

Strangeness and charmness content of the nucleon from overlap fermions on 2 + 1-flavor domain-wall fermion configurations

(χ QCD Collaboration)

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Abstract

We present a calculation of the strangeness and charmness contents $\langle N|\bar{s}s|N\rangle$ and $\langle N|\bar{c}c|N\rangle$ of the nucleon from dynamical lattice QCD with 2 + 1 flavors. The calculation is performed with overlap valence quarks on 2+1-flavor domain-wall fermion gauge configurations. The configurations are generated by the RBC collaboration on a $24^3 \times 64$ lattice with sea-quark mass $am_l = 0.005$, $am_s = 0.04$, and inverse lattice spacing $a^{-1} = 1.73$ GeV. Both actions have chiral symmetry which is essential in avoiding contamination due to the operator mixing with other flavors. The nucleon propagator and the quark loops are both computed with stochastic grid sources, while low-mode substitution and low-mode averaging methods are used respectively which substantially improve the signal-to-noise ratio. We obtain the strangeness matrix element $f_{T_s} = m_s \langle N|\bar{s}s|N\rangle / M_N = 0.0334(62)$, and the charmness content $f_{T_c} = m_c \langle N|\bar{c}c|N\rangle / M_N = 0.094(31)$ which is resolved from zero by 3 σ precision for the first time.

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I. INTRODUCTION

The strangeness and charmness content of the nucleon are of fundamental importance to our understanding of the sea-quark contribution to nucleon structure. In particular, the sea-quark contribution to the scalar u/d quark content $\langle N|\bar{u}u|N\rangle$ is crucial to furthering our understanding of the pion-nucleon sigma term. In addition to their relevance to nucleon structure, the strangeness and charmness content of the nucleon have drawn recent interest due to their relevance in dark matter searches [1–3]. One popular candidate for dark matter is a weakly interacting massive particle (WIMP). The scalar (spin-independent) effective four-fermion interaction between the WIMP and the quarks, such as the neutralino-nucleon scattering in the context of the minimal supersymmetric standard model, is $\alpha_{3q_i}\bar{\chi}\chi\bar{q}_i q_i$ [1, 2]. Additionally, the scalar neutralino-nucleus coupling is found to be much larger than the axial-vector coupling [4]. In this case, the scalar contribution to the total χ -nucleus cross section is

$$\sigma = \frac{4m_r^2}{\pi}[Zf_p + (A - Z)f_n]^2, \quad (1)$$

where m_r is the reduced χ -nucleus mass and

$$\frac{f_N}{m_N} = \sum_{q=u,d,s} f_{T_q}^N \frac{\alpha_{3q}}{m_q} + \sum_{Q=c,b,t} f_{T_Q}^N \frac{\alpha_{3Q}}{m_Q}, \quad (2)$$

where $N = p, n$, and

$$f_{T_q} = \frac{m_q \langle N|\bar{q}q|N\rangle}{m_N} \quad (3)$$

$$f_{T_Q} = \frac{m_Q \langle N|\bar{Q}Q|N\rangle}{m_N} \quad (4)$$

are the contributions for the light (q) and heavy (Q) quarks respectively. From the trace anomaly for the nucleon mass

$$m_N = \sum_{q=u,d,s} m_q \langle N|\bar{q}q|N\rangle + \sum_{Q=c,b,t} m_Q \langle N|\bar{Q}Q|N\rangle - \frac{7\alpha_s}{8\pi} \langle N|G_{\mu\nu}G_{\mu\nu}|N\rangle \quad (5)$$

and the heavy-quark expansion [5], one can relate the glue condensate in the nucleon to the heavy-quark condensate. Equation (5) becomes

$$m_N = \sum_{q=u,d,s} m_q \langle N|\bar{q}q|N\rangle + \frac{27}{2}m_Q \langle N|\bar{Q}Q|N\rangle \quad (6)$$

and Eq. (2) is written as

$$\frac{f_N}{m_N} = \sum_{q=u,d,s} f_{T_q}^N \frac{\alpha_{3q}}{m_q} + \frac{2}{27} \bar{f}_{T_Q}^N \sum_{Q=c,b,t} \frac{\alpha_{3Q}}{m_Q}, \quad (7)$$

where $\bar{f}_{T_Q}^N = 1 - \sum_{q=u,d,s} f_{T_q}^N$. This expression is most often used in the analysis for dark matter searches [1–3]. Since the couplings α_{3q} and α_{3Q} contain many terms that are proportional to the quark mass, e.g. through Higgs exchange, we see from Eqs. (2) and (7) that the total spin-independent neutralino-nucleus cross section is mainly proportional to f_{T_q} and f_{T_Q} . Thus, it is important to determine them precisely.

At energy scales comparable to Λ_{QCD} , perturbative calculations of f_{T_s} and f_{T_c} in the nucleon are prohibitively difficult due to the nonperturbative nature of QCD. However, these sea-quark matrix elements are accessible to lattice calculations. The first lattice calculations of f_{T_s} were done with Wilson fermions on quenched lattices and with heavy dynamical Wilson fermion configurations [6–8]. Those calculations gave relatively large values for f_{T_s} on the order of $\sim 0.19(1)$ [7]. More recent calculations with Wilson-clover dynamical fermions also yield large values ($f_{T_s} \sim 0.1\text{--}0.46$) [9–11], whereas calculations with fermions incorporating chiral symmetry result in much smaller f_{T_s} on the order of a few percent [12–19].

The large value for f_{T_s} found using Wilson-type fermions is due to the additive renormalization of the quark mass due to the lattice-spacing-dependent chiral symmetry breaking. As a consequence, there is mixing between the $\bar{u}u$ and $\bar{d}d$ operators and the $\bar{s}s$ operator [20] [12]. This leads to a subtraction term from $\langle N | \bar{s}s | N \rangle$ which is proportional to the matrix element $\langle N | \bar{u}u + \bar{d}d | N \rangle$. Since the latter involves the valence contribution, the subtraction turns out to be large. For example, it is found that

$$y = \frac{2 \langle N | \bar{s}s | N \rangle}{\langle N | \bar{u}u + \bar{d}d | N \rangle} \quad (8)$$

is changed from $y = 0.53(12)$ to $y = -0.28(33)$ after the subtractions for the $N_f = 2$ lattice with the nonperturbatively improved clover fermions [20]. Similarly, $y = 0.336(3)$ becomes $y = 0.059(37)(28)$ after subtraction for the $N_f = 2$, $32^3 \times 64$ lattice with Wilson-clover fermions [21]. An alternative way of evaluating the strangeness matrix element is to apply the Feynman-Hellman theorem and take the derivative with respect to m_π^2 and m_K^2 instead of the strange quark mass [22]. This approach avoids the additive mass part of the subtraction [12] and leads to a small $f_{T_s} = 0.033(16)(4)(2)$ [22].

To avoid the large systematic errors caused by explicit chiral symmetry breaking with Wilson-type fermions, we instead adopt overlap fermions with exact chiral symmetry on the lattice for the valence and the quark loop, for which the quark mass receives no additive renormalization.

In addition to having small $O(a^2)$ discretization errors [23, 24], the overlap fermion that we use for the valence quarks in the nucleon can also be used for the light and charm quarks in the loop insertion with small $O(m^2a^2)$ error [25, 26]. This allows us to calculate both f_{T_s} and f_{T_c} .

