, m and n will be discrete measures with disjoint support:

$$m = u - u_{xx} = 2\sum_{k=1}^{N} m_k\, \delta_{x_k}, \qquad n = v - v_{xx} = 2\sum_{k=1}^{N} n_k\, \delta_{x_k}. \tag{1.12}$$

This ansatz satisfies the PDE (1.2) if and only if the functions $x_k(t)$, $m_k(t)$ and $n_k(t)$ satisfy the following system of ODEs:

$$\begin{aligned} \dot{x}_k &= u(x_k)\, v(x_k), \\ \dot{m}_k &= m_k\big(u(x_k)\, v_x(x_k) - 2u_x(x_k) v(x_k)\big), \\ \dot{n}_k &= n_k\big(u_x(x_k)\, v(x_k) - 2u(x_k) v_x(x_k)\big), \end{aligned} \tag{1.13}$$

for $k = 1, 2, \dots, N$. (Here we use the shorthand notation

$$u(x_k) = \sum_{i=1}^{N} m_i\, e^{-|x_k - x_i|}$$

and

$$u_x(x_k) = \sum_{i=1}^{N} m_i\, e^{-|x_k - x_i|} \operatorname{sgn}(x_k - x_i).$$

If $u(x) = \sum m_i\, e^{-|x-x_i|}$, then the derivative u_x is undefined at the points x_k where $m_k \neq 0$, but here $\operatorname{sgn} 0 = 0$ by definition, so $u_x(x_k)$ really denotes the average of the one-sided (left and right) derivatives at those points. Note that the conditions $m_k = 0$ and $m_k \neq 0$ both are preserved by the ODEs. Similar remarks apply to v, of course.)

Knowing the solution of the inverse spectral problem for (1.3a)+(1.4a) in this discrete case makes it possible to explicitly determine the solutions to the peakon ODEs (1.13). Details about these peakon solutions and their dynamics will be published in a separate paper; here we will focus on the approximation-theoretical aspects of the inverse spectral problem. (But see Remark 4.12.)

We will only deal with the special case where the discrete measures are *interlacing*, meaning that there are $N = 2K$ sites

$$x_1 < x_2 < \cdots < x_{2K},$$

with the measure m supported on the odd-numbered sites x_{2a-1}, and the measure n supported on the even-numbered sites x_{2a}; see Figure 1 and Remark 3.1. The general formulas for recovering the positions x_k and the weights m_{2a-1} and n_{2a} are given in Corollary 4.5; they are written out more explicitly for illustration in Example 4.10 (the case $K = 2$) and Example 4.11 (the case $K = 3$). The case $K = 1$ is somewhat degenerate, and is treated separately in Section 4.3.

Appendix C contains an index of the notation used in this article.

2 Forward spectral problem

2.1 Transformation to a finite interval

Let us start by giving a precise definition of the spectral problem to be studied. The time dependence in the two Lax pairs for the Geng–Xue equation will be of no interest to us in this paper, so we consider t as fixed and omit it in the notation. The equations which govern the x dependence in the two Lax pairs are (1.3a) and (1.4a), respectively. Consider the first of these:

$$\frac{\partial}{\partial x}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}0 & zn(x) & 1\\ 0 & 0 & zm(x)\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \qquad \text{for } x \in \mathbf{R}, \tag{1.3a}$$

where $m(x)$ and $n(x)$ are given. Our main interest lies in the discrete case, when m and n are actually not functions but discrete measures as in (1.12), but we will not specialize to that case until Section 3.

There is a useful change of variables, similar to the one used for Novikov's equation [17], which produces a slightly simpler differential equation on a finite interval:

$$\begin{aligned}
y &= \tanh x, \\
\phi_1(y) &= \psi_1(x) \cosh x - \psi_3(x) \sinh x, \\
\phi_2(y) &= z\, \psi_2(x), \\
\phi_3(y) &= z^2\, \psi_3(x)/\cosh x, \\
g(y) &= m(x)\, \cosh^3 x, \\
h(y) &= n(x)\, \cosh^3 x, \\
\lambda &= -z^2.
\end{aligned} \tag{2.1}$$

Under this transformation (with $z \neq 0$), equation (1.3a) is equivalent to

$$\frac{\partial}{\partial y} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & h(y) & 0 \\ 0 & 0 & g(y) \\ -\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \qquad \text{for } -1 < y < 1. \tag{2.2a}$$

(Notice that the 1 in the upper right corner of the matrix has been removed by the transformation. When $h = g$, equation (2.2a) reduces to the *dual cubic string* studied in [17].) In order to define a spectrum we impose the following boundary conditions on the differential equation (2.2a):

$$\phi_2(-1) = \phi_3(-1) = 0, \qquad \phi_3(1) = 0. \tag{2.2b}$$

By the *eigenvalues* of the problem (2.2) we then of course mean those values of λ for which (2.2a) has nontrivial solutions satisfying (2.2b).

The same transformation (2.1) applied to the twin Lax equation (1.4a) leads to the same equation except that g and h are interchanged. The spectrum of this twin equation will in general be different. To be explicit, the second spectrum is defined by the differential equation

$$\frac{\partial}{\partial y} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & g(y) & 0 \\ 0 & 0 & h(y) \\ -\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \qquad \text{for } -1 < y < 1, \tag{2.3a}$$

again with boundary conditions

$$\phi_2(-1) = \phi_3(-1) = 0, \qquad \phi_3(1) = 0. \tag{2.3b}$$

Remark 2.1. Via the transformation (2.1), every concept pertaining to the original Lax equations (1.3a) and (1.4a) will have a counterpart in terms of the transformed equations (2.2a) and (2.3a), and vice versa. In the main text, we will work with (2.2a) and (2.3a) on the finite interval. However, a few things are more conveniently dealt with directly in terms of the original equations (1.3a) and (1.4a) on the real line; these are treated in Appendix B. More specifically, we prove there that the spectra defined above are real and simple, and we also obtain expressions for certain quantities that will be constants of motion for the Geng–Xue peakon dynamics.

Remark 2.2. Transforming the boundary conditions (2.2b) back to the real line via (2.1) yields

$$\lim_{x\to-\infty}\psi_2(x)=\lim_{x\to-\infty}e^{x}\psi_3(x)=0,\qquad \lim_{x\to+\infty}e^{-x}\psi_3(x)=0. \tag{2.4}$$

Each eigenvalue $\lambda\neq 0$ of (2.2) corresponds to a pair of eigenvalues $z=\pm\sqrt{-\lambda}$ of (1.3a)+(2.4). As an exceptional case, $\lambda=0$ is an eigenvalue of (2.2), but $z=0$ is not an eigenvalue of (1.3a)+(2.4); this is an artifact caused by the transformation (2.1) being singular for $z=0$. When talking about eigenvalues below, we will refer to λ rather than z.

In Section 2.3 below we will also encounter the condition $\phi_1(-1)=1$; this translates into

$$\lim_{x\to-\infty}e^{-x}\big(\psi_1(x;z)-\psi_3(x;z)\big)=2. \tag{2.5}$$

2.2 Transition matrices

Let

$$\mathcal{A}(y;\lambda)=\begin{pmatrix}0 & h(y) & 0\\ 0 & 0 & g(y)\\ -\lambda & 0 & 0\end{pmatrix},\quad \widetilde{\mathcal{A}}(y;\lambda)=\begin{pmatrix}0 & g(y) & 0\\ 0 & 0 & h(y)\\ -\lambda & 0 & 0\end{pmatrix} \tag{2.6}$$

denote the coefficient matrices appearing in the spectral problems (2.2) and (2.3), respectively. To improve readability, we will often omit the dependence on y in the notation, and write the differential equations simply as

$$\frac{\partial\Phi}{\partial y}=\mathcal{A}(\lambda)\Phi,\qquad \frac{\partial\Phi}{\partial y}=\widetilde{\mathcal{A}}(\lambda)\Phi, \tag{2.7}$$

respectively, where $\Phi=(\phi_1,\phi_2,\phi_3)^T$. Plenty of information about this pair of equations can be deduced from the following modest observation:

Lemma 2.3. *The matrices $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ satisfy*

$$\widetilde{\mathcal{A}}(\lambda)=-J\mathcal{A}(-\lambda)^TJ,\quad \textit{where}\quad J=\begin{pmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}=J^T=J^{-1}. \tag{2.8}$$

Proof. A one-line calculation. ☐

Definition 2.4 (Involution σ)**.** Let σ denote the following operation on the (loop) group of invertible complex 3×3 matrices $X(\lambda)$ depending on the parameter $\lambda\in\mathbf{C}$:

$$X(\lambda)^\sigma=JX(-\lambda)^{-T}J. \tag{2.9}$$

(We use the customary abbreviation $X^{-T}=(X^T)^{-1}=(X^{-1})^T$.)

Remark 2.5. It is easily verified that σ is a group homomorphism and an involution:

$$\big(X(\lambda)Y(\lambda)\big)^\sigma=X(\lambda)^\sigma Y(\lambda)^\sigma,\qquad \big(X(\lambda)^\sigma\big)^\sigma=X(\lambda).$$

Definition 2.6 (Fundamental matrices and transition matrices)**.** Let $U(y;\lambda)$ be the fundamental matrix of (2.2a) and $\widetilde{U}(y;\lambda)$ its counterpart for (2.3a); i.e., they are the unique solutions of the matrix ODEs

$$\frac{\partial U}{\partial y} = \mathcal{A}(y;\lambda)\,U, \qquad U(-1;\lambda) = I, \tag{2.10}$$

and

$$\frac{\partial \widetilde{U}}{\partial y} = \widetilde{\mathcal{A}}(y;\lambda)\,\widetilde{U}, \qquad \widetilde{U}(-1;\lambda) = I, \tag{2.11}$$

respectively, where I is the 3×3 identity matrix. The fundamental matrices evaluated at the right endpoint $y = 1$ will be called the *transition matrices* and denoted by

$$S(\lambda) = U(1;\lambda), \qquad \widetilde{S}(\lambda) = \widetilde{U}(1;\lambda). \tag{2.12}$$

Remark 2.7. The fundamental matrix contains the solution of any initial value problem: $\Phi(y) = U(y;\lambda)\Phi(-1)$ is the unique solution to the ODE $d\Phi/dy = \mathcal{A}(\lambda)\Phi$ satisfying given initial data $\Phi(-1)$ at the left endpoint $y = -1$. In particular, the value of the solution at the right endpoint $y = 1$ is $\Phi(1) = S(\lambda)\Phi(-1)$.

Theorem 2.8. *For all* $y \in [-1,1]$,

$$\det U(y;\lambda) = \det \widetilde{U}(y;\lambda) = 1 \tag{2.13}$$

and

$$\widetilde{U}(y;\lambda) = U(y;\lambda)^\sigma. \tag{2.14}$$

In particular, $\det S(\lambda) = \det \widetilde{S}(\lambda) = 1$*, and* $\widetilde{S}(\lambda) = S(\lambda)^\sigma$*.*

Proof. Equation (2.13) follows from Liouville's formula, since $\mathcal{A}$ is trace-free:

$$\det U(y;\lambda) = \big(\det U(-1;\lambda)\big) \exp \int_{-1}^{y} \operatorname{tr} \mathcal{A}(\xi;\lambda)\, d\xi = (\det I)\exp 0 = 1,$$

and similarly for $\widetilde{U}$. To prove (2.14), note first that

$$\begin{aligned}\frac{\partial U(\lambda)^{-1}}{\partial y} &= -U(\lambda)^{-1}\,\frac{\partial U(\lambda)}{\partial y}\,U(\lambda)^{-1} \\ &= -U(\lambda)^{-1}\,\mathcal{A}(\lambda)\,U(\lambda)\,U(\lambda)^{-1} = -U(\lambda)^{-1}\,\mathcal{A}(\lambda),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{\partial}{\partial y} U(\lambda)^\sigma &= \frac{\partial}{\partial y}\Big(J\,U(-\lambda)^{-T}\,J\Big) \\ &= J\left(\frac{\partial U(-\lambda)^{-1}}{\partial y}\right)^T J \\ &= J\left(-U(-\lambda)^{-1}\,\mathcal{A}(-\lambda)\right)^T J \\ &= -J\,\mathcal{A}(-\lambda)^T U(-\lambda)^{-T} J.\end{aligned}$$

With the help of Lemma 2.3 this becomes

$$\begin{aligned}\frac{\partial}{\partial y} U(\lambda)^{\sigma} &= \widetilde{\mathcal{A}}(\lambda)\, J\, U(-\lambda)^{-T} J \\ &= \widetilde{\mathcal{A}}(\lambda)\, U(\lambda)^{\sigma}.\end{aligned}$$

Since $U(\lambda)^{\sigma} = I = \widetilde{U}(\lambda)$ when $y = -1$, we see that $U(\lambda)^{\sigma}$ and $\widetilde{U}(\lambda)$ satisfy the same ODE and the same initial condition; hence they are equal for all y by uniqueness. □

Corollary 2.9. *The transition matrices $S(\lambda)$ and $\widetilde{S}(\lambda)$ satisfy $JS(\lambda)^T J\widetilde{S}(-\lambda) = I$ and $\widetilde{S}(-\lambda) = J\big(\operatorname{adj} S(\lambda)\big)^T J$, where* adj *denotes the adjugate (cofactor) matrix. In detail, this means that*

$$\begin{pmatrix} S_{33} & -S_{23} & S_{13} \\ -S_{32} & S_{22} & -S_{12} \\ S_{31} & -S_{21} & S_{11} \end{pmatrix}_{\lambda} \begin{pmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} & \widetilde{S}_{31} \\ \widetilde{S}_{21} & \widetilde{S}_{22} & \widetilde{S}_{32} \\ \widetilde{S}_{31} & \widetilde{S}_{32} & \widetilde{S}_{33} \end{pmatrix}_{-\lambda} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.15}$$

and

$$\widetilde{S}(-\lambda) = \begin{pmatrix} S_{11}S_{22} - S_{12}S_{21} & S_{11}S_{23} - S_{21}S_{13} & S_{12}S_{23} - S_{13}S_{22} \\ S_{11}S_{32} - S_{12}S_{31} & S_{11}S_{33} - S_{13}S_{31} & S_{12}S_{33} - S_{13}S_{32} \\ S_{21}S_{32} - S_{22}S_{31} & S_{21}S_{33} - S_{23}S_{31} & S_{22}S_{33} - S_{23}S_{32} \end{pmatrix}_{\lambda}. \tag{2.16}$$

(The subscripts $\pm\lambda$ indicate the point where the matrix entries are evaluated.)

2.3 Weyl functions

Consider next the boundary conditions $\phi_2(-1) = \phi_3(-1) = 0 = \phi_3(1)$ in the two spectral problems (2.2) and (2.3). Fix some value of $\lambda \in \mathbf{C}$, and let $\Phi = (\phi_1, \phi_2, \phi_3)^T$ be a solution of $d\Phi/dy = \mathcal{A}(\lambda)\Phi$ satisfying the boundary conditions at the left endpoint: $\phi_2(-1) = \phi_3(-1) = 0$. For normalization, we can take $\phi_1(-1) = 1$; then the solution Φ is unique, and its value at the right endpoint is given by the first column of the transition matrix: $\Phi(1) = S(\lambda)\Phi(-1) = S(\lambda)(1,0,0)^T = (S_{11}(\lambda), S_{21}(\lambda), S_{31}(\lambda))^T$. This shows that the boundary condition at the right endpoint, $\phi_3(1) = 0$, is equivalent to $S_{31}(\lambda) = 0$. In other words: λ is an eigenvalue of the first spectral problem (2.2) if and only if $S_{31}(\lambda) = 0$.

We define the following two *Weyl functions* for the first spectral problem using the entries from the first column of $S(\lambda)$:

$$W(\lambda) = -\frac{S_{21}(\lambda)}{S_{31}(\lambda)}, \qquad Z(\lambda) = -\frac{S_{11}(\lambda)}{S_{31}(\lambda)}. \tag{2.17}$$

The entries of $S(\lambda)$ depend analytically on the parameter λ, so the Weyl functions will be meromorphic, with poles (or possibly removable singularities) at the eigenvalues. The signs in (2.17) (and also in (2.18), (2.24), (2.26) below) are chosen so that the residues at these poles will be positive when g and h are positive; see in particular Theorem 3.10.

Similarly, λ is an eigenvalue of the twin spectral problem (2.3) if and only if $\widetilde{S}_{31}(\lambda) = 0$, and we define corresponding Weyl functions

$$\widetilde{W}(\lambda) = -\frac{\widetilde{S}_{21}(\lambda)}{\widetilde{S}_{31}(\lambda)}, \qquad \widetilde{Z}(\lambda) = -\frac{\widetilde{S}_{11}(\lambda)}{\widetilde{S}_{31}(\lambda)}. \tag{2.18}$$

Theorem 2.10. *The Weyl functions satisfy the relation*

$$Z(\lambda) + W(\lambda)\widetilde{W}(-\lambda) + \widetilde{Z}(-\lambda) = 0. \tag{2.19}$$

Proof. The $(3,1)$ entry in the matrix equality (2.15) is

$$S_{31}(\lambda)\widetilde{S}_{11}(-\lambda) - S_{21}(\lambda)\widetilde{S}_{21}(-\lambda) + S_{11}(\lambda)\widetilde{S}_{31}(-\lambda) = 0.$$

Division by $-S_{31}(\lambda)\widetilde{S}_{31}(-\lambda)$ gives the desired result. □

2.4 Adjoint spectral problems

Let us define a bilinear form on vector-valued functions

$$\Phi(y) = \begin{pmatrix} \phi_1(y) \\ \phi_2(y) \\ \phi_3(y) \end{pmatrix}$$

with $\phi_1, \phi_2, \phi_3 \in L^2(-1,1)$:

$$\begin{aligned} \langle \Phi, \Omega \rangle &= \int_{-1}^{1} \Phi(y)^T J\, \Omega(y)\, dy \\ &= \int_{-1}^{1} \big(\phi_1(y)\omega_3(y) - \phi_2(y)\omega_2(y) + \phi_3(y)\omega_1(y)\big)\, dy. \end{aligned} \tag{2.20}$$

Lemma 2.3 implies that

$$\Big(A(\lambda)\Phi\Big)^T J\, \Omega = -\Phi^T J \Big(\widetilde{A}(-\lambda)\Omega\Big),$$

which, together with an integration by parts, leads to

$$\left\langle \left(\tfrac{d}{dy} - A(\lambda)\right)\Phi, \Omega \right\rangle = \big[\Phi^T J\, \Omega\big]_{y=-1}^{1} - \left\langle \Phi, \left(\tfrac{d}{dy} - \widetilde{A}(-\lambda)\right)\Omega \right\rangle. \tag{2.21}$$

Now, if Φ satisfies the boundary conditions $\phi_2(-1) = \phi_3(-1) = 0 = \phi_3(1)$, then what remains of the boundary term $\big[\Phi^T J\, \Omega\big]_{-1}^{1}$ is

$$\phi_1(1)\omega_3(1) - \phi_2(1)\omega_2(1) - \phi_1(-1)\omega_3(-1),$$

and this can be killed by imposing the conditions $\omega_3(-1) = 0 = \omega_2(1) = \omega_3(1)$. Consequently, when acting on differentiable L^2 functions with such boundary conditions, respectively, the operators $\frac{d}{dy} - A(\lambda)$ and $-\frac{d}{dy} + \widetilde{A}(-\lambda)$ are adjoint to each other with respect to the bilinear form $\langle \cdot, \cdot \rangle$:

$$\left\langle \left(\tfrac{d}{dy} - A(\lambda)\right)\Phi, \Omega \right\rangle = \left\langle \Phi, \left(-\tfrac{d}{dy} + \widetilde{A}(-\lambda)\right)\Omega \right\rangle.$$

This calculation motivates the following definition.

Definition 2.11. The *adjoint problem* to the spectral problem (2.2) is

$$\frac{\partial \Omega}{\partial y} = \widetilde{\mathcal{A}}(-\lambda)\Omega, \tag{2.22a}$$

$$\omega_3(-1) = 0 = \omega_2(1) = \omega_3(1). \tag{2.22b}$$

Proposition 2.12. *Let $\Omega(y)$ be the unique solution of (2.22a) which satisfies the boundary conditions at the right endpoint, $\omega_2(1) = \omega_3(1) = 0$, together with $\omega_1(1) = 1$ (for normalization). Then, at the left endpoint $y = -1$, we have*

$$\Omega(-1) = \widetilde{S}(-\lambda)^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} = JS(\lambda)^T J\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}S_{33}(\lambda)\\-S_{32}(\lambda)\\S_{31}(\lambda)\end{pmatrix}. \tag{2.23}$$

Proof. Since (2.22a) agrees with the twin ODE (2.3a) except for the sign of λ, the twin transition matrix with λ negated, $\widetilde{S}(-\lambda)$, will relate boundary values of (2.22a) at $y = -1$ to boundary values at $y = +1$: $(1,0,0)^T = \Omega(1) = \widetilde{S}(-\lambda)\Omega(-1)$. The rest follows from Corollary 2.9. □

Corollary 2.13. *The adjoint problem (2.22) has the same spectrum as (2.2).*

Proof. By (2.23), the remaining boundary condition $\omega_3(-1) = 0$ for (2.22) is equivalent to $S_{31}(\lambda) = 0$, which, as we saw in the previous section, is also the condition for λ to be an eigenvalue of (2.2). □

We define Weyl functions for the adjoint problem as follows, using the entries from the third row of $S(\lambda)$ appearing in (2.23):

$$W^*(\lambda) = -\frac{S_{32}(\lambda)}{S_{31}(\lambda)}, \qquad Z^*(\lambda) = -\frac{S_{33}(\lambda)}{S_{31}(\lambda)}. \tag{2.24}$$

To complete the picture, we note that there is of course also an adjoint problem for the twin spectral problem (2.3), namely

$$\frac{\partial \Omega}{\partial y} = \mathcal{A}(-\lambda)\Omega, \tag{2.25a}$$

$$\omega_3(-1) = 0 = \omega_2(1) = \omega_3(1). \tag{2.25b}$$

A similar calculation as above shows that the eigenvalues are given by the zeros of $\widetilde{S}_{31}(\lambda)$, and hence they are the same as for (2.3). We define the twin adjoint Weyl functions as

$$\widetilde{W}^*(\lambda) = -\frac{\widetilde{S}_{32}(\lambda)}{\widetilde{S}_{31}(\lambda)}, \qquad \widetilde{Z}^*(\lambda) = -\frac{\widetilde{S}_{33}(\lambda)}{\widetilde{S}_{31}(\lambda)}. \tag{2.26}$$

Theorem 2.14. *The adjoint Weyl functions satisfy the relation*

$$Z^*(\lambda) + W^*(\lambda)\widetilde{W}^*(-\lambda) + \widetilde{Z}^*(-\lambda) = 0. \tag{2.27}$$

Proof. Since a matrix commutes with its inverse, we can equally well multiply the factors in (2.15) in the opposite order: $\widetilde{S}(-\lambda) \cdot JS(\lambda)^T J = I$. Division of the $(3,1)$ entry in this identity by $-S_{31}(\lambda)\widetilde{S}_{31}(-\lambda)$ gives the result. □

3 The discrete case

We now turn to the discrete case (1.12), when $m(x)$ and $n(x)$ are discrete measures (linear combinations of Dirac deltas) with disjoint supports. More specifically, we will study the *interlacing* discrete case where there are $N = 2K$ sites numbered in ascending order,

$$x_1 < x_2 < \cdots < x_{2K},$$

with the measure m supported on the odd-numbered sites x_{2a-1}, and the measure n supported on the even-numbered sites x_{2a}. That is, we take $m_2 = m_4 = \cdots = 0$ and $n_1 = n_3 = \cdots = 0$, so that

$$\begin{aligned} m &= 2\sum_{k=1}^{N} m_k\, \delta_{x_k} = 2\sum_{a=1}^{K} m_{2a-1}\, \delta_{x_{2a-1}}, \\ n &= 2\sum_{k=1}^{N} n_k\, \delta_{x_k} = 2\sum_{a=1}^{K} n_{2a}\, \delta_{x_{2a}}. \end{aligned} \tag{3.1}$$

We will also assume that the nonzero m_k and n_k are *positive*; this will be needed in order to prove that the eigenvalues $\lambda = -z^2$ are positive. The setup is illustrated in Figure 1.

Given such a configuration, consisting of the $4K$ numbers $\{x_k, m_{2a-1}, n_{2a}\}$, we are going to define a set of $4K$ *spectral variables*, consisting of $2K-1$ *eigenvalues* $\lambda_1, \dots, \lambda_K$ and $\mu_1, \dots, \mu_{K-1}$, together with $2K+1$ *residues* $a_1, \dots, a_K$, $b_1, \dots, b_{K-1}$, b_∞ and b_∞^*. In Section 4 we will show that this correspondence is a bijection onto the set of spectral variables with simple positive ordered eigenvalues and positive residues (Theorem 4.8), and give explicit formulas for the inverse map (Corollary 4.5).

Remark 3.1. Non-interlacing cases can be reduced to the interlacing case by introducing auxiliary weights at additional sites so that the problem becomes interlacing, and then letting these weights tend to zero in the solution of the interlacing inverse problem; the details will be published in another paper.

Remark 3.2. The case $K = 1$ is somewhat degenerate, and also rather trivial. It is dealt with separately in Section 4.3. In what follows, we will (mostly without comment) assume that $K \geq 2$ whenever that is needed in order to make sense of the formulas.

Under the transformation (2.1), the discrete measures m and n on $\mathbf{R}$ are mapped into discrete measures g and h, respectively, supported at the points

$$y_k = \tanh x_k \tag{3.2}$$

in the finite interval $(-1, 1)$. The formulas $g(y) = m(x) \cosh^3 x$ and $h(y) = n(x) \cosh^3 x$ from (2.1) should be interpreted using the relation $\delta(x - x_k) dx = \delta(y - y_k) dy$, leading to $\delta_{x_k}(x) = \delta_{y_k}(y) \frac{dy}{dx}(x_k) = \delta_{y_k}(y)/\cosh^2 x_k$. Since we will be working a lot with these measures, it will be convenient to change the numbering a little, and call the weights $g_1, g_2, \dots, g_K$ and $h_1, h_2, \dots, h_K$ rather than $g_1, g_3, \dots, g_{2K-1}$ and $h_2, h_4, \dots, h_{2K}$; see Figure 2. With this numbering, we get

$$g = \sum_{a=1}^{K} g_a\, \delta_{y_{2a-1}}, \qquad h = \sum_{a=1}^{K} h_a\, \delta_{y_{2a}}, \tag{3.3}$$

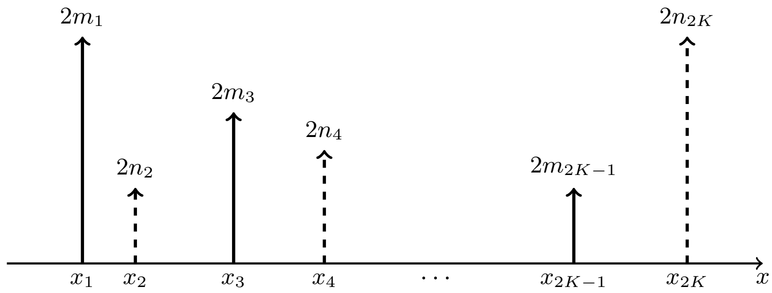

Figure 1. Notation for the measures m and n on the real line $\mathbf{R}$ in the interlacing discrete case (3.1).

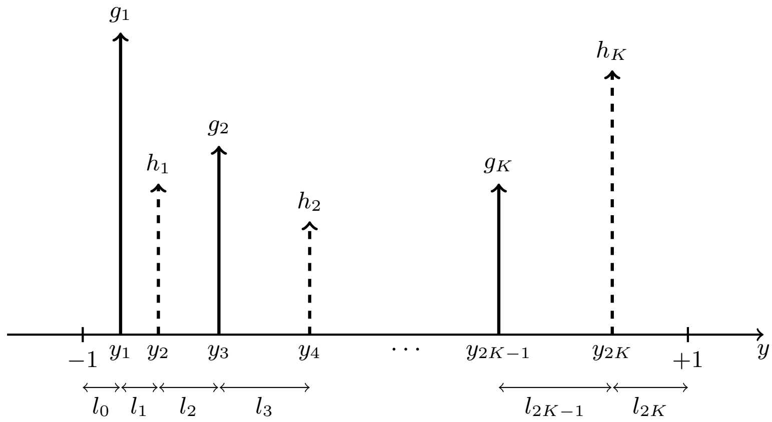

Figure 2. Notation for the measures g and h on the finite interval $(-1, 1)$ in the interlacing discrete case (3.3).

where

$$g_a = 2m_{2a-1} \cosh x_{2a-1}, \qquad h_a = 2n_{2a} \cosh x_{2a}. \tag{3.4}$$

3.1 The first spectral problem

The ODE (2.2a), $\partial_y \Phi = \mathcal{A}(y; \lambda)\Phi$, is easily solved explicitly in the discrete case. Since g and h are zero between the points y_k, the ODE reduces to

$$\frac{\partial}{\partial y} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

in those intervals; that is, ϕ_1 and ϕ_2 are constant in each interval $y_k < y < y_{k+1}$ (for $0 \le k \le 2K$, where we let $y_0 = -1$ and $y_{2K+1} = +1$), while ϕ_3 is piecewise a polynomial in y of degree one. The total change in the value of ϕ_3 over the interval is given by the product of the length of the interval, denoted

$$l_k = y_{k+1} - y_k, \tag{3.5}$$

and the slope of the graph of ϕ_3; this slope is $-\lambda$ times the constant value of ϕ_1 in the interval. In other words:

$$\Phi(y_{k+1}^-) = L_k(\lambda)\Phi(y_k^+), \tag{3.6}$$

where the propagation matrix L_k is defined by

$$L_k(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda l_k & 0 & 1 \end{pmatrix}. \tag{3.7}$$

At the points y_k, the ODE forces the derivative $\partial_y \Phi$ to contain a Dirac delta, and this imposes jump conditions on Φ. These jump conditions will be of different type depending on whether k is even or odd, since that affects whether the Dirac delta is encountered in entry $(1,2)$ or $(2,3)$ in the coefficient matrix

$$\mathcal{A}(y;\lambda) = \begin{pmatrix} 0 & h(y) & 0 \\ 0 & 0 & g(y) \\ -\lambda & 0 & 0 \end{pmatrix}.$$

More precisely, when $k = 2a$ is even, we get a jump condition of the form

$$\Phi(y_k^+) - \Phi(y_k^-) = \begin{pmatrix} 0 & h_a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi(y_k).$$

This implies that ϕ_2 and ϕ_3 don't jump at even-numbered y_k, and the continuity of ϕ_2 in particular implies that the jump in ϕ_1 has a well-defined value $h_a \phi_2(y_{2a})$.

When $k = 2a-1$ is odd, the condition is

$$\Phi(y_k^+) - \Phi(y_k^-) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g_a \\ 0 & 0 & 0 \end{pmatrix} \Phi(y_k).$$

Thus, ϕ_1 and ϕ_3 are continuous at odd-numbered y_k, and the jump in ϕ_2 has a well-defined value $g_a \phi_3(y_{2a-1})$.

This step-by-step construction of $\Phi(y)$ is illustrated in Figure 3 when $\Phi(-1) = (1,0,0)^T$; as we have already seen, this particular case is of interest in connection with the spectral problem (2.2) where we have boundary conditions $\phi_2(-1) = \phi_3(-1) = 0 = \phi_3(1)$.