For a heavy quark of flavor Q , it is shown [5] that, to leading order in the heavy quark expansion, the matrix element $m_Q \langle N | \bar{Q}Q | N \rangle$ is related to the glue condensate in the nucleon,

$$\sigma_Q \equiv m_Q \langle N | \bar{Q}Q | N \rangle \rightarrow -\frac{\alpha_s}{12\pi} \langle N | GG | N \rangle. \quad (9)$$

If c, b , and t quarks are all considered to be heavy and the light quark contributions are ignored, the combination of the trace anomaly and Eq. (9) predicts

$$\sigma_Q = \frac{2}{27} m_N = 70 \text{ MeV}. \quad (10)$$

It would be interesting to check this prediction by a direct lattice calculation, since the heavy quarks provide a significant contribution to the WIMP-on-nucleon cross section. In previous calculations, the charmness content of the nucleon $\langle N | \bar{c}c | N \rangle = 0.056(27)$ has been obtained with the MILC HISQ configurations [16] with a 2σ signal. In this work, we calculate the strangeness $\langle N | \bar{s}s | N \rangle$ and charmness content $\langle N | \bar{c}c | N \rangle$ of the nucleon in lattice QCD with overlap valence quarks on 2+1-flavor domain-wall fermion gauge configurations.

The calculation is done on a $24^3 \times 64$ lattice with sea-quark mass $m_l = 0.005$, $m_s = 0.04$, and lattice spacing $a^{-1} = 1.73 \text{ GeV}$. We obtain the strangeness content $f_{T_s} = m_s \langle N | \bar{s}s | N \rangle / M_N = 0.0334(62)$ which has more than 5σ precision. Compared with previous lattice calculations either with disconnected insertion or via the Feynman-Hellman theorem [9, 10, 12, 13, 15–19, 21, 22], the present calculation has the smallest error. We also obtain the charmness content $f_{T_c} = m_c \langle N | \bar{c}c | N \rangle / M_N = 0.094(31)$. This is the first time the charmness has been calculated with more than 3σ precision.

Our paper is organized as follows. The overlap formulation is briefly summarized in Sec. II. The technical details for calculating the nucleon two-point functions and the quark loops are described in Sec. III and Sec. IV respectively. They are combined to calculate

the disconnected three-point functions in Sec. V. Finally, the numerical results with chiral extrapolation are presented in Sec. VI and the conclusions are given in Sec. VII.

II. OVERLAP FERMIONS

We adopt the overlap fermion formulation for the valence quarks in the nucleon correlation functions as well as for the quark loops. The inversion of overlap fermions using deflation of low eigenmodes and the construction of meson and nucleon two-point functions with low-mode substitution have been detailed previously [26]. Deflation with low eigenmodes and hypercubic smearing speed up the inversion by a factor of ~ 50 for the $24^3 \times 64$ lattice that we use for this calculation and low-mode substitution (LMS) improves the errors of the meson and nucleon correlators by a factor of $\sim 3 - 4$ [26]. The overlap quark propagators are calculated on gauge configurations with $2 + 1$ -flavors of dynamical domain-wall fermions (DWF). As we mentioned in the Introduction, we adopt both the overlap and DWF fermion formalisms since they preserve chiral symmetry via the Ginsparg-Wilson relation. Due to the high degree of chiral symmetry, the calculation of the quark content is free of the problems that plague the Wilson-type fermions as outlined in the above discussion. As an additional advantage, the $O(m^2 a^2)$ discretization errors are small for the overlap fermion. This allows us to compute the charmness contribution to the nucleon in addition to the strangeness contribution.

The overlap operator [27] is defined as

$$D_{ov}(\rho) = 1 + \gamma_5 \varepsilon(\gamma_5 D_w(\rho)), \quad (11)$$

where ε is the matrix sign function and $D_w(\rho)$ is the usual Wilson fermion operator, except with a negative mass parameter $-\rho = 1/2\kappa - 4$ in which $\kappa_c < \kappa < 0.25$. We set $\kappa = 0.2$ in our calculation, corresponding to $\rho = 1.5$. The massive overlap Dirac operator is defined as

$$\begin{aligned} D_m &= \rho D_{ov}(\rho) + m \left(1 - \frac{D_{ov}(\rho)}{2}\right) \\ &= \rho + \frac{m}{2} + \left(\rho - \frac{m}{2}\right) \gamma_5 \varepsilon(\gamma_5 D_w(\rho)). \end{aligned} \quad (12)$$

To accommodate the chiral transformation, it is usually convenient to use the chirally regulated field $\hat{\psi} = (1 - \frac{1}{2}D_{ov})\psi$ in lieu of ψ in the interpolation field and the currents. This

leads to an effective propagator

$$G \equiv D_{\text{eff}}^{-1} \equiv \left(1 - \frac{D_{ov}}{2}\right) D_m^{-1} = \frac{1}{D_c + m}, \quad (13)$$

where $D_c = \frac{\rho D_{ov}}{1 - D_{ov}/2}$ is chiral, i.e. $\{\gamma_5, D_c\} = 0$ [28]. It is worthwhile to point out that this effective propagator has the same form as that in the continuum [25]. In other words, the inverse of the propagator is a chirally invariant massless Dirac operator plus the quark mass term. As long as the $O(m^2 a^2)$ error is small, this formulation is suitable for both light and heavy quarks.

We adopt the Zolotarev approximation to evaluate the matrix sign function. This entails two nested conjugate gradient loops to calculate the propagator of the overlap fermion. For each conjugate gradient loop, we use deflation with low eigenmodes to speed up the inversion. The details are given in Ref. [26]. Due to the normality of D_{ov} , i.e. $D_{ov}^\dagger D_{ov} = D_{ov} D_{ov}^\dagger$ and the Ginsparg-Wilson relation $\{\gamma_5, D_{ov}\} = D_{ov} \gamma_5 D_{ov}$, the eigenvalues of D_{ov} are on a unit circle with the center at unity. The real and chiral modes are at 0 and 2. Others on the circle are paired with conjugate eigenvalues. In other words, if v_i is an eigenvector

$$D_{ov} v_i = \lambda_i v_i, \quad (14)$$

then its conjugate partner $\gamma_5 v_i$ is also an eigenvector with eigenvalue λ_i^* ,

$$D_{ov} \gamma_5 v_i = \lambda_i^* \gamma_5 v_i. \quad (15)$$

To compute the quark propagator, we first find a few hundred pairs of the lowest eigenvectors of the massless overlap operator in addition to the zero modes. Once we have obtained these lowest eigenvectors, we can solve the high-mode part of the quark propagator by projecting out the low modes from the source

$$D_{eff} G^H \eta = \left(1 - \sum_i \left(v_i v_i^\dagger + \gamma_5 v_i v_i^\dagger \gamma_5\right) \left(1 - \frac{1}{2} \delta_{\lambda_i, 0}\right)\right) \eta, \quad (16)$$

where η is the source vector and the factor $1 - \frac{1}{2} \delta_{\lambda_i, 0}$ takes care of the zero modes which are either left-handed or right-handed.

In contrast, the low-mode part of the effective quark propagator can be constructed with eigenvectors directly,

$$G^L = \sum_i \left[\frac{(1 - \frac{\lambda_i}{2}) v_i v_i^\dagger}{\rho \lambda_i + m(1 - \frac{\lambda_i}{2})} + \frac{(1 - \frac{\lambda_i^*}{2}) \gamma_5 v_i v_i^\dagger \gamma_5}{\rho \lambda_i^* + m(1 - \frac{\lambda_i^*}{2})} \right] \left(1 - \frac{1}{2} \delta_{\lambda_i, 0}\right). \quad (17)$$

For a given source η , the total effective quark propagator is

$$G\eta = G^H\eta + G^L\eta. \quad (18)$$

However, we should point out that since G^L is constructed from the eigenmodes rather than by inverting a source vector, we can compute the any-to-any propagator for any source and sink location with little additional computation. We shall use this fact to carry out the low-mode substitution to replace $G^L\eta$ in Eq. (18) with G^L for a source with given grid points which greatly improves the nucleon correlator. This will be explained in more detail in the next section.