With the notation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 1 & x & \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.8}$$

the jump conditions take the form

$$\Phi(y_{2a}^+) = \begin{bmatrix} h_a \\ 0 \end{bmatrix} \Phi(y_{2a}^-), \qquad \Phi(y_{2a-1}^+) = \begin{bmatrix} 0 \\ g_a \end{bmatrix} \Phi(y_{2a-1}^-).$$

(For the purposes of this paper, the top right entry of $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ might as well have been set equal to zero; we have defined it as $\frac{1}{2}xy$ only to make $\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right]$ agree with

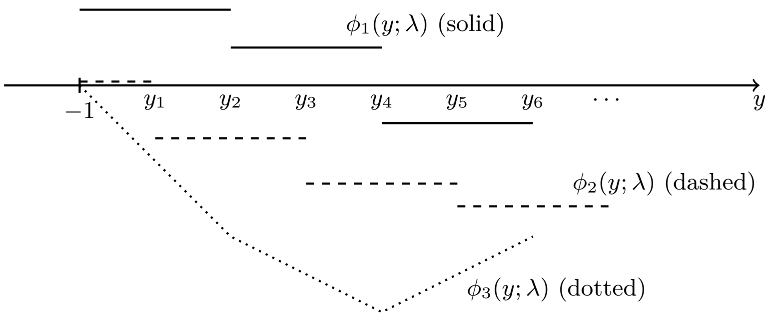

Figure 3. Structure of the solution to the initial value problem $\partial_y \Phi = \mathcal{A}(y;\lambda)\Phi$ with $\Phi(-1;\lambda) = (1,0,0)^T$, in the discrete interlacing case. The components ϕ_1 and ϕ_2 are piecewise constant, while ϕ_3 is continuous and piecewise linear, with slope equal to $-\lambda$ times the value of ϕ_1. At the odd-numbered sites y_{2a-1}, the value of ϕ_2 jumps by $g_a\phi_3(y_{2a-1})$. At the even-numbered sites y_{2a}, the value of ϕ_1 jumps by $h_a\phi_2(y_{2a})$. The parameter λ is an eigenvalue of the spectral problem (2.2) iff it is a zero of $\phi_3(1;\lambda)$, which is a polynomial in λ of degree $K+1$, with constant term zero. This picture illustrates a case where λ and the weights g_a and h_a are all positive.

a jump matrix appearing in our earlier work [23, 17].) We can thus write the transition matrix $S(\lambda)$ as a product of $1+4K$ factors,

$$S(\lambda) = L_{2K}(\lambda)\begin{bmatrix} h_K \\ 0 \end{bmatrix} L_{2K-1}(\lambda)\begin{bmatrix} 0 \\ g_K \end{bmatrix} L_{2K-2}(\lambda)\cdots\begin{bmatrix} h_1 \\ 0 \end{bmatrix} L_1(\lambda)\begin{bmatrix} 0 \\ g_1 \end{bmatrix} L_0(\lambda). \tag{3.9}$$

For later use in connection with the inverse problem, we also consider the partial products $T_j(\lambda)$ containing the leftmost $1+4j$ factors (for $j=0,\dots,K$); put differently, $T_{K-j}(\lambda)$ is obtained by omitting all factors after $L_{2j}(\lambda)$ in the product for $S(\lambda)$:

$$T_{K-j}(\lambda) = L_{2K}(\lambda)\cdots\begin{bmatrix} h_{j+1} \\ 0 \end{bmatrix} L_{2j+1}(\lambda)\begin{bmatrix} 0 \\ g_{j+1} \end{bmatrix} L_{2j}(\lambda). \tag{3.10}$$

Thus $S(\lambda) = T_K(\lambda)$, and $T_{K-j}(\lambda)$ depends on $(g_{j+1},\dots,g_K)$, $(h_{j+1},\dots,h_K)$, $(l_{2j},\dots,l_{2K})$.

Proposition 3.3. *The entries of $T_j(\lambda)$ are polynomials in λ, with degrees as follows:*

$$\deg T_j(\lambda) = \begin{pmatrix} j & j-1 & j-1 \\ j & j-1 & j-1 \\ j+1 & j & j \end{pmatrix} \qquad (j\ge 1). \tag{3.11}$$

The constant term in each entry is given by

$$T_{K-j}(0) = \begin{bmatrix} h_K \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ g_K \end{bmatrix}\cdots\begin{bmatrix} h_{j+1} \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ g_{j+1} \end{bmatrix} = \begin{pmatrix} 1 & \sum_{a>j} h_a & \sum_{a\ge b>j} h_a g_b \\ 0 & 1 & \sum_{a>j} g_a \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.12}$$

For those entries whose constant term is zero, the coefficient of λ^1 *is given by*

$$\frac{dT_{K-j}}{d\lambda}(0) = \begin{pmatrix} * & * & * \\ -\sum_{a>j}\sum_{k=2j}^{2a-2} g_a l_k & * & * \\ -\sum_{k=2j}^{2K} l_k & -\sum_{a>j}\sum_{k=2a}^{2K} h_a l_k & * \end{pmatrix}. \tag{3.13}$$

The highest coefficient in the $(3,1)$ *entry is given by*

$$(T_j)_{31}(\lambda) = (-\lambda)^{j+1}\left(\prod_{m=K-j}^{K} l_{2m}\right)\left(\prod_{a=K+1-j}^{K} g_a h_a\right) + \cdots. \tag{3.14}$$

Proof. Equation (3.12) follows at once from setting $\lambda = 0$ in (3.10). Next, group the factors in fours (except for the lone first factor $L_{2K}(\lambda)$) so that (3.10) takes the form $T_{K-j} = L_{2K}\, t_K\, t_{K-1} \cdots t_{j+1}$, where

$$t_a(\lambda) = \begin{bmatrix} h_a \\ 0 \end{bmatrix} L_{2a-1}(\lambda) \begin{bmatrix} 0 \\ g_a \end{bmatrix} L_{2a-2}(\lambda) = \begin{pmatrix} 1 & h_a & h_a g_a \\ 0 & 1 & g_a \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} l_{2a-2} h_a g_a \\ l_{2a-2} g_a \\ l_{2a-1} + l_{2a-2} \end{pmatrix} (1,0,0).$$

The degree count (3.11) follows easily by considering the highest power of λ arising from multiplying these factors, and (3.14) also falls out of this. Differentiating $T_{K-j+1}(\lambda) = T_{K-j}(\lambda)\, t_j(\lambda)$ and letting $\lambda = 0$ gives

$$\frac{dT_{K-j+1}}{d\lambda}(0) = \frac{dT_{K-j}}{d\lambda}(0) \begin{pmatrix} 1 & h_j & h_j g_j \\ 0 & 1 & g_j \\ 0 & 0 & 1 \end{pmatrix} - T_{K-j}(0) \begin{pmatrix} l_{2j-2} h_j g_j & 0 & 0 \\ l_{2j-2} g_j & 0 & 0 \\ l_{2j-1} + l_{2j-2} & 0 & 0 \end{pmatrix}. \tag{3.15}$$

With the help of (3.12) one sees that the $(3,1)$ entry of this equality reads

$$(T'_{K-j+1})_{31}(0) = (T'_{K-j})_{31}(0) - (l_{2j-1} + l_{2j-2}), \tag{3.16}$$

the $(3,2)$ entry is

$$(T'_{K-j+1})_{32}(0) = (T'_{K-j})_{32}(0) + h_j (T_{K-j})'_{31}(0), \tag{3.17}$$

and the $(2,1)$ entry is

$$(T'_{K-j+1})_{21}(0) = (T'_{K-j})_{21}(0) - l_{2j-2} g_j - \left(\sum_{a>j} g_a\right)(l_{2j-1} + l_{2j-2}). \tag{3.18}$$

Solving these recurrences, with the inital conditions coming from $T'_0(0) = L'_{2K}(0)$ (i.e., $-l_{2K}$ in the $(3,1)$ position, zero elsewhere), gives equation (3.13). □

We also state the result for the important special case $S(\lambda) = T_K(\lambda)$. (In the $(3,1)$ entry, $\sum_{k=0}^{2K} l_k = 2$ is the length of the whole interval $[-1,1]$.)

Corollary 3.4. *The entries of* $S(\lambda)$ *are polynomials in* λ*, with*

$$\deg S(\lambda) = \begin{pmatrix} K & K-1 & K-1 \\ K & K-1 & K-1 \\ K+1 & K & K \end{pmatrix}, \tag{3.19}$$

$$S(0) = \begin{pmatrix} 1 & \sum_a h_a & \sum_{a\geq b} h_a g_b \\ 0 & 1 & \sum_a g_a \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.20}$$

$$S'(0) = \begin{pmatrix} * & * & * \\ -\sum_{a=1}^{K}\sum_{k=0}^{2a-2} g_a l_k & * & * \\ -\sum_{k=0}^{2K} l_k & -\sum_{a=1}^{K}\sum_{k=2a}^{2K} h_a l_k & * \end{pmatrix}, \tag{3.21}$$

$$S_{31}(\lambda) = (-\lambda)^{K+1}\left(\prod_{m=0}^{K} l_{2m}\right)\left(\prod_{a=1}^{K} g_a h_a\right) + \cdots. \tag{3.22}$$

Remark 3.5. The ideas that we use go back to Stieltjes's memoir on continued fractions [29] and its relation to an inhomogeneous string problem, especially its inverse problem, discovered by Krein in the 1950s. A comprehensive account of the inverse string problem can be found in [11], especially Section 5.9. The connection to Stieltjes continued fractions is explained in [13, Supplement II] and in [1]. Briefly stated, if $\phi(y;\lambda)$ satisfies the string equation

$$-\phi_{yy} = \lambda g(y)\phi, \qquad -1 < y < 1, \qquad \phi(-1;\lambda) = 0,$$

with a discrete mass distribution $g(y) = \sum_{j=1}^{n} g_j \delta_{y_j}$, then the Weyl function $W(\lambda) = \frac{\phi_y(1;\lambda)}{\phi(1;\lambda)}$ admits the continued fraction expansion

$$W(z) = \cfrac{1}{l_n + \cfrac{1}{-zg_n + \cfrac{1}{l_{n-1} + \cfrac{1}{\ddots + \cfrac{1}{-zg_2 + \cfrac{1}{l_1 + \cfrac{1}{-zg_1 + \cfrac{1}{l_0}}}}}}}}$$

(where $l_j = y_{j+1} - y_j$), whose convergents (Padé approximants) $T_{2j}(\lambda) = \frac{P_{2j}(\lambda)}{Q_{2j}(\lambda)}$ satisfy

$$P_{2j}(\lambda) = (-1)^j g_n \left(\prod_{k=n-j+1}^{n-1} l_k g_k\right)\lambda^j + \cdots,$$

$$Q_{2j}(\lambda) = (-1)^j \left(\prod_{k=n-j+1}^{n} l_k g_k\right)\lambda^j + \cdots.$$

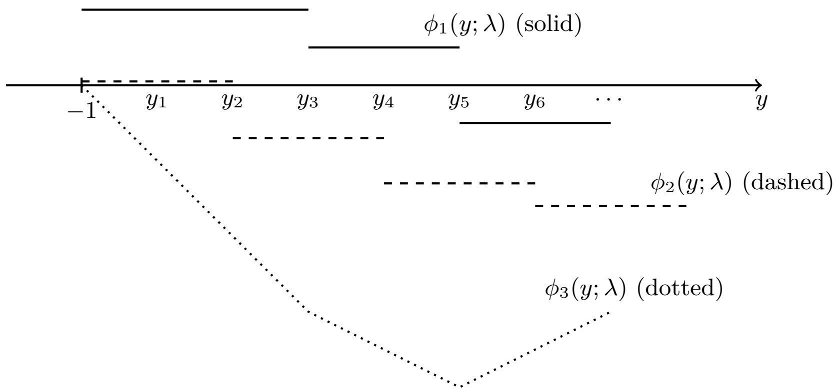

Figure 4. Structure of the solution to the twin problem $\partial_y \Phi = \widetilde{\mathcal{A}}(y;\lambda)\Phi$ with $\Phi(-1;\lambda) = (1,0,0)^T$, in the discrete interlacing case. The differences compared to Figure 3 are the following: At the odd-numbered sites y_{2a-1}, the value of ϕ_1 (not ϕ_2) jumps by $g_a \phi_2(y_{2a-1})$. At the even-numbered sites y_{2a}, the value of ϕ_2 (not ϕ_1) jumps by $h_a \phi_3(y_{2a})$. The parameter λ is an eigenvalue of the twin spectral problem (2.3) iff it is a zero of $\phi_3(1;\lambda)$, which is a polynomial in λ of degree K (not $K+1$), with constant term zero. Note that the first mass g_1 has no influence here. (Indeed, since $\phi_2(y_1;\lambda) = 0$, there is no jump in ϕ_1 at $y = y_1$, regardless of the value of g_1.)

3.2 The second spectral problem

For the twin ODE (2.3a), $\partial_y \Phi = \widetilde{\mathcal{A}}(y;\lambda)\Phi$, where the measures g and h are swapped, the construction is similar. The only difference is that the weights g_a at the odd-numbered sites will occur in the type of jump condition that we previously had for the weights h_a at the even-numbered sites (and vice versa). Thus, the transition matrix is in this case

$$\widetilde{S}(\lambda) = L_{2K}(\lambda) \begin{bmatrix} 0 \\ h_K \end{bmatrix} L_{2K-1}(\lambda) \begin{bmatrix} g_K \\ 0 \end{bmatrix} L_{2K-2}(\lambda) \cdots \begin{bmatrix} 0 \\ h_1 \end{bmatrix} L_1(\lambda) \begin{bmatrix} g_1 \\ 0 \end{bmatrix} L_0(\lambda). \quad (3.23)$$

This solution is illustrated in Figure 4 for the initial condition $\Phi(-1) = (1,0,0)^T$. It is clear that it behaves a bit differently, since the first weight g_1 has no influence on this solution Φ, and therefore not on the second spectrum either. (The first column in $\left[\begin{smallmatrix} g_1 \\ 0 \end{smallmatrix}\right] L_0(\lambda)$ does not depend on g_1.)

Let $\widetilde{T}_j(\lambda)$ be the partial product containing the first $1+4j$ factors in the product for $\widetilde{S}(\lambda)$; in other words,

$$\widetilde{T}_{K-j}(\lambda) = L_{2K}(\lambda) \cdots \begin{bmatrix} 0 \\ h_{j+1} \end{bmatrix} L_{2j+1}(\lambda) \begin{bmatrix} g_{j+1} \\ 0 \end{bmatrix} L_{2j}(\lambda). \quad (3.24)$$

Proposition 3.6. *The entries of $\widetilde{T}_j(\lambda)$ are polynomials in λ, satisfying*

$$\widetilde{T}_1(\lambda) = \begin{pmatrix} 1 & g_K & 0 \\ -\lambda h_K (l_{2K-1} + l_{2K-2}) & 1 - \lambda h_K g_K l_{2K-1} & h_K \\ -\lambda (l_{2K} + l_{2K-1} + l_{2K-2}) & -\lambda g_K (l_{2K} + l_{2K-1}) & 1 \end{pmatrix}, \quad (3.25)$$

$$\deg \widetilde{T}_j(\lambda) = \begin{pmatrix} j-1 & j-1 & j-2 \\ j & j & j-1 \\ j & j & j-1 \end{pmatrix} \qquad (j \geq 2), \quad (3.26)$$

$$\widetilde{T}_{K-j}(0) = \begin{bmatrix} 0 \\ h_K \end{bmatrix} \begin{bmatrix} g_K \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ h_{j+1} \end{bmatrix} \begin{bmatrix} g_{j+1} \\ 0 \end{bmatrix} = \begin{pmatrix} 1 & \sum_{a>j} g_a & \sum_{a>b>j} g_a h_b \\ 0 & 1 & \sum_{a>j} h_a \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.27}$$

$$\frac{d\widetilde{T}_{K-j}}{d\lambda}(0) = \begin{pmatrix} * & * & * \\ -\sum_{a>j} \sum_{k=2j}^{2a-1} h_a l_k & * & * \\ -\sum_{k=2j}^{2K} l_k & -\sum_{a>j} \sum_{k=2a-1}^{2K} g_a l_k & * \end{pmatrix}. \tag{3.28}$$

For $0 \le j \le K-2$, the highest coefficients in the $(2,1)$ and $(3,1)$ entries are given by

$$(\widetilde{T}_{K-j})_{21}(\lambda) = (-\lambda)^{K-j} \left(\prod_{m=j+2}^{K} l_{2m-1} \right) (l_{2j+1} + l_{2j})\, h_K \left(\prod_{a=j}^{K-1} g_{a+1} h_a \right) + \cdots, \tag{3.29}$$

$$(\widetilde{T}_{K-j})_{31}(\lambda) = (-\lambda)^{K-j} (l_{2K} + l_{2K-1}) \left(\prod_{m=j+2}^{K-1} l_{2m-1} \right) (l_{2j+1} + l_{2j}) \left(\prod_{a=1}^{K-1} g_{a+1} h_a \right) + \cdots, \tag{3.30}$$

where $\prod_{m=K}^{K-1} = 1$. Moreover, $\widetilde{T}_j$ and T_j are related by the involution σ (see Definition 2.4):

$$\widetilde{T}_j(\lambda) = T_j(\lambda)^\sigma. \tag{3.31}$$

Proof. The degree count and the coefficients are obtained like in the proof of Proposition 3.3, although the details are a bit more involved in this case. (Group the factors in $\widetilde{T}_j$ as follows: $L_{2K}(\lambda)$ times a pair of factors, times a number a quadruples of the same form as $t_a(\lambda)$ in the proof of Proposition 3.3 but with h_a and g_a replaced by g_{a+1} and h_a respectively, times a final pair at the end.)

The σ-relation (3.31) can be seen as yet another manifestation of Theorem 2.8, and (since σ is a group homomorphism) it also follows directly from the easily verified formulas $L_k(\lambda)^\sigma = L_k(\lambda)$ and $\begin{bmatrix} x \\ y \end{bmatrix}^\sigma = \begin{bmatrix} y \\ x \end{bmatrix}$. ☐

We record the results in particular for the case $\widetilde{S}(\lambda) = \widetilde{T}_K(\lambda)$:

Corollary 3.7. *The entries of $\widetilde{S}(\lambda)$ are polynomials in λ, satisfying*

$$\deg \widetilde{S}(\lambda) = \begin{pmatrix} K-1 & K-1 & K-2 \\ K & K & K-1 \\ K & K & K-1 \end{pmatrix}, \tag{3.32}$$

$$\widetilde{S}(0) = \begin{pmatrix} 1 & \sum_a g_a & \sum_{a>b} g_a h_b \\ 0 & 1 & \sum_a h_a \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.33}$$

$$\widetilde{S}'(0) = \begin{pmatrix} * & * & * \\ -\sum_{a=1}^{K}\sum_{k=0}^{2a-1} h_a l_k & * & * \\ -2 & -\sum_{a=1}^{K}\sum_{k=2a-1}^{2K} g_a l_k & * \end{pmatrix}. \tag{3.34}$$

(The interpretation of (3.32) when $K = 1$ is that the $(1,3)$ entry is the zero polynomial.) The leading terms of $\widetilde{S}_{21}(\lambda)$ and $\widetilde{S}_{31}(\lambda)$ are given by

$$\widetilde{S}_{21}(\lambda) = (-\lambda)^K \left(\prod_{m=2}^{K} l_{2m-1}\right)(l_0 + l_1)\, h_K \left(\prod_{a=1}^{K-1} g_{a+1}h_a\right) + \cdots, \tag{3.35}$$

$$\widetilde{S}_{31}(\lambda) = (-\lambda)^K (l_{2K} + l_{2K-1}) \left(\prod_{m=2}^{K-1} l_{2m-1}\right)(l_0 + l_1) \left(\prod_{a=1}^{K-1} g_{a+1}h_a\right) + \cdots, \tag{3.36}$$

with the exception of the case $K = 1$ where we simply have $\widetilde{S}_{31}(\lambda) = -2\lambda$. (The empty product $\prod_{m=2}^{1} l_{2m-1}$ is omitted from $\widetilde{S}_{31}$ in the case $K = 2$, and from $\widetilde{S}_{21}$ in the case $K = 1$.)

3.3 Weyl functions and spectral measures

Since the entries of $S(\lambda)$ are polynomials, the Weyl functions $W = -S_{21}/S_{31}$ and $Z = -S_{11}/S_{31}$ are rational functions in the discrete case. They have poles at the eigenvalues of the spectral problem (2.2). Likewise, the twin Weyl functions $\widetilde{W} = -\widetilde{S}_{21}/\widetilde{S}_{31}$, $\widetilde{Z} = -\widetilde{S}_{11}/\widetilde{S}_{31}$ are rational functions, with poles at the eigenvalues of the twin spectral problem (2.3).

Theorem 3.8. *If all g_k and h_k are positive, then both spectra are nonnegative and simple. The eigenvalues of (2.2) and (2.3) will be denoted by*

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_K \qquad \text{(zeros of } S_{31}\text{)}, \tag{3.37}$$

$$0 = \mu_0 < \mu_1 < \cdots < \mu_{K-1} \qquad \text{(zeros of } \widetilde{S}_{31}\text{)}. \tag{3.38}$$

Proof. This is proved in the appendix; see Theorem B.1. (It is clear that if the zeros of the polynomials $S_{31}(\lambda)$ and $\widetilde{S}_{31}(\lambda)$ are real, then they can't be negative, since the coefficients in the polynomials have alternating signs and all terms therefore have the same sign if $\lambda < 0$. However, it's far from obvious that the zeros are real, much less simple. These facts follow from properties of oscillatory matrices, belonging to the beautiful theory of oscillatory kernels due to Gantmacher and Krein; see [13, Ch. II].) □

Remark 3.9. Taking Propositions 3.4 and 3.7 into account, we can thus write

$$S_{31}(\lambda) = -2\lambda \prod_{i=1}^{K}\left(1 - \frac{\lambda}{\lambda_i}\right), \qquad \widetilde{S}_{31}(\lambda) = -2\lambda \prod_{j=1}^{K-1}\left(1 - \frac{\lambda}{\mu_j}\right). \tag{3.39}$$

Theorem 3.10. *If all g_k and h_k are positive, then the Weyl functions have partial fraction decompositions*

$$W(\lambda) = \sum_{i=1}^{K} \frac{a_i}{\lambda - \lambda_i}, \tag{3.40a}$$

$$\widetilde{W}(\lambda) = -b_\infty + \sum_{j=1}^{K-1} \frac{b_j}{\lambda - \mu_j}, \tag{3.40b}$$

$$Z(\lambda) = \frac{1}{2\lambda} + \sum_{i=1}^{K} \frac{c_i}{\lambda - \lambda_i}, \tag{3.40c}$$

$$\widetilde{Z}(\lambda) = \frac{1}{2\lambda} + \sum_{j=1}^{K-1} \frac{d_j}{\lambda - \mu_j}, \tag{3.40d}$$

where a_i, b_j, b_∞, c_i, d_j are positive, and where W and $\widetilde{W}$ determine Z and $\widetilde{Z}$ through the relations

$$c_i = a_i b_\infty + \sum_{j=1}^{K-1} \frac{a_i b_j}{\lambda_i + \mu_j}, \qquad d_j = \sum_{i=1}^{K} \frac{a_i b_j}{\lambda_i + \mu_j}. \tag{3.41}$$

Proof. The form of the decompositions follows from Propositions 3.4 and 3.7 (polynomial degrees), together with Theorem 3.8 (all poles are simple). In $W = -S_{21}/S_{31}$ the factor λ cancels, so there is no residue at $\lambda = 0$, and similarly for $\widetilde{W} = -\widetilde{S}_{21}/\widetilde{S}_{31}$ (which however is different from W in that the degree of the numerator equals the degree of the denominator; hence the constant term $-b_\infty$). The residue of $Z = -S_{11}/S_{31}$ at $\lambda = 0$ is $-S_{11}(0)/S_{31}'(0) = 1/2$ by Corollary 3.4, and similarly for $\widetilde{Z}(\lambda)$.

From the expressions (3.35) and (3.36) for the highest coefficient of $\widetilde{S}_{21}$ and $\widetilde{S}_{31}$ we obtain (for $K \geq 2$)

$$b_\infty = -\lim_{\lambda\to\infty} \widetilde{W}(\lambda) = \lim_{\lambda\to\infty} \frac{\widetilde{S}_{21}(\lambda)}{\widetilde{S}_{31}(\lambda)} = \frac{h_K l_{2K-1}}{l_{2K} + l_{2K-1}}, \tag{3.42}$$

which shows that $b_\infty > 0$. (In the exceptional case $K = 1$ we have instead $-\widetilde{W}(\lambda) = \frac{1}{2} h_1 (l_2 + l_1) = b_\infty > 0$.)

The proof that a_i and b_j are positive will be given at the end of Section 3.5. It will then follow from (3.41) that c_i and d_j are positive as well.

To prove (3.41), recall the relation $Z(\lambda) + W(\lambda)\widetilde{W}(-\lambda) + \widetilde{Z}(-\lambda) = 0$ from Theorem 2.10. Taking the residue at $\lambda = \lambda_i$ on both sides yields

$$c_i + a_i \widetilde{W}(-\lambda_i) + 0 = 0.$$

Taking instead the residue at $\lambda = \mu_j$ in $Z(-\lambda) + W(-\lambda)\widetilde{W}(\lambda) + \widetilde{Z}(\lambda) = 0$, we obtain

$$0 + W(-\mu_j) b_j + d_j = 0.$$

□

Definition 3.11 (Spectral measures)**.** Let α and β denote the discrete measures

$$\alpha = \sum_{i=1}^{K} a_i \delta_{\lambda_i}, \qquad \beta = \sum_{j=1}^{K-1} b_j \delta_{\mu_j}, \tag{3.43}$$

where a_i and b_j are the residues in $W(\lambda)$ and $\widetilde{W}(\lambda)$ from (3.40a) and (3.40b).

We can write W and $\widetilde{W}$ in terms of these spectral measures α and β, and likewise for Z and $\widetilde{Z}$ if we use (3.41):

$$W(\lambda) = \int \frac{d\alpha(x)}{\lambda - x}, \tag{3.44a}$$

$$\widetilde{W}(\lambda) = \int \frac{d\beta(y)}{\lambda - y} - b_\infty, \tag{3.44b}$$

$$Z(\lambda) = \frac{1}{2\lambda} + \iint \frac{d\alpha(x) d\beta(y)}{(\lambda - x)(x + y)} + b_\infty W(\lambda), \tag{3.44c}$$

$$\widetilde{Z}(\lambda) = \frac{1}{2\lambda} + \iint \frac{d\alpha(x) d\beta(y)}{(x + y)(\lambda - y)}. \tag{3.44d}$$

(Note the appearance here of the Cauchy kernel $1/(x + y)$.)

We have now completed the spectral characterization of the boundary value problems (2.2a) and (2.3a). The remainder of Section 3 is devoted to establishing some basic facts which will be needed for formulating and solving the inverse problem in Section 4.

3.4 Rational approximations to the Weyl functions

The Weyl functions $W(\lambda)$ and $Z(\lambda)$ are defined using entries of the transition matrix $S(\lambda)$. Next, we will see how entries of the matrices $T_j(\lambda)$ (partial products of $S(\lambda)$; see (3.10)) produce rational approximations to the Weyl functions. We have chosen here to work with the second column of $T_j(\lambda)$, since it seems to be the most convenient for the inverse problem, but this choice is by no means unique; many other similar approximation results could be derived.

Theorem 3.12. *Fix some j with $1 \le j \le K$, write $T(\lambda) = T_j(\lambda)$ for simplicity, and consider the polynomials*

$$Q(\lambda) = -T_{32}(\lambda), \quad P(\lambda) = T_{22}(\lambda), \quad R(\lambda) = T_{12}(\lambda). \tag{3.45}$$

Then the following properties hold:

$$\deg Q = j, \qquad \deg P = j - 1, \qquad \deg R = j - 1, \tag{3.46}$$

$$Q(0) = 0, \qquad P(0) = 1, \tag{3.47}$$

and, as $\lambda \to \infty$,

$$W(\lambda)Q(\lambda) - P(\lambda) = \mathcal{O}\left(\frac{1}{\lambda}\right), \tag{3.48a}$$

$$Z(\lambda)Q(\lambda) - R(\lambda) = \mathcal{O}\left(\frac{1}{\lambda}\right), \tag{3.48b}$$

$$R(\lambda) + P(\lambda)\widetilde{W}(-\lambda) + Q(\lambda)\widetilde{Z}(-\lambda) = \mathcal{O}\left(\frac{1}{\lambda^j}\right). \tag{3.48c}$$

(For $j = K$, the right-hand side of (3.48c) can be replaced by zero.)

Proof. Equations (3.46) and (3.47) were already proved in Proposition 3.3. With the notation used in that proof, the first column of the transition matrix $S(\lambda)$ is given by

$$\begin{pmatrix} S_{11}(\lambda) \\ S_{21}(\lambda) \\ S_{31}(\lambda) \end{pmatrix} = \underbrace{L_{2K}(\lambda)\, t_K(\lambda) \cdots t_{K+1-j}(\lambda)}_{=T(\lambda)} \underbrace{t_{K-j}(\lambda) \cdots t_1(\lambda) \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)}_{= \left(\begin{smallmatrix} a_1(\lambda) \\ a_2(\lambda) \\ a_3(\lambda) \end{smallmatrix}\right)},$$

where a_1, a_2, a_3 have degree at most $K-j$ in λ. Hence,

$$\begin{aligned} WQ - P &= -\frac{S_{21}}{S_{31}} (-T_{32}) - T_{22} = \frac{T_{32} S_{21} - T_{22} S_{31}}{S_{31}} \\ &= \frac{T_{32}(T_{21}, T_{22}, T_{23}) \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right) - T_{22}(T_{31}, T_{32}, T_{33}) \left(\begin{smallmatrix} a_1 \\ a_2 \\ a_3 \end{smallmatrix}\right)}{S_{31}} \\ &= \frac{-a_1 \begin{vmatrix} T_{21} & T_{22} \\ T_{31} & T_{32} \end{vmatrix} + a_3 \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix}}{S_{31}} = \frac{-a_1 (T^{-1})_{31} + a_3 (T^{-1})_{11}}{S_{31}}, \end{aligned}$$

where the last step uses that $\det T(\lambda) = 1$ (since each factor in T has determinant one). By (3.31), $T^{-1}(\lambda) = J \widetilde{T}(-\lambda)^T J$, where $\widetilde{T}(\lambda)$ is shorthand for $\widetilde{T}_j(\lambda)$ (defined by (3.24)). In particular, $(T^{-1})_{31}(\lambda) = \widetilde{T}_{31}(-\lambda)$ and $(T^{-1})_{11}(\lambda) = \widetilde{T}_{33}(-\lambda)$, so

$$W(\lambda) Q(\lambda) - P(\lambda) = \frac{-a_1(\lambda)\, \widetilde{T}_{31}(-\lambda) + a_3(\lambda)\, \widetilde{T}_{33}(-\lambda)}{S_{31}(\lambda)}.$$

By (3.19) and (3.26) we have

$$\deg S_{31} = K + 1, \qquad \deg \widetilde{T}_{31} = j, \qquad \deg \widetilde{T}_{33} = j - 1,$$

which shows that $WQ - P = \mathcal{O}\left(\lambda^{(K-j)+j-(K+1)}\right) = \mathcal{O}\left(\lambda^{-1}\right)$ as $\lambda \to \infty$.

The proof that $ZQ - R = \mathcal{O}\left(\lambda^{-1}\right)$ is entirely similar.

To prove (3.48c), we start from

$$\begin{pmatrix} \widetilde{S}_{11}(\lambda) \\ \widetilde{S}_{21}(\lambda) \\ \widetilde{S}_{31}(\lambda) \end{pmatrix} = \widetilde{T}(\lambda) \begin{pmatrix} b_1(\lambda) \\ b_2(\lambda) \\ b_3(\lambda) \end{pmatrix},$$

where b_1, b_2, b_3 have degree at most $K-j$. Using again $\widetilde{T}(\lambda) = T(\lambda)^\sigma = J T(-\lambda)^{-T} J$, we obtain

$$\begin{aligned} -b_2(-\lambda) &= -(0,1,0) J T(\lambda)^T J \begin{pmatrix} \widetilde{S}_{11}(-\lambda) \\ \widetilde{S}_{21}(-\lambda) \\ \widetilde{S}_{31}(-\lambda) \end{pmatrix} \\ &= (0,1,0) T(\lambda)^T \begin{pmatrix} \widetilde{S}_{31}(\lambda) \\ -\widetilde{S}_{21}(\lambda) \\ \widetilde{S}_{11}(-\lambda) \end{pmatrix} \\ &= \big(R(\lambda), P(\lambda), -Q(\lambda)\big) \begin{pmatrix} 1 \\ \widetilde{W}(\lambda) \\ -\widetilde{Z}(-\lambda) \end{pmatrix} \widetilde{S}_{31}(\lambda). \end{aligned}$$

Since $\widetilde{S}_{31}$ has degree K by (3.32), we find that $R(\lambda)+P(\lambda)\widetilde{W}(-\lambda)+Q(\lambda)\widetilde{Z}(-\lambda) = -b_2(-\lambda)/\widetilde{S}_{31}(\lambda) = \mathcal{O}\left(\lambda^{(K-j)-K}\right) = \mathcal{O}\left(\lambda^{-j}\right)$. (When $j = K$ we have $b_2(\lambda) = 0$.) □

Remark 3.13. Loosely speaking, the approximation conditions (3.48) say that

$$\frac{P(\lambda)}{Q(\lambda)} \approx W(\lambda), \quad \frac{R(\lambda)}{Q(\lambda)} \approx Z(\lambda),$$

and moreover these approximate Weyl functions satisfy

$$\tfrac{R}{Q}(\lambda) + \tfrac{P}{Q}(\lambda)\widetilde{W}(-\lambda) + \widetilde{Z}(-\lambda) \approx 0$$

in place of the exact relation

$$Z(\lambda) + W(\lambda)\widetilde{W}(-\lambda) + \widetilde{Z}(-\lambda) = 0$$

from Theorem 2.10. We say that the triple (Q, P, R) provides a Type I Hermite–Padé approximation of the functions W and Z, and simultaneously a Type II Hermite–Padé approximation of the functions $\widetilde{W}$ and $\widetilde{Z}$; see Section 5 in [6].

We will see in Section 4 that for given Weyl functions and a given order of approximation j, the properties in Theorem 3.12 are enough to determine the polynomials Q, P, R uniquely. This is the key to the inverse problem, together with the following simple proposition. We will need to consider Q, P, R for different values of j, and we will write Q_j, P_j, R_j to indicate this. As a somewhat degenerate case not covered by Theorem 3.12 (the degree count (3.46) fails), we have

$$Q_0(\lambda) = 0, \qquad P_0(\lambda) = 1, \qquad R_0(\lambda) = 0, \tag{3.49}$$

coming from the second column of $T_0(\lambda) = L_{2K}(\lambda)$.

Proposition 3.14. *If all $Q_j(\lambda)$ and $R_j(\lambda)$ are known, then the weights h_j and their positions y_{2j} can be determined:*

$$h_j = R_{K-j+1}(0) - R_{K-j}(0), \tag{3.50}$$

$$(1 - y_{2j})h_j = Q'_{K-j+1}(0) - Q'_{K-j}(0), \tag{3.51}$$

for $j = 1, \dots, K$.

Proof. By definition, $Q_j = -(T_j)_{32}$ and $R_j = (T_j)_{12}$, and Proposition 3.3 says that $R_{K-j}(0) = \sum_{a>j} h_a$ and $Q'_{K-j}(0) = \sum_{a>j} \sum_{k=2a}^{2K} h_a l_k$, for $0 \le j \le K-1$. The statement follows. (Note that $\sum_{k=2j}^{2K} l_k = 1 - y_{2j}$.) □

In order to access the weights g_j and their positions y_{2j-1} we will exploit the symmetry of the setup, via the adjoint problem; see Section 3.6.

3.5 Adjoint Weyl functions

Recall the adjoint Weyl functions defined by (2.24) and (2.26),

$$W^* = -S_{32}/S_{31}, \quad Z^* = -S_{33}/S_{31}, \quad \widetilde{W}^* = -\widetilde{S}_{32}/\widetilde{S}_{31}, \quad \widetilde{Z}^* = -\widetilde{S}_{33}/\widetilde{S}_{31},$$

which have the same denominators as the ordinary Weyl functions

$$W = -S_{21}/S_{31}, \quad Z = -S_{11}/S_{31}, \quad \widetilde{W} = -\widetilde{S}_{21}/\widetilde{S}_{31}, \quad Z = -\widetilde{S}_{11}/\widetilde{S}_{31},$$

but different numerators. Since the transition matrices $S(\lambda)$ and $\widetilde{S}(\lambda)$ both have the property that the $(2,1)$ and $(3,2)$ entries have the same degree, and the $(1,1)$ and $(3,3)$ entries have the same degree (see Propositions 3.4 and 3.7), the adjoint Weyl functions will have partial fraction decompositions of exactly the same form as their non-starred counterparts (cf. Theorem 3.10), with the same poles but different residues:

$$W^*(\lambda) = \sum_{i=1}^{K} \frac{a_i^*}{\lambda - \lambda_i}, \tag{3.52a}$$

$$\widetilde{W}^*(\lambda) = -b_\infty^* + \sum_{j=1}^{K-1} \frac{b_j^*}{\lambda - \mu_j}, \tag{3.52b}$$

$$Z^*(\lambda) = \frac{1}{2\lambda} + \sum_{i=1}^{K} \frac{c_i^*}{\lambda - \lambda_i}, \tag{3.52c}$$

$$\widetilde{Z}^*(\lambda) = \frac{1}{2\lambda} + \sum_{j=1}^{K-1} \frac{d_j^*}{\lambda - \mu_j}. \tag{3.52d}$$

Just like in the proof of Theorem 3.10, it follows from Theorem 2.14 that

$$c_i^* = a_i^* b_\infty^* + \sum_{j=1}^{K-1} \frac{a_i^* b_j^*}{\lambda_i + \mu_j}, \qquad d_j^* = \sum_{i=1}^{K} \frac{a_i^* b_j^*}{\lambda_i + \mu_j}, \tag{3.53}$$

so that Z^* and $\widetilde{Z}^*$ are determined by W^* and $\widetilde{W}^*$. Moreover, there is the following connection between the ordinary Weyl functions and their adjoints.

Theorem 3.15. *Assume that $K \geq 2$. The residues of W and W^* satisfy*

$$a_k a_k^* = \frac{\lambda_k \prod_{j=1}^{K-1} \left(1 + \frac{\lambda_k}{\mu_j}\right)}{2 \prod_{\substack{i=1 \\ i \neq k}}^{K} \left(1 - \frac{\lambda_k}{\lambda_i}\right)^2}, \qquad k = 1, \dots, K. \tag{3.54}$$

Likewise, the residues of $\widetilde{W}$ and $\widetilde{W}^*$ satisfy

$$b_k b_k^* = \frac{\mu_k \prod_{i=1}^{K} \left(1 + \frac{\mu_k}{\lambda_i}\right)}{2 \prod_{\substack{j=1 \\ j \neq k}}^{K-1} \left(1 - \frac{\mu_k}{\mu_j}\right)^2}, \qquad k = 1, \dots, K-1. \tag{3.55}$$

(The empty product appearing when $K = 2$ should be omitted; thus, $b_1 b_1^* = \frac{1}{2}\mu_1 \prod_{i=1}^{2}(1 + \mu_1/\lambda_i)$ in this case.) Moreover,

$$b_\infty b_\infty^* = \frac{l_1 l_3 \cdots l_{2K-1}}{l_0 l_2 l_4 \cdots l_{2K}} \times \left(\prod_{j=1}^{K-1} \mu_j\right) \Bigg/ \left(\prod_{i=1}^{K} \lambda_i\right). \tag{3.56}$$

Proof. We first prove (3.54). From (2.16) we have $\widetilde{S}_{31}(-\lambda) = S_{21}(\lambda)S_{32}(\lambda) - S_{22}(\lambda)S_{31}(\lambda)$. Evaluation at $\lambda = \lambda_k$ kills S_{31}, so

$$\widetilde{S}_{31}(-\lambda_k) = S_{21}(\lambda_k)S_{32}(\lambda_k).$$

Since the poles of W and W^* are simple, the residues are given by $a_k = -S_{21}(\lambda_k)/S_{31}'(\lambda_k)$ and $a_k^* = -S_{32}(\lambda_k)/S_{31}'(\lambda_k)$. Multiplication yields

$$a_k a_k^* = \frac{S_{21}(\lambda_k)S_{32}(\lambda_k)}{S_{31}'(\lambda_k)^2} = \frac{\widetilde{S}_{31}(-\lambda_k)}{S_{31}'(\lambda_k)^2},$$

and insertion of the expressions for S_{31} and $\widetilde{S}_{31}$ from (3.39) finishes the job.

The proof of equation (3.55) is similar.

As for (3.56), we saw in (3.42) that

$$b_\infty = \frac{h_K l_{2K-1}}{l_{2K} + l_{2K-1}}.$$

In the same way, or by using the symmetry transformation (3.62) described in the next section, one shows that

$$b_\infty^* = \frac{g_1 l_1}{l_0 + l_1}.$$

Combining $S_{31}(\lambda) = -2\lambda \prod_{i=1}^{K}(1 - \lambda/\lambda_i)$ with the expression (3.22) for the highest coefficient of S_{31} yields

$$\prod_{i=1}^{K} \lambda_i = \frac{1}{2}\left(\prod_{m=0}^{K} l_{2m}\right)\left(\prod_{a=1}^{K} g_a h_a\right),$$

and similarly we find by comparing $\widetilde{S}_{31}(\lambda) = -2\lambda \prod_{j=1}^{K-1}(1 - \lambda/\mu_j)$ to (3.36) that

$$\prod_{j=1}^{K-1} \mu_j = \frac{1}{2}(l_{2K} + l_{2K-1})\left(\prod_{m=2}^{K-1} l_{2m-1}\right)(l_0 + l_1)\left(\prod_{a=1}^{K-1} g_{a+1} h_a\right).$$

Equation (3.56) follows. □

Remark 3.16. When $K = 1$, we have

$$a_1 a_1^* = \frac{2}{\lambda_1} \tag{3.57}$$

as shown in (4.53), while (3.56) breaks down for the same reason that (3.42) did; by (4.49), (4.50) and (4.51), we have instead

$$b_\infty b_\infty^* = \frac{(l_0 + l_1)(l_1 + l_2)}{2 l_0 l_2 \lambda_1} \tag{3.58}$$

in this case.

Remark 3.17. Theorem 3.15 shows that W and $\widetilde{W}$ together determine W^*, since $a_1^*, \dots, a_K^*$ can be computed from (3.54) if one knows $\{a_k, b_k, b_\infty, \lambda_i, \mu_j\}$. But they only *almost* determine $\widetilde{W}^*$; the residues $b_1^*, \dots, b_{K-1}^*$ can be computed from (3.55), but the constant b_∞^* is not determined! This turns out to be highly significant for the inverse spectral problem: the Weyl functions W and $\widetilde{W}$ don't contain enough information to recover the first weight g_1 and its position y_1; for this we need to know the value of b_∞^* as well.

We can now prove the positivity of the residues a_i and b_j in Theorem 3.10. (The notation introduced in this proof will not be used elsewhere, and is omitted from the index of notation in Appendix C.)

Proof of Theorem 3.10, continued. We consider the residues $\{a_i\}_{i=1}^K$ first. For $K = 1$ we have $S_{21}(\lambda) = -g_1 l_0 \lambda$ and $S_{31}(\lambda) = -2\lambda + g_1 h_1 l_0 l_2 \lambda^2$, so that

$$W(\lambda) = -\frac{S_{21}(\lambda)}{S_{31}(\lambda)} = \frac{\frac{1}{h_1 l_2}}{\lambda - \frac{2}{g_1 h_1 l_0 l_2}};$$

hence $a_1 = \frac{1}{h_1 l_2} > 0$. We now proceed by induction on K. Suppose that the residues a_i are positive when $K = m-1$, and consider the case $K = m \geq 2$. Because of (3.54), no a_i can ever be zero as long as all masses are positive, and therefore it is sufficient to verify that all a_i are positive when the last pair of masses are given by $g_m = h_m = \varepsilon$ with $\varepsilon > 0$ small; since the residues depend continuously on the masses, they will keep their signs as g_m and h_m are allowed to vary arbitrarily over all positive values. From (3.9) we get

$$\begin{pmatrix} S_{11}(\lambda,\varepsilon) \\ S_{21}(\lambda,\varepsilon) \\ S_{31}(\lambda,\varepsilon) \end{pmatrix} = L_{2m}(\lambda) \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} L_{2m-1}(\lambda) \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} L_{2m-2}(\lambda) \cdots \begin{bmatrix} h_1 \\ 0 \end{bmatrix} L_1(\lambda) \begin{bmatrix} 0 \\ g_1 \end{bmatrix} L_0(\lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where we consider all positions and all masses except $g_m = h_m = \varepsilon$ as fixed, and treat the $S_{ij}(\lambda,\varepsilon)$ as polynomials in two variables. The spectral data defined by these polynomials will then of course also be considered as functions of ε: $\{\lambda_i(\varepsilon), a_i(\varepsilon)\}_{i=1}^m$. (As we will soon see, the largest eigenvalue $\lambda_m(\varepsilon)$ has a pole of order 2 at $\varepsilon = 0$, while the other eigenvalues are analytic functions of ε.) The first four factors in the product above are

$$L_{2m}(\lambda) \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} L_{2m-1}(\lambda) \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} = \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 \\ 0 & 1 & \varepsilon \\ -(l_{2m} + l_{2m-1})\lambda & -\varepsilon l_{2m}\lambda & 1 - \varepsilon^2 l_{2m}\lambda \end{pmatrix}.$$

We denote the product of the remaining factors by $(s_{11}(\lambda), s_{21}(\lambda), s_{31}(\lambda))^T$; these polynomials have the same form as S_{11}, S_{21} and S_{23} (see Corollary 3.4), but with $m-1$ instead of m, so their degrees are one step lower, and they only depend on $\{g_k, h_k\}_{k=1}^{m-1}$ and $\{l_k\}_{k=0}^{2m-2}$, not on l_{2m-1}, l_{2m} and $g_m = h_m = \varepsilon$. We thus have

$$\begin{aligned} \begin{pmatrix} S_{11}(\lambda,\varepsilon) \\ S_{21}(\lambda,\varepsilon) \\ S_{31}(\lambda,\varepsilon) \end{pmatrix} &= \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 \\ 0 & 1 & \varepsilon \\ -(l_{2m} + l_{2m-1})\lambda & -\varepsilon l_{2m}\lambda & 1 - \varepsilon^2 l_{2m}\lambda \end{pmatrix} \begin{pmatrix} s_{11}(\lambda) \\ s_{21}(\lambda) \\ s_{31}(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} S_{11}(\lambda,0) \\ S_{21}(\lambda,0) \\ S_{31}(\lambda,0) \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon & \varepsilon^2 \\ 0 & 0 & \varepsilon \\ 0 & -\varepsilon l_{2m}\lambda & -\varepsilon^2 l_{2m}\lambda \end{pmatrix} \begin{pmatrix} s_{11}(\lambda) \\ s_{21}(\lambda) \\ s_{31}(\lambda) \end{pmatrix}. \end{aligned} \tag{3.59}$$

The polynomials $S_{ij}(\lambda, 0)$ define the spectral data for the case $K = m-1$ (since the final pair of masses is absent when $\varepsilon = 0$); in particular we know from Theorem 3.8 that $S_{31}(\lambda, 0)$ has a zero at $\lambda = 0$, and that the other $m-1$ zeros are positive and simple. If $\lambda = \lambda_i \neq 0$ is one of these other zeros, then at the point $(\lambda, \varepsilon) = (\lambda_i, 0)$ we therefore have $S_{31} = 0$ and $\partial S_{31}/\partial\lambda \neq 0$, so by the Implicit Function Theorem there is an analytic function $\lambda_i(\varepsilon)$, defined around $\varepsilon = 0$, such that $\lambda_i(0) = \lambda_i$ and $S_{31}(\lambda_i(\varepsilon), \varepsilon) = 0$. It follows that for $i = 1, \dots, m-1$, the residue

$$a_i(\varepsilon) = \operatorname*{res}_{\lambda=\lambda_i(\varepsilon)} W(\lambda, \varepsilon) = -\frac{S_{21}(\lambda_i(\varepsilon), \varepsilon)}{\dfrac{\partial S_{31}}{\partial \lambda}(\lambda_i(\varepsilon), \varepsilon)}$$

depends analytically on ε too, and it is therefore positive for small $\varepsilon > 0$, since it is positive for $\varepsilon = 0$ by the induction hypothesis. This settles part of our claim.

It remains to show that the last residue $a_m(\varepsilon)$ is positive. As a first step, we show that $\lambda_m(\varepsilon)$ has a pole of order 2 at $\varepsilon = 0$. For convenience, let

$$f(\lambda, \varepsilon) = \frac{S_{31}(\lambda, \varepsilon)}{\lambda};$$

this is a polynomial of degree m in λ, and $\lambda_m(\varepsilon)$ is the largest root of the equation $f(\lambda, \varepsilon) = 0$. From (3.59) we have

$$f(\lambda, \varepsilon) = f(\lambda, 0) - l_{2m}\Big(\varepsilon s_{21}(\lambda) + \varepsilon^2 s_{31}(\lambda)\Big).$$

Using Corollary 3.4, we see that the leading terms of $f(\lambda, 0) = S_{31}(\lambda, 0)/\lambda$ and $l_{2m}s_{31}(\lambda)$ are $(-1)^m C_1 \lambda^{m-1}$ and $(-1)^m C_2 \lambda^m$, respectively, with

$$C_1 = \left(\prod_{r=0}^{m-2} l_{2r}\right)(l_{2m-2} + l_{2m-1} + l_{2m})\left(\prod_{a=1}^{m-1} g_a h_a\right) > 0,$$
$$C_2 = \left(\prod_{r=0}^{m} l_{2r}\right)\left(\prod_{a=1}^{m-1} g_a h_a\right) > 0.$$

(The precise form of these constants is not very important, only their positivity.) Moreover, $s_{21}(\lambda)$ has degree $m-1$. Thus

$$\begin{aligned} f(\lambda, \varepsilon) &= f(\lambda, 0) - l_{2m}\Big(\varepsilon s_{21}(\lambda) + \varepsilon^2 s_{31}(\lambda)\Big) \\ &= (-1)^{m+1} C_2 \varepsilon^2 \lambda^m + p(\lambda, \varepsilon), \end{aligned}$$

with a polynomial $p(\lambda, \varepsilon)$ of degree $m-1$ in λ. Since $p(\lambda, 0) = f(\lambda, 0)$ has leading term $(-1)^m C_1 \lambda^{m-1}$, we see that

$$\varepsilon^{2m-2} p(\kappa\varepsilon^{-2}, \varepsilon) = (-1)^m C_1 \kappa^{m-1} + (\text{terms containing } \varepsilon).$$

Hence, the equation $f(\lambda, \varepsilon) = 0$, of which $\lambda_m(\varepsilon)$ is the largest root, can be written in terms of the new variable $\kappa = \lambda\varepsilon^2$ as

$$\begin{aligned} 0 &= (-1)^{m+1} \varepsilon^{2m-2} f(\lambda, \varepsilon) \\ &= C_2 \varepsilon^{2m} \lambda^m + \varepsilon^{2m-2}(-1)^{m+1} p(\lambda, \varepsilon) \\ &= C_2 \kappa^m + \varepsilon^{2m-2}(-1)^{m+1} p(\kappa\varepsilon^{-2}, \varepsilon) \\ &= C_2 \kappa^m - C_1 \kappa^{m-1} + \varepsilon\, q(\kappa, \varepsilon), \end{aligned}$$

for some two-variable polynomial $q(\kappa,\varepsilon)$. As before, the Implicit Function Theorem shows that this equation has an analytic solution $\kappa(\varepsilon)$ with $\kappa(0) = C_1/C_2$, which corresponds to a meromorphic zero of $f(\lambda,\varepsilon)$ with a pole of order 2, as claimed:

$$\lambda_m(\varepsilon) = \frac{\kappa(\varepsilon)}{\varepsilon^2} = \frac{C_1/C_2 + \mathcal{O}(\varepsilon)}{\varepsilon^2}.$$

Finally, the corresponding residue is

$$a_m(\varepsilon) = \operatorname*{res}_{\lambda=\lambda_m(\varepsilon)} W(\lambda,\varepsilon) = -\frac{S_{21}(\lambda_m(\varepsilon),\varepsilon)}{\dfrac{\partial S_{31}}{\partial \lambda}(\lambda_m(\varepsilon),\varepsilon)}.$$

The derivative of the polynomial S_{31} at its largest zero has the same sign as the leading term of S_{31}, namely $(-1)^{m+1}$. As for the sign of S_{21}, we have from (3.59) that

$$S_{21}(\lambda,\varepsilon) = S_{21}(\lambda,0) + \varepsilon s_{31}(\lambda),$$

where $S_{21}(\lambda,0)$ and $s_{31}(\lambda)$ have degrees $m-1$ and m, respectively. When this is evaluated at $\lambda = \lambda_m(\varepsilon) \sim \frac{C_1}{C_2}\varepsilon^{-2}$, the two terms on the right-hand side are of order ε^{2m-2} and ε^{2m-1}, respectively, so the dominant behavior as $\varepsilon \to 0^+$ comes from the leading term of $s_{31}(\lambda)$:

$$S_{21}(\lambda_m(\varepsilon),\varepsilon) \sim \varepsilon(-1)^m \frac{C_2}{l_{2m}} \left(\frac{C_1/C_2}{\varepsilon^2}\right)^m.$$

In particular, the sign of $S_{21}(\lambda_m(\varepsilon),\varepsilon)$ is $(-1)^m$, and it follows that $a_m(\varepsilon) > 0$, which is what we wanted to show. This concludes the proof of positivity for the residues a_i.

The proof for the residues $\{b_j\}_{j=1}^{K-1}$ is similar. In the base case $K=1$ there is nothing to show. Assume that they are positive for $K = m-1$, and consider the case $K = m \geq 2$. We have from (3.23)

$$\begin{pmatrix} \widetilde{S}_{11}(\lambda,\varepsilon) \\ \widetilde{S}_{21}(\lambda,\varepsilon) \\ \widetilde{S}_{31}(\lambda,\varepsilon) \end{pmatrix} = L_{2m}(\lambda) \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} L_{2m-1}(\lambda) \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} L_{2m-2}(\lambda) \cdots \begin{bmatrix} 0 \\ h_1 \end{bmatrix} L_1(\lambda) \begin{bmatrix} g_1 \\ 0 \end{bmatrix} L_0(\lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Splitting off the first four factors

$$L_{2m}(\lambda) \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} L_{2m-1}(\lambda) \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} = \begin{pmatrix} 1 & \varepsilon & 0 \\ -\varepsilon l_{2m-1}\lambda & 1-\varepsilon^2 l_{2m-1}\lambda & \varepsilon \\ -(l_{2m}+l_{2m-1})\lambda & -\varepsilon(l_{2m}+l_{2m-1})\lambda & 1 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \widetilde{S}_{11}(\lambda,\varepsilon) \\ \widetilde{S}_{21}(\lambda,\varepsilon) \\ \widetilde{S}_{31}(\lambda,\varepsilon) \end{pmatrix} = \begin{pmatrix} \widetilde{S}_{11}(\lambda,0) \\ \widetilde{S}_{21}(\lambda,0) \\ \widetilde{S}_{31}(\lambda,0) \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon & 0 \\ -\varepsilon l_{2m-1}\lambda & -\varepsilon^2 l_{2m-1}\lambda & \varepsilon \\ 0 & -\varepsilon(l_{2m}+l_{2m-1})\lambda & 0 \end{pmatrix} \begin{pmatrix} \widetilde{s}_{11}(\lambda) \\ \widetilde{s}_{21}(\lambda) \\ \widetilde{s}_{31}(\lambda) \end{pmatrix}, \tag{3.60}$$

where the degrees on the left-hand side are $(m-1,m,m)$, while both 3×1 matrices appearing on the right-hand side have degrees $(m-2,m-1,m-1)$ (cf. Corollary 3.7). The eigenvalues $\{\mu_j(\varepsilon)\}_{j=1}^{m-1}$ are the zeros of the polynomial

$$\widetilde{f}(\lambda,\varepsilon) = \frac{\widetilde{S}_{31}(\lambda,\varepsilon)}{\lambda} = \frac{\widetilde{S}_{31}(\lambda,0)}{\lambda} - \varepsilon(l_{2m}+l_{2m-1})\widetilde{s}_{21}(\lambda).$$

As above, it follows easily that $\{\mu_j(\varepsilon)\}_{j=1}^{m-2}$ are analytic, and that the corresponding residues $\{b_j(\varepsilon)\}_{j=1}^{m-2}$ are positive. The largest zero $\mu_{m-1}(\varepsilon)$ has a pole of order 1 at $\varepsilon = 0$, as we now show. By Corollary 3.7, the leading terms of $\widetilde{S}_{31}(\lambda, 0)$ and $(l_{2m} + l_{2m-1})\widetilde{s}_{21}(\lambda)$ are $(-1)^{m-1}\widetilde{C}_1\lambda^{m-1}$ and $(-1)^{m-1}\widetilde{C}_2\lambda^{m-1}$, respectively, with some positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$. (For the record, these constants are

$$\widetilde{C}_1 = \left(\sum_{a=2m-3}^{2m} l_a\right)\left(\prod_{r=2}^{m-2} l_{2r-1}\right)(l_0 + l_1)\left(\prod_{a=1}^{m-2} g_{a+1}h_a\right) > 0,$$

$$\widetilde{C}_2 = (l_{2m} + l_{2m-1})\left(\prod_{r=2}^{m-1} l_{2r-1}\right)(l_0 + l_1)\, h_{m-1}\left(\prod_{a=1}^{m-2} g_{a+1}h_a\right) > 0.$$

Special case: $\widetilde{C}_1 = 2$ if $m = 2$. The empty product $\prod_{r=2}^{1} l_{2r-1}$ is omitted in $\widetilde{C}_1$ when $m = 3$ and in $\widetilde{C}_2$ when $m = 2$.) Hence,

$$\begin{aligned}\widetilde{f}(\lambda, \varepsilon) &= \widetilde{f}(\lambda, 0) - \varepsilon(l_{2m} + l_{2m-1})\widetilde{s}_{21}(\lambda)\\ &= (-1)^m\widetilde{C}_2\varepsilon\lambda^{m-1} + \widetilde{p}(\lambda, \varepsilon),\end{aligned}$$

with a polynomial $\widetilde{p}(\lambda, \varepsilon)$ of degree $m-2$ in λ, such that $\widetilde{p}(\lambda, 0) = \widetilde{f}(\lambda, 0)$ has leading term $(-1)^{m-1}\widetilde{C}_1\lambda^{m-2}$, so that

$$\varepsilon^{m-2}\widetilde{p}(\widetilde{\kappa}\varepsilon^{-1}, \varepsilon) = (-1)^{m-1}\widetilde{C}_1\widetilde{\kappa}^{m-1} + (\text{terms containing } \varepsilon).$$

The equation $\widetilde{f}(\lambda, \varepsilon) = 0$, of which $\mu_{m-1}(\varepsilon)$ is the largest root, can therefore be written in terms of the new variable $\widetilde{\kappa} = \lambda\varepsilon$ as

$$\begin{aligned}0 &= (-1)^m\varepsilon^{m-2}\widetilde{f}(\lambda, \varepsilon)\\ &= \widetilde{C}_2\varepsilon^{m-1}\lambda^{m-1} + \varepsilon^{m-2}(-1)^m\widetilde{p}(\lambda, \varepsilon)\\ &= \widetilde{C}_2\widetilde{\kappa}^m + \varepsilon^{m-2}(-1)^m\widetilde{p}(\widetilde{\kappa}\varepsilon^{-1}, \varepsilon)\\ &= \widetilde{C}_2\widetilde{\kappa}^m - \widetilde{C}_1\widetilde{\kappa}^{m-1} + \varepsilon\,\widetilde{q}(\widetilde{\kappa}, \varepsilon),\end{aligned}$$

for some two-variable polynomial $\widetilde{q}(\widetilde{\kappa}, \varepsilon)$. The Implicit Function Theorem gives an analytic function $\widetilde{\kappa}(\varepsilon)$ with $\widetilde{\kappa}(0) = \widetilde{C}_1/\widetilde{C}_2$, and, as claimed,

$$\mu_{j-1}(\varepsilon) = \frac{\widetilde{\kappa}(\varepsilon)}{\varepsilon} = \frac{\widetilde{C}_1/\widetilde{C}_2 + \mathcal{O}(\varepsilon)}{\varepsilon}.$$

The corresponding residue is

$$b_{m-1}(\varepsilon) = \operatorname*{res}_{\lambda=\mu_{m-1}(\varepsilon)} \widetilde{W}(\lambda, \varepsilon) = -\frac{\widetilde{S}_{21}(\mu_{m-1}(\varepsilon), \varepsilon)}{\dfrac{\partial\widetilde{S}_{31}}{\partial\lambda}(\mu_{m-1}(\varepsilon), \varepsilon)}.$$

The leading term of S_{31} determines the sign of the derivative $\partial S_{31}/\partial\lambda$ at the largest zero, namely $(-1)^m$. From (3.60),

$$\widetilde{S}_{21}(\lambda, \varepsilon) = \widetilde{S}_{21}(\lambda, 0) - \varepsilon l_{2m-1}\lambda\widetilde{s}_{11}(\lambda) - \varepsilon^2 l_{2m-1}\lambda\widetilde{s}_{21}(\lambda) + \varepsilon\widetilde{s}_{31}(\lambda),$$

and when evaluating this at $\lambda = \mu_{m-1}(\varepsilon) \sim \frac{\widetilde{C}_1}{\widetilde{C}_2}\varepsilon^{-1}$, the last three terms on the right-hand side are of order ε^{2-m}, so the contribution of order ε^{1-m} from the first term $\widetilde{S}_{21}(\lambda, 0)$ is the dominant one as $\varepsilon \to 0^+$, and it has the sign $(-1)^{m-1}$. It follows that $b_{m-1}(\varepsilon) > 0$, and the proof is complete. □

3.6 Symmetry

For solutions of the differential equation (2.2a), $\frac{\partial \Phi}{\partial y} = \mathcal{A}(\lambda)\Phi$, the transition matrix $S(\lambda)$ propagates initial values at the left endpoint to final values at the right endpoint:

$$\Phi(+1) = S(\lambda)\Phi(-1).$$

The transition matrix depends of course not only on λ but also on $g(y)$ and $h(y)$, which in our discrete setup means the point masses g_j and h_j interlacingly positioned at the sites y_k with $l_k = y_{k+1} - y_k$; let us write

$$\begin{aligned} S(\lambda) &= L_{2K}(\lambda)\begin{bmatrix} h_K \\ 0 \end{bmatrix} L_{2K-1}(\lambda)\begin{bmatrix} 0 \\ g_K \end{bmatrix} L_{2K-2}(\lambda)\cdots\begin{bmatrix} h_1 \\ 0 \end{bmatrix} L_1(\lambda)\begin{bmatrix} 0 \\ g_1 \end{bmatrix} L_0(\lambda) \\ &= S(\lambda; l_0, \dots, l_{2K}; g_1, h_1, \dots, g_K, h_K) \end{aligned}$$

to indicate this.

For the adjoint equation (2.22a), $\frac{\partial \Omega}{\partial y} = \widetilde{\mathcal{A}}(-\lambda)\Omega$, we saw in Proposition 2.12 that the matrix $\widetilde{S}(-\lambda)^{-1} = JS(\lambda)^T J$ propagates values in the opposite direction, from initial values at the right endpoint to final values at the left endpoint. If we denote this matrix by $S^*(\lambda)$, we thus have

$$\Omega(-1) = S^*(\lambda)\Omega(+1).$$

When going from right to left, one encounters the point masses in the opposite order compared to when going from left to right, and the following theorem shows that the solution $\Omega(y)$ reacts just like $\Phi(y)$ does when encountering a mass, except for a difference in sign.

Theorem 3.18. *The adjoint transition matrix is given by*

$$S^*(\lambda) = S(\lambda; l_{2K}, \dots, l_0; -h_K, -g_K, \dots, -h_1, -g_1). \tag{3.61}$$

(And similarly with tildes for the twin problems.)

Proof. Use $J = J^T = J^{-1}$ together with $JL_k(\lambda)^T J = L_k(\lambda)$ and $J\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]^T J = \left[\begin{smallmatrix} -y \\ -x \end{smallmatrix}\right]$ to obtain

$$\begin{aligned} S^*(\lambda) &= JS(\lambda)^T J \\ &= J\left(L_{2K}(\lambda)\begin{bmatrix} h_K \\ 0 \end{bmatrix}\cdots\begin{bmatrix} 0 \\ g_1 \end{bmatrix} L_0(\lambda)\right)^T J \\ &= \left(JL_0(\lambda)^T J\right)\left(J\begin{bmatrix} 0 \\ g_1 \end{bmatrix}^T J\right)\cdots\left(J\begin{bmatrix} h_K \\ 0 \end{bmatrix}^T J\right)\left(JL_{2K}(\lambda)^T J\right) \\ &= L_0(\lambda)\begin{bmatrix} -g_1 \\ 0 \end{bmatrix}\cdots\begin{bmatrix} 0 \\ -h_K \end{bmatrix} L_{2K}(\lambda). \end{aligned}$$

□

Remark 3.19. The adjoint Weyl functions W^* and Z^* are defined from the first column in $S^* = JS^T J$,

$$(S^*_{11}, S^*_{11}, S^*_{11})^T = (S_{33}, -S_{32}, S_{31})^T,$$

in almost the same way as W and Z are defined from the first column in S, but there is a slight sign difference in W^* since we have defined all Weyl functions so that they will have positive residues:

$$W = -S_{21}/S_{31}, \qquad Z = -S_{11}/S_{31},$$

but

$$W^* = -S_{32}/S_{31} = +S_{21}^*/S_{31}^*, \qquad Z^* = -S_{33}/S_{31} = -S_{11}^*/S_{31}^*.$$

As a consequence, we see for example that if

$$a_k = F(l_0, \dots, l_{2K}; g_1, h_1, \dots, g_K, h_K)$$

indicates how the residue a_k in W depends on the configuration of the masses, then

$$-a_k^* = F(l_{2K}, \dots, l_0; -h_K, -g_K, \dots, -h_1, -g_1),$$

with the same function F, will determine the corresponding residue in W^*.

Remark 3.20. In Section 4.2 we will use Theorem 3.12 and Proposition 3.14 to derive formulas for recovering the weights h_j and their positions y_{2j} from the Weyl functions W and $\widetilde{W}$. Because of the symmetry properties described here, the same formulas can then be used to recover the weights g_j and their positions y_{2j-1} from the adjoint Weyl functions W^* and $\widetilde{W}^*$, by substituting

$$\begin{aligned} a_i &\mapsto -a_i^*, & l_k &\mapsto l_{2K-k}, \\ b_j &\mapsto -b_j^*, & g_j &\mapsto -h_{K+1-j}, \\ b_\infty &\mapsto -b_\infty^*, & h_j &\mapsto -g_{K+1-j}. \end{aligned} \tag{3.62}$$

Note that $1 - y_m = \sum_{k=m}^{2K} l_k$ is to be replaced by $\sum_{k=m}^{2K} l_{2K-k} = \sum_{s=0}^{2K-m} l_s = 1 + y_{2K+1-m}$.

4 The inverse spectral problem

To summarize what we have seen so far, the Weyl functions $W(\lambda)$, $Z(\lambda)$, $\widetilde{W}(\lambda)$, $\widetilde{Z}(\lambda)$ encode much of the information about our twin spectral problems. In particular, in the discrete interlacing case with positive weights, the Weyl functions are rational functions in the spectral variable λ, with poles at the (positive and simple) eigenvalues of the spectral problems, and the functions Z and $\widetilde{Z}$ are completely determined by W and $\widetilde{W}$ (which in turn are of course determined by the given discrete measures m and n that define the whole setup).