III. THE NUCLEON TWO-POINT CORRELATION FUNCTION

Various attempts have been made to improve the statistics of hadronic two-point correlation functions such as using a smeared source, a volume source with fixed gauge, and all-to-all propagators. The computation of all-to-all propagators usually involves noise sources on different sites. However, the quark propagator from one site can be contaminated by those from neighboring sites. For example, when constructing the nucleon correlation function, the three quark propagators may be from the same source site or from different source sites. The latter case is not gauge-invariant and will introduce noise after averaging over a finite number of configurations. Fig. 1 shows the gauge-invariant and the noninvariant parts of the correlation function.

The signal-to-noise issue has been examined [26] for the connected hadron correlators from the noise source on a time slice and it was found that the noise wall source was worse than the point source for all mesons except the pion. It is worse still for the nucleon. In this case, the signal-to-noise ratio is

$$\frac{C(t, \vec{p} = 0)}{\sigma} \approx \sqrt{\frac{N}{V_3}} e^{-(m_N - 3/2m_\pi)t}, \quad (19)$$

where N is the number of noises and V_3 is the three-volume of the time slice. In addition to the usual exponential suppression in time, there is a prefactor which reduces the signal-to-noise ratio further by a factor of $\sqrt{V_3}$. The situation can be ameliorated by reducing the noise contamination from neighboring sites with less source points. This introduces the idea of a noise grid source with support on some uniformly spaced grid points on a time

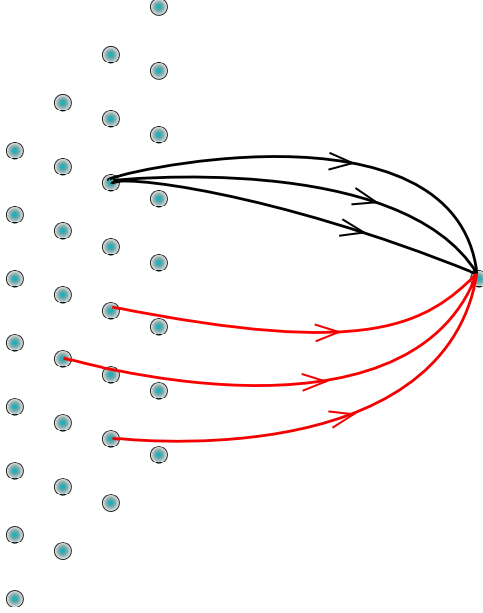


FIG. 1. (Color online) Diagram to illustrate the signal and noise of the nucleon correlation function with a Z_3 noise grid source on a time slice. The upper part with three quarks originating from the same spatial site is an example of the gauge-invariant signal and the lower one is an example of the gauge-noninvariant noise with three quarks originating from different spatial sites which will be suppressed by gauge average and noise average.

slice, but this does not fundamentally alter the conclusion that, given the same computer time, any noise source is worse than the point source for the meson and nucleon. These observations suggest a new algorithm for the grid noise with low-mode substitution (LMS) to reduce the variance from noise contamination while simultaneously addressing the low-mode correlation [26]. The idea is to replace the low-mode part of the quark propagator $G^L\eta$ estimated from noise sources by the exact G^L in Eq. (17) when all three quarks are in the low modes or when two quarks are in the low modes and the third is in the high modes – in constructing the baryon correlators to reduce the noise contributions shown in Fig. 1. It turns out that this LMS is quite successful. It reverses the above-mentioned trend that a noise source is worse than a point source and instead reduces the errors compared to the point source [26].

The Z_3 grid noise is

$$\eta(x) = \sum_{i \in \mathcal{G}} \theta_i \delta_{x,i}, \quad (20)$$

where \mathcal{G} is a sparse grid of lattice sites on a time slice, and θ_i is the Z_3 random phase on site i with the property $\theta_i^3 = 1$ and $\langle \theta_i \theta_j \theta_k \rangle = \delta_{ij} \delta_{jk}$. Tests on the $32^3 \times 64$ lattice with a Z_3 noise on 64 evenly spaced grid points on a time slice and with LMS reveal that for light quarks (pion masses at 200 – 300 MeV) the errors of the meson and nucleon masses can be reduced by a factor of 3 to 4 as compared to the point source [26].

Since the nucleon has a physical size of ~ 0.6 – 0.8 fm as deduced from its axial and electromagnetic form factors, a smeared source for the nucleon usually leads to a reduction of errors for the nucleon mass from the point source. In view of this, we introduce a smeared-grid source to increase the overlap with the nucleon ground-state wave function and diminish the contribution from the radially excited states and the collateral πN scattering states. We adopt the gauge-invariant spatial Gaussian smearing [29] on the grid source,

$$\eta^S(x') = S(x', x)\eta(x), \quad (21)$$

where $S(x', x)$ is the smearing operator. By design, the smearing operator $S(x', x)$ should produce a Gaussian distribution with a Klein-Gordon propagator. It is computed as an iteration of many small smearing steps,

$$S(x', x) = \left(1 - \frac{3w^2}{2n}\right)^n \left[1 + \frac{w^2}{4n - 6w^2} \sum_{i=1}^3 \left(U_i(x', t)\delta_{x', x-\hat{i}} + U_i^\dagger(x' - \hat{i}, t)\delta_{x', x+\hat{i}}\right)\right]^n, \quad (22)$$

where w is the input width for a Gaussian distribution and n is the number of smearing steps. The corresponding quark propagator with a smeared source is

$$\begin{aligned} G(y, \eta^S) &= D^{-1}(y, x')\eta^S(x'), \\ &= \sum_{i \in \mathcal{G}} \theta_i D^{-1}(y, x')S(x', x)\delta_{x,i}. \end{aligned} \quad (23)$$

To check if the actual width from the smearing procedure is consistent with the input width parameter w , we first define the actual smearing size by

$$r = \sqrt{\frac{\sum_x x^2 \rho(x)}{\sum_x \rho(x)}}, \quad (24)$$

where, for each spatial position x , $\rho(x)$ is the norm over spin and color of the smeared source vector which is created from a point source vector $\delta(x)$. Then we plot r vs w for different step sizes n in Fig. 2. We see that for a given n , there is a range of w where the resultant smearing size r has a nearly linear relation with w . Beyond that range, r flattens off as w

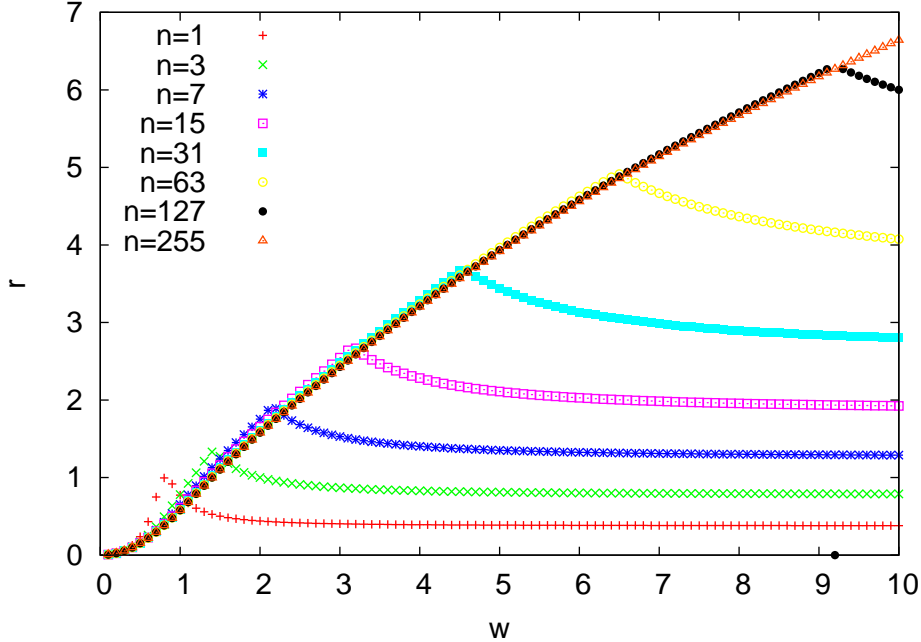


FIG. 2. The plot shows the relation between the input width parameter w and the output radius of the gauge-invariant Gaussian smearing r for different iteration numbers n .

increases. As n increases, the range for the near-linear relation expands. Even though we do not have evidence that the smeared source, when its r is much smaller than the input w , leads to ill physical effects, we think it is safe to keep the linear relationship. Consequently, we will use sufficient n for a prescribed width w so that r and w are in the linear range.

We choose the parameters $w = 4$ and $n = 100$ for this work which give a smearing size $r \sim 3$. The smeared distribution for a typical configuration is plotted in Fig. 3 which is indeed close to a Gaussian shape. This particular choice of w and n is aimed at increasing the overlap with the nucleon wave function while simultaneously minimizing contamination from different grid sites.

In addition to the smearing source, we also placed two sources on the time slices $t = 0$ and $t = 32$ and calculated the inversion simultaneously. As we shall see later in the calculation of the nucleon mass and quark scalar matrix elements, the time window for the fitting range is between $t = 6$ to $t = 14$ which is far from the two sources such that the contamination from the time backward propagating $S_{11}(1/2^-)$ state from the second source at a distance of 32 time slices away is negligible. This approach nearly doubles our statistics without

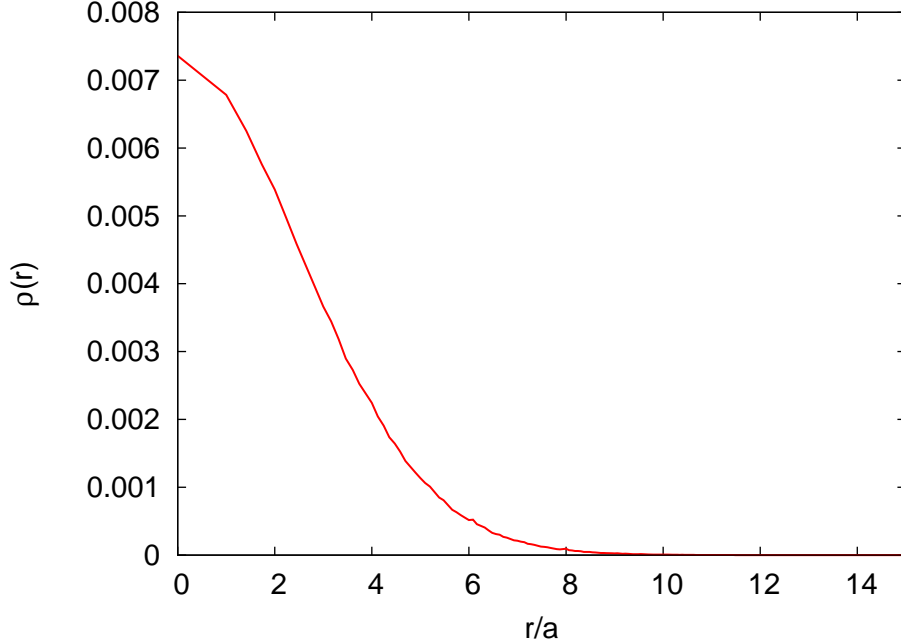


FIG. 3. The plot shows the profile of the smeared source vector with $w = 4$ and $n = 100$ on a $24^3 \times 64$ configuration.

computational overhead.

The local interpolating operator of the nucleon is taken to be [30]

$$\begin{aligned}\chi_\alpha(x) &= \epsilon^{abc} \psi_\alpha^{(u)a}(x) \psi_\beta^{(u)b}(x) (\tilde{C})_{\beta\gamma} \psi_\gamma^{(d)c}(x) \\ \bar{\chi}_{\alpha'}(x) &= -\epsilon^{a'b'c'} \bar{\psi}_{\gamma'}^{(d)c'} (\tilde{C})_{\gamma'\beta'} \bar{\psi}_{\beta'}^{(u)b'}(x) \bar{\psi}_{\alpha'}^{(u)a'}(x),\end{aligned}\quad (25)$$

where $\tilde{C} = \gamma_2 \gamma_4 \gamma_5$ in the Pauli-Sakurai gamma-matrix convention. The color indices are denoted with latin letters and the Dirac indices are denoted with greek letters. The nucleon correlation function is constructed as

$$\begin{aligned}C(y, x; \Gamma; G^{(u)}, G^{(d)}, G^{(u)}) &= \langle \epsilon^{abc} \epsilon^{a'b'c'} \left[\text{tr} \left(\Gamma G^{(u)aa'}(y, x) \underline{G}^{(d)bb'}(y, x) G^{(u)cc'}(y, x) \right) \right. \\ &\quad \left. + \text{tr} \left(\Gamma G^{(u)aa'}(y, x) \right) \text{tr} \left(\underline{G}^{(d)bb'}(y, x) G^{(u)cc'}(y, x) \right) \right] \rangle, \quad (26)\end{aligned}$$

where the $G^{(u/d)aa'}(y, x)$ stands for the u/d quark propagator from the site x to y and the color indices from a' to a . The average is taken over the different gauge configurations and noise sources. \underline{G} is defined as $(\tilde{C}G\tilde{C}^{-1})^T$. The trace and the transpose operations only act on Dirac indices. When the masses of the u and d quarks are set to be equal, $G^{(d)}$ and $G^{(u)}$

are the same propagator. The correlation function $C(G_1, G_2, G_3)$ is a linear functional of the three functions G_1, G_2 , and G_3 .

From Eq. (23), we obtain the quark propagator as a summation of the propagators from different grid sites with different Z_3 phases:

$$G(y, \eta^S) = \sum_{i \in \mathcal{G}} \theta_i G_i(y), \quad (27)$$

where

$$G_i(y) \equiv D^{-1}(y, x') S(x', x_i). \quad (28)$$

The nucleon correlation function can be written as

$$\begin{aligned} C(G, G, G) &= \langle \hat{C}(\sum_i \theta_i G_i, \sum_j \theta_j G_j, \sum_k \theta_k G_k) \rangle \\ &= \sum_{i,j,k} \langle \theta_i \theta_j \theta_k \hat{C}(G_i, G_j, G_k) \rangle \\ &= \sum_{i,j,k} \delta_{ij} \delta_{jk} \langle \hat{C}(G_i, G_j, G_k) \rangle \\ &\longrightarrow \sum_i \langle \hat{C}(G_i, G_i, G_i) \rangle, \end{aligned} \quad (29)$$

where \hat{C} denotes the correlator in a gauge configuration. $\hat{C}(G_i, G_i, G_i)$ is the correlator of a point or a smeared source at site x_i for a gauge configuration. Thus, given a sufficient number of noise vectors and/or gauge configurations, this effectively increases the statistics of the correlator by the number of the grid points as compared to that of the point source.