The measures depend on the $4K$ parameters

$$x_1, x_2, \dots, x_{2K-1}, x_{2K}, \qquad m_1, m_3, \dots, m_{2K-1}, \qquad n_2, n_4, \dots, n_{2K}$$

(or equivalently $\{y_k, g_{2a-1}, h_{2a}\}$), while the Weyl function W depends on the $2K$ parameters

$$\lambda_1, \dots, \lambda_K, \qquad a_1, \dots, a_K,$$

and its twin $\widetilde{W}$ on the $2K-1$ parameters

$$\mu_1, \dots, \mu_{K-1}, \qquad b_1, \dots, b_{K-1}, \qquad b_\infty.$$

To get an inverse spectral problem where the number of spectral data matches the number of parameters to reconstruct, we therefore need to supplement W and $\widetilde{W}$ by one extra piece of information, and a suitable choice turns out to be the coefficient b_∞^* defined by (3.52b). We will show in this section how to recover the discrete interlacing measures m and n (or, equivalently, their counterparts g and h on the finite interval) from this set of spectral data $\{\lambda_i, a_i, \mu_j, b_j, b_\infty, b_\infty^*\}$ that they give rise to. Moreover, we will show that the *necessary* constraints ($0 < \lambda_1 < \cdots < \lambda_K$, $0 < \mu_1 < \cdots < \mu_{K-1}$, and all a_i, b_j, b_∞, b_∞^* positive) are also *sufficient* for such a set of numbers to be the spectral data of a unique pair of interlacing discrete measures m and n.

4.1 Approximation problem

As we mentioned in Section 3.4, the properties in Theorem 3.12 are enough to determine the polynomials Q, P, R uniquely, and this fact will be proved here.

Theorem 4.1. *Let b_∞ be a positive constant. Let α and β be compactly supported measures on the positive real axis, with moments*

$$\alpha_k = \int x^k d\alpha(x), \qquad \beta_k = \int y^k d\beta(y), \tag{4.1}$$

and bimoments (with respect to the Cauchy kernel $\frac{1}{x+y}$)

$$I_{km} = \iint \frac{x^k y^m}{x+y} d\alpha(x) d\beta(y). \tag{4.2}$$

Define W, $\widetilde{W}$, Z, $\widetilde{Z}$ by the formulas (3.44) (repeated here for convenience):

$$W(\lambda) = \int \frac{d\alpha(x)}{\lambda - x}, \tag{3.44a}$$

$$\widetilde{W}(\lambda) = \int \frac{d\beta(y)}{\lambda - y} - b_\infty, \tag{3.44b}$$

$$Z(\lambda) = \frac{1}{2\lambda} + \iint \frac{d\alpha(x) d\beta(y)}{(\lambda - x)(x+y)} + b_\infty W(\lambda), \tag{3.44c}$$

$$\widetilde{Z}(\lambda) = \frac{1}{2\lambda} + \iint \frac{d\alpha(x) d\beta(y)}{(x+y)(\lambda - y)}. \tag{3.44d}$$

Fix a positive integer j. (If α and β are supported at infinitely many points, then j can be arbitary. In the discrete case with $\alpha = \sum_{i=1}^K a_i \delta_{\lambda_i}$ and $\beta = \sum_{i=1}^{K-1} b_i \delta_{\mu_i}$, we restrict j to the interval $1 \le j \le K$.)

Then there are unique polynomials $Q(\lambda) = Q_j(\lambda)$, $P(\lambda) = P_j(\lambda)$, $R(\lambda) = R_j(\lambda)$ satisfying the conditions of Theorem 3.12 (also repeated here for convenience):

$$\deg Q = j, \qquad \deg P = j-1, \qquad \deg R = j-1, \tag{3.46}$$

$$Q(0) = 0, \qquad P(0) = 1, \tag{3.47}$$

and, as $\lambda \to \infty$*,*

$$W(\lambda)Q(\lambda) - P(\lambda) = \mathcal{O}\left(\frac{1}{\lambda}\right), \tag{3.48a}$$

$$Z(\lambda)Q(\lambda) - R(\lambda) = \mathcal{O}\left(\frac{1}{\lambda}\right), \tag{3.48b}$$

$$R(\lambda) + P(\lambda)\widetilde{W}(-\lambda) + Q(\lambda)\widetilde{Z}(-\lambda) = \mathcal{O}\left(\frac{1}{\lambda^j}\right). \tag{3.48c}$$

These polynomials are given by

$$Q(\lambda) = \lambda\, p(\lambda), \tag{4.3a}$$

$$P(\lambda) = \int \frac{Q(\lambda) - Q(x)}{\lambda - x}\, d\alpha(x), \tag{4.3b}$$

$$R(\lambda) = \iint \frac{Q(\lambda) - Q(x)}{(\lambda - x)(x + y)}\, d\alpha(x) d\beta(y) + \tfrac{1}{2} p(\lambda) + b_\infty P(\lambda), \tag{4.3c}$$

where

$$p(\lambda) = \frac{\det \begin{pmatrix} 1 & I_{10} & \dots & I_{1,j-2} \\ \lambda & I_{20} & \dots & I_{2,j-2} \\ \vdots & \vdots & & \vdots \\ \lambda^{j-1} & I_{j0} & \dots & I_{j,j-2} \end{pmatrix}}{\det \begin{pmatrix} \alpha_0 & I_{10} & \dots & I_{1,j-2} \\ \alpha_1 & I_{20} & \dots & I_{2,j-2} \\ \vdots & \vdots & & \vdots \\ \alpha_{j-1} & I_{j0} & \dots & I_{j,j-2} \end{pmatrix}}. \tag{4.3d}$$

(If $j = 1$*, equation (4.3d) should be read as* $p(\lambda) = 1/\alpha_0$*.)*

In particular, we have (using notation from Section A.3)

$$Q'(0) = p(0) = \frac{\det \begin{pmatrix} I_{20} & \dots & I_{2,j-2} \\ \vdots & & \vdots \\ I_{j0} & \dots & I_{j,j-2} \end{pmatrix}}{\det \begin{pmatrix} \alpha_0 & I_{10} & \dots & I_{1,j-2} \\ \alpha_1 & I_{20} & \dots & I_{2,j-2} \\ \vdots & \vdots & & \vdots \\ \alpha_{j-1} & I_{j0} & \dots & I_{j,j-2} \end{pmatrix}} = \frac{\mathcal{J}^{20}_{j-1,j-1}}{\mathcal{J}^{01}_{j,j-1}} \tag{4.4}$$

(to be read as $Q'(0) = 1/\alpha_0$ if $j = 1$), and

$$R(0) = \iint \frac{p(x)}{x+y} d\alpha(x) d\beta(y) + \tfrac{1}{2} p(0) + b_\infty$$
$$= \frac{\det \begin{pmatrix} I_{00} + \frac{1}{2} & I_{10} & \dots & I_{1,j-2} \\ I_{10} & I_{20} & \dots & I_{2,j-2} \\ \vdots & \vdots & & \vdots \\ I_{j-1,0} & I_{j0} & \dots & I_{j,j-2} \end{pmatrix}}{\det \begin{pmatrix} \alpha_0 & I_{10} & \dots & I_{1,j-2} \\ \alpha_1 & I_{20} & \dots & I_{2,j-2} \\ \vdots & \vdots & & \vdots \\ \alpha_{j-1} & I_{j0} & \dots & I_{j,j-2} \end{pmatrix}} + b_\infty = \frac{\mathcal{K}_j}{\mathcal{J}^{01}_{j,j-1}} + b_\infty \tag{4.5}$$

(to be read as $R(0) = (I_{00} + \frac{1}{2})/\alpha_0 + b_\infty$ if $j = 1$).

Proof. A bit of notation first: define projection operators acting on (formal or convergent) Laurent series $f(\lambda) = \sum_{k \in \mathbf{Z}} c_k \lambda^k$ as follows:

$$\Pi_{\ge 0} f = \sum_{k \ge 0} c_k \lambda^k, \qquad \Pi_{>0} f = \sum_{k > 0} c_k \lambda^k, \qquad \Pi_{<0} f = \sum_{k < 0} c_k \lambda^k. \tag{4.6}$$

Note that we can expand $W(\lambda)$ in a Laurent series with negative powers,

$$W(\lambda) = \int \frac{d\alpha(x)}{\lambda - x} = \frac{1}{\lambda} \int \sum_{k \ge 0} \left(\frac{x}{\lambda} \right)^k d\alpha(x) = \sum_{k \ge 0} \frac{\alpha_k}{\lambda^{k+1}},$$

and similarly for the other Weyl functions.

We see at once that the conditions (3.48a) and (3.48b) determine the polynomials P and R uniquely, by projection on nonnegative powers, if the polynomial Q is known:

$$P = \Pi_{\ge 0}[QW], \qquad R = \Pi_{\ge 0}[QZ]. \tag{4.7}$$

Inserting this into (3.48c) gives

$$\Pi_{\ge 0}[QZ](\lambda) + \Pi_{\ge 0}[QW](\lambda)\, \widetilde{W}(-\lambda) + Q(\lambda) \widetilde{Z}(-\lambda) = \mathcal{O}\left(\frac{1}{\lambda^j} \right).$$

Writing $\Pi_{\ge 0} = \mathrm{id} - \Pi_{<0}$ produces

$$Q(\lambda) Z(\lambda) + Q(\lambda) W(\lambda) \widetilde{W}(-\lambda) + Q(\lambda) \widetilde{Z}(-\lambda) \\ - \Pi_{<0}[QZ](\lambda) - \Pi_{<0}[QW](\lambda)\, \widetilde{W}(-\lambda) = \mathcal{O}\left(\frac{1}{\lambda^j} \right),$$

where the first three terms cancel thanks to the identity $Z(\lambda) + W(\lambda)\widetilde{W}(-\lambda) + \widetilde{Z}(-\lambda) = 0$ which follows from the definitions (3.44) by a short calculation (cf. also (2.19)). This leaves

$$\Pi_{<0}[QZ](\lambda) + \Pi_{<0}[QW](\lambda)\, \widetilde{W}(-\lambda) = \mathcal{O}\left(\frac{1}{\lambda^j} \right). \tag{4.8}$$

Next, note that

$$Q(\lambda)W(\lambda) = Q(\lambda)\int \frac{d\alpha(x)}{\lambda - x} = \int \frac{Q(\lambda) - Q(x)}{\lambda - x}\, d\alpha(x) + \int \frac{Q(x)}{\lambda - x}\, d\alpha(x),$$

where the first term is a polynomial in λ (since $Q(\lambda) - Q(x)$ vanishes when $\lambda = x$ and therefore contains $\lambda - x$ as a factor), and the second term is $\mathcal{O}(1/\lambda)$ as $\lambda \to \infty$. Thus, the first and second term are $\Pi_{\geq 0}[QW]$ and $\Pi_{<0}[QW]$, respectively, which gives on the one hand the claimed integral representation for P,

$$P(\lambda) = \Pi_{\geq 0}[QW](\lambda) = \int \frac{Q(\lambda) - Q(x)}{\lambda - x}\, d\alpha(x), \tag{4.9}$$

and on the other hand, multiplying the negative projection by $\widetilde{W}(-\lambda)$,

$$\begin{aligned}\Pi_{<0}[QW](\lambda)\,\widetilde{W}(-\lambda) &= \left(\int \frac{Q(x)}{\lambda - x}\, d\alpha(x)\right)\left(\int \frac{d\beta(y)}{-\lambda - y} - b_\infty\right) \\ &= -b_\infty \int \frac{Q(x)}{\lambda - x}\, d\alpha(x) - \iint \frac{Q(x)}{(\lambda - x)(\lambda + y)}\, d\alpha(x) d\beta(y). \end{aligned}\tag{4.10}$$

Similarly,

$$\begin{aligned}Q(\lambda)Z(\lambda) = \frac{Q(\lambda)}{2\lambda} &+ \iint \frac{Q(\lambda) - Q(x)}{(\lambda - x)(x + y)}\, d\alpha(x) d\beta(y) \\ &+ \iint \frac{Q(x)}{(\lambda - x)(x + y)}\, d\alpha(x) d\beta(y) + b_\infty Q(\lambda) W(\lambda),\end{aligned}$$

where the first term is a polynomial in λ since we require $Q(0) = 0$, likewise the second term is a polynomial (by the same argument as above), and the third term is $\mathcal{O}(1/\lambda)$ as $\lambda \to \infty$. Thus, we obtain from the first two terms, together with the contribution to nonnegative powers from the fourth term, the claimed integral representation for R,

$$R(\lambda) = \Pi_{\geq 0}[QZ](\lambda) = \frac{Q(\lambda)}{2\lambda} + \iint \frac{Q(\lambda) - Q(x)}{(\lambda - x)(x + y)}\, d\alpha(x) d\beta(y) + b_\infty P(\lambda), \tag{4.11}$$

and from the third term, together with the contribution to negative powers from the fourth term,

$$\Pi_{<0}[QZ](\lambda) = \iint \frac{Q(x)}{(\lambda - x)(x + y)}\, d\alpha(x) d\beta(y) + b_\infty \int \frac{Q(x)}{\lambda - x}\, d\alpha(x). \tag{4.12}$$

Inserting (4.10) and (4.12) into (4.8) gives

$$\iint \frac{Q(x)}{(\lambda - x)(x + y)}\, d\alpha(x) d\beta(y) - \iint \frac{Q(x)}{(\lambda - x)(\lambda + y)}\, d\alpha(x) d\beta(y) = \mathcal{O}\left(\frac{1}{\lambda^j}\right),$$

which simplifies to

$$\iint \frac{Q(x)}{(x + y)(\lambda + y)}\, d\alpha(x) d\beta(y) = \mathcal{O}\left(\frac{1}{\lambda^j}\right). \tag{4.13}$$

Since $Q(0) = 0$, we write $Q(x) = x\, p(x)$, where p is a polynomial of degree $j-1$. Upon expanding $1/(\lambda + y) = \sum_{k \geq 0} y^k \lambda^{-(k+1)}$, the condition (4.13) takes the form

$$\iint \frac{p(x)\, xy^k}{x+y}\, d\alpha(x) d\beta(y) = 0, \qquad 0 \leq k \leq j-2. \tag{4.14}$$

This imposes $j-1$ linear equations for the j coefficients in $p(x) = p_0 + p_1 x + \cdots + p_{j-1} x^{j-1}$:

$$(p_0, \ldots, p_{j-1}) \begin{pmatrix} I_{10} & \ldots & I_{1,j-2} \\ I_{20} & \ldots & I_{2,j-2} \\ \vdots & & \vdots \\ I_{j0} & \ldots & I_{j,j-2} \end{pmatrix} = (0, \ldots, 0).$$

Adding an extra column,

$$(p_0, \ldots, p_{j-1}) \begin{pmatrix} I_{10} & \ldots & I_{1,j-2} & I_{1,j-1} \\ I_{20} & \ldots & I_{2,j-2} & I_{2,j-1} \\ \vdots & & \vdots & \vdots \\ I_{j0} & \ldots & I_{j,j-2} & I_{j,j-1} \end{pmatrix} = (0, \ldots, 0, *),$$

we see that the row vector $(p_0, \ldots, p_{j-1})$ is proportional to the last row of the inverse of the bimoment matrix in question (which is invertible by the assumption about infinitely many points of support, or by the restriction on j in the discrete case). Hence, by Cramer's rule,

$$p(x) = C \det \begin{pmatrix} I_{10} & \ldots & I_{1,j-2} & 1 \\ I_{20} & \ldots & I_{2,j-2} & x \\ \vdots & & \vdots & \vdots \\ I_{j0} & \ldots & I_{j,j-2} & x^{j-1} \end{pmatrix}.$$

The constant C is determined by the remaining normalization condition $P(0) = 1$; from (4.9) we get

$$\begin{aligned} 1 = P(0) &= \int \frac{Q(0) - Q(x)}{0 - x}\, d\alpha(x) = \int p(x)\, d\alpha(x) \\ &= C \det \begin{pmatrix} I_{10} & \ldots & I_{1,j-2} & \int 1\, d\alpha(x) \\ I_{20} & \ldots & I_{2,j-2} & \int x\, d\alpha(x) \\ \vdots & & \vdots & \vdots \\ I_{j0} & \ldots & I_{j,j-2} & \int x^{j-1}\, d\alpha(x) \end{pmatrix} \\ &= C \det \begin{pmatrix} I_{10} & \ldots & I_{1,j-2} & \alpha_0 \\ I_{20} & \ldots & I_{2,j-2} & \alpha_1 \\ \vdots & & \vdots & \vdots \\ I_{j0} & \ldots & I_{j,j-2} & \alpha_{j-1} \end{pmatrix}. \end{aligned}$$

Finally, the expression (4.4) for $Q'(0)$ follows at once upon setting $\lambda = 0$ in $Q'(\lambda) = p(\lambda) + \lambda p'(\lambda)$ and using the determinantal expression (4.3d) for $p(\lambda)$; the last term in (4.4) represents an evaluation of the determinants in terms of certain integrals (which will be sums when the measures α and β are discrete). This is explained in the appendix; see Section A.3, in particular equations (A.18) and (A.21). The expression (4.5) for $R(0)$ is also immediate from the formula (4.3c) for $R(\lambda)$, since $(Q(0) - Q(x))/(0 - x) = Q(x)/x = p(x)$. (The symbol $\mathcal{K}_j$ is just notation for the determinant in the numerator; it doesn't seem to have a simple direct integral representation, but we will mainly be interested in the difference between $R_j(0)$ and $R_{j+1}(0)$, which equation (A.27) takes care of.) $\square$

Remark 4.2. The polynomial $p(x)$ in Theorem 4.1 is proportional to $p_{j-1}(x)$, where $\{p_n(x), q_n(y)\}_{n\geq 0}$ are the normalized Cauchy biorthogonal polynomials with respect to the measures $x\,d\alpha(x)$ and $d\beta(y)$. This can be seen either directly from the biorthogonality condition (4.14), or by comparing the numerators in the formula (4.3d) for p and the formula (A.4) (with $I_{a+1,b}$ instead of I_{ab}) for p_n.

4.2 Recovery formulas for the weights and their positions

Most of the work is now done, and we can at last state the solution to the inverse problem of recovering the weights g_j and h_j and their positions y_k from the spectral data encoded in the Weyl functions. The answer will be given in terms of the integrals $\mathcal{J}_{nm}^{rs}$ defined by equation (A.15) in Section A.3 in the appendix. Since α and β are discrete measures here, these integrals are in fact sums; see (A.31) in Section A.4.

Theorem 4.3. *The weights and positions of the even-numbered point masses are given by the formulas*

$$h_K = \frac{I_{00} + \frac{1}{2}}{\alpha_0} + b_\infty, \tag{4.15}$$

$$(1 - y_{2K})h_K = \frac{1}{\alpha_0}, \tag{4.16}$$

and, for $j = 2, \dots, K$,

$$h_{K+1-j} = \frac{\mathcal{J}_{j-1,j-1}^{10}\big(\mathcal{J}_{j,j-1}^{00} + \frac{1}{2}\mathcal{J}_{j-1,j-2}^{11}\big)}{\mathcal{J}_{j-1,j-2}^{01}\mathcal{J}_{j,j-1}^{01}}, \tag{4.17}$$

$$(1 - y_{2(K+1-j)})h_{K+1-j} = \frac{\mathcal{J}_{j-1,j-2}^{11}\mathcal{J}_{j-1,j-1}^{10}}{\mathcal{J}_{j-1,j-2}^{01}\mathcal{J}_{j,j-1}^{01}}. \tag{4.18}$$

Proof. We use Proposition 3.14 together with (4.4) and (4.5) from Theorem 4.1. The rightmost mass is special; we have

$$h_K = R_1(0) - R_0(0) = \left(\frac{I_{00} + \frac{1}{2}}{\alpha_0} + b_\infty\right) - 0$$

and

$$(1 - y_{2K})h_K = Q_1'(0) - Q_0'(0) = \frac{1}{\alpha_0} - 0.$$

For the other masses we get

$$h_{K+1-j} = R_j(0) - R_{j-1}(0) = \left(\frac{\mathcal{K}_j}{\mathcal{J}_{j,j-1}^{01}} + b_\infty\right) - \left(\frac{\mathcal{K}_{j-1}}{\mathcal{J}_{j-1,j-2}^{01}} + b_\infty\right),$$

which equals (4.17) according to (A.27), and

$$(1 - y_{2(K+1-j)})h_{K+1-j} = Q_j'(0) - Q_{j-1}'(0) = \frac{\mathcal{J}^{20}_{j-1,j-1}}{\mathcal{J}^{01}_{j,j-1}} - \frac{\mathcal{J}^{20}_{j-2,j-2}}{\mathcal{J}^{01}_{j-1,j-2}},$$

which equals (4.18) according to (A.24). □

The symmetry described in Remark 3.20 immediately provides formulas for the odd-numbered point masses. We let $(\mathcal{J}^*)^{rs}_{nm}$ denote the integral $\mathcal{J}^{rs}_{nm}$ evaluated using the measures

$$\alpha^* = \sum_{i=1}^{K} a_i^* \delta_{\lambda_i} \qquad \text{and} \qquad \beta^* = \sum_{j=1}^{K-1} b_i^* \delta_{\mu_j} \tag{4.19}$$

in place of α and β, and similarly for the moments $\alpha_r^* = (\mathcal{J}^*)^{rs}_{10}$ and $\beta_s^* = (\mathcal{J}^*)^{rs}_{01}$, and the Cauchy bimoments $I^*_{rs} = (\mathcal{J}^*)^{rs}_{11}$. Then the symmetry transformation (3.62) also entails the substitution

$$\mathcal{J}^{rs}_{nm} \mapsto (-1)^{n+m} (\mathcal{J}^*)^{rs}_{nm} \tag{4.20}$$

(including as special cases $\alpha_k \mapsto -\alpha_k^*$, $\beta_k \mapsto -\beta_k^*$, and $I_{ab} \mapsto I^*_{ab}$).

Corollary 4.4. *The weights and positions of the odd-numbered point masses are given by the formulas*

$$g_1 = \frac{I^*_{00} + \frac{1}{2}}{\alpha_0^*} + b^*_\infty, \tag{4.21}$$

$$(1 + y_1) g_1 = \frac{1}{\alpha_0^*}, \tag{4.22}$$

and, for $j = 2, \ldots, K$,

$$g_j = \frac{(\mathcal{J}^*)^{10}_{j-1,j-1}\big((\mathcal{J}^*)^{00}_{j,j-1} + \frac{1}{2}(\mathcal{J}^*)^{11}_{j-1,j-2}\big)}{(\mathcal{J}^*)^{01}_{j-1,j-2}(\mathcal{J}^*)^{01}_{j,j-1}}, \tag{4.23}$$

$$(1 + y_{2j-1}) g_j = \frac{(\mathcal{J}^*)^{11}_{j-1,j-2}(\mathcal{J}^*)^{10}_{j-1,j-1}}{(\mathcal{J}^*)^{01}_{j-1,j-2}(\mathcal{J}^*)^{01}_{j,j-1}}. \tag{4.24}$$

Corollary 4.5. *The corresponding weights and positions on the real line are given by the formulas*

$$x_{2K} = \tfrac{1}{2} \ln 2(I_{00} + b_\infty \alpha_0), \tag{4.25}$$

$$n_{2K} = \frac{1}{\alpha_0} \sqrt{\frac{I_{00} + b_\infty \alpha_0}{2}}, \tag{4.26}$$

$$x_1 = -\tfrac{1}{2} \ln 2(I^*_{00} + b^*_\infty \alpha^*_0), \tag{4.27}$$

$$m_1 = \frac{1}{\alpha_0^*} \sqrt{\frac{I^*_{00} + b^*_\infty \alpha^*_0}{2}}, \tag{4.28}$$

and, for $j = 2, \ldots, K$*,*

$$x_{2(K+1-j)} = \tfrac{1}{2} \ln \left(\frac{2\,\mathcal{J}^{00}_{j,j-1}}{\mathcal{J}^{11}_{j-1,j-2}} \right), \tag{4.29}$$

$$n_{2(K+1-j)} = \frac{\mathcal{J}^{10}_{j-1,j-1}}{\mathcal{J}^{01}_{j-1,j-2}\mathcal{J}^{01}_{j,j-1}} \sqrt{\frac{\mathcal{J}^{00}_{j,j-1}\mathcal{J}^{11}_{j-1,j-2}}{2}}, \tag{4.30}$$

$$x_{2j-1} = -\tfrac{1}{2} \ln \left(\frac{2\,(\mathcal{J}^*)^{00}_{j,j-1}}{(\mathcal{J}^*)^{11}_{j-1,j-2}} \right), \tag{4.31}$$

$$m_{2j-1} = \frac{(\mathcal{J}^*)^{10}_{j-1,j-1}}{(\mathcal{J}^*)^{01}_{j-1,j-2}(\mathcal{J}^*)^{01}_{j,j-1}} \sqrt{\frac{(\mathcal{J}^*)^{00}_{j,j-1}(\mathcal{J}^*)^{11}_{j-1,j-2}}{2}}. \tag{4.32}$$

In terms of non-starred quantities (together with b^*_∞*), the odd-numbered variables take the form*

$$x_{2(K+1-j)-1} = \tfrac{1}{2} \ln \left(\frac{2\,\mathcal{J}^{00}_{jj}}{\mathcal{J}^{11}_{j-1,j-1}} \right), \tag{4.33}$$

$$m_{2(K+1-j)-1} = \frac{\mathcal{J}^{01}_{j,j-1}}{\mathcal{J}^{10}_{jj}\,\mathcal{J}^{10}_{j-1,j-1}} \sqrt{\frac{\mathcal{J}^{11}_{j-1,j-1}\mathcal{J}^{00}_{jj}}{2}}, \tag{4.34}$$

for $j = 1, \ldots, K-1$*, and*

$$x_1 = \tfrac{1}{2} \ln \left(\frac{2\,\mathcal{J}^{00}_{K,K-1}}{\mathcal{J}^{11}_{K-1,K-2} + \dfrac{2b^*_\infty L}{M}\,\mathcal{J}^{10}_{K-1,K-1}} \right) \tag{4.35}$$

$$m_1 = \frac{M/L}{\mathcal{J}^{10}_{K-1,K-1}} \sqrt{\frac{\mathcal{J}^{00}_{K,K-1}}{2} \left(\mathcal{J}^{11}_{K-1,K-2} + \frac{2b^*_\infty L}{M}\,\mathcal{J}^{10}_{K-1,K-1} \right)}, \tag{4.36}$$

where $L = \prod_{i=1}^{K} \lambda_i$ *and* $M = \prod_{j=1}^{K-1} \mu_j$*.*

Proof. Since $y_k = \tanh x_k = (e^{2x_k} - 1)/(e^{2x_k} + 1)$, we have

$$\exp(2x_k) = \frac{1 + y_k}{1 - y_k} = \frac{2}{1 - y_k} - 1.$$

Moreover, $h_j = 2n_{2j}\cosh x_{2j} = n_{2j}(e^{2x_{2j}} + 1)e^{-x_{2j}} = n_{2j}\left(\frac{2}{1-y_{2j}}\right)e^{-x_{2j}}$ implies that

$$2n_{2j}\exp(-x_{2j}) = (1 - y_{2j})h_j.$$

Now it is just a matter of plugging in the formulas from Theorem 4.3 and solving for even-numbered x_{2j} and n_{2j}. For example:

$$\begin{aligned} \tfrac{1}{2}\exp(2x_{2(K+1-j)}) &= \frac{1}{1 - y_{2(K+1-j)}} - \frac{1}{2} = \frac{h_{K+1-j}}{(1 - y_{2(K+1-j)})h_{K+1-j}} - \frac{1}{2} \\ &= \frac{\mathcal{J}^{10}_{j-1j-1}\big(\mathcal{J}^{00}_{j,j-1} + \frac{1}{2}\mathcal{J}^{11}_{j-1,j-2}\big)}{\mathcal{J}^{11}_{j-1,j-2}\mathcal{J}^{10}_{j-1,j-1}} - \frac{1}{2} = \frac{\mathcal{J}^{00}_{j,j-1}}{\mathcal{J}^{11}_{j-1,j-2}}. \end{aligned}$$

The odd-numbered x_{2j-1} and m_{2j-1} are dealt with similarly, using the formulas from Corollary 4.4 together with

$$\exp(-2x_k) = \frac{1-y_k}{1+y_k} = \frac{2}{1+y_k} - 1$$

and

$$2m_{2j-1}\exp(x_{2j-1}) = (1+y_{2j-1})g_j.$$

In order to translate starred to non-starred, we use Lemma A.3 (with $A = K$ and $B = K-1$):

$$\begin{aligned}(\mathcal{J}^*)^{00}_{j,j-1} &= \frac{L^{2j-(j-1)+0-1} M^{2(j-1)-j+0-1} \mathcal{J}^{1-0,1-0}_{K-j,(K-1)-(j-1)}}{2^{j+(j-1)} \mathcal{J}^{00}_{K,K-1}} \\ &= \frac{L^j M^{j-3} \mathcal{J}^{11}_{K-j,K-j}}{2^{2j-1} \mathcal{J}^{00}_{K,K-1}},\end{aligned}$$

and similarly for the other $(\mathcal{J}^*)^{rs}_{nm}$ occurring in the formulas for x_{2j-1} and m_{2j-1}. All the factors L, M and $\mathcal{J}^{00}_{K,K-1}$ cancel in the quotients, except in the formulas for x_1 and m_1 where we have

$$I^*_{00} = (\mathcal{J}^*)^{00}_{11} = \frac{\mathcal{J}^{11}_{K-1,K-2}}{4\,\mathcal{J}^{00}_{K,K-1}}$$

and

$$\alpha^*_0 = (\mathcal{J}^*)^{0s}_{10} = (\mathcal{J}^*)^{01}_{10} = \frac{L^1 M^{-1} \mathcal{J}^{10}_{K-1,K-1}}{2\,\mathcal{J}^{00}_{K,K-1}}.$$

□

Remark 4.6. A more compact way of writing the solution is to state the formulas in terms of the following quantities (where $r = K+1-j$ throughout):

$$\begin{aligned}\tfrac{1}{2}\exp 2x_{2K} &= I_{00} + b_\infty \alpha_0, \\ \tfrac{1}{2}\exp 2x_{2r} &= \frac{\mathcal{J}^{00}_{j,j-1}}{\mathcal{J}^{11}_{j-1,j-2}}, \qquad j = 2,\dots,K, \\ \tfrac{1}{2}\exp 2x_{2r-1} &= \frac{\mathcal{J}^{00}_{jj}}{\mathcal{J}^{11}_{j-1,j-1}}, \qquad j = 1,\dots,K-1, \\ \tfrac{1}{2}\exp 2x_1 &= \frac{\mathcal{J}^{00}_{K,K-1}}{\mathcal{J}^{11}_{K-1,K-2} + \dfrac{2b^*_\infty L}{M}\mathcal{J}^{10}_{K-1,K-1}}\end{aligned} \tag{4.37}$$

and

$$\begin{aligned}2n_{2K}\exp(-x_{2K}) &= \frac{1}{\alpha_0}, \\ 2n_{2r}\exp(-x_{2r}) &= \frac{\mathcal{J}^{11}_{j-1,j-2}\mathcal{J}^{10}_{j-1,j-1}}{\mathcal{J}^{01}_{j-1,j-2}\mathcal{J}^{01}_{j,j-1}}, \qquad j = 2,\dots,K, \\ 2m_{2r-1}\exp(-x_{2r-1}) &= \frac{\mathcal{J}^{11}_{j-1,j-1}\mathcal{J}^{01}_{j,j-1}}{\mathcal{J}^{10}_{jj}\mathcal{J}^{10}_{j-1,j-1}}, \qquad j = 1,\dots,K-1, \\ 2m_1\exp(-x_1) &= \frac{M\,\mathcal{J}^{11}_{K-1,K-2}}{L\,\mathcal{J}^{10}_{K-1,K-1}} + 2b^*_\infty.\end{aligned} \tag{4.38}$$

We now know that the set of spectral data computed from the interlacing discrete measures m and n allows us to reconstruct these measures uniquely, and we also know (Theorem 3.8, Theorem 3.10, equation (3.56)) that the eigenvalues are positive and simple and that the residues are positive (provided that the point masses in m and n are positive). Next we will show that there are no further constraints on the spectral data, i.e., any set of such numbers are the spectral data of a unique pair of interlacing discrete measures. It will be convenient to introduce a bit of terminology first.