Since we compute the quark propagator by splitting it into the low-mode and the high-mode pieces

$$\begin{aligned} G &= G^H + G^L \\ &= G^H + \sum_i \theta_i G_i^L, \end{aligned} \quad (30)$$

the nucleon correlation function can be split into contributions from the low modes and the

high modes

$$\begin{aligned}
C(G, G, G) &= C(G^H + \sum_i \theta_i G_i^L, G^H + \sum_j \theta_j G_j^L, G^H + \sum_k \theta_k G_k^L) \\
&= C(G^H, G^H, G^H) + \sum_i C(\theta_i G_i^L, \theta_i G_i^L, \theta_i G_i^L) \\
&\quad + \sum_i C(\theta_i G_i^L, G^H, G^H) + \sum_i C(G^H, \theta_i G_i^L, G^H) + \sum_i C(G^H, G^H, \theta_i G_i^L) \\
&\quad + \sum_i C(\theta_i G_i^L, \theta_i G_i^L, G^H) + \sum_i C(\theta_i G_i^L, G^H, \theta_i G_i^L) + \sum_i C(G^H, \theta_i G_i^L, \theta_i G_i^L) \\
&\quad + \sum_{i \neq j} C(\theta_i G_i^L, \theta_j G_j^L, G^H) + \sum_{i \neq j} C(\theta_i G_i^L, G^H, \theta_j G_j^L) + \sum_{i \neq j} C(G^H, \theta_i G_i^L, \theta_j G_j^L) \\
&\quad + \sum_{i \neq j \text{ or } j \neq k \text{ or } k \neq i} C(\theta_i G_i^L, \theta_j G_j^L, \theta_k G_k^L), \tag{31}
\end{aligned}$$

where both G^H and $\sum_i \theta_i G_i^L$ are noise-estimated propagators. We denote the latter explicitly because we will modify it below to improve the signal-to-noise ratio.

We first note that the last four terms with a summation over different Z_3 noises, i.e. terms with $\sum_{i \neq j}$ and $\sum_{i \neq j \text{ or } j \neq k \text{ or } k \neq i}$ are pure noise. They can be dropped without changing the expectation value of the noise estimate. For the purely high-mode contribution, as well as the mixed terms containing two G^H and a single G^L , one needs to compute them with the Z_3 noise technique. However, for the correlators either involving three G^L or with two G^L and one G^H , one can implement the LMS [26] to improve the signal. For the case where all the propagators are purely from the low modes, we substitute the noise-estimated one [the second term on the right side of Eq. (31)] with the exact expression without noise, i.e. $\sum_i C(G_i^L, G_i^L, G_i^L)$ which is the sum of the low-mode contribution to point-source correlators over the grid points. For those terms containing only one G^H and two G^L , we replace the noise-estimated correlator [the sixth through eighth terms on the right side of Eq. (31)] with the one where the sources of the two G^L are placed at the same grid site i and multiplied with θ_i^2 . This ties them with G^H , which contains a θ , to form the less noisy multiple-point nucleon correlator over the grid points. (See the upper part of Fig. 1.) Since we replaced two noise-estimated G^L with one noise-estimated $G^L \otimes G^L$, this should yield a gain in the signal-to-noise ratio by $\sqrt{V_3}$, where V_3 is the number of grid points.

Therefore, the nucleon correlator with LMS is

$$\begin{aligned}
C_{LMS}(G, G, G) &= C(G^H, G^H, G^H) + C(G^L, G^H, G^H) + G(G^H, G^L, G^H) + G(G^H, G^H, G^L) \\
&+ \langle \sum_i \theta_i^2 [\hat{C}(G_i^L, G_i^L, G^H) + \hat{C}(G_i^L, G^H, G_i^L) + \hat{C}(G^H, G_i^L, G_i^L)] \rangle \\
&+ \sum_i C(G_i^L, G_i^L, G_i^L). \tag{32}
\end{aligned}$$

We should point out that this is different from low-mode averaging [31, 32] in that the noise-estimated low-mode propagators in LMS are substituted with the exact ones whenever possible and matched with the high-mode propagators at the grid points, whereas low-mode averaging would replace the noise-estimated low-mode part with the average over all space-time points.

Ideally, we would compute a separate set of quark propagators from each source point on the grid source, eliminating those gauge-dependent contributions to the two-point function (which contribute only noise) where the propagators come from different source points while still gaining the benefits of the extra sources. However, this is prohibitively expensive, as it requires additional inversions. As mentioned previously, the grid source is chosen in order to achieve some of the benefits of using multiple spatial sources while requiring no additional inversions, but noise is introduced by those contributions which mix the various source locations. However, the purely low-mode part requires no inversions to calculate, so it can be computed exactly without noise from these spurious contributions. Since this low-mode contribution dominates the correlation function at large time separation, the use of the low modes to compute it exactly allows for a substantial reduction in the error of the nucleon mass.

In Fig. 4, we plot the effective masses of the nucleon correlation functions from the point source, the noise-grid source, the noise-grid source with smearing and folding of the two correlators from sources on two time slices at $t = 0$ and $t = 32$, and the variation calculation with the point and the noise-grid source. The fitted masses are tabulated in Table I. They are calculated on the ensemble of 50 $2 + 1$ flavor DWF configurations on the $24^3 \times 64$ lattice with $am_l = 0.005$. The overlap propagator is computed with the valence quark mass at $am \sim 0.016$ close to the light sea mass which corresponds to $m_\pi \sim 330$ MeV. We see the fitted nucleon mass for the point source has a 12% error. The Z_3 noise grid source with LMS reduces the error by a factor of 2.8 down to 4.6%. Replacing the point grid with a

smearred grid and putting the sources on two time slices reduces the error further to 1.8%. Finally, a variational calculation with a second nucleon interpolation field and smearred grid sources gives an error of 1.1%. For this, the second nucleon interpolation field we use is $\chi'_\alpha(x) = \epsilon^{abc}\gamma_5\psi_\alpha^{(u)a}(x)\psi_\beta^{(u)b}(x)C_{\beta\gamma}\psi_\gamma^{(d)c}(x)$ and each quark in the interpolation field can be a point-grid or a smearred-grid source. The combination of point-grid and smearred-grid sources gives eight operators. Of these eight, we need to consider the symmetric combination between the two u quarks so that they are spatially symmetric. This reduces the choice to six. We calculate all of the six combinations for each of the two interpolation fields (which differ in their Dirac structure) and during the analysis choose four of the twelve which have a good result for the ground state and a decent result for the lowest excited state. The operators of the 4×4 variational calculation are obtained by choosing χ or χ' and choosing all three quarks smearred or just one quark smearred.

It is worth emphasizing that replacing the point source by a point-grid source with LMS, with no further improvements, reduces the error by a factor of 2.8, improving the statistics by a factor of 8 with no added expense. For comparison, we note a recent study of the all-mode-averaging (AMA) method [33] on the same $24^3 \times 64$ lattices which found that to achieve the same error on the nucleon mass with AMA on 32 smearred sources would cost 1/4 of the computer time of a single smearred source without AMA. This implies that AMA, given the same resources, would improve the statistics by a factor of 4.

The variational approach using two different interpolating fields combined with a mixed point/smearred source does reduce the variance, but not enough to justify the increased computational cost required for the additional point-source inversion. Overall, we find that the use of smearred Z_3 noise-grid sources on two time slices is the most efficient approach; this reduces the error of the nucleon mass by a factor of 7 compared to that of the point source while adding only the slight overhead of LMS to the computational cost.

	Point	Z_3 -grid + LMS	Z_3 -grid + LMS + Smear + Folding	Z_3 -grid + LMS + Variation
Nucleon mass (GeV)	1.13(14)	1.08(5)	1.14(2)	1.12(1)

TABLE I. Comparison of nucleon masses from several methods.