Definition 4.7. Let $\mathcal{P} \subset \mathbf{R}^{4K}$ (the "pure peakon sector") be the set of tuples

$$\mathbf{p} = (x_1, \dots, x_{2K}; m_1, n_2, \dots, m_{2K-1}, n_{2K})$$

satisfying

$$x_1 < \dots < x_{2K}, \quad \text{all } m_{2a-1} > 0, \quad \text{all } n_{2a} > 0,$$

and let $\mathcal{R} \subset \mathbf{R}^{4K}$ (the "set of admissible spectral data") be the set of tuples

$$\mathbf{r} = (\lambda_1, \dots, \lambda_K; \mu_1, \dots, \mu_{K-1}; a_1, \dots, a_K; b_1, \dots, b_{K-1}; b_\infty, b_\infty^*)$$

satisfying

$$0 < \lambda_1 < \dots < \lambda_K, \quad 0 < \mu_1 < \dots < \mu_{K-1}, \quad \text{all } a_i, b_j, b_\infty, b_\infty^* > 0.$$

The **forward spectral map** taking a point $\mathbf{p} \in \mathcal{P}$, representing a pair of interlacing discrete measures

$$m = 2\sum_{a=1}^{K} m_{2a-1}\, \delta_{x_{2a-1}} \quad \text{and} \quad n = 2\sum_{a=1}^{K} n_{2a}\, \delta_{x_{2a}},$$

to the corresponding spectral data $\mathbf{r} \in \mathcal{R}$ (as described in Section 3) will be denoted by

$$\mathcal{S} \colon \mathcal{P} \to \mathcal{R}. \tag{4.39}$$

The formulas in Corollary 4.5 (or Remark 4.6) define a function

$$\mathcal{T} \colon \mathcal{R} \to R^{4K} \tag{4.40}$$

which we will call the **inverse spectral map**.

Theorem 4.8. *For $K \geq 2$, the function $\mathcal{S}$ maps $\mathcal{P}$ bijectively onto $\mathcal{R}$, and $\mathcal{T} \colon \mathcal{R} \to \mathcal{P}$ is the inverse map. (See Section 4.3 for the case $K = 1$.)*

Proof. To begin with, $\mathcal{T}$ maps $\mathcal{R}$ into $\mathcal{P}$; this is the content of Lemma 4.9 below. By Corollary 4.5, $\mathcal{T} \circ \mathcal{S} = \mathrm{id}_{\mathcal{P}}$. Thus $\mathcal{S}$ is a homeomorphism onto its range. It remains to show that the range of $\mathcal{S}$ is all of $\mathcal{R}$ and that $\mathcal{S} \circ \mathcal{T} = \mathrm{id}_{\mathcal{R}}$. For this, it is most convenient to use the alternative description of the forward spectral map given in Appendix B, where the spectral data are defined directly in terms of $\{x_k, m_{2a-1}, n_{2a}\}$ without going via the transformation to the finite interval $[-1, 1]$. (At the beginning of Appendix B there is a summary comparing the two descriptions.) The eigenvalues λ_i and μ_j, as well as the residues a_i, b_j and b_∞, are all uniquely determined by certain polynomials $A(\lambda)$, $\widetilde{A}(\lambda)$, $B(\lambda)$ and $\widetilde{B}(\lambda)$ with the property that their coefficients are polynomials in the variables

$\{m_{2a-1}\, e^{\pm x_{2a-1}}, n_{2a}\, e^{\pm x_{2a}}\}$; see (B.11), (B.12), (B.19) and (B.20). Let us write $A(\lambda; \mathbf{p})$ (etc.) to indicate this dependence of the coefficients on the masses and positions. It is clear from the symmetry of the problem that b_∞^* could also be definied similarly (although we have chosen not to name and write out the corresponding polynomials, instead using (B.6) as the definition).

Now, since Remark 4.6 exhibits the variables $\{m_{2a-1}\, e^{\pm x_{2a-1}}, n_{2a}\, e^{\pm x_{2a}}\}$ as rational functions of the spectral variables $\mathbf{r}$, the coefficients in the polynomial $A(\lambda; \mathcal{T}(\mathbf{r}))$ are also rational functions of $\mathbf{r}$. Since $\mathcal{T} \circ \mathcal{S} = \mathrm{id}_{\mathcal{P}}$, we know that these coefficients agree with the coefficients of $\prod_{i=1}^{K}(1 - \lambda/\lambda_k)$ (see (B.3)) for each $\mathbf{r}$ in the range of $\mathcal{S}$ (which is an open set in $\mathcal{R}$ since $\mathcal{S}$ is a homeomorphism). Hence $A(\lambda; \mathcal{T}(\mathbf{r})) = \prod_{i=1}^{K}(1 - \lambda/\lambda_k)$ identically as a rational function of $\mathbf{r}$, and in particular this identity holds for any $\mathbf{r} \in \mathcal{R}$. The same argument works for the other polynomials involved in defining the spectral variables, and therefore $\mathcal{S} \circ \mathcal{T} = \mathrm{id}_{\mathcal{R}}$, as desired. ☐

Lemma 4.9. *The function $\mathcal{T}$ maps $\mathcal{R}$ into $\mathcal{P}$, i.e., the formulas in Corollary 4.5 give **positive** masses $m_{2a-1} > 0$ and $n_{2a} > 0$, and **ordered** positions $x_1 < \cdots < x_{2K}$, for any spectral data in the admissible set $\mathcal{R}$.*

Proof. Positivity of m_{2j-1} and n_{2j} is obvious. To show that the positions x_k are ordered, we will use the formulas (4.37) for $q_k = \frac{1}{2}\exp 2x_k$ and show that $q_1 < \cdots < q_{2K}$.

The outermost intervals present no problems, since

$$q_{2K} - q_{2K-1} = (I_{00} + b_\infty \alpha_0) - \frac{\mathcal{J}_{11}^{00}}{\mathcal{J}_{00}^{11}} = b_\infty \alpha_0 > 0$$

and

$$\frac{1}{q_1} - \frac{1}{q_2} = \frac{\mathcal{J}_{K-1,K-2}^{11} + \dfrac{2b_\infty^* L}{M}\, \mathcal{J}_{K-1,K-1}^{10}}{\mathcal{J}_{K,K-1}^{00}} - \frac{\mathcal{J}_{K-1,K-2}^{11}}{\mathcal{J}_{K,K-1}^{00}} = \frac{2b_\infty^* L}{M}\, \frac{\mathcal{J}_{K-1,K-1}^{10}}{\mathcal{J}_{K,K-1}^{00}} > 0.$$

As for the other distances, the differences

$$q_{2r} - q_{2r-1} = \frac{\mathcal{J}_{j,j-1}^{00}}{\mathcal{J}_{j-1,j-2}^{11}} - \frac{\mathcal{J}_{jj}^{00}}{\mathcal{J}_{j-1,j-1}^{11}}, \qquad j = 2, \dots, K-1,$$

and

$$q_{2r-1} - q_{2r-2} = \frac{\mathcal{J}_{jj}^{00}}{\mathcal{J}_{j-1,j-1}^{11}} - \frac{\mathcal{J}_{j+1,j}^{00}}{\mathcal{J}_{j,j-1}^{11}}, \qquad j = 1, \dots, K-1,$$

are all positive according to (the fairly technical) Lemma A.4 in Section A.4. ☐

Example 4.10. Let us explicitly write out the solution formulas for the inverse problem in the case $K = 2$, by expanding the sums $\mathcal{J}_{nm}^{rs}$ (including $I_{00} = \mathcal{J}_{11}^{00}$) as explained in Section A.4; recall that the lower indices n and m give the number of factors a_i and b_j in each term, and also determine the dimensions of the accompanying Vandermonde-like factors Ψ_{IJ} (see (A.33)), while the upper indices r and s are the powers to which the additional factors λ_i and μ_j appear. The spectral data are

$$\lambda_1, \lambda_2, \mu_1, a_1, a_2, b_1, b_\infty, b_\infty^*,$$

and we want to recover

$$x_1, x_2, x_3, x_4, m_1, n_2, m_3, n_4.$$

In terms of the quantities from Remark 4.6, we get

$$\begin{aligned}
\tfrac{1}{2}e^{2x_4} &= I_{00} + b_\infty \alpha_0 = \frac{a_1 b_1}{\lambda_1+\mu_1} + \frac{a_2 b_1}{\lambda_2+\mu_1} + b_\infty(a_1+a_2),\\
\tfrac{1}{2}e^{2x_3} &= \frac{\mathcal{J}_{11}^{00}}{\mathcal{J}_{00}^{11}} = \frac{I_{00}}{1} = \frac{a_1 b_1}{\lambda_1+\mu_1} + \frac{a_2 b_1}{\lambda_2+\mu_1},\\
\tfrac{1}{2}e^{2x_2} &= \frac{\mathcal{J}_{21}^{00}}{\mathcal{J}_{10}^{11}} = \frac{\dfrac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)}a_1a_2b_1}{\lambda_1 a_1 + \lambda_2 a_2},\\
\tfrac{1}{2}e^{2x_1} &= \frac{\mathcal{J}_{21}^{00}}{\mathcal{J}_{10}^{11} + \dfrac{2b_\infty^* L}{M}\mathcal{J}_{11}^{10}}\\
&= \frac{\dfrac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)}a_1a_2b_1}{\lambda_1 a_1 + \lambda_2 a_2 + \dfrac{2b_\infty^*\lambda_1\lambda_2}{\mu_1}\left(\dfrac{\lambda_1 a_1 b_1}{\lambda_1+\mu_1} + \dfrac{\lambda_2 a_2 b_1}{\lambda_2+\mu_1}\right)}
\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}
2n_4 e^{-x_4} &= \frac{1}{\alpha_0} = \frac{1}{a_1+a_2},\\
2m_3 e^{-x_3} &= \frac{\mathcal{J}_{00}^{11}\mathcal{J}_{10}^{01}}{\mathcal{J}_{11}^{10}\mathcal{J}_{00}^{10}} = \frac{1\cdot\mathcal{J}_{10}^{01}}{\mathcal{J}_{11}^{10}\cdot 1} = \frac{a_1+a_2}{\dfrac{\lambda_1 a_1 b_1}{\lambda_1+\mu_1} + \dfrac{\lambda_2 a_2 b_1}{\lambda_2+\mu_1}},\\
2n_2 e^{-x_2} &= \frac{\mathcal{J}_{10}^{11}\mathcal{J}_{11}^{10}}{\mathcal{J}_{10}^{01}\mathcal{J}_{21}^{01}} = \frac{(\lambda_1 a_1 + \lambda_2 a_2)\left(\dfrac{\lambda_1 a_1 b_1}{\lambda_1+\mu_1} + \dfrac{\lambda_2 a_2 b_1}{\lambda_2+\mu_1}\right)}{(a_1+a_2)\,\dfrac{\mu_1(\lambda_1-\lambda_2)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)}a_1a_2b_1},\\
2m_1 e^{-x_1} &= \frac{M\,\mathcal{J}_{10}^{11}}{L\,\mathcal{J}_{11}^{10}} + 2b_\infty^* = \frac{\mu_1(\lambda_1 a_1 + \lambda_2 a_2)}{\lambda_1\lambda_2\left(\dfrac{\lambda_1 a_1 b_1}{\lambda_1+\mu_1} + \dfrac{\lambda_2 a_2 b_1}{\lambda_2+\mu_1}\right)} + 2b_\infty^*.
\end{aligned} \tag{4.42}$$

Example 4.11. Similarly, in the case $K=3$ the spectral data are

$$\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, a_1, a_2, a_3, b_1, b_2, b_\infty, b_\infty^*,$$

and we want to recover

$$x_1, x_2, x_3, x_4, x_5, x_6, m_1, n_2, m_3, n_4, m_5, n_6.$$

The solution is

$$
\begin{aligned}
\tfrac{1}{2}e^{2x_6} &= I_{00} + b_\infty \alpha_0 = \sum_{i=1}^{3}\sum_{j=1}^{2} \frac{a_i b_j}{\lambda_i + \mu_j} + b_\infty (a_1 + a_2 + a_3),\\
\tfrac{1}{2}e^{2x_5} &= \frac{\mathcal{J}_{11}^{00}}{\mathcal{J}_{00}^{11}} = I_{00} = \sum_{i=1}^{3}\sum_{j=1}^{2} \frac{a_i b_j}{\lambda_i + \mu_j},\\
\tfrac{1}{2}e^{2x_4} &= \frac{\mathcal{J}_{21}^{00}}{\mathcal{J}_{10}^{11}} = \frac{\displaystyle\sum_{I=12,13,23}\sum_{j=1}^{2} \frac{(\lambda_{i_1}-\lambda_{i_2})^2}{(\lambda_{i_1}+\mu_j)(\lambda_{i_2}+\mu_j)}\, a_{i_1} a_{i_2} b_j}{\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3},\\
\tfrac{1}{2}e^{2x_3} &= \frac{\mathcal{J}_{22}^{00}}{\mathcal{J}_{11}^{11}} = \frac{\displaystyle\sum_{I=12,13,23} \frac{(\lambda_{i_1}-\lambda_{i_2})^2(\mu_1-\mu_2)^2}{\prod_{i\in I}\prod_{j=1}^{2}(\lambda_i+\mu_j)}\, a_{i_1} a_{i_2} b_1 b_2}{\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{2} \frac{\lambda_i \mu_j}{\lambda_i + \mu_j}\, a_i b_j},\\
\tfrac{1}{2}e^{2x_2} &= \frac{\mathcal{J}_{32}^{00}}{\mathcal{J}_{21}^{11}} = \frac{\displaystyle \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2(\mu_1-\mu_2)^2}{\prod_{i=1}^{3}\prod_{j=1}^{2}(\lambda_i+\mu_j)}\, a_1 a_2 a_3 b_1 b_2}{\displaystyle\sum_{I=12,13,23}\sum_{j=1}^{2} \frac{(\lambda_{i_1}-\lambda_{i_2})^2 \lambda_{i_1}\lambda_{i_2}\mu_j}{(\lambda_{i_1}+\mu_j)(\lambda_{i_2}+\mu_j)}\, a_{i_1} a_{i_2} b_j},\\
\tfrac{1}{2}e^{2x_1} &= \frac{\mathcal{J}_{32}^{00}}{\mathcal{J}_{21}^{11} + \dfrac{2b_\infty^* L}{M}\, \mathcal{J}_{22}^{10}}\\
&= \frac{\mathcal{J}_{32}^{00}}{\mathcal{J}_{21}^{11} + \dfrac{2b_\infty^* \lambda_1\lambda_2\lambda_3}{\mu_1\mu_2} \displaystyle\sum_{I=12,13,23} \frac{(\lambda_{i_1}-\lambda_{i_2})^2(\mu_1-\mu_2)^2\lambda_{i_1}\lambda_{i_2}}{\prod_{i\in I}\prod_{j=1}^{2}(\lambda_i+\mu_j)}\, a_{i_1} a_{i_2} b_1 b_2}
\end{aligned}
\tag{4.43}
$$

and

$$
\begin{aligned}
2n_6 e^{-x_6} &= \frac{1}{\alpha_0} = \frac{1}{a_1 + a_2 + a_3},\\
2m_5 e^{-x_5} &= \frac{\mathcal{J}_{00}^{11}\mathcal{J}_{10}^{01}}{\mathcal{J}_{11}^{10}\mathcal{J}_{00}^{10}} = \frac{\mathcal{J}_{10}^{01}}{\mathcal{J}_{11}^{10}} = \frac{a_1 + a_2 + a_3}{\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{2} \frac{\lambda_i}{\lambda_i + \mu_j}\, a_i b_j},\\
2n_4 e^{-x_4} &= \frac{\mathcal{J}_{10}^{11}\mathcal{J}_{11}^{10}}{\mathcal{J}_{10}^{01}\mathcal{J}_{21}^{01}},\\
2m_3 e^{-x_3} &= \frac{\mathcal{J}_{11}^{11}\mathcal{J}_{21}^{01}}{\mathcal{J}_{22}^{10}\mathcal{J}_{11}^{10}},\\
2n_2 e^{-x_2} &= \frac{\mathcal{J}_{21}^{11}\mathcal{J}_{22}^{10}}{\mathcal{J}_{21}^{01}\mathcal{J}_{32}^{01}},\\
2m_1 e^{-x_1} &= \frac{M\,\mathcal{J}_{21}^{11}}{L\,\mathcal{J}_{22}^{10}} + 2b_\infty^* = \frac{\mu_1\mu_2\,\mathcal{J}_{21}^{11}}{\lambda_1\lambda_2\lambda_3\,\mathcal{J}_{22}^{10}} + 2b_\infty^*.
\end{aligned}
\tag{4.44}
$$

(The last few right-hand sides are too large to write in expanded form here, but explicit expressions for the sums $\mathcal{J}^{rs}_{nm}$ are written out in Example A.2.)

Remark 4.12. Using the Lax pairs for the Geng–Xue equation, it is not difficult to show (details will be published elsewhere) that the peakon ODEs (1.13) induce the following time dependence for the spectral variables:

$$\dot{\lambda}_i = 0, \quad \dot{a}_i = \frac{a_i}{\lambda_i}, \quad \dot{\mu}_j = 0, \quad \dot{b}_j = \frac{b_j}{\mu_j}, \quad \dot{b}_\infty = 0, \quad \dot{b}^*_\infty = 0. \tag{4.45}$$

This means that the formulas in Corollary 4.5 give the solution to the peakon ODEs (1.13) in the interlacing case, if we let the variables $\{\lambda_i, \mu_j, b_\infty, b^*_\infty\}$ be constant, and let $\{a_i, b_j\}$ have the time dependence

$$a_i(t) = a_i(0)\, e^{t/\lambda_i}, \qquad b_j(t) = b_j(0)\, e^{t/\mu_j}, \tag{4.46}$$

and the coefficients derived in Section B.3 are constants of motion.

In particular, (4.41) and (4.42) give the solution to the $2+2$ interlacing peakon ODEs

$$\begin{aligned}
\dot{x}_1 &= (m_1 + m_3 E_{13})(n_2 E_{12} + n_4 E_{14}),\\
\dot{x}_2 &= (m_1 E_{12} + m_3 E_{23})(n_2 + n_4 E_{24}),\\
\dot{x}_3 &= (m_1 E_{13} + m_3)(n_2 E_{23} + n_4 E_{34}),\\
\dot{x}_4 &= (m_1 E_{14} + m_3 E_{34})(n_2 E_{24} + n_4),\\
\frac{\dot{m}_1}{m_1} &= (m_1 + m_3 E_{13})(n_2 E_{12} + n_4 E_{14}) - 2 m_3 E_{13}(n_2 E_{12} + n_4 E_{14}),\\
\frac{\dot{n}_2}{n_2} &= (-m_1 E_{12} + m_3 E_{23})(n_2 + n_4 E_{24}) - 2(m_1 E_{12} + m_3 E_{23}) n_4 E_{24},\\
\frac{\dot{m}_3}{m_3} &= (m_1 E_{13} + m_3)(-n_2 E_{23} + n_4 E_{34}) + 2 m_1 E_{13}(n_2 E_{23} + n_4 E_{34}),\\
\frac{\dot{n}_4}{n_4} &= (-m_1 E_{14} - m_3 E_{34})(n_2 E_{24} + n_4) + 2(m_1 E_{14} + m_3 E_{34}) n_2 E_{24},
\end{aligned} \tag{4.47}$$

where $E_{ij} = e^{-|x_i - x_j|} = e^{x_i - x_j}$ for $i < j$, and (4.43) and (4.44) give the solution to the corresponding ODEs for the case $K = 3$. Likewise, (4.55) below gives the solution to the $1+1$ peakon ODEs

$$\dot{x}_1 = \dot{x}_2 = \frac{\dot{m}_1}{m_1} = -\frac{\dot{n}_2}{n_2} = m_1 n_2 E_{12}. \tag{4.48}$$

However, in this last case the equations are rather trivial, and all the heavy machinery is not really required. Indeed, $m_1 n_2 E_{12}$ is a constant of motion, so direct integration gives $x_1(t) = x_1(0) + ct$, $x_2(t) = x_2(0) + ct$, $m_1(t) = m_1(0)e^{ct}$, $n_2(t) = n_2(0)e^{-ct}$, where $c = m_1(0)\, n_2(0)\, e^{x_1(0) - x_2(0)}$.

4.3 The case $K = 1$

As already mentioned in Remark 3.2, the case $K = 1$ is degenerate. We have

$$\begin{aligned}
S(\lambda) &= L_2(\lambda) \begin{bmatrix} h_1 \\ 0 \end{bmatrix} L_1(\lambda) \begin{bmatrix} 0 \\ g_1 \end{bmatrix} L_0(\lambda)\\
&= \begin{pmatrix} 1 - \lambda g_1 h_1 l_0 & h_1 & g_1 h_1 \\ -\lambda g_1 l_0 & 1 & g_1 \\ \lambda^2 g_1 h_1 l_0 l_2 - 2\lambda & -\lambda h_1 l_2 & 1 - \lambda g_1 h_1 l_2 \end{pmatrix}
\end{aligned}$$

and

$$\widetilde{S}(\lambda) = L_2(\lambda)\begin{bmatrix}0\\h_1\end{bmatrix}L_1(\lambda)\begin{bmatrix}g_1\\0\end{bmatrix}L_0(\lambda)$$
$$= \begin{pmatrix} 1 & g_1 & 0 \\ -\lambda h_1(l_0+l_1) & 1-\lambda g_1 h_1 l_1 & h_1 \\ -2\lambda & -\lambda g_1(l_1+l_2) & 1 \end{pmatrix},$$

and it follows (cf. (3.40)) that

$$W(\lambda) = -\frac{S_{21}(\lambda)}{S_{31}(\lambda)} = \frac{g_1 l_0}{\lambda g_1 h_1 l_0 l_2 - 2} = \frac{a_1}{\lambda - \lambda_1},$$

where

$$\lambda_1 = \frac{2}{g_1 h_1 l_0 l_2}, \qquad a_1 = \frac{1}{h_1 l_2}, \tag{4.49}$$

and

$$\widetilde{W}(\lambda) = -\frac{\widetilde{S}_{21}(\lambda)}{\widetilde{S}_{31}(\lambda)} = -\frac{h_1(l_0+l_1)}{2} = -b_\infty,$$

where

$$b_\infty = \frac{h_1(l_0+l_1)}{2} = \frac{h_1(2-l_2)}{2}. \tag{4.50}$$

Moreover (cf. (3.52)),

$$\widetilde{W}^*(\lambda) = -\frac{\widetilde{S}_{32}(\lambda)}{\widetilde{S}_{31}(\lambda)} = -\frac{g_1(l_1+l_2)}{2} = -b_\infty^*,$$

where

$$b_\infty^* = \frac{g_1(l_1+l_2)}{2} = \frac{g_1(2-l_0)}{2}. \tag{4.51}$$

We also have

$$W^*(\lambda) = -\frac{S_{32}(\lambda)}{S_{31}(\lambda)} = \frac{h_1 l_2}{\lambda g_1 h_1 l_0 l_2 - 2} = \frac{a_1^*}{\lambda - \lambda_1},$$

where

$$a_1^* = \frac{1}{g_1 l_0}, \tag{4.52}$$

so that (cf. Theorem 3.15)

$$a_1 a_1^* = \frac{1}{h_1 l_2}\cdot\frac{1}{g_1 l_0} = \frac{\lambda_1}{2}. \tag{4.53}$$

From these equations it follows that

$$\begin{aligned} \frac{1}{l_0} &= a_1^* b_\infty^* + \frac{1}{2}, \\ \frac{1}{l_2} &= a_1 b_\infty + \frac{1}{2}, \\ g_1 &= b_\infty^* + \frac{1}{2a_1^*}, \\ h_1 &= b_\infty + \frac{1}{2a_1}. \end{aligned} \tag{4.54}$$

Mapping back to the real line using $-1+l_0 = y_1 = \tanh x_1$, $1-l_2 = y_2 = \tanh x_2$, $g_1 = 2m_1 \cosh x_1$ and $h_1 = 2n_2 \cosh x_2$, we get

$$\begin{aligned} \tfrac{1}{2} e^{2x_2} &= a_1 b_\infty, \\ \tfrac{1}{2} e^{-2x_1} &= a_1^* b_\infty^*, \\ 2n_2 e^{-x_2} &= \frac{1}{a_1}, \\ 2m_1 e^{x_1} &= \frac{1}{a_1^*}, \end{aligned} \tag{4.55}$$

which recovers x_1, x_2, m_1, n_2 from the spectral data λ_1, a_1, b_∞, b_∞^* (and $a_1^* = (2a_1)^{-1}\lambda_1$).

It is clear that all the spectral variables are positive if m_1 and n_2 are positive, but there is an additional constraint (not present for $K \geq 2$): the ordering requirement $x_1 < x_2$ is fulfilled if and only if

$$1 < e^{2(x_2 - x_1)} = 4a_1 a_1^* b_\infty b_\infty^* = 2\lambda_1 b_\infty b_\infty^*. \tag{4.56}$$

In the terminology of Definition 4.7, the range of the forward spectral map $\mathcal{S}$ for $K = 1$ is not all of $\mathcal{R}$, but only the subset where $\lambda_1 b_\infty b_\infty^* > \frac{1}{2}$.

5 Concluding remarks

In this paper we have studied a third order non-selfadjoint boundary value problem coming from the Lax pair(s) of the nonlinear integrable PDE (1.2) put forward by Geng and Xue [14]. We have given a complete solution of the forward and inverse spectral problems in the case of two positive interlacing discrete measures. The main motivation for this is the explicit construction of peakons, a special class of weak solutions to the PDE; more details about this will be given in a separate paper. This inverse problem is closely related to the inverse problems for the *discrete cubic string* appearing in connection with peakon solutions to the Degasperis–Procesi equation (1.7), and for the *discrete dual cubic string* playing the corresponding role for Novikov's equation (1.5) (see [23] and [17], respectively), but it has the interesting new feature of involving two Lax pairs and two independent spectral measures.

A Cauchy biorthogonal polynomials

The theory of Cauchy biorthogonal polynomials, developed by Bertola, Gekhtman and Szmigielski [6, 5, 4, 7], provides a conceptual framework for understanding the approximation problems and determinants that appear in this paper. In Sections A.1 and A.2 below, we recall a few of the basic definitions and properties, just to give a flavour of the theory and put our results in a wider context. Section A.3 is the crucial one for the purpose of this paper; it contains determinant evaluations (and also defines notation) used in the main text.

A.1 Definitions

Let α and β be measures on the positive real axis, with finite moments

$$\alpha_k = \int x^k d\alpha(x), \qquad \beta_k = \int y^k d\beta(y), \tag{A.1}$$

and finite bimoments with respect to the Cauchy kernel $1/(x+y)$,

$$I_{ab} = \iint \frac{x^a y^b}{x+y} d\alpha(x) d\beta(y). \tag{A.2}$$

According to (A.17) below, the matrix $(I_{ab})_{a,b=0}^{n-1}$ has positive determinant D_n for every n, provided that α and β have infinitely many points of support. Then there are unique polynomials $(p_n(x))_{n=0}^{\infty}$ and $(q_n(y))_{n=0}^{\infty}$ such that

(i) $\deg p_n = \deg q_n = n$ for all n,

(ii) the biorthogonality condition

$$\int \frac{p_i(x)\, q_j(y)}{x+y} d\alpha(x) d\beta(y) = \delta_{ij} \tag{A.3}$$

holds for all i and j (where δ_{ij} is the Kronecker delta),

(iii) for each n, the leading coefficient of p_n is positive and equal to the leading coefficient of q_n.

These polynomials are given by the determinantal formulas

$$p_n(x) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} I_{00} & I_{01} & \cdots & I_{0,n-1} & 1 \\ I_{10} & I_{11} & \cdots & I_{1,n-1} & x \\ \vdots & \vdots & & \vdots & \vdots \\ I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n-1} & x^{n-1} \\ I_{n0} & I_{n1} & \cdots & I_{n,n-1} & x^n \end{vmatrix}, \tag{A.4}$$

$$q_n(y) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} I_{00} & I_{01} & \cdots & I_{0,n-1} & I_{0n} \\ I_{10} & I_{11} & \cdots & I_{1,n-1} & I_{1n} \\ \vdots & \vdots & & \vdots & \vdots \\ I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n-1} & I_{n-1,n} \\ 1 & y & \cdots & y^{n-1} & y^n \end{vmatrix}. \tag{A.5}$$

If either α or β (or both) is a discrete measure, the determinant D_n will be zero for all sufficiently large n, and then there will only be finitely many biorthogonal polynomials; cf. Section A.4.

A.2 Four-term recursion

One basic property of the Cauchy kernel which underlies much of the theory is the following:

$$\begin{aligned} I_{a+1,b} + I_{a,b+1} &= \iint \frac{x^{a+1} y^b}{x+y} d\alpha(x) d\beta(y) + \iint \frac{x^a y^{b+1}}{x+y} d\alpha(x) d\beta(y) \\ &= \iint x^a y^b d\alpha(x) d\beta(y) = \int x^a d\alpha(x) \int y^b d\beta(y) = \alpha_a \beta_b. \end{aligned} \tag{A.6}$$

For example, if $X = (X_{ij})_{i,j\geq 0}$ and $Y = (Y_{ij})_{i,j\geq 0}$ are the semi-infinite Hessenberg matrices (lower triangular plus an extra diagonal above the main one) defined by

$$x \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = X \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix}, \qquad y \begin{pmatrix} q_0(y) \\ q_1(y) \\ \vdots \end{pmatrix} = Y \begin{pmatrix} q_0(y) \\ q_1(y) \\ \vdots \end{pmatrix},$$

then it is straightforward to show that (A.6) implies

$$X_{nm} + Y_{mn} = \pi_n \eta_m, \tag{A.7}$$

where the numbers

$$\pi_n = \int p_n(x) d\alpha(x) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} I_{00} & I_{01} & \cdots & \alpha_0 \\ I_{10} & I_{11} & \cdots & \alpha_1 \\ \vdots & \vdots & & \vdots \\ I_{n0} & I_{n1} & \cdots & \alpha_n \end{vmatrix} \tag{A.8}$$

are positive by (A.19) below, and similarly for $\eta_m = \int q_m(y) d\beta(y)$. Since the matrix

$$L = \begin{pmatrix} -\pi_0^{-1} & \pi_1^{-1} & 0 & 0 & \cdots \\ 0 & -\pi_1^{-1} & \pi_2^{-1} & 0 & \\ 0 & 0 & -\pi_2^{-1} & \pi_3^{-1} & \\ \vdots & & & \ddots & \ddots \end{pmatrix} \tag{A.9}$$

kills the vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots)^T$, we have $0 = L\boldsymbol{\pi}\boldsymbol{\eta}^T = L(X+Y^T) = LX+LY^T$. If $M_{[a,b]}$ denotes the set of matrices which are zero outside of the band of diagonals number a to b inclusive (with the main diagonal as number zero, and subdiagonals labelled by negative numbers), then $LX \in M_{[0,1]} \cdot M_{[-\infty,1]} = M_{[-\infty,2]}$ and $LY^T \in M_{[0,1]} \cdot M_{[-1,\infty]} = M_{[-1,\infty]}$. But since their sum is zero, it follows that LX and LY^T are both in $M_{[-1,2]}$ (four-banded). Thus

$$x\, L \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix} = LX \begin{pmatrix} p_0(x) \\ p_1(x) \\ \vdots \end{pmatrix}, \tag{A.10}$$

with $L \in M_{[0,1]}$ and $LX \in M_{[-1,2]}$, is a four-term recurrence satisfied by the polynomials p_n.

The same argument applied to $X^T + Y = \boldsymbol{\eta}\boldsymbol{\pi}^T$ shows that the polynomials q_n satisfy a corresponding four-term recurrence

$$y\, \widetilde{L} \begin{pmatrix} q_0(y) \\ q_1(y) \\ \vdots \end{pmatrix} = \widetilde{L} Y \begin{pmatrix} q_0(y) \\ q_1(y) \\ \vdots \end{pmatrix}, \tag{A.11}$$

where $\widetilde{L}$ is like L except for $\boldsymbol{\pi}$ being replaced by $\boldsymbol{\eta}$.

Further generalizations of familiar properties from the theory of classical orthogonal polynomials include Christoffel–Darboux-like identities, interlacing of zeros, characterization by Hermite–Padé and Riemann–Hilbert problems, and connections to random matrix models; see [6, 5, 4, 7] for more information.

A.3 Determinant identities

For $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, let

$$\Delta(x) = \Delta(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j) \tag{A.12}$$

and

$$\Gamma(x) = \Gamma(x_1, \dots, x_n) = \prod_{i<j} (x_i + x_j), \tag{A.13}$$

where the right-hand sides are interpreted as 1 (empty products) if $n = 0$ or $n = 1$. Moreover, for $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$, let

$$\Gamma(x; y) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i + y_j). \tag{A.14}$$

Finally, let σ_n be the sector in $\mathbf{R}^n_+$ defined by the inequalities $0 < x_1 < \dots < x_n$. With this notation in place, we define

$$\mathcal{J}^{rs}_{nm} = \int_{\sigma_n \times \sigma_m} \frac{\Delta(x)^2 \Delta(y)^2 \left(\prod_{i=1}^n x_i\right)^r \left(\prod_{j=1}^m y_j\right)^s}{\Gamma(x; y)} d\alpha^n(x) d\beta^m(y), \tag{A.15}$$

for n and m positive. We also consider the degenerate cases

$$\begin{aligned}
\mathcal{J}^{rs}_{n0} &= \int_{\sigma_n} \Delta(x)^2 \Big(\prod_{i=1}^{n} x_i\Big)^r d\alpha^n(x) \qquad (n > 0), \\
\mathcal{J}^{rs}_{0m} &= \int_{\sigma_m} \Delta(y)^2 \Big(\prod_{j=1}^{m} y_j\Big)^s d\beta^m(y) \qquad (m > 0), \\
\mathcal{J}^{rs}_{00} &= 1.
\end{aligned} \tag{A.16}$$

Note that $\mathcal{J}^{rs}_{10} = \alpha_r$ and $\mathcal{J}^{rs}_{01} = \beta_s$.

When α and β are discrete measures, the integrals $\mathcal{J}^{rs}_{nm}$ reduce to sums; see Section A.4 below.

Many types of determinants involving the bimoments $I_{ab} = \iint \frac{x^a y^b}{x+y} d\alpha(x) d\beta(y)$ can be evaluated in terms of such integrals; these formulas are similar in spirit to Heine's formula for Hankel determinants of moments $\alpha_k = \int x^k d\alpha(x)$, encountered in the theory of orthogonal polynomials:

$$\det(\alpha_{i+j})_{i,j=0}^{n-1} = \int_{\sigma_n} \Delta(x)^2 d\alpha^n(x) = \mathcal{J}^{00}_{n0}.$$

Here we have collected a few such formulas (all formulated for determinants of size $n \times n$). To begin with, specializing Theorem 2.1 in [6] to the case of the Cauchy kernel $K(x, y) = 1/(x + y)$, we get the most basic bimoment determinant identity,

$$D_n = \det(I_{ij})_{i,j=0}^{n-1} = \begin{vmatrix} I_{00} & \dots & I_{0,n-1} \\ \vdots & & \vdots \\ I_{n-1,0} & \dots & I_{n-1,n-1} \end{vmatrix} = \mathcal{J}^{00}_{nn}. \tag{A.17}$$

Applying (A.17) with the measure $x^r\, d\alpha(x)$ in place of $d\alpha(x)$ and $y^s\, d\beta(y)$ in place of $d\beta(y)$ gives

$$\det(I_{r+i,s+j})_{i,j=0}^{n-1} = \mathcal{J}_{nn}^{rs}. \tag{A.18}$$

Proposition 3.1 in [6] says that

$$\begin{vmatrix} I_{00} & \dots & I_{0,n-2} & \alpha_0 \\ \vdots & & \vdots & \vdots \\ I_{n-1,0} & \dots & I_{n-1,n-2} & \alpha_{n-1} \end{vmatrix} = \mathcal{J}_{n,n-1}^{00}. \tag{A.19}$$

(For $n = 1$, the left-hand side should be read as the 1×1 determinant with the single entry α_0; this agrees with $\mathcal{J}_{10}^{00} = \alpha_0$.) By the same trick, we find from this that

$$\begin{vmatrix} I_{rs} & \dots & I_{r,s+n-2} & \alpha_r \\ \vdots & & \vdots & \vdots \\ I_{r+n-1,s} & \dots & I_{r+n-1,s+n-2} & \alpha_{r+n-1} \end{vmatrix} = \mathcal{J}_{n,n-1}^{rs}. \tag{A.20}$$

We also need the following identity:

$$\begin{vmatrix} \alpha_0 & I_{10} & \dots & I_{1,n-2} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1} & I_{n0} & \dots & I_{n,n-2} \end{vmatrix} = \mathcal{J}_{n,n-1}^{01} \tag{A.21}$$

Proof of (A.21). From (A.20) we have

$$\mathcal{J}_{n,n-1}^{01} = \begin{vmatrix} I_{01} & \dots & I_{0,n-1} & \alpha_0 \\ \vdots & & \vdots & \vdots \\ I_{n-1,1} & \dots & I_{n-1,n-1} & \alpha_{n-1} \end{vmatrix}.$$

We rewrite the bimoments as $I_{jk} = \alpha_j \beta_{k-1} - I_{j+1,k-1}$ (using (A.6)), and then subtract β_{k-1} times the last column from the other columns $k = 1, \dots, n-1$. This transforms the determinant into

$$\begin{vmatrix} -I_{10} & \dots & -I_{1,n-2} & \alpha_0 \\ \vdots & & \vdots & \vdots \\ -I_{n0} & \dots & -I_{n,n-2} & \alpha_{n-1} \end{vmatrix}$$

without changing its value. Now move the column of α's to the left; on its way, it passes each of the other columns, thereby cancelling all the minus signs. □

Other useful formulas follow from the Desnanot–Jacobi identity, also known as Lewis Carroll's identity (or as a special case of Sylvester's identity [12, Section II.3]): if X is an $n \times n$ determinant (with $n \geq 2$), then

$$XY = X^{nn} X^{11} - X^{1n} X^{n1}, \tag{A.22}$$

where X^{ij} is the subdeterminant of X obtained by removing row i and column j, and where $Y = (X^{nn})^{11}$ is the "central" subdeterminant of X obtained by removing the first and last row as well as the first and last column (for $n = 2$, we take $Y = 1$ by definition). For example, applying this identity to the bimoment determinant (A.18) (of size $n+1$ instead of n) gives

$$\mathcal{J}_{n+1,n+1}^{rs} \mathcal{J}_{n-1,n-1}^{r+1,s+1} = \mathcal{J}_{nn}^{rs} \mathcal{J}_{nn}^{r+1,s+1} - \mathcal{J}_{nn}^{r+1,s} \mathcal{J}_{nn}^{r,s+1}, \tag{A.23}$$

and from (A.21) we get

$$\mathcal{J}^{01}_{n+1,n}\mathcal{J}^{20}_{n-1,n-1} = \mathcal{J}^{01}_{n,n-1}\mathcal{J}^{20}_{nn} - \mathcal{J}^{11}_{n,n-1}\mathcal{J}^{10}_{nn} \tag{A.24}$$

(for $n \geq 1$, in both cases).

Another identity, which holds for arbitrary z_i, w_i, X_{ij}, is

$$\begin{vmatrix} z_1 & X_{11} & \dots & X_{1,n-1} \\ z_2 & X_{21} & \dots & X_{2,n-1} \\ \vdots & \vdots & & \vdots \\ z_n & X_{n1} & \dots & X_{n,n-1} \end{vmatrix} \begin{vmatrix} w_1 & X_{11} & \dots & X_{1,n-1} & X_{1n} \\ w_2 & X_{21} & \dots & X_{2,n-1} & X_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ w_n & X_{n1} & \dots & X_{n,n-1} & X_{nn} \\ w_{n+1} & X_{n+1,1} & \dots & X_{n+1,n-1} & X_{n+1,n+1} \end{vmatrix}$$
$$= \begin{vmatrix} w_1 & X_{11} & \dots & X_{1,n-1} \\ w_2 & X_{21} & \dots & X_{2,n-1} \\ \vdots & \vdots & & \vdots \\ w_n & X_{n1} & \dots & X_{n,n-1} \end{vmatrix} \begin{vmatrix} z_1 & X_{11} & \dots & X_{1,n-1} & X_{1n} \\ z_2 & X_{21} & \dots & X_{2,n-1} & X_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ z_n & X_{n1} & \dots & X_{n,n-1} & X_{nn} \\ z_{n+1} & X_{n+1,1} & \dots & X_{n+1,n-1} & X_{n+1,n+1} \end{vmatrix} -$$
$$\begin{vmatrix} X_{11} & \dots & X_{1,n-1} & X_{1n} \\ X_{21} & \dots & X_{2,n-1} & X_{2n} \\ \vdots & & \vdots & \vdots \\ X_{n1} & \dots & X_{n,n-1} & X_{nn} \end{vmatrix} \begin{vmatrix} z_1 & w_1 & X_{11} & \dots & X_{1,n-1} \\ z_2 & w_2 & X_{21} & \dots & X_{2,n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ z_n & w_n & X_{n1} & \dots & X_{n,n-1} \\ z_{n+1} & w_{n+1} & X_{n+1,1} & \dots & X_{n+1,n-1} \end{vmatrix}. \tag{A.25}$$

Indeed, the coefficients of z_{n+1} on the right-hand side cancel, and the coefficients of the other variables z_j on both sides agree, which can be seen by applying the Desnanot–Jacobi identity to the second determinant on the left-hand side with its jth row moved to the top.

In the text – see equation (4.5) – we encounter the $n \times n$ determinant

$$\mathcal{K}_n = \begin{vmatrix} I_{00} + \frac{1}{2} & I_{10} & \dots & I_{1,n-2} \\ I_{10} & I_{20} & \dots & I_{2,n-2} \\ \vdots & \vdots & & \vdots \\ I_{n-1,0} & I_{n0} & \dots & I_{n,n-2} \end{vmatrix}, \tag{A.26}$$

which satisfies the recurrence

$$\frac{\mathcal{K}_{n+1}}{\mathcal{J}^{01}_{n+1,n}} = \frac{\mathcal{K}_n}{\mathcal{J}^{01}_{n,n-1}} + \frac{\mathcal{J}^{10}_{nn}(\mathcal{J}^{00}_{n+1,n} + \frac{1}{2}\mathcal{J}^{11}_{n,n-1})}{\mathcal{J}^{01}_{n,n-1}\mathcal{J}^{01}_{n+1,n}}. \tag{A.27}$$

Proof of (A.27). By taking $z_i = \alpha_{i-1}$, $w_i = I_{i0} + \frac{1}{2}\delta_{i0}$ and $X_{ij} = I_{i,j-1}$ in (A.25), and using (A.18) and (A.21), we find that $\mathcal{J}^{01}_{n,n-1}\mathcal{K}_{n+1} = \mathcal{K}_n\mathcal{J}^{01}_{n+1,n} - \mathcal{J}^{10}_{nn}Z$, where

$$Z = \begin{vmatrix} \alpha_0 & I_{00} + \frac{1}{2} & I_{10} & I_{11} & \dots & I_{1,n-2} \\ \alpha_1 & I_{10} & I_{20} & I_{21} & \dots & I_{2,n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_n & I_{n0} & I_{n+1,0} & I_{n+1,1} & \dots & I_{n+1,n-2} \end{vmatrix}.$$

Using again the trick of rewriting the bimoments (except in column 2) as $I_{j+1,k} = \alpha_j\beta_k - I_{j,k+1}$, subtracting β_{k-3} times the first column from column k

(for $k = 3, \dots, n$), and moving the first column to the right, we see that

$$\begin{aligned} Z &= -\begin{vmatrix} I_{00} + \frac{1}{2} & I_{01} & I_{02} & \dots & I_{0,n-1} & \alpha_0 \\ I_{10} & I_{11} & I_{12} & \dots & I_{1,n-1} & \alpha_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ I_{n0} & I_{n,1} & I_{n,2} & \dots & I_{n,n-1} & \alpha_n \end{vmatrix} \\ &= -(\mathcal{J}^{00}_{n+1,n} + \tfrac{1}{2}\mathcal{J}^{11}_{n,n-1}). \end{aligned}$$

(The last equality follows from (A.20).) Consequently,

$$\mathcal{J}^{01}_{n,n-1}\mathcal{K}_{n+1} = \mathcal{K}_n\mathcal{J}^{01}_{n+1,n} + \mathcal{J}^{10}_{nn}(\mathcal{J}^{00}_{n+1,n} + \tfrac{1}{2}\mathcal{J}^{11}_{n,n-1}),$$

which is equivalent to (A.27). ☐

Remark A.1. It was shown in [23, Lemma 4.10] that when $\alpha = \beta$, the factorization

$$\mathcal{J}^{10}_{nn} = \frac{1}{2^n}\left(\int_{\sigma_n} \frac{\Delta(x)^2}{\Gamma(x)}\, d\alpha^n(x)\right)^2 \tag{A.28}$$

holds. We are not aware of anything similar in the general case with $\alpha \neq \beta$.

A.4 The discrete case

Consider next the bimoments I_{ab} and the integrals $\mathcal{J}^{rs}_{nm}$ defined by (A.15) in the case when α and β are discrete measures, say

$$\alpha = \sum_{i=1}^{A} a_i \delta_{\lambda_i}, \qquad \beta = \sum_{j=1}^{B} b_j \delta_{\mu_j}. \tag{A.29}$$

(In the setup in the main text, we have $A = K$ and $B = K - 1$.) Then the bimoments become

$$I_{ab} = \iint \frac{x^a y^b}{x+y} d\alpha(x) d\beta(y) = \sum_{i=1}^{A}\sum_{j=1}^{B} \frac{\lambda_i^a \mu_j^b}{\lambda_i + \mu_j}\, a_i b_j, \tag{A.30}$$

and likewise the integrals $\mathcal{J}^{rs}_{nm}$ turn into sums:

$$\mathcal{J}^{rs}_{nm} = \sum_{I \in \binom{[A]}{n}} \sum_{J \in \binom{[B]}{m}} \Psi_{IJ}\, \lambda_I^r a_I\, \mu_J^s b_J. \tag{A.31}$$

Here $\binom{[A]}{n}$ denotes the set of n-element subsets $I = \{i_1 < i_2 < \dots < i_n\}$ of the integer interval $[A] = \{1, 2, \dots, A\}$, and similarly for $\binom{[B]}{m}$. Morever,

$$\lambda_I^r a_I\, \mu_J^s b_J = \Big(\prod_{i\in I} \lambda_i^r a_i\Big)\Big(\prod_{j\in J} \mu_j^s b_j\Big) \tag{A.32}$$

and

$$\Psi_{IJ} = \frac{\Delta_I^2 \widetilde{\Delta}_J^2}{\Gamma_{IJ}}, \tag{A.33}$$

where we use the shorthand notation

$$\begin{aligned}\Delta_I^2 &= \Delta(\lambda_{i_1},\dots,\lambda_{i_n})^2 = \prod_{\substack{a,b\in I\\ a<b}} (\lambda_a-\lambda_b)^2,\\ \widetilde{\Delta}_J^2 &= \Delta(\mu_{j_1},\dots,\mu_{j_m})^2 = \prod_{\substack{a,b\in J\\ a<b}} (\mu_a-\mu_b)^2,\\ \Gamma_{IJ} &= \Gamma(\lambda_{i_1},\dots,\lambda_{i_n};\mu_{j_1},\dots,\mu_{j_m}) = \prod_{i\in I,\, j\in J} (\lambda_i+\mu_j).\end{aligned} \tag{A.34}$$

For later use, we also introduce the symbol

$$\Delta_{I_1 I_2}^2 = \prod_{i_1\in I_1,\, i_2\in I_2} (\lambda_{i_1}-\lambda_{i_2})^2, \tag{A.35}$$

and similarly for $\widetilde{\Delta}_{J_1 J_2}$. Empty products (as in Δ_I^2 when I is a singleton or the empty set) are taken to be 1 by definition. When needed for the sake of clarity, we will write $\Psi_{I,J}$ instead of Ψ_{IJ}, etc.

For (positive) measures on the positive real line (a_i, λ_i, b_j, μ_j positive), we thus have $\mathcal{J}_{nm}^{rs} > 0$ for $0 \le n \le A$ and $0 \le m \le B$, otherwise $\mathcal{J}_{nm}^{rs} = 0$.

Example A.2. Below we have listed the nonzero $\mathcal{J}_{nm}^{00}$ in the case $A = 3$ and $B = 2$ (the more general sum $\mathcal{J}_{nm}^{rs}$ is obtained by replacing each a_i and b_j in $\mathcal{J}_{nm}^{00}$ by $\lambda_i^r a_i$ and $\mu_j^s b_j$, respectively):

$$\begin{aligned}\mathcal{J}_{00}^{00} &= 1,\\ \mathcal{J}_{10}^{00} &= a_1+a_2+a_3,\\ \mathcal{J}_{20}^{00} &= (\lambda_1-\lambda_2)^2 a_1a_2 + (\lambda_1-\lambda_3)^2 a_1a_3 + (\lambda_2-\lambda_3)^2 a_2a_3,\\ \mathcal{J}_{30}^{00} &= (\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2 a_1a_2a_3,\end{aligned}$$

$$\begin{aligned}\mathcal{J}_{01}^{00} &= b_1+b_2,\\ \mathcal{J}_{11}^{00} &= I_{00}\\ &= \frac{1}{\lambda_1+\mu_1}a_1b_1 + \frac{1}{\lambda_2+\mu_1}a_2b_1 + \frac{1}{\lambda_3+\mu_1}a_3b_1\\ &\quad + \frac{1}{\lambda_1+\mu_2}a_1b_2 + \frac{1}{\lambda_2+\mu_2}a_2b_2 + \frac{1}{\lambda_3+\mu_2}a_3b_2,\\ \mathcal{J}_{21}^{00} &= \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)}a_1a_2b_1 + \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\mu_2)(\lambda_2+\mu_2)}a_1a_2b_2\\ &\quad + \frac{(\lambda_1-\lambda_3)^2}{(\lambda_1+\mu_1)(\lambda_3+\mu_1)}a_1a_3b_1 + \frac{(\lambda_1-\lambda_3)^2}{(\lambda_1+\mu_2)(\lambda_3+\mu_2)}a_1a_3b_2\\ &\quad + \frac{(\lambda_2-\lambda_3)^2}{(\lambda_2+\mu_1)(\lambda_2+\mu_1)}a_2a_3b_1 + \frac{(\lambda_2-\lambda_3)^2}{(\lambda_2+\mu_2)(\lambda_2+\mu_2)}a_2a_3b_2,\\ \mathcal{J}_{31}^{00} &= \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)(\lambda_3+\mu_1)}a_1a_2a_3b_1\\ &\quad + \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2}{(\lambda_1+\mu_2)(\lambda_2+\mu_2)(\lambda_3+\mu_2)}a_1a_2a_3b_2,\end{aligned}$$

$$\begin{aligned}
\mathcal{J}_{02}^{00} &= (\mu_1-\mu_2)^2 b_1 b_2,\\
\mathcal{J}_{12}^{00} &= \frac{(\mu_1-\mu_2)^2}{(\lambda_1+\mu_1)(\lambda_1+\mu_2)} a_1 b_1 b_2 + \frac{(\mu_1-\mu_2)^2}{(\lambda_2+\mu_1)(\lambda_2+\mu_2)} a_2 b_1 b_2\\
&\quad + \frac{(\mu_1-\mu_2)^2}{(\lambda_3+\mu_1)(\lambda_3+\mu_2)} a_3 b_1 b_2,\\
\mathcal{J}_{22}^{00} &= \frac{(\lambda_1-\lambda_2)^2(\mu_1-\mu_2)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)(\lambda_1+\mu_2)(\lambda_2+\mu_2)} a_1 a_2 b_1 b_2\\
&\quad + \frac{(\lambda_1-\lambda_3)^2(\mu_1-\mu_2)^2}{(\lambda_1+\mu_1)(\lambda_3+\mu_1)(\lambda_1+\mu_2)(\lambda_3+\mu_2)} a_1 a_3 b_1 b_2\\
&\quad + \frac{(\lambda_2-\lambda_3)^2(\mu_1-\mu_2)^2}{(\lambda_2+\mu_1)(\lambda_3+\mu_1)(\lambda_2+\mu_2)(\lambda_3+\mu_2)} a_2 a_3 b_1 b_2,\\
\mathcal{J}_{32}^{00} &= \frac{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_2-\lambda_3)^2(\mu_1-\mu_2)^2}{(\lambda_1+\mu_1)(\lambda_2+\mu_1)(\lambda_3+\mu_1)(\lambda_1+\mu_2)(\lambda_2+\mu_2)(\lambda_3+\mu_2)} a_1 a_2 a_3 b_1 b_2.
\end{aligned}$$

Lemma A.3. *Let $(\mathcal{J}^*)_{nm}^{rs}$ denote the integral $\mathcal{J}_{nm}^{rs}$ evaluated using the measures*

$$\alpha^* = \sum_{i=1}^{A} a_i^* \delta_{\lambda_i} \qquad \text{and} \qquad \beta^* = \sum_{j=1}^{B} b_i^* \delta_{\mu_j} \tag{A.36}$$

in place of α and β, and suppose that the four measures are related as in Theorem 3.15:

$$a_k a_k^* = \frac{\lambda_k \prod_{j=1}^{B}\left(1+\frac{\lambda_k}{\mu_j}\right)}{2\prod_{i=1, i\neq k}^{A}\left(1-\frac{\lambda_k}{\lambda_i}\right)^2}, \qquad b_k b_k^* = \frac{\mu_k \prod_{i=1}^{A}\left(1+\frac{\mu_k}{\lambda_i}\right)}{2\prod_{j=1, j\neq k}^{B}\left(1-\frac{\mu_k}{\mu_j}\right)^2}. \tag{A.37}$$

Then

$$\begin{aligned}
(\mathcal{J}^*)_{nm}^{rs} &= \frac{1}{2^{n+m}}\left(\prod_{i=1}^{A}\lambda_i\right)^{2n-m+r-1}\left(\prod_{j=1}^{B}\mu_j\right)^{2m-n+s-1}\frac{\mathcal{J}_{A-n,B-m}^{1-r,1-s}}{\mathcal{J}_{AB}^{00}}\\
&= \frac{\mathcal{J}_{A-n,B-m}^{1-r,1-s}}{2^{n+m}\mathcal{J}_{AB}^{m-2n+1-r,n-2m+1-s}}.
\end{aligned} \tag{A.38}$$

Proof. This is a fairly straightforward computation. To begin with, let $L = \prod_{i=1}^{A}\lambda_i$ and $M = \prod_{j=1}^{B}\mu_j$, and write (A.37) as

$$a_k a_k^* = \frac{L^2\,\Gamma_{\{k\},[B]}}{2\lambda_k M \prod_{i=1, i\neq k}^{A}(\lambda_i-\lambda_k)^2}, \qquad b_k b_k^* = \frac{M^2\,\Gamma_{[A],\{k\}}}{2\mu_k L \prod_{j=1, j\neq k}^{B}(\mu_j-\mu_k)^2}.$$

If $I \in \binom{[A]}{n}$, we therefore have

$$a_I a_I^* = \prod_{i\in I} a_i a_i^* = \frac{1}{\lambda_I}\left(\frac{L^2}{2M}\right)^n \frac{\Gamma_{I,[B]}}{\prod_{i\in I}\left(\prod_{t=1, t\neq i}^{A}(\lambda_t-\lambda_i)^2\right)}.$$

(The factor in front is $\lambda_I = \prod_{i \in I} \lambda_i$.) In the denominator, the factor $(\lambda_p - \lambda_q)^2$ will appear twice if p and q both belong to I, once if one of them does, and not at all if both belong to $[A] \setminus I$. Thus we can write

$$a_I a_I^* = \frac{1}{\lambda_I} \left(\frac{L^2}{2M} \right)^n \frac{\Gamma_{I,[B]}\, \Delta^2_{[A]\setminus I}}{\Delta_I^2\, \Delta^2_{[A]}}.$$

Similarly,

$$b_J b_J^* = \frac{1}{\mu_J} \left(\frac{M^2}{2L} \right)^m \frac{\Gamma_{[A],J}\, \widetilde{\Delta}^2_{[B]\setminus J}}{\widetilde{\Delta}_J^2\, \widetilde{\Delta}^2_{[B]}}.$$

Putting this into the definition of $(\mathcal{J}^*)^{rs}_{nm}$ gives

$$\begin{aligned} (\mathcal{J}^*)^{rs}_{nm} &= \sum_{I \in \binom{[A]}{n}} \sum_{J \in \binom{[B]}{m}} \Psi_{IJ}\, \lambda_I^r a_I^*\, \mu_J^s b_J^* \\ &= \sum_I \sum_J \left(\frac{\Delta_I^2 \widetilde{\Delta}_J^2}{\Gamma_{I,J}}\, \lambda_I^r\, \frac{1}{\lambda_I a_I} \left(\frac{L^2}{2M} \right)^n \frac{\Gamma_{I,[B]}\, \Delta^2_{[A]\setminus I}}{\Delta_I^2\, \Delta^2_{[A]}} \times \right. \\ &\qquad \left. \mu_J^s\, \frac{1}{\mu_J b_J} \left(\frac{M^2}{2L} \right)^m \frac{\Gamma_{[A],J}\, \widetilde{\Delta}^2_{[B]\setminus J}}{\widetilde{\Delta}_J^2\, \widetilde{\Delta}^2_{[B]}} \right) \end{aligned}$$

Now note that

$$\begin{aligned} \Gamma_{I,[B]} &= \Gamma_{I,J}\, \Gamma_{I,[B]\setminus J}, \\ \Gamma_{[A],J} &= \Gamma_{I,J}\, \Gamma_{[A]\setminus I,J}, \\ \Gamma_{[A],[B]} &= \Gamma_{I,J}\, \Gamma_{[A]\setminus I,J}\, \Gamma_{I,[B]\setminus J}\, \Gamma_{[A]\setminus I,[B]\setminus J}, \end{aligned}$$

which implies

$$\frac{\Gamma_{I,[B]}\, \Gamma_{[A],J}}{\Gamma_{I,J}} = \frac{\Gamma_{[A],[B]}}{\Gamma_{[A]\setminus I,[B]\setminus J}}.$$

Thus,

$$
\begin{aligned}
(\mathcal{J}^*)^{rs}_{nm} &= \sum_I \sum_J \Bigg(\frac{\lambda_I^{r-1}\mu_J^{s-1}L^{2n-m}M^{2m-n}}{2^{n+m}a_I b_J} \times \\
&\qquad \frac{\Gamma_{[A],[B]}}{\Delta^2_{[A]}\widetilde{\Delta}^2_{[B]}} \times \frac{\Delta^2_{[A]\setminus I}\widetilde{\Delta}^2_{[B]\setminus J}}{\Gamma_{[A]\setminus I,[B]\setminus J}} \Bigg) \\
&= \sum_I \sum_J \Bigg(\left(\frac{L}{\lambda_{[A]\setminus I}}\right)^{r-1} \left(\frac{M}{\mu_{[B]\setminus J}}\right)^{s-1} \frac{L^{2n-m}M^{2m-n}}{2^{n+m}} \times \\
&\qquad \frac{a_{[A]\setminus I}b_{[B]\setminus J}}{a_{[A]}b_{[B]}} \times \frac{\Psi_{[A]\setminus I,[B]\setminus J}}{\Psi_{[A],[B]}} \Bigg) \\
&= \frac{L^{2n-m+r-1}M^{2m-n+s-1}}{2^{n+m}\Psi_{[A],[B]}a_{[A]}b_{[B]}} \times \\
&\qquad \sum_I \sum_J \Psi_{[A]\setminus I,[B]\setminus J} \left(\lambda_{[A]\setminus I}\right)^{1-r} a_{[A]\setminus I} \left(\mu_{[B]\setminus J}\right)^{1-s} b_{[B]\setminus J} \\
&= \frac{L^{2n-m+r-1}M^{2m-n+s-1}}{2^{n+m}\mathcal{J}^{00}_{AB}} \sum_{U\in\binom{[A]}{A-n}} \sum_{V\in\binom{[B]}{B-m}} \Psi_{UV}\,\lambda_U^{1-r}a_U\,\mu_V^{1-s}b_V \\
&= \frac{L^{2n-m+r-1}M^{2m-n+s-1}}{2^{n+m}\mathcal{J}^{00}_{AB}}\,\mathcal{J}^{1-r,1-s}_{A-n,B-m},
\end{aligned}
$$

as claimed. □

The following lemma is needed in the proof of Lemma A.4.

Lemma A.4. *Suppose (as in the main text) that the number of point masses in* α *and* β *are* K *and* $K-1$*:*

$$\alpha = \sum_{i=1}^{K} a_i \delta_{\lambda_i}, \qquad \beta = \sum_{j=1}^{K-1} b_j \delta_{\mu_j}.$$

Then the quantities $\mathcal{J}^{rs}_{nm}$ *satisfy*

$$\mathcal{J}^{00}_{j,j-1}\mathcal{J}^{11}_{j-1,j-1} - \mathcal{J}^{00}_{jj}\mathcal{J}^{11}_{j-1,j-2} > 0, \qquad j = 2,\dots,K-1, \tag{A.39}$$

and

$$\mathcal{J}^{00}_{jj}\mathcal{J}^{11}_{j,j-1} - \mathcal{J}^{00}_{j+1,j}\mathcal{J}^{11}_{j-1,j-1} > 0, \qquad j = 1,\dots,K-1. \tag{A.40}$$

Proof. In (A.39) we let $j = m+1$ for convenience ($1 \le m \le K-2$), and expand the left-hand side using (A.31):

$$
\begin{aligned}
&\mathcal{J}^{00}_{m+1,m}\mathcal{J}^{11}_{mm} - \mathcal{J}^{00}_{m+1,m+1}\mathcal{J}^{11}_{m,m-1} \\
&= \left(\sum_{A\in\binom{[K]}{m+1}} \sum_{C\in\binom{[K-1]}{m}} \Psi_{AC}\, a_A\, b_C \right)\left(\sum_{B\in\binom{[K]}{m}} \sum_{D\in\binom{[K-1]}{m}} \Psi_{BD}\, \lambda_B\, \mu_D\, a_B\, b_D \right) \\
&\quad - \left(\sum_{A\in\binom{[K]}{m+1}} \sum_{C\in\binom{[K-1]}{m+1}} \Psi_{AC}\, a_A\, b_C \right)\left(\sum_{B\in\binom{[K]}{m}} \sum_{D\in\binom{[K-1]}{m-1}} \Psi_{BD}\, \lambda_B\, \mu_D\, a_B\, b_D \right) \\
&= \sum_{A\in\binom{[K]}{m+1}} \sum_{B\in\binom{[K]}{m}} \sum_{C\in\binom{[K-1]}{m}} \sum_{D\in\binom{[K-1]}{m}} \Psi_{AC}\Psi_{BD}\, \lambda_B\, \mu_D\, a_A\, a_B\, b_C\, b_D \\
&\quad - \sum_{A\in\binom{[K]}{m+1}} \sum_{B\in\binom{[K]}{m}} \sum_{C\in\binom{[K-1]}{m+1}} \sum_{D\in\binom{[K-1]}{m-1}} \Psi_{AC}\Psi_{BD}\, \lambda_B\, \mu_D\, a_A\, a_B\, b_C\, b_D.
\end{aligned}
\tag{A.41}
$$

Let us denote the summand by

$$
f(A, B, C, D) = \Psi_{AC}\Psi_{BD}\, \lambda_B\, \mu_D\, a_A\, a_B\, b_C\, b_D
$$

for simplicity.

Choosing two subsets A and B of a set is equivalent to first choosing $R = A\cap B$ and then choosing two disjoint sets $X = A \setminus (A\cap B)$ and $Y = B \setminus (A\cap B)$ among the remaining elements. If $|A| = m+1$ and $|B| = m$, then

$$
|A \cap B| = m - k, \qquad |A \setminus (A\cap B)| = k+1, \qquad |B \setminus (A \cap B)| = k,
$$

for some $k \in \{0, 1, \ldots, m\}$. Thus

$$
\sum_{A\in\binom{[K]}{m+1}} \sum_{B\in\binom{[K]}{m}} f(A, B, C, D) = \sum_{k=0}^{m} \sum_{R\in\binom{[K]}{m-k}} \sum_{\substack{X\in\binom{[K]\setminus R}{k+1} \\ Y\in\binom{[K]\setminus R}{k} \\ X\cap Y=\emptyset}} f(R+X, R+Y, C, D),
$$

where we write $R + X$ rather than $R \cup X$, in order to indicate that it is a union of *disjoint* sets. (If $K - (m - k) < 2k + 1$, then the innermost sum is empty.) With similar rewriting for C and D, (A.41) becomes

$$
\begin{aligned}
&\sum_{k=0}^{m}\sum_{l=0}^{m-1} \sum_{R\in\binom{[K]}{m-k}} \sum_{S\in\binom{[K-1]}{m-l}} \sum_{\substack{X\in\binom{[K]\setminus R}{k+1} \\ Y\in\binom{[K]\setminus R}{k} \\ X\cap Y=\emptyset}} \sum_{\substack{Z\in\binom{[K-1]\setminus S}{l} \\ W\in\binom{[K-1]\setminus S}{l} \\ Z\cap W=\emptyset}} f(R+X, R+Y, S+Z, S+W) \\
-&\sum_{k=0}^{m}\sum_{l=0}^{m-1} \sum_{R\in\binom{[K]}{m-k}} \sum_{S\in\binom{[K-1]}{m-l}} \sum_{\substack{X\in\binom{[K]\setminus R}{k+1} \\ Y\in\binom{[K]\setminus R}{k} \\ X\cap Y=\emptyset}} \sum_{\substack{Z\in\binom{[K-1]\setminus S}{l+1} \\ W\in\binom{[K-1]\setminus S}{l-1} \\ Z\cap W=\emptyset}} f(R+X, R+Y, S+Z, S+W).
\end{aligned}
\tag{A.42}
$$

(The innermost sum on the second line is empty when $l = 0$.) Now, since

$$
\begin{aligned}
\Psi_{R+X,S+Z}\Psi_{R+Y,S+W} &= \frac{\Delta^2_{R+X}\widetilde{\Delta}^2_{S+Z}}{\Gamma_{R+X,S+Z}}\,\frac{\Delta^2_{R+Y}\widetilde{\Delta}^2_{S+W}}{\Gamma_{R+Y,S+W}} \\
&= \frac{\Delta^2_R\Delta^2_X\Delta^2_{RX}\widetilde{\Delta}^2_S\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_{SZ}}{\Gamma_{RS}\Gamma_{RZ}\Gamma_{XS}\Gamma_{XZ}}\,\frac{\Delta^2_R\Delta^2_Y\Delta^2_{RY}\widetilde{\Delta}^2_S\widetilde{\Delta}^2_W\widetilde{\Delta}^2_{SW}}{\Gamma_{RS}\Gamma_{RW}\Gamma_{YS}\Gamma_{YW}} \\
&= \Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XW}\Gamma_{YZ}\,\frac{\Delta^4_R\Delta^2_{R,X+Y}\widetilde{\Delta}^4_S\widetilde{\Delta}^2_{S,Z+W}}{\Gamma_{R+X+Y,S+Z+W}\Gamma_{RS}},
\end{aligned}
$$

we have

$$
\begin{aligned}
&f(R+X,R+Y,S+Z,S+W) \\
&= \Psi_{R+X,S+Z}\Psi_{R+Y,S+W}\,\lambda_{R+Y}\,\mu_{S+W}\,a_{R+X}\,a_{R+Y}\,b_{S+Z}\,b_{S+W} \\
&= \left(\Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XW}\Gamma_{YZ}\,\lambda_Y\,\mu_W\right)\times \\
&\quad \left(\frac{\Delta^4_R\Delta^2_{R,X+Y}\widetilde{\Delta}^4_S\widetilde{\Delta}^2_{S,Z+W}}{\Gamma_{R+X+Y,S+Z+W}\Gamma_{RS}}\,\lambda_R\,\mu_S\,a^2_R\,a_{X+Y}\,b^2_S\,b_{Z+W}\right),
\end{aligned}
$$

where the first factor depends on the sets X, Y, Z and W individually, while the second factor involves only their unions $U = X+Y$ and $V = Z+W$. Therefore we can write (A.42) as

$$
\begin{aligned}
&\sum_{k=0}^{m}\sum_{l=0}^{m-1}\sum_{R\in\binom{[K]}{m-k}}\sum_{S\in\binom{[K-1]}{m-l}}\sum_{U\in\binom{[K]\setminus R}{2k+1}}\sum_{V\in\binom{[K-1]\setminus S}{2l}}\left(\frac{\Delta^4_R\Delta^2_{RU}\widetilde{\Delta}^4_S\widetilde{\Delta}^2_{SV}}{\Gamma_{R+U,S+V}\Gamma_{RS}}\times\right. \\
&\quad \lambda_R\,\mu_S\,a^2_R\,a_U\,b^2_S\,b_V\Bigg(\sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l}}\Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W \\
&\qquad\left.-\sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l+1\\|W|=l-1}}\Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W\Bigg)\right)
\end{aligned}
\tag{A.43}
$$

This is in fact positive (which is what we wanted to prove), because of the following identity for the expression in brackets: if $|U| = 2k+1$ and $|V| = 2l$, then

$$
\begin{aligned}
&\sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l}}\Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W \\
&\qquad -\sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l+1\\|W|=l-1}}\Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W \\
&= \sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l}}\Delta^2_X\Delta^2_Y\widetilde{\Delta}^2_Z\widetilde{\Delta}^2_W\Gamma_{XZ}\Gamma_{YW}\lambda_Y\mu_W.
\end{aligned}
\tag{A.44}
$$

(Note the change from $\Gamma_{XW}\Gamma_{YZ}$ on the left to $\Gamma_{XZ}\Gamma_{YW}$ on the right.)