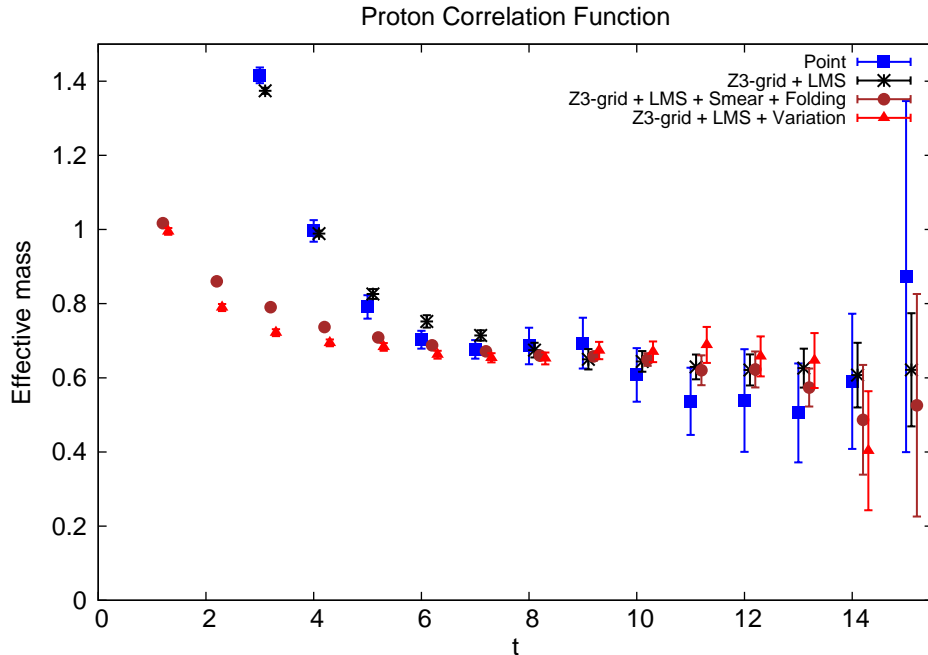


FIG. 4. The comparison of proton effective masses for the point source, the Z_3 point-grid source, the Z_3 smeared-grid source and variation. The fitted results are given in Table I.

IV. QUARK LOOPS

The quark loop with zero momentum on a time slice is defined as $\sum_{\vec{x}} \text{tr} \Gamma D^{-1}(\vec{x}, \vec{x})$ where the trace is over color and spin and Γ is a γ matrix. It is computed for the quark propagator which begins and ends at the same site (or more generally a neighboring site if a point-split current or smearing function is involved). It is needed for meson correlators with the quark-antiquark annihilation channels and the disconnected insertions in nucleon form-factor calculations. Since the quark-loop calculation involves the sum of quark propagators originating from all lattice sites on a time slice, it is not practical to calculate it with the usual point-source inversion. Instead, stochastic estimation is usually used for such calculations. Taking the noise source in Eq. (20) where the grid includes all lattice points, the loop at the position i for each gauge configuration is then estimated as

$$\begin{aligned}
L(i) &= \langle \theta_i^* D^{-1}(i, x) \eta(x) \rangle \\
&= \langle \theta_i^* \sum_{j \in \mathcal{G}} \theta_j D^{-1}(i, x) \delta_{x,j} \rangle \\
&\longrightarrow D^{-1}(i, i),
\end{aligned} \tag{33}$$

where the average is taken over the noise. When the noise source $\eta(x)$ is diluted in color and Dirac indices, the loop $L(i)$ is a matrix in color-spin space. For a finite number of noises, there is noise contribution from different sites which is not a gauge-invariant quantity and is suppressed by gauge averaging in addition to noise averaging. It has been shown that Z_N noise is optimal since it gives the minimum variance [34, 35]. Here we shall use the Z_4 noise which, in addition to calculating loops, can be applied to the evaluation of quark annihilation involving two quarks and two antiquarks in the interpolation operators, such as in meson-meson scattering calculations.

In the nucleon three-point function involving the loops in the disconnected insertion (DI), the quark loop including the external current needs to be Fourier transformed to give a definite momentum transfer. Because of the translational invariance after gauge averaging, one does not need to put the noise on all spatial points at a given time slice. Instead, one could select an evenly spaced grid separated by Δ_x sites in each of the spatial directions. In this case, the Fourier transformation of $f(x)$ on a lattice with periodic boundary conditions can project to a low-momentum q in the x direction via the relation $\sum_{i \in \mathcal{G}} e^{-iqx_i} f(x_i)$. Besides the definite momentum q , it involves a mixture of the next higher momentum being $q_H = q \pm \frac{L}{\Delta_x} p_\ell$, where L is the spatial lattice dimension and p_ℓ is the unit of lattice momentum, i.e. $p_\ell = \frac{2\pi}{L}$. Other higher momenta can also be mixed in, but we will not discuss them here. In the nucleon DI calculation, the source and sink momenta of the nucleon propagator can have a definite momentum (the grid source for the nucleon propagator with $\Delta_x = 6$ will have a zero-momentum source mixed with $p = 4p_\ell$). As long as these mixed momenta are taken into account in selecting the desired momentum transfer, the mixed q_H loop will be suppressed due to momentum conservation. Should it happen that the contribution from the mixed high-momentum loop with q_H does not vanish due to the finite noise and gauge configurations, it will involve the intermediate state with energy $E = \sqrt{m_N^2 + q_H^2}$. Its contribution is exponentially suppressed with a factor $\sim e^{-\Delta E t}$ where $\Delta E = \sqrt{m_N^2 + q_H^2} - \sqrt{m_N^2 + q^2}$. Since $q_H > q$ and thus $\Delta E \geq 0$, the suppression factor can be substantial for the range of t where the signal of the DI calculation is to be obtained. For the present work, the scalar content is a forward matrix element which requires $q = 0$. This makes it easier to consider the possible q_H contamination.

In our present calculation of the scalar matrix elements on the $24^3 \times 64$ lattice, we shall use the grid with $(\Delta_x, \Delta_y, \Delta_z, \Delta_t) = (4, 4, 4, 2)$ for the high modes with odd-even dilution

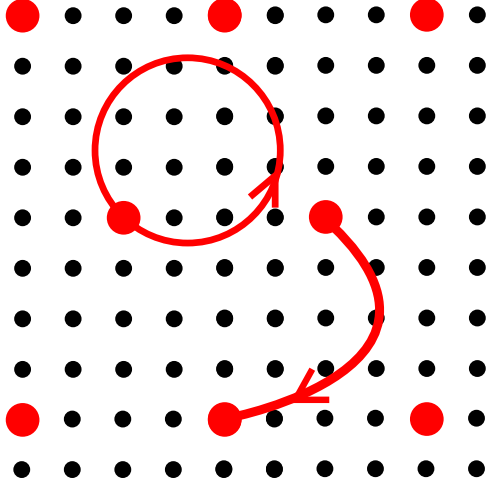


FIG. 5. The diagram shows the finite-noise contribution of the loops. The red dots are the grid sources on even sites. The closed loop shown is an example of the signal and the open-jawed curve is an example of a noise contribution which is suppressed by gauge and noise averaging and by the finite-distance separation of the quark propagator.

as well as dilution in time. This entails the calculation of four noise propagators (two for odd-even and two for time dilution). We show in Fig. 5 a cartoon of the even-odd dilution and the time dilution. The nearest neighbor which can give rise to noise contribution is at a Euclidean distance of $d = \sqrt{4^2 + 2^2} \approx 4.5$ which is reasonably far; as such, we do not do the unbiased subtraction as has been done previously for noises which have support on all lattice points [7, 36], where the nearest neighbor is a single lattice site away. Should there be noise contribution from the higher mixed momentum q_H despite suppression due to momentum conservation as discussed above, it will be further suppressed by $e^{-\Delta E t}$. With $q_H = 2\pi/4$, $q = 0$ and $m_N \sim 0.66$, ΔE is 1.04. As a result, the suppression factor for $t \geq 6$ where the strangeness matrix element will be extracted is less than $e^{-6.24} = 0.002$ which renders this noise contribution negligible.