To prove (A.44), we can take $U = [2k{+}1]$ and $V = [2l]$ without loss of generality. If $l = 0$, the identity is trivial, since both sides reduce to $\sum_{X+Y=U} \Delta_X^2 \Delta_Y^2 \lambda_Y$. The rest of the proof concerns the case $l \geq 1$.

When $k = 0$ and $l = 1$, both sides reduce to $\lambda_1(\mu_1 + \mu_2) + 2\mu_1\mu_2$. For fixed (k, l) with $k \geq 1$, evaluation at $\lambda_{2k} = \lambda_{2k+1} = c$ gives on both sides $2c \prod_{i=1}^{2k-1}(\lambda_i - c)^2 \prod_{j=1}^{2l}(c + \mu_j)$ times the corresponding expression with $k - 1$ instead of k. Provided that the identity for $(k-1, l)$ is true, our (k, l) identity therefore holds when $\lambda_{2k} = \lambda_{2k+1}$, and in fact (because of the symmetry) whenever any two λ_i are equal. This implies that the difference between the left-hand side and the right-hand side is divisible by Δ_U^2. (Any polynomial p in the variables λ_i which vanishes whenever two λ_i are equal is divisible by Δ_U. If in addition p is a *symmetric* polynomial, then p/Δ_U is antisymmetric and therefore again vanishes whenever two λ_i are equal; hence p/Δ_U is divisible by Δ_U.) Considered as polynomials in λ_1, the difference has degree $2k + l$ and Δ_U^2 has degree $4k$. If $l < 2k$, then $2k + l < 4k$, and in this case we can conclude that the difference must be identically zero. To summarize: if the $(k-1, l)$ identity is true and $l < 2k$, then the (k, l) identity is also true.

Similarly, for fixed (k, l) with $l \geq 2$, evaluation at $\mu_{2l-1} = \mu_{2l} = c$ gives $2c \prod_{i=1}^{2k+1}(\lambda_i + c) \prod_{j=1}^{2l-2}(\mu_j - c)^2$ times the corresponding identity with $l - 1$ instead of l. As polynomials in μ_1, the difference between the left-hand side and the right-hand side has degree $2l + k$ and $\widetilde{\Delta}_V^2$ has degree $4l - 2$. The same argument as above shows that if the $(k, l-1)$ identity is true and $k < 2l - 2$, then the (k, l) identity is also true.

Since any integer pair (k, l) with $k \geq 0$ and $l \geq 1$, except $(k, l) = (0, 1)$, satisfies at least one of the inequalities $l < 2k$ or $k < 2l - 2$, we can work our way down to the already proved base case $(0, 1)$ from any other (k, l) by decreasing either k or l by one in each step. This concludes the proof of (A.44), and thereby (A.39) is also proved.

The proof of (A.40) is similar: the left-hand side expands to

$$\begin{aligned}
&\sum_{k=0}^{j} \sum_{l=1}^{j} \sum_{R \in \binom{[K]}{j-k}} \sum_{S \in \binom{[K-1]}{j-l}} \sum_{\substack{X \in \binom{[K] \setminus R}{k} \\ Y \in \binom{[K] \setminus R}{k} \\ X \cap Y = \emptyset}} \sum_{\substack{Z \in \binom{[K-1] \setminus S}{l} \\ W \in \binom{[K-1] \setminus S}{l-1} \\ Z \cap W = \emptyset}} f(R+X, R+Y, S+Z, S+W) \\
&- \sum_{k=0}^{j} \sum_{l=1}^{j} \sum_{R \in \binom{[K]}{j-k}} \sum_{S \in \binom{[K-1]}{j-l}} \sum_{\substack{X \in \binom{[K] \setminus R}{k+1} \\ Y \in \binom{[K] \setminus R}{k-1} \\ X \cap Y = \emptyset}} \sum_{\substack{Z \in \binom{[K-1] \setminus S}{l} \\ W \in \binom{[K-1] \setminus S}{l-1} \\ Z \cap W = \emptyset}} f(R+X, R+Y, S+Z, S+W),
\end{aligned} \tag{A.45}$$

which equals

$$\sum_{k=0}^{j}\sum_{l=1}^{j}\sum_{R\in\binom{[K]}{j-k}}\sum_{S\in\binom{[K-1]}{j-l}}\sum_{U\in\binom{[K]\setminus R}{2k}}\sum_{V\in\binom{[K-1]\setminus S}{2l-1}}\Bigg(\frac{\Delta_R^4\Delta_{RU}^2\widetilde{\Delta}_S^4\widetilde{\Delta}_{SV}^2}{\Gamma_{R+U,S+V}\Gamma_{RS}}\times$$
$$\lambda_R\,\mu_S\,a_R^2\,a_U\,b_S^2\,b_V\Bigg(\sum_{\substack{X+Y=U\\|X|=k\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l-1}}\Delta_X^2\Delta_Y^2\widetilde{\Delta}_Z^2\widetilde{\Delta}_W^2\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W$$
$$-\sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k-1}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l-1}}\Delta_X^2\Delta_Y^2\widetilde{\Delta}_Z^2\widetilde{\Delta}_W^2\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W\Bigg)\Bigg), \tag{A.46}$$

which is positive, since for $|U|=2k$ and $|V|=2l-1$ the identity

$$\sum_{\substack{X+Y=U\\|X|=k\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l-1}}\Delta_X^2\Delta_Y^2\widetilde{\Delta}_Z^2\widetilde{\Delta}_W^2\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W$$
$$-\sum_{\substack{X+Y=U\\|X|=k+1\\|Y|=k-1}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l-1}}\Delta_X^2\Delta_Y^2\widetilde{\Delta}_Z^2\widetilde{\Delta}_W^2\Gamma_{XW}\Gamma_{YZ}\lambda_Y\mu_W \tag{A.47}$$
$$=\sum_{\substack{X+Y=U\\|X|=k\\|Y|=k}}\sum_{\substack{Z+W=V\\|Z|=l\\|W|=l-1}}\Delta_X^2\Delta_Y^2\widetilde{\Delta}_Z^2\widetilde{\Delta}_W^2\Gamma_{XZ}\Gamma_{YW}\lambda_Y\mu_W$$

holds; it is proved using the same technique as above. □

B The forward spectral problem on the real line

Consider the Lax equations (1.3a) and (1.4a) in the interlacing discrete case (3.1). In what follows, we will start from scratch and analyze these equations directly on real line, without passing to the finite interval $(-1,1)$ via the transformation (2.1). As we will see, this leads in a natural way to the definition of certain polynomials $\{A_k(\lambda),B_k(\lambda),C_k(\lambda)\}_{k=0}^N$. Since the two approaches are equivalent (cf. Remark 2.1), these polynomials are of course related to the quantitites defined in Section 3. Before delving into the details, let us just state these relations, for the sake of comparison.

In the interval $y_k<y<y_{k+1}$, the wave function $\Phi(y;\lambda)$ is given by

$$\begin{pmatrix}\phi_1(y;\lambda)\\ \phi_2(y;\lambda)\\ \phi_3(y;\lambda)\end{pmatrix}=\begin{pmatrix}A_k(\lambda)-\lambda C_k(\lambda)\\ -2\lambda B_k(\lambda)\\ -\lambda(1+y)A_k(\lambda)+\lambda^2(1-y)C_k(\lambda)\end{pmatrix}. \tag{B.1}$$

Hence, letting $y\to 1^-$ we obtain (with (A,B,C) as synonyms for (A_N,B_N,C_N))

$$\begin{pmatrix}\phi_1(1;\lambda)\\ \phi_2(1;\lambda)\\ \phi_3(1;\lambda)\end{pmatrix}=\begin{pmatrix}A(\lambda)-\lambda C(\lambda)\\ -2\lambda B(\lambda)\\ -2\lambda A(\lambda)\end{pmatrix}. \tag{B.2}$$

As for the spectra and the Weyl functions, we have

$$A(\lambda) = \prod_{k=1}^{K} \left(1 - \frac{\lambda}{\lambda_k}\right), \qquad \widetilde{A}(\lambda) = \prod_{k=1}^{K-1} \left(1 - \frac{\lambda}{\mu_k}\right), \tag{B.3}$$

$$W(\lambda) = -\frac{\phi_2(1;\lambda)}{\phi_3(1;\lambda)} = -\frac{B(\lambda)}{A(\lambda)}, \qquad Z(\lambda) = -\frac{\phi_1(1;\lambda)}{\phi_3(1;\lambda)} = \frac{1}{2\lambda} - \frac{C(\lambda)}{A(\lambda)}, \tag{B.4}$$

$$\widetilde{W}(\lambda) = -\frac{\widetilde{\phi}_2(1;\lambda)}{\widetilde{\phi}_3(1;\lambda)} = -\frac{\widetilde{B}(\lambda)}{\widetilde{A}(\lambda)}, \qquad \widetilde{Z}(\lambda) = -\frac{\widetilde{\phi}_1(1;\lambda)}{\widetilde{\phi}_3(1;\lambda)} = \frac{1}{2\lambda} - \frac{\widetilde{C}(\lambda)}{\widetilde{A}(\lambda)}. \tag{B.5}$$

The residues a_i, b_j and b_∞ are defined from the Weyl functions as before; see Theorem 3.10. One can also define similar polynomials corresponding to the adjoint Weyl functions, in order to define b_∞^* as in (3.52), but it is perhaps more convenient to define b_∞^* using the relations (3.56) and (3.58), which in this setting take the form

$$b_\infty b_\infty^* = \begin{cases} \dfrac{1}{2}\left(\displaystyle\prod_{i=1}^{K} \frac{(1-E_{2i-1,2i}^2)}{\lambda_i\, E_{2i-1,2i}^2}\right)\left(\displaystyle\prod_{j=1}^{K-1} \frac{\mu_j}{(1-E_{2j,2j+1}^2)}\right), & K \geq 2, \\ \dfrac{1}{2\lambda_1 E_{12}^2}, & K = 1, \end{cases} \tag{B.6}$$

where $E_{ij} = e^{-|x_i - x_j|} = e^{x_i - x_j}$ for $i < j$.

B.1 Setup

To begin with, recall equation (1.3a), which determines $\Psi(x;z)$:

$$\begin{aligned} \partial_x \psi_1(x;z) &= z n(x) \psi_2(x;z) + \psi_3(x;z), \\ \partial_x \psi_2(x;z) &= z m(x) \psi_3(x;z), \\ \partial_x \psi_3(x;z) &= \psi_1(x;z), \end{aligned} \tag{B.7}$$

Away from the points x_k where the distributions m and n are supported, this reduces to

$$\partial_x \psi_1 = \psi_3, \qquad \partial_x \psi_2 = 0, \qquad \partial_x \psi_3 = \psi_1,$$

so $\psi_2(x;z)$ is piecewise constant, and $\psi_1(x;z)$ and $\psi_3(x;z)$ are piecewise linear combinations of e^x and e^{-x}. It is convenient to write this as

$$\begin{pmatrix} \psi_1(x;z) \\ \psi_2(x;z) \\ \psi_3(x;z) \end{pmatrix} = \begin{pmatrix} A_k e^x + z^2 C_k e^{-x} \\ 2z B_k \\ A_k e^x - z^2 C_k e^{-x} \end{pmatrix}, \qquad x_k < x < x_{k+1}, \tag{B.8}$$

where the coefficients $\{A_k, B_k, C_k\}_{k=0}^N$ may depend on z but not on x. (Here we set $x_0 = -\infty$ and $x_{N+1} = +\infty$, so that the x axis splits into $N+1$ intervals $x_k < x < x_{k+1}$ numbered by $k = 0, 1, \ldots, N$.) Then the conditions (2.4) and (2.5) for $\Psi(x;z)$ at $\pm\infty$ translate into

$$B_0 = C_0 = 0 = A_N, \qquad A_0 = 1, \tag{B.9}$$

respectively. So we impose $(A_0, B_0, C_0) = (1, 0, 0)$ (i.e., $\Psi(x;z) = (e^x, 0, e^x)^T$ for $x < x_1$) and investigate for which z the condition $A_N(z) = 0$ is satisfied; the corresponding values $\lambda = -z^2$ will be the eigenvalues considered in the main text (cf. Remark 2.2).

The pieces (B.8) are stitched together by evaluating equations (B.7) at the sites $x = x_k$. Since the Dirac delta is the distributional derivative of the Heaviside step function, a jump in ψ_i at x_k will give rise to a Dirac delta term δ_{x_k} in $\partial_x \psi_i$, whose coefficient must match that of the corresponding Dirac delta coming from m or n on the right-hand side of (B.7). Denoting jumps by $\big[f(x_k)\big] = f(x_k^+) - f(x_k^-)$, we find at the odd-numbered sites $x = x_k = x_{2a-1}$ (where m is supported) the jump conditions

$$\begin{aligned} \big[\psi_1(x_k;z)\big] &= 0, \\ \big[\psi_2(x_k;z)\big] &= 2zm_k\psi_3(x_k), \\ \big[\psi_3(x_k;z)\big] &= 0, \end{aligned}$$

while at the even-numbered sites $x = x_k = x_{2a}$ (where n is supported) we get

$$\begin{aligned} \big[\psi_1(x_k;z)\big] &= 2zn_k\psi_2(x_k), \\ \big[\psi_2(x_k;z)\big] &= 0, \\ \big[\psi_3(x_k;z)\big] &= 0. \end{aligned}$$

Upon expressing the left and right limits $\psi_i(x_k^\pm)$ using (B.8), these jump conditions translate into linear equations relating (A_k, B_k, C_k) to $(A_{k-1}, B_{k-1}, C_{k-1})$. Solving for (A_k, B_k, C_k) yields

$$\begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix} = S_k(-z^2) \begin{pmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{pmatrix}, \tag{B.10}$$

with the jump matrix S_k (not to be confused with the transition matrix $S(\lambda)$ defined by (2.12) and used in the main text) defined by

$$S_k(\lambda) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ m_k e^{x_k} & 1 & \lambda m_k e^{-x_k} \\ 0 & 0 & 1 \end{pmatrix}, & k = 2a-1, \\ \begin{pmatrix} 1 & -2\lambda n_k e^{-x_k} & 0 \\ 0 & 1 & 0 \\ 0 & 2n_k e^{x_k} & 1 \end{pmatrix}, & k = 2a, \end{cases} \tag{B.11}$$

for $a = 1, \dots, K$. Starting with $(A_0, B_0, C_0) = (1, 0, 0)$ we obtain in the rightmost interval $x > x_N$ polynomials $(A_N, B_N, C_N) = (A(\lambda), B(\lambda), C(\lambda))$ in the variable $\lambda = -z^2$:

$$\begin{pmatrix} A(\lambda) \\ B(\lambda) \\ C(\lambda) \end{pmatrix} = S_{2K}(\lambda) S_{2K-1}(\lambda) \cdots S_2(\lambda) S_1(\lambda) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{B.12}$$

Between the factors $S_{2K}(\lambda)$ and $S_1(\lambda)$ in the matrix product there are $K-1$

pairs of factors of the form

$$S_{2a+1}(\lambda)S_{2a}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ m_{2a+1}e^{x_{2a+1}} & 1 & 0 \\ 0 & 2n_{2a}e^{x_{2a}} & 1 \end{pmatrix} +$$

$$\lambda \begin{pmatrix} 0 & -2n_{2a}e^{-x_{2a}} & 0 \\ 0 & 2m_{2a+1}n_{2a}(e^{x_{2a}-x_{2a+1}} - e^{x_{2a+1}-x_{2a}}) & m_{2a+1}e^{-x_{2a+1}} \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(B.13)}$$

each such pair depending linearly on λ. The factor $S_1(\lambda)\,(1,0,0)^T$ does not depend on λ, and therefore the vector $\big(A(\lambda), B(\lambda), C(\lambda)\big)^T$ equals $S_{2K}(\lambda)$ times a vector whose entries have degree $K-1$ in λ. Since λ only appears in the top row of $S_{2K}(\lambda)$, we see that $B(\lambda)$ and $C(\lambda)$ are polynomials of degree $K-1$, while $A(\lambda)$ is of degree K. We will name the coefficients in these polynomials as follows:

$$\begin{aligned} A(\lambda) &= 1 - 2\lambda[A]_1 + \cdots + (-2\lambda)^K [A]_K, \\ B(\lambda) &= [B]_0 - 2\lambda[B]_1 + \cdots + (-2\lambda)^{K-1}[B]_{K-1}, \\ C(\lambda) &= [C]_0 - 2\lambda[C]_1 + \cdots + (-2\lambda)^{K-1}[C]_{K-1}. \end{aligned} \quad \text{(B.14)}$$

These coefficients can be computed explicitly in terms of the positions x_k and the weights m_k and n_k by carefully studying what happens when multiplying out the matrix product $S_{2K}(\lambda)S_{2K-1}(\lambda)\cdots S_2(\lambda)S_1(\lambda)\,(1,0,0)^T$. For example, using the abbreviation

$$E_{ab} = e^{-|x_a - x_b|} \qquad (= e^{x_a - x_b} \text{ when } a < b) \quad \text{(B.15)}$$

we have

$$\begin{aligned} [A]_1 &= \sum_{1 \le i < j \le N} m_i n_j E_{ij}, \\ [A]_K &= m_1 n_2 E_{12}\,(1 - E_{23}^2)\, m_3 n_4 E_{34}\,(1 - E_{45}^2)\, m_5 n_6 E_{56} \cdots \\ &\qquad \cdots (1 - E_{N-2,N-1}^2)\, m_{N-1} n_N E_{N-1,N}, \\ [B]_0 &= B(0) = \sum_{1 \le i < N} m_i e^{x_i}. \end{aligned} \quad \text{(B.16)}$$

(Recall that $N = 2K$. Note also that since m_{2a} and n_{2a-1} are zero, only the terms with i odd and j even contribute to the sums.) Later we will show a simpler way to read off all the coefficients in $A(\lambda)$; see (B.37) in Section B.3.

For the second Lax equation (1.4a) things are similar, except that the roles of m and n are swapped. This leads to

$$\begin{pmatrix} \widetilde{\psi}_1(x;z) \\ \widetilde{\psi}_2(x;z) \\ \widetilde{\psi}_3(x;z) \end{pmatrix} = \begin{pmatrix} \widetilde{A}_k e^x + z^2 \widetilde{C}_k e^{-x} \\ 2z\widetilde{B}_k \\ \widetilde{A}_k e^x - z^2 \widetilde{C}_k e^{-x} \end{pmatrix}, \qquad x_k < x < x_{k+1}, \quad \text{(B.17)}$$

and

$$\begin{pmatrix} \widetilde{A}_k \\ \widetilde{B}_k \\ \widetilde{C}_k \end{pmatrix} = \widetilde{S}_k(-z^2) \begin{pmatrix} \widetilde{A}_{k-1} \\ \widetilde{B}_{k-1} \\ \widetilde{C}_{k-1} \end{pmatrix}, \quad \text{(B.18)}$$

where

$$\widetilde{S}_k(\lambda) = \begin{cases} \begin{pmatrix} 1 & -2\lambda m_k e^{-x_k} & 0 \\ 0 & 1 & 0 \\ 0 & 2m_k e^{x_k} & 1 \end{pmatrix}, & k = 2a-1, \\ \\ \begin{pmatrix} 1 & 0 & 0 \\ n_k e^{x_k} & 1 & \lambda n_k e^{-x_k} \\ 0 & 0 & 1 \end{pmatrix}, & k = 2a. \end{cases} \tag{B.19}$$

Starting again with $(\widetilde{A}_0, \widetilde{B}_0, \widetilde{C}_0) = (1,0,0)$, we have in the rightmost interval $(\widetilde{A}_N, \widetilde{B}_N, \widetilde{C}_N) = (\widetilde{A}(\lambda), \widetilde{B}(\lambda), \widetilde{C}(\lambda))$, where

$$\begin{pmatrix} \widetilde{A}(\lambda) \\ \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) \end{pmatrix} = \widetilde{S}_{2K}(\lambda)\widetilde{S}_{2K-1}(\lambda)\cdots\widetilde{S}_2(\lambda)\widetilde{S}_1(\lambda)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{B.20}$$

Because of the asymmetry between m and n coming from the interlacing, the variable λ appears in a slightly different way here; in this case we have K pairs of factors $\widetilde{S}_{2a}\widetilde{S}_{2a-1}$, each of a similar form as the pairs $S_{2a+1}S_{2a}$ that we computed earlier:

$$\begin{aligned} \widetilde{S}_{2a}(\lambda)\widetilde{S}_{2a-1}(\lambda) = &\begin{pmatrix} 1 & 0 & 0 \\ n_{2a}e^{x_{2a}} & 1 & 0 \\ 0 & 2m_{2a-1}e^{x_{2a-1}} & 1 \end{pmatrix} + \\ &\lambda \begin{pmatrix} 0 & -2m_{2a-1}e^{-x_{2a-1}} & 0 \\ 0 & 2n_{2a}m_{2a-1}(e^{x_{2a-1}-x_{2a}} - e^{x_{2a}-x_{2a-1}}) & n_{2a}e^{-x_{2a}} \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.21}$$

From this we see that $\widetilde{S}_2\widetilde{S}_1\,(1,0,0)^T$ is independent of λ, so that the degrees of $\widetilde{A}(\lambda)$, $\widetilde{B}(\lambda)$ and $\widetilde{C}(\lambda)$ are at most $K-1$. The leftmost pair $\widetilde{S}_{2K}\widetilde{S}_{2K-1}$ has no λ in its bottom row, so $\widetilde{C}(\lambda)$ is in fact only of degree $K-2$. Naming the coefficients as

$$\begin{aligned} \widetilde{A}(\lambda) &= 1 - 2\lambda[\widetilde{A}]_1 + \cdots + (-2\lambda)^{K-1}[\widetilde{A}]_{K-1}, \\ \widetilde{B}(\lambda) &= [B]_0 - 2\lambda[\widetilde{B}]_1 + \cdots + (-2\lambda)^{K-1}[\widetilde{B}]_{K-1}, \\ \widetilde{C}(\lambda) &= [C]_0 - 2\lambda[\widetilde{C}]_1 + \cdots + (-2\lambda)^{K-2}[\widetilde{C}]_{K-2}, \end{aligned} \tag{B.22}$$

we have for example

$$\begin{aligned} [\widetilde{A}]_1 &= \sum_{1<j<i<N} n_j m_i E_{ji}, \\ [\widetilde{A}]_{K-1} &= n_2 m_3 E_{23}\,(1-E_{34}^2)\, n_4 m_5 E_{45}\,(1-E_{56}^2)\, n_6 m_7 E_{67}\cdots \\ &\qquad \cdots (1-E_{N-3,N-2}^2)\, n_{N-2} m_{N-1} E_{N-2,N-1}, \\ [\widetilde{B}]_0 &= \widetilde{B}(0) = \sum_{1<j\le N} n_j e^{x_j}, \\ [\widetilde{B}]_{K-1} &= [\widetilde{A}]_{K-1}\, n_N e^{x_N}(1-E_{N-1,N}^2), \end{aligned} \tag{B.23}$$

where (as in (B.16)) only terms with i odd and j even contribute to the sums. See (B.40) in Section B.3 for an easy way to read off all the coefficients of $\widetilde{A}(\lambda)$.

B.2 Positivity and simplicity of the spectra

By construction, the zeros of $A(\lambda)$ and $\widetilde{A}(\lambda)$ are exactly the nonzero eigenvalues $\lambda_1, \dots, \lambda_K$ and $\mu_1, \dots, \mu_{K-1}$ treated in the main text, so the following theorem implies Theorem 3.8:

Theorem B.1. *If all nonzero weights m_{2a-1} and n_{2a} are positive, then the polynomials $A(\lambda)$ and $\widetilde{A}(\lambda)$ have positive simple zeros $\lambda_1, \dots, \lambda_K$ and $\mu_1, \dots, \mu_{K-1}$, respectively.*

Proof. We will rewrite the two spectral problems, (1.3a) with boundary conditions $B_0 = C_0 = 0 = A_N$, and its twin (1.4a) with boundary conditions $\widetilde{B}_0 = \widetilde{C}_0 = 0 = \widetilde{A}_N$, as matrix eigenvalue problems. (Recall that A and $\widetilde{A}$ are just aliases for A_N and $\widetilde{A}_N$, respectively.)

For the first problem, elimination of ψ_1 from (1.3a) gives $\partial_x \psi_2 = zm\psi_3$ and $(\partial_x^2 - 1)\psi_3 = zn\psi_2$, which, considering the boundary conditions in the form (2.4), we can write as

$$\psi_2(x) = z \int_{-\infty}^{x} \psi_3(y)\, dm(y), \quad \psi_3(x) = -z \int_{-\infty}^{\infty} \frac{1}{2} e^{-|x-y|} \psi_2(y)\, dn(y). \tag{B.24}$$

Evaluating the first equation at the even-numbered x_k and the second equation at the odd-numbered x_k, we find the $2K \times 2K$ eigenvalue problem

$$\begin{pmatrix} \psi_{2,\text{even}} \\ \psi_{3,\text{odd}} \end{pmatrix} = z \begin{pmatrix} 0 & 2(I+\mathcal{L})\mathcal{M} \\ -\mathcal{E}\mathcal{N} & 0 \end{pmatrix} \begin{pmatrix} \psi_{2,\text{even}} \\ \psi_{3,\text{odd}} \end{pmatrix}, \tag{B.25}$$

where

$$\begin{aligned}
\psi_{2,\text{even}} &= \big(\psi_2(x_2), \psi_2(x_4), \dots, \psi_2(x_{2K})\big)^T, \\
\psi_{3,\text{odd}} &= \big(\psi_3(x_1), \psi_3(x_3), \dots, \psi_3(x_{2K-1})\big)^T, \\
I &= K \times K \text{ identity matrix}, \\
\mathcal{L} &= \text{strictly lower triangular } K \times K \text{ matrix with } \mathcal{L}_{ij} = 1 \text{ for } i > j, \\
\mathcal{E} &= (e^{-|x_{2i-1} - x_{2j}|})_{i,j=1}^K = (E_{2i-1,2j})_{i,j=1}^K \quad \text{(using the notation of (B.15))}, \\
\mathcal{M} &= \operatorname{diag}(m_1, m_3, \dots, m_{2K-1}), \\
\mathcal{N} &= \operatorname{diag}(n_2, n_4, \dots, n_{2K}).
\end{aligned}$$

Eliminating $\psi_{3,\text{odd}}$ we can write this as a $K \times K$ eigenvalue problem in terms of $\psi_{2,\text{even}}$ alone:

$$\psi_{2,\text{even}} = 2\lambda\, (I+\mathcal{L})\mathcal{M}\mathcal{E}\mathcal{N}\, \psi_{2,\text{even}} \qquad (\lambda = -z^2). \tag{B.26}$$

As we've seen earlier, the eigenvalues are given precisely by the zeros of $A(\lambda)$, and since $A(0) = 1$ we must therefore have

$$A(\lambda) = \det(I - 2\lambda\, (I+\mathcal{L})\mathcal{M}\mathcal{E}\mathcal{N}). \tag{B.27}$$

Now, for positive numbers $\{m_{2k-1}, n_{2k}\}_{k=1}^K$, the matrix $(I+\mathcal{L})\mathcal{M}\mathcal{E}\mathcal{N}$ is oscillatory, since $I+\mathcal{L}$ is nonsingular and totally nonnegative (being the path matrix for the planar network illustrated in Figure 5), and since $\mathcal{M}\mathcal{E}\mathcal{N}$ is totally positive ($\mathcal{E}$ being a submatrix of the totally positive matrix $(E_{ij})_{i,j=1}^{2K}$). This implies that

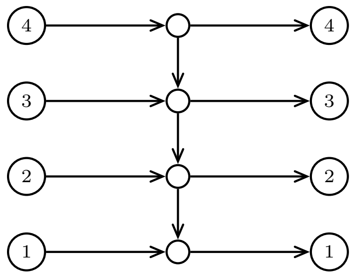

Figure 5. A planar network (illustrated in the case $K=4$) for which $I+\mathcal{L}$ is the path matrix. What this means is that matrix entry (i,j) equals the number of paths from source node i on the left to sink node j on the right; in this case there is one path if $i \geq j$ and none if $i<j$.

its eigenvalues, which up to an unimportant factor of 2 are the zeros of A, are positive and simple. (See, for example, our earlier papers [23, 17] for a summary of the relevant results from the theory of total positivity used here, and for further references.)