To further improve the loop calculation, we use low-mode averaging (LMA) for the low-mode contribution by summing over the spatial volume on a time slice, while the high-mode contribution is calculated on the grid as described above.

$$L = \sum_i L_L(i) + \sum_j \frac{L^3}{\Delta_x \Delta_y \Delta_z} L_H(j), \quad (34)$$

where L_H is scaled from the sum of the grid points to that of the full space volume to match

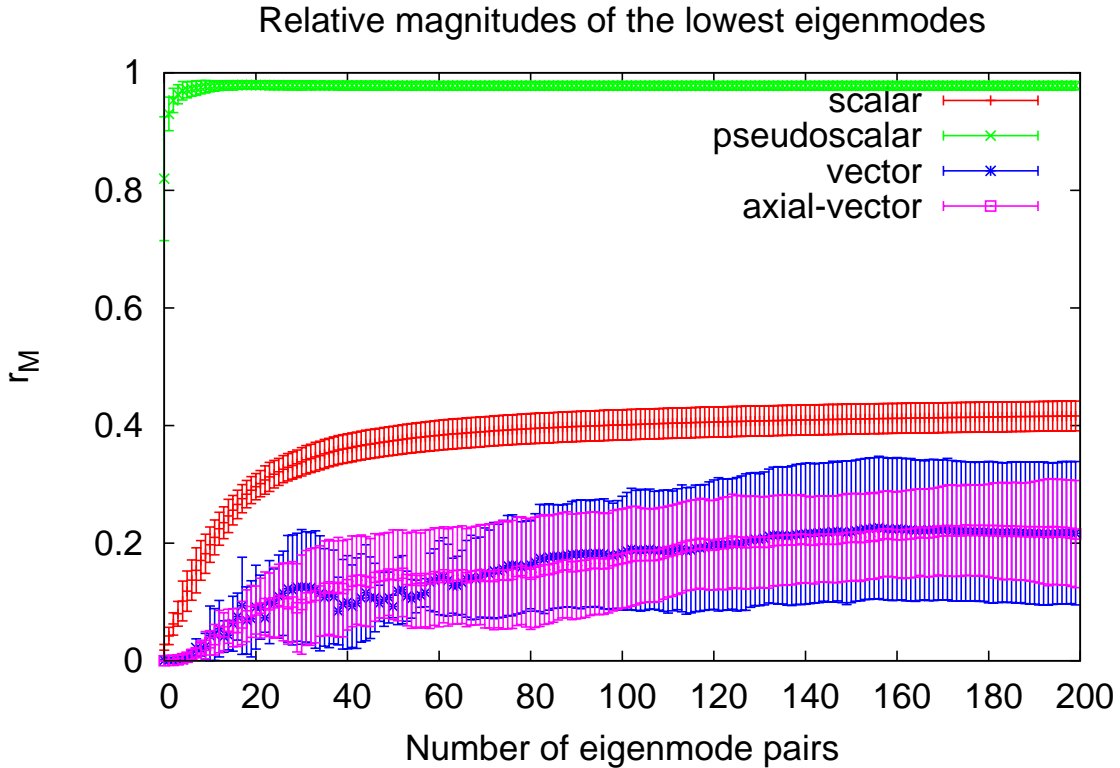


FIG. 6. The plot shows the relative low-mode magnitude r_M defined in Eq. (35) as a function of the accumulated pairs of eigenmodes plus the zero modes. The corresponding pion mass is about 330 MeV for both the quark in the loop and the light sea-quark. The error bars denote the standard deviation from five gauge configurations.

with $L_L(i)$ from LMA.

It is interesting to find out the individual contributions to the quark loops from the low modes and the high modes. To this end, we measure the following quantity to gauge the magnitude of the low-mode contribution relative to the sum of those of the low modes and the high modes,

$$r_M = \frac{|L^{low}|}{|L^{low}| + |L^{high}|}. \quad (35)$$

The norms of the low- and the high-mode parts of the loops are used to avoid the singular situation when the full loop is near zero, especially for the vector and axial-vector channels, where the vacuum expectation values of the loops are zero. The ratio is taken before averaging over the time slices so as to be more relevant to the DI calculation where the loop is to be correlated with the nucleon propagator in a limited time range. We used

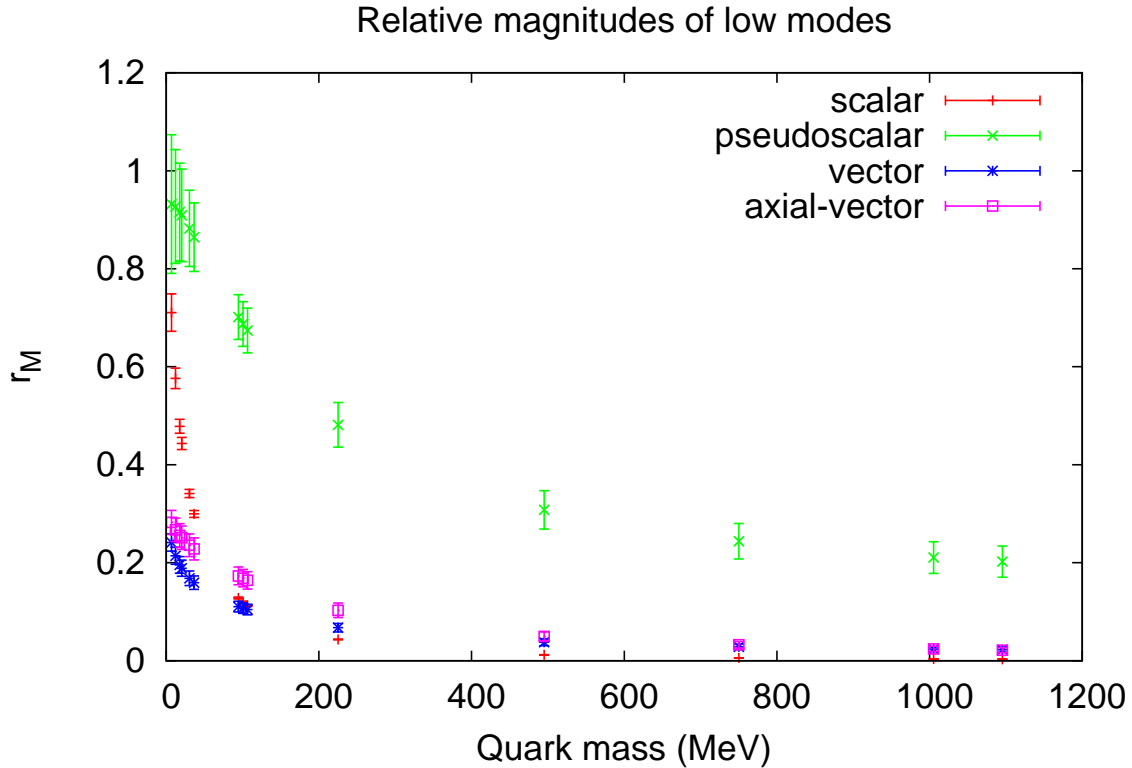


FIG. 7. The plot shows the relative low-mode magnitude r_M with different bare quark masses for the loop. The error bars denote the standard deviation over gauge configurations. This study is carried out on 100 configurations.

five configurations in this study.