For the second spectral problem we swap m and n and obtain

$$\widetilde{\psi}_2(x) = z\int_{-\infty}^{x} \widetilde{\psi}_3(y)\,dn(y), \quad \widetilde{\psi}_3(x) = -z\int_{-\infty}^{\infty} \frac{1}{2}e^{-|x-y|}\widetilde{\psi}_2(y)\,dm(y). \tag{B.28}$$

These integral equations are evaluated the other way around (at odd-numbered and even-numbered x_k, respectively); this yields

$$\begin{pmatrix} \widetilde{\psi}_{2,\text{odd}} \\ \widetilde{\psi}_{3,\text{even}} \end{pmatrix} = z\begin{pmatrix} 0 & 2\mathcal{L}\mathcal{N} \\ -\mathcal{E}^T\mathcal{M} & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\psi}_{2,\text{odd}} \\ \widetilde{\psi}_{3,\text{even}} \end{pmatrix}, \tag{B.29}$$

which in terms of $\widetilde{\psi}_{2,\text{odd}}$ alone becomes

$$\widetilde{\psi}_{2,\text{odd}} = 2\lambda\,\mathcal{L}\mathcal{N}\mathcal{E}^T\mathcal{M}\,\widetilde{\psi}_{2,\text{odd}} \qquad (\lambda=-z^2). \tag{B.30}$$

Thus,

$$\widetilde{A}(\lambda) = \det(I - 2\lambda\,\mathcal{L}\mathcal{N}\mathcal{E}^T\mathcal{M}). \tag{B.31}$$

The previous argument doesn't quite work for dealing with the zeros of $\widetilde{A}(\lambda)$, since $\mathcal{L}$ is singular and one therefore cannot draw the conclusion that the matrix $\mathcal{L}\mathcal{N}\mathcal{E}^T\mathcal{M}$ appearing in (B.31) is oscillatory (only totally nonnegative, which is not enough to show simplicity of the zeros). However, a slightly modified argument does the trick. Note that the first row and the last column of the $2K\times 2K$ matrix in (B.29) are zero. Thus $\widetilde{\psi}_2(x_1)=0$ if (B.29) is satisfied, and the value of $\widetilde{\psi}_3(x_{2K})$ doesn't really enter into the problem either (it appears only in the left-hand side, and is automatically determined by all the other quantities in the equation). Therefore (B.29) has nontrivial solutions if and only if there are nontrivial solutions to the following truncated $(2K-2)\times(2K-2)$ problem obtained by removing the masses m_1 and n_{2K} (i.e., by deleting the first and last row and the first and last column):

$$\begin{pmatrix} \widetilde{\psi}'_{2,\text{odd}} \\ \widetilde{\psi}'_{3,\text{even}} \end{pmatrix} = z\begin{pmatrix} 0 & 2(I'+\mathcal{L}')\mathcal{N}' \\ -(\mathcal{E}')^T\mathcal{M}' & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\psi}'_{2,\text{odd}} \\ \widetilde{\psi}'_{3,\text{even}} \end{pmatrix}, \tag{B.32}$$

where

$$\begin{aligned}
\widetilde{\psi}'_{2,\text{odd}} &= \big(\widetilde{\psi}_2(x_3), \widetilde{\psi}_2(x_5), \dots, \widetilde{\psi}_2(x_{2K-1})\big)^T, \\
\widetilde{\psi}'_{3,\text{even}} &= \big(\widetilde{\psi}_3(x_2), \widetilde{\psi}_3(x_4), \dots, \widetilde{\psi}_3(x_{2K-2})\big)^T, \\
I' &= (K-1) \times (K-1) \text{ identity matrix}, \\
\mathcal{L}' &= \text{strictly lower triangular } (K-1) \times (K-1) \text{ with } \mathcal{L}'_{ij} = 1 \text{ for } i > j, \\
\mathcal{E}' &= \mathcal{E} \text{ with its first row and last column removed}, \\
\mathcal{M}' &= \operatorname{diag}(m_3, m_5, \dots, m_{2K-1}), \\
\mathcal{N}' &= \operatorname{diag}(n_2, n_4, \dots, n_{2K-2}).
\end{aligned}$$

In terms of $\widetilde{\psi}'_{2,\text{odd}}$ alone, this becomes

$$\widetilde{\psi}'_{2,\text{odd}} = 2\lambda\,(I' + \mathcal{L}')\mathcal{N}'(\mathcal{E}')^T \mathcal{M}' \widetilde{\psi}'_{2,\text{odd}},$$

and the conclusion is that

$$A(\lambda) = \det(I' - 2\lambda\,(I + \mathcal{L}')\mathcal{N}'(\mathcal{E}')^T \mathcal{M}' \widetilde{\psi}'_{2,\text{odd}}), \tag{B.33}$$

where $(I' + \mathcal{L}')\mathcal{N}'(\mathcal{E}')^T \mathcal{M}'$ is an oscillatory $(K-1) \times (K-1)$ matrix (by the previous argument). This shows that $\widetilde{A}(\lambda)$ has positive simple zeros too. □

B.3 Expressions for the coefficients of A and $\widetilde{A}$

From equations (B.27) and (B.31) we can extract nice and fairly explicit representations of the coefficients of the polynomials $A(\lambda)$ and $\widetilde{A}(\lambda)$. These coefficients are of particular interest, since they turn out to be constants of motion for the peakon solutions to the Geng–Xue equation. (It is not hard to show, using the Lax pairs, that $A(\lambda)$ and $\widetilde{A}(\lambda)$ are independent of time; the details will be published in a separate paper about peakons.)

First a bit of notation: $\binom{S}{k}$ will denote the set of k-element subsets of a set S, and $[K]$ is the set $\{1, 2, \dots, K\}$. For a matrix X and index sets $I = \{i_1 < \cdots < i_m\}$ and $J = \{j_1 < \cdots < j_n\}$, we write X_{IJ} for the submatrix obtained from X by taking elements from the rows indexed by I and the columns indexed by J; in other words, $X_{IJ} = \big(X_{i_a j_b}\big)_{\substack{a=1,\dots,m \\ b=1,\dots,n}}$.

To begin with, (B.27) says that $A(\lambda) = \det(I - 2\lambda\,(I + \mathcal{L})\mathcal{M}\mathcal{E}\mathcal{N})$, which shows that the quantity $[A]_k$ from (B.14) (the coefficient of $(-2\lambda)^k$ in $A(\lambda)$) equals the sum of the principal $k \times k$ minors in $(I + \mathcal{L})\mathcal{M}\mathcal{E}\mathcal{N}$:

$$[A]_k = \sum_{J \in \binom{[K]}{k}} \det\big((I + \mathcal{L})\mathcal{M}\mathcal{E}\mathcal{N}\big)_{JJ}. \tag{B.34}$$

A general fact is that for any $K \times K$ matrix X and for any fixed $J \in \binom{[K]}{k}$, we have the identity

$$\det\big((I + \mathcal{L})X\big)_{JJ} = \sum_{\substack{I \in \binom{[K]}{k} \\ I \preccurlyeq J}} \det X_{IJ}, \tag{B.35}$$

with summation over all index sets I of size k that are "half-strictly interlacing" with J:

$$I \preccurlyeq J \quad \Longleftrightarrow \quad i_1 \le j_1 < i_2 \le j_2 < \cdots < i_k \le j_k. \tag{B.36}$$

(This is similar to, but much simpler than, the "Canada Day Theorem" about certain sums of minors of *symmetric* matrices, which appeared in the context of Novikov peakons [17, 15].) Equation (B.35) can be proved by expanding $\det((I+\mathcal{L})X)_{JJ}$ with the Cauchy–Binet formula and computing the minors of $I+\mathcal{L}$ by applying the Lindström–Gessel–Viennot Lemma to the planar network for $I+\mathcal{L}$ in Figure 5. (We briefly recall the statement of this lemma: if X is the (weighted) path matrix of a planar network G, then the minor $\det X_{IJ}$ equals the number of vertex-disjoint path families (or the weighted sum over such families) connecting the sources indexed by I to the sinks indexed by J.) Alternatively, one can do row operations directly, as follows:

$$\begin{aligned}
\det((I+\mathcal{L})X)_{JJ} &= \det\Bigl(((I+\mathcal{L})X)_{j_r j_s}\Bigr)_{r,s=1}^{k} \\
&= \det\Bigl(\sum_{m=1}^{j_r} X_{m j_s}\Bigr)_{r,s=1}^{k} \\
&= \det\Bigl(\sum_{m=j_{r-1}+1}^{j_r} X_{m j_s}\Bigr)_{r,s=1}^{k} \\
&= \sum_{i_1=1}^{j_1} \sum_{i_2=j_1+1}^{j_2} \cdots \sum_{i_k=j_{k-1}+1}^{j_k} \det\Bigl(X_{i_r j_s}\Bigr)_{r,s=1}^{k} \\
&= \sum_{\substack{i_1 \le j_1 \\ j_1 < i_2 \le j_2 \\ j_2 < i_3 \le j_3 \\ \cdots \\ j_{k-1} < i_k \le j_k}} \det X_{IJ}.
\end{aligned}$$

(In the second line, we used the definition of $\mathcal{L}$. In the third line, we have subtracted from each row the row above it; $j_0 = 0$ by definition. Next, the summation index is renamed from m to i_r in row r; this lets us use multilinearity to bring the sums outside of the determinant.) Applying this fact to (B.34), we obtain the desired representation

$$[A]_k = \sum_{\substack{I,J \in \binom{[K]}{k} \\ I \preccurlyeq J}} \det(\mathcal{N}\mathcal{E}\mathcal{M})_{IJ}, \tag{B.37}$$

This is useful, since these determinants can be evaluated using the Lindström–Gessel–Viennot Lemma on the planar network shown in Figure 6; see Example B.2 below.

In an entirely similar way one derives the identity

$$\det(\mathcal{L}X)_{JJ} = \sum_{\substack{I \in \binom{[K]}{k} \\ I \curlyeqprec J}} \det X_{IJ}, \tag{B.38}$$

with the other type of "half-strictly interlacing" relation

$$I \curlyeqprec J \quad \Longleftrightarrow \quad i_1 < j_1 \le i_2 < j_2 \le \cdots \le i_k < j_k. \tag{B.39}$$

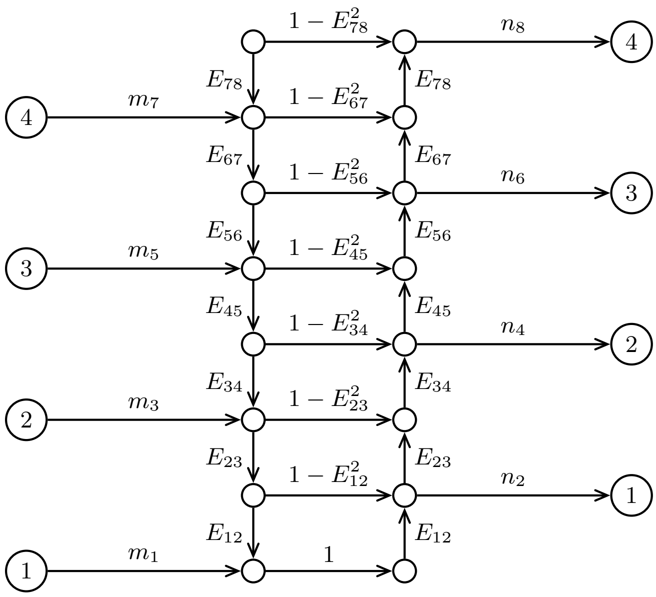

Figure 6. A weighted planar network (illustrated in the case $K = 4$) for which $\mathcal{MEN}$ is the weighted path matrix; this means that the (i, j) entry is the weighted sum of all paths from source i to sink j, each path being counted with a weight equal to the product of its edge weights.

Indeed, we can use the network for $\mathcal{L}$ in Figure 7, or do row operations:

$$\begin{aligned}
\det(\mathcal{L}X)_{JJ} &= \det\Big((\mathcal{L}X)_{j_r j_s} \Big)_{r,s=1}^{k} \\
&= \det\Big(\sum_{m=1}^{j_r - 1} X_{m j_s} \Big)_{r,s=1}^{k} \\
&= \det\Big(\sum_{m=j_{r-1}}^{j_r - 1} X_{m j_s} \Big)_{r,s=1}^{k} \\
&= \sum_{i_1=1}^{j_1-1} \sum_{i_2=j_1}^{j_2-1} \cdots \sum_{i_k=j_{k-1}}^{j_k-1} \det\Big(X_{i_r j_s} \Big)_{r,s=1}^{k} \\
&= \sum_{\substack{i_1 < j_1 \\ j_1 \le i_2 < j_2 \\ j_2 \le i_3 < j_3 \\ \cdots \\ j_{k-1} \le i_k < j_k}} \det X_{IJ}.
\end{aligned}$$

From (B.31) we then see that the coefficients defined by (B.22) are given by

$$[\widetilde{A}]_k = \sum_{J \in \binom{[K]}{k}} \det\big(\mathcal{L}\mathcal{N}\mathcal{E}^T\mathcal{M}\big)_{JJ},$$

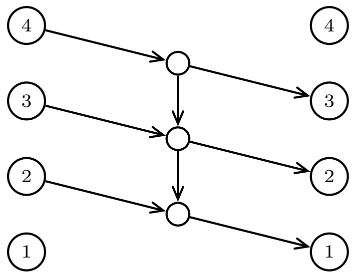

Figure 7. A planar network (illustrated in the case $K = 4$) for which $\mathcal{L}$ is the path matrix. There is one path from source node i to sink node j if $i > j$ and none otherwise.

which by the identity above amounts to

$$[\widetilde{A}]_k = \sum_{\substack{I,J \in \binom{[K]}{k} \\ I \preccurlyeq J}} \det(\mathcal{N}\mathcal{E}^T\mathcal{M})_{IJ} = \sum_{\substack{I,J \in \binom{[K]}{k} \\ I \preccurlyeq J}} \det(\mathcal{M}\mathcal{E}\mathcal{N})_{JI}. \tag{B.40}$$

Again, we can use the network for $\mathcal{MEN}$ in Figure 6 to read off these determinants (but note that the transposition in the last step of (B.40) has the effect that now the sources are index by J and the sinks by I).

Example B.2. Consider the case $K = 4$. We compute the coefficients $[A]_k$ using (B.37) and the planar network in Figure 6.

Note identities such as $E_{12}E_{23}E_{34} = E_{14}$ and $(1-E_{23}^2)+E_{23}\cdot(1-E_{12}^2)\cdot E_{23} = 1 - E_{13}^2$, which are used repeatedly when computing path weights. For example, the determinant corresponding to $I = \{1, 3\}$ and $J = \{1, 4\}$ is found by locating all pairs of path connecting source 1 to sink 1 and source 3 to sink 4 in Figure 6, and having no vertices in common. There is only one path $1 \to 1$, and its weight is $m_1 \cdot 1 \cdot E_{12} \cdot n_2$. Then there are three paths $3 \to 4$ not touching this first path, and their weights are

$$m_5 \cdot (1 - E_{45}^2) \cdot E_{56} \cdot E_{67} \cdot E_{78} \cdot n_8,$$
$$m_5 \cdot E_{45} \cdot (1 - E_{34}^2) \cdot E_{45} \cdot E_{56} \cdot E_{67} \cdot E_{78} \cdot n_8,$$
$$m_5 \cdot E_{45} \cdot E_{34} \cdot (1 - E_{23}^2) \cdot E_{34} \cdot E_{45} \cdot E_{56} \cdot E_{67} \cdot E_{78} \cdot n_8,$$

or, in other words,

$$m_5(1 - E_{45}^2)E_{58}n_8,$$
$$m_5(E_{45}^2 - E_{35}^2)E_{58}n_8,$$
$$m_5(E_{35}^2 - E_{25}^2)E_{58}n_8.$$

Multiplying each of these by the first weight $m_1E_{12}n_2$ gives the weights of the three vertex-disjoint path pairs $13 \to 14$, which we add up to obtain the determinant (according to the Lindström–Gessel–Viennot Lemma):

$$\det(\mathcal{MEN})_{13,14} = m_1E_{12}n_2 \cdot m_5(1 - E_{25}^2)E_{58}n_8.$$

The formulas for the coefficients $[A]_k$ found in this way are

$$\begin{aligned}[A]_1 &= \sum_{i\le j} \det(\mathcal{MEN})_{ij} = \sum_{i\le j} (\mathcal{MEN})_{ij} \\ &= m_1E_{12}n_2 + m_1E_{14}n_4 + m_1E_{16}n_6 + m_1E_{18}n_8 + m_3E_{34}n_4 \\ &\quad + m_3E_{36}n_6 + m_3E_{38}n_8 + m_5E_{56}n_6 + m_5E_{58}n_8 + m_7E_{78}n_8,\end{aligned} \tag{B.41a}$$

$$\begin{aligned}[A]_2 &= \sum_{i_1\le j_1<i_2\le j_2} \det(\mathcal{MEN})_{i_1i_2,j_1j_2} \\ &= \det(\mathcal{MEN})_{12,12} + \det(\mathcal{MEN})_{12,13} + \det(\mathcal{MEN})_{12,14} \\ &\quad + \det(\mathcal{MEN})_{13,13} + \det(\mathcal{MEN})_{13,14} + \det(\mathcal{MEN})_{14,14} \\ &\quad + \det(\mathcal{MEN})_{13,23} + \det(\mathcal{MEN})_{13,24} + \det(\mathcal{MEN})_{14,24} \\ &\quad + \det(\mathcal{MEN})_{14,34} + \det(\mathcal{MEN})_{23,23} + \det(\mathcal{MEN})_{23,24} \\ &\quad + \det(\mathcal{MEN})_{24,24} + \det(\mathcal{MEN})_{24,34} + \det(\mathcal{MEN})_{34,34} \\ &= m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{34}n_4 + m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{36}n_6 \\ &\quad + m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{38}n_8 + m_1E_{12}n_2 \cdot m_5(1-E_{25}^2)E_{56}n_6 \\ &\quad + m_1E_{12}n_2 \cdot m_5(1-E_{25}^2)E_{58}n_8 + m_1E_{12}n_2 \cdot m_7(1-E_{27}^2)E_{78}n_8 \\ &\quad + m_1E_{14}n_4 \cdot m_5(1-E_{45}^2)E_{56}n_6 + m_1E_{14}n_4 \cdot m_5(1-E_{45}^2)E_{58}n_8 \\ &\quad + m_1E_{14}n_4 \cdot m_7(1-E_{47}^2)E_{78}n_8 + m_1E_{16}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8 \\ &\quad + m_3E_{34}n_4 \cdot m_5(1-E_{45}^2)E_{56}n_6 + m_3E_{34}n_4 \cdot m_5(1-E_{45}^2)E_{58}n_8 \\ &\quad + m_3E_{34}n_4 \cdot m_7(1-E_{47}^2)E_{78}n_8 + m_3E_{36}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8 \\ &\quad + m_5E_{56}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8,\end{aligned} \tag{B.41b}$$

$$\begin{aligned}[A]_3 &= \sum_{i_1\le j_1<i_2\le j_2<i_3\le j_3} \det(\mathcal{MEN})_{i_1i_2i_3,j_1j_2j_3} \\ &= \det(\mathcal{MEN})_{123,123} + \det(\mathcal{MEN})_{123,124} + \det(\mathcal{MEN})_{124,124} \\ &\quad + \det(\mathcal{MEN})_{124,134} + \det(\mathcal{MEN})_{134,134} + \det(\mathcal{MEN})_{134,234} \\ &\quad + \det(\mathcal{MEN})_{234,234} \\ &= m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{34}n_4 \cdot m_5(1-E_{45}^2)E_{56}n_6 \\ &\quad + m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{34}n_4 \cdot m_5(1-E_{45}^2)E_{58}n_8 \\ &\quad + m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{34}n_4 \cdot m_7(1-E_{47}^2)E_{78}n_8 \\ &\quad + m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{36}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8 \\ &\quad + m_1E_{12}n_2 \cdot m_5(1-E_{25}^2)E_{56}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8 \\ &\quad + m_1E_{14}n_4 \cdot m_5(1-E_{45}^2)E_{56}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8 \\ &\quad + m_3E_{34}n_4 \cdot m_5(1-E_{45}^2)E_{56}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8,\end{aligned} \tag{B.41c}$$

and

$$\begin{aligned}[A]_4 &= \sum_{i_1\le j_1<i_2\le j_2<i_3\le j_3<i_4\le j_4} \det(\mathcal{MEN})_{i_1i_2i_3i_4,j_1j_2j_3j_4} \\ &= \det(\mathcal{MEN})_{1234,1234} \\ &= m_1E_{12}n_2 \cdot m_3(1-E_{23}^2)E_{34}n_4 \\ &\quad \cdot m_5(1-E_{45}^2)E_{56}n_6 \cdot m_7(1-E_{67}^2)E_{78}n_8.\end{aligned} \tag{B.41d}$$

(Cf. the expressions (B.16) for the lowest and highest coefficients $[A]_1$ and $[A]_K$ in general.)

Similarly, we compute the coefficients $[\widetilde{A}]_k$ using (B.40):

$$\begin{aligned}[\widetilde{A}]_1 &= \sum_{i<j} \det(\mathcal{M}\mathcal{E}\mathcal{N})_{ji} = \sum_{i<j} (\mathcal{M}\mathcal{E}\mathcal{N})_{ji} \\ &= m_3 E_{23} n_2 + m_5 E_{25} n_2 + m_7 E_{27} n_2 \\ &\quad + m_5 E_{45} n_4 + m_7 E_{47} n_4 + m_7 E_{67} n_6,\end{aligned} \tag{B.42a}$$

$$\begin{aligned}[\widetilde{A}]_2 &= \sum_{i_1<j_1\le i_2<j_2} \det(\mathcal{M}\mathcal{E}\mathcal{N})_{j_1 j_2, i_1 i_2} \\ &= \det(\mathcal{M}\mathcal{E}\mathcal{N})_{23,12} + \det(\mathcal{M}\mathcal{E}\mathcal{N})_{24,12} + \det(\mathcal{M}\mathcal{E}\mathcal{N})_{24,13} \\ &\quad + \det(\mathcal{M}\mathcal{E}\mathcal{N})_{34,13} + \det(\mathcal{M}\mathcal{E}\mathcal{N})_{34,23} \\ &= m_3 E_{23} n_2 \cdot m_5 E_{45} (1 - E_{34}^2) n_4 \\ &\quad + m_3 E_{23} n_2 \cdot m_7 E_{47} (1 - E_{34}^2) n_4 \\ &\quad + m_3 E_{23} n_2 \cdot m_7 E_{67} (1 - E_{36}^2) n_6 \\ &\quad + m_5 E_{25} n_2 \cdot m_7 E_{67} (1 - E_{56}^2) n_6 \\ &\quad + m_5 E_{45} n_4 \cdot m_7 E_{67} (1 - E_{56}^2) n_6\end{aligned} \tag{B.42b}$$

and

$$\begin{aligned}[\widetilde{A}]_3 &= \sum_{i_1<j_1\le i_2<j_2\le i_3<j_3} \det(\mathcal{M}\mathcal{E}\mathcal{N})_{j_1 j_2 j_3, i_1 i_2 i_3} \\ &= \det(\mathcal{M}\mathcal{E}\mathcal{N})_{234,123} \\ &= m_3 E_{23} n_2 \cdot m_5 E_{45} (1 - E_{34}^2) n_4 \cdot m_7 E_{67} (1 - E_{56}^2) n_6.\end{aligned} \tag{B.42c}$$

(Cf. the expressions (B.23) for $[\widetilde{A}]_1$ and $[\widetilde{A}]_{K-1}$ in general.)

C Guide to notation

For the convenience of the reader, here is an index of the notation used in this article.

$m(x)$, $n(x)$, z, $\Psi(x;z) = (\psi_1, \psi_2, \psi_3)^T$	Section 1
Spectral problems for Ψ	(1.3a) + (2.4)
	(1.4a) + (2.4)
$g(y)$, $h(y)$, $\lambda = -z^2$, $\Phi(y;\lambda) = (\phi_1, \phi_2, \phi_3)^T$	(2.1)
Spectral problems for Φ	(2.2), (2.3)
Coefficient matrices $\mathcal{A}(y;\lambda)$, $\widetilde{\mathcal{A}}(y;\lambda)$	(2.6)
$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	(2.8)
Involution $X(\lambda)^\sigma = J X(-\lambda)^{-T} J$	(2.9)
Fundamental matrices $U(y;\lambda)$, $\widetilde{U}(y;\lambda)$	(2.10), (2.11)
Transition matrices	
$S(\lambda) = U(1;\lambda)$, $\widetilde{S}(\lambda) = \widetilde{U}(1;\lambda)$	(2.12)
Weyl functions	
$W = -S_{21}/S_{31}$, $Z = -S_{11}/S_{31}$	(2.17)

Twin Weyl functions	
$\widetilde{W} = -\widetilde{S}_{21}/\widetilde{S}_{31}$, $Z = -\widetilde{S}_{11}/\widetilde{S}_{31}$	(2.17)
Bilinear form $\langle \Phi, \Omega \rangle = \int_{-1}^{1} \Phi(y)^T J\, \Omega(y)\, dy$	(2.20)
Adjoint spectral problems for	
$\Omega = \Omega(y;\lambda) = (\omega_1, \omega_2, \omega_3)$	(2.22), (2.25)
Adjoint Weyl functions	
$W^* = -S_{32}/S_{31}$, $Z^* = -S_{33}/S_{31}$	(2.24)
Twin adjoint Weyl functions	
$\widetilde{W}^* = -\widetilde{S}_{32}/\widetilde{S}_{31}$, $\widetilde{Z}^* = -\widetilde{S}_{33}/\widetilde{S}_{31}$	(2.24)
Discrete interlacing measures	
$m = m_1\delta_{x_1} + m_3\delta_{x_3} + \cdots + m_{N-1}\delta_{x_{N-1}}$	
$n = n_2\delta_{x_2} + n_4\delta_{x_4} + \cdots + n_N\delta_{x_N}$	
with $x_1 < x_2 < \cdots < x_N$, $N = 2K$	(3.1)
Transformed measures	
$g = g_1\delta_{y_1} + g_2\delta_{y_3} + \cdots + g_K\delta_{y_{2K-1}}$	
$h = h_1\delta_{y_2} + h_2\delta_{y_4} + \cdots + h_K\delta_{y_{2K}}$	(3.3)
with $y_k = \tanh x_k$,	(3.2)
$g_a = 2m_{2a-1}\cosh x_{2a-1}$, $h_a = 2n_{2a}\cosh x_{2a}$	(3.4)
Interval lengths $l_k = y_{k+1} - y_k$	(3.5)
Propagation matrices	
$L_k(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\lambda l_k & 0 & 1 \end{pmatrix}$, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 1 & x & \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$	(3.7), (3.8)
Transition matrix in the discrete case	
$S(\lambda) = L_{2K}(\lambda)\begin{bmatrix} h_K \\ 0 \end{bmatrix} L_{2K-1}(\lambda)\begin{bmatrix} 0 \\ g_K \end{bmatrix}\cdots$	(3.9)
and its partial products $T_j(\lambda)$	(3.10)
Twin transition matrix	
$\widetilde{S}(\lambda) = L_{2K}(\lambda)\begin{bmatrix} 0 \\ h_K \end{bmatrix} L_{2K-1}(\lambda)\begin{bmatrix} g_K \\ 0 \end{bmatrix}\cdots$	(3.23)
and its partial products $\widetilde{T}_j(\lambda)$	(3.24)
Eigenvalues	
$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_K$	
$0 = \mu_0 < \mu_1 < \cdots < \mu_{K-1}$	Theorem 3.8
Residues of Weyl functions	
a_i, b_j, b_∞, c_i, d_j $(1 \le i \le K,\ 1 \le j \le K-1)$	Theorem 3.10
Spectral measures	
$\alpha = \sum_{i=1}^{K} a_i\delta_{\lambda_i}$, $\beta = \sum_{j=1}^{K-1} b_j\delta_{\mu_j}$	(3.43)
Weyl functions as integrals	
$W(\lambda) = \int \frac{d\alpha(x)}{\lambda - x}$, etc.	(3.44)
Entries of $T(\lambda) = T_j(\lambda)$ (for some fixed j)	
$Q = -T_{32}$, $P = T_{22}$, $R = T_{12}$	
$(W \approx P/Q,\ Z \approx R/Q)$	(3.45)
Residues of adjoint Weyl functions	
a_i^*, b_j^*, b_∞^*, c_i^*, d_j^*	(3.52)
Adjoint transition matrix	
$S^*(\lambda) = \widetilde{S}(-\lambda)^{-1} = JS(\lambda)^T J$	Theorem 3.18
Adjoint Weyl functions in terms of S^*	
$W^* = +S_{21}^*/S_{31}^*$, $Z^* = -S_{11}^*/S_{31}^*$	Remark 3.19
Moments and bimoments of spectral measures	
$\alpha_k = \int x^k d\alpha(x)$, $\beta_k = \int y^k d\beta(y)$,	(4.1); see also (A.1)

$I_{km} = \iint \frac{x^k y^m}{x+y} d\alpha(x) d\beta(y)$	(4.2); see also (A.2)
Determinant $\mathcal{K}_n$ involving bimoments	(4.5); see also (A.26)
Spectral map	
Pure peakon sector $\mathcal{P} \subset \mathbf{R}^{4K}$	
Admissible spectral data $\mathcal{R} \subset \mathbf{R}^{4K}$	
Forward map $\mathcal{S} \colon \mathcal{P} \to \mathcal{R}$	
Inverse map $\mathcal{T} \colon \mathcal{R} \to \mathcal{P}$	Definition 4.7
Cauchy biorthogonal polynomials $p_n(x)$, $q_n(y)$	(A.4), (A.5)
Vandermonde-type expression	
$\Delta(x) = \Delta(x_1, \dots, x_n) = \prod_{i<j} (x_i - x_j)$	(A.12)
$\Gamma(x) = \Gamma(x_1, \dots, x_n) = \prod_{i<j} (x_i + x_j)$	(A.13)
$\Gamma(x; y) = \Gamma(x_1, \dots, x_n; y_1, \dots, y_m)$ $= \prod_{i=1}^n \prod_{j=1}^m (x_i + y_j)$	(A.14)
Generalized Heine-type integrals	
$\mathcal{J}_{nm}^{rs} = \int\limits_{\sigma_n \times \sigma_m} \frac{\Delta(x)^2 \Delta(y)^2 \left(\prod x_i\right)^r \left(\prod y_j\right)^s d\alpha^n(x) d\beta^m(y)}{\Gamma(x;y)}$ where $\sigma_n = \{x \in \mathbf{R}^n : 0 < x_1 < \dots < x_n\}$	(A.15)
Degenerate cases $\mathcal{J}_{0m}^{rs}$, $\mathcal{J}_{n0}^{rs}$, $\mathcal{J}_{00}^{rs}$	(A.16)
The basic bimoment determinant	
$D_n = \det(I_{ij})_{i,j=0}^{n-1} = \mathcal{J}_{nn}^{00}$	(A.17)
General discrete setup	
$\alpha = \sum_{i=1}^A a_i \delta_{\lambda_i}$, $\beta = \sum_{j=1}^B b_j \delta_{\mu_j}$	Section A.4
($A = K$, $B = K - 1$ in the main text)	
Heine-type integrals as sums in the discrete case	
$\mathcal{J}_{nm}^{rs} = \sum_{I \in \binom{[A]}{n}} \sum_{J \in \binom{[B]}{m}} \Psi_{IJ}\, \lambda_I^r a_I\, \mu_J^s b_J$	(A.31)
where	
$[A] = \{1, 2, \dots, A\}$	
$\binom{[A]}{n}$ = set of n-element subsets of $[A]$	
$\lambda_I^r a_I\, \mu_J^s b_J = \left(\prod_{i \in I} \lambda_i^r a_i\right) \left(\prod_{j \in J} \mu_j^s b_j\right)$	(A.32)
$\Psi_{IJ} = \frac{\Delta_I^2 \widetilde{\Delta}_J^2}{\Gamma_{IJ}}$	(A.33)
$\Delta_I^2 = \Delta(\lambda_{i_1}, \dots, \lambda_{i_n})^2$,	
$\widetilde{\Delta}_J^2 = \Delta(\mu_{j_1}, \dots, \mu_{j_m})^2$,	
$\Gamma_{IJ} = \Gamma(\lambda_{i_1}, \dots, \lambda_{i_n}; \mu_{j_1}, \dots, \mu_{j_m})$	(A.34)
$\Delta_{I_1 I_2}^2 = \prod_{i_1 \in I_1,\, i_2 \in I_2} (\lambda_{i_1} - \lambda_{i_2})^2$	(A.35)
$\mathcal{J}_{nm}^{00}$ written out in the case $A = 3$, $B = 2$	Example A.2
$(\mathcal{J}^*)_{nm}^{rs}$	Lemma A.3
Polynomials $A_k(\lambda)$, $B_k(\lambda)$, $C_k(\lambda)$	(B.8), (B.10)
Jump matrix $S_k(\lambda)$	(B.11)
$\big(A(\lambda), B(\lambda), C(\lambda)\big) = \big(A_N(\lambda), B_N(\lambda), C_N(\lambda)\big)$	(B.12)
Coefficients $[A]_i$, $[B]_i$, $[C]_i$ in A, B, C	(B.14)
$E_{ab} = e^{-\lvert x_a - x_b \rvert}$	(B.15)
Polynomials $\widetilde{A}_k(\lambda)$, $\widetilde{B}_k(\lambda)$, $\widetilde{C}_k(\lambda)$	(B.17), (B.18)
Jump matrix $\widetilde{S}_k(\lambda)$	(B.19)
$\big(\widetilde{A}(\lambda), \widetilde{B}(\lambda), \widetilde{C}(\lambda)\big) = \big(\widetilde{A}_N(\lambda), \widetilde{B}_N(\lambda), \widetilde{C}_N(\lambda)\big)$	(B.20)
Coefficients $[\widetilde{A}]_i$, $[\widetilde{B}]_i$, $[\widetilde{C}]_i$ in $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$	(B.22)
$\psi_{2,\mathrm{even}}$, $\psi_{3,\mathrm{odd}}$, $\mathcal{L}$, $\mathcal{E}$, $\mathcal{M}$, $\mathcal{N}$	(B.25)

$\widetilde{\psi}_{2,\text{odd}}$, $\widetilde{\psi}_{3,\text{even}}$	(B.29)
$\widetilde{\psi}'_{2,\text{even}}$, $\widetilde{\psi}'_{3,\text{odd}}$, $\mathcal{L}'$, $\mathcal{E}'$, $\mathcal{M}'$, $\mathcal{N}'$	(B.32)
$[K] = \{1, 2, \dots, K\}$	
$\binom{S}{k}$, the set of k-element subsets of a set S	
Index sets $I = \{i_1 < \dots < i_m\}$, $J = \{j_1 < \dots < j_n\}$	
Submatrix $X_{IJ} = \left(X_{i_a j_b}\right)_{\substack{a=1,\dots,m \\ b=1,\dots,n}}$	Section B.3
"Half-strictly interlacing" relations:	
$I \preccurlyeq J \iff i_1 \le j_1 < i_2 \le j_2 < \dots < i_k \le j_k$	(B.36)
$I \precsim J \iff i_1 < j_1 \le i_2 < j_2 \le \dots \le i_k < j_k$	(B.39)

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