The relative magnitudes of the low modes, defined as r_M in Eq. (35), in terms of the accumulated pairs of low eigenmodes in addition to the zero modes are shown in Fig. 6 for different currents. This is for the case of a quark mass which corresponds to a pion mass of ~ 330 MeV. We see that r_M of the pseudoscalar loop is dominated by a few lowest eigenmodes, and all the higher modes contribute only a few percent. This is not surprising, since the space-time integral of the pseudoscalar density times the quark mass is just the topological charge of the configuration which is totally determined by the zero modes. For the scalar loop, both the L^{low} and L^{high} are positive on all time slices and for all configurations. Therefore, the relative magnitude r_M is the relative low-mode contribution. In this case, $\sim 40\%$ of the scalar loop is contributed by the lowest 50 pairs of eigenmodes. Even at this low quark mass, most of the contribution comes from the high modes. In the vector and

axial-vector channels, the relative low-mode magnitude is small with a large variance.

To study the quark-mass dependence, we show r_M in Fig. 7 with 200 pairs of the lowest eigenmodes plus zero modes as a function of the quark mass. The relative magnitude of the low modes in the pseudoscalar channel dominates in the light quark region and decreases with quark mass. The r_M for the scalar, vector, and axial loops decrease with quark mass faster than that of the pseudoscalar loop and become very small for $m > 500$ MeV. This hints that for the charmness in the nucleon, the high modes may play a substantial role.

V. DISCONNECTED INSERTION OF THE THREE-POINT CORRELATION FUNCTION

In order to extract the strangeness and charmness contributions to the nucleon given in Eq. (3), we compute the ratio of the disconnected three-point function to the two-point function, defined as

$$R(t, t', t_0) = \frac{\langle \chi_N(t) \bar{q}q(t') \bar{\chi}_N(t_0) \rangle - \langle \chi_N(t) \bar{\chi}_N(t_0) \rangle \langle \bar{q}q(t') \rangle}{\langle \chi_N(t) \bar{\chi}_N(t_0) \rangle}. \quad (36)$$

The matrix element for the quark content in the nucleon can be extracted from this ratio,

$$\begin{aligned} f_{T_q} &= \frac{m_q}{m_N} \langle N | \bar{q}q | N \rangle \\ &= \frac{m_q}{m_N} \lim_{\substack{t'-t_0 \rightarrow \infty \\ t-t' \rightarrow \infty}} R(t, t', t_0). \end{aligned} \quad (37)$$

For the disconnected insertion, one can write the above ratio in terms of the nucleon propagator and the quark loop as in Sec. IV,

$$R(t, t', t_0) = \frac{\langle \hat{C}(t, t_0) (L(t') - \langle L(t') \rangle) \rangle}{\langle \hat{C}(t, t_0) \rangle}. \quad (38)$$

This approach is illustrated in Fig. 8.

We show in Fig. 9 $R(t, t', t_0)$ as a function of t' from $t_0 = 0$ to $t = 12$ for the case where the quark mass in the loop is 0.063, which corresponds to the strange quark mass, and the quark mass in the nucleon propagator is 0.016, which corresponds to $m_\pi = 330$ MeV. This calculation is done on 176 configurations. One can see that the error bars are quite large, and it appears to be difficult to determine a plateau region from which to extract the matrix element.

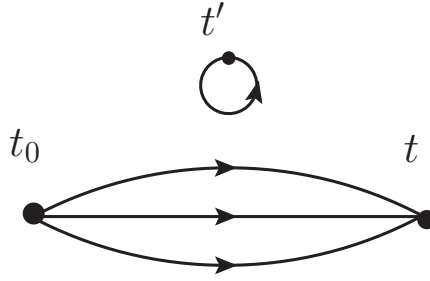


FIG. 8. The sketch illustrates the disconnected insertion of the three-point function.

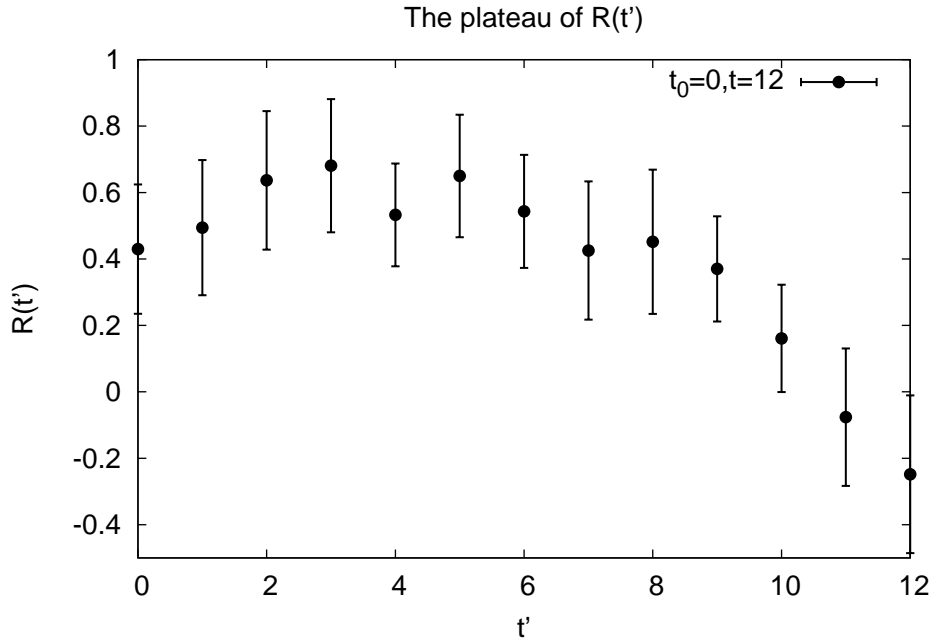


FIG. 9. The plot shows the behavior of $R(t, t', t_0)$ as a function of t' for fixed t_0 and t at 0 and 12. The expected plateau in the middle of the time window is rather noisy.

One can improve the statistics by using the summed ratio method [37]. As illustrated in Fig. 10, $R(t, t', t_0)$ is summed over t' between $t_0 + 1$ and $t - 1$ inclusive

$$R'(t, t_0) = \sum_{t'=t_0+1}^{t-1} R(t, t', t_0). \quad (39)$$

Previous calculations of the quark momentum fraction $\langle x \rangle$ in the DI [38], have studied various choices for the domain of t' . They found that choosing t' between $t_0 + 1$ and $t - 1$

inclusive produced less noise than other choices. For this reason, we have chosen to adopt the same domain in our calculations.

The ratio method is a means to incorporate information from multiple values of $t - t_0$ and t' , thus reducing the statistical error, without needing to explicitly consider the points at which t' might suffer from excited-state pollution because they are too close to t_0 and t . The method relies on the fact that when the value of R is summed from source to sink, the contribution from these contaminated points near the end does not depend on the distance from source to sink so long as that distance is large enough. When the propagator length is increased, the additional contribution to the sum comes from a single point in the center, within the plateau region. Thus, by examining how much the summed ratio increases when the propagator length is increased, we can get an estimate for the plateau value of R that incorporates information from multiple values of t' and t .

It was shown [37, 38] that the ratio $R'(t, t_0)$ has a linear behavior in a region where t is large enough to have a plateau in $R(t, t', t_0)$ and thus

$$R'(t, t_0)_{t \gg t_0} \rightarrow \text{const} + t \langle N | \bar{q}q | N \rangle. \quad (40)$$

We fit the value of the summed ratio R' to constant-plus-linear over the range in which t is large enough for R to have a plateau; the slope in R' is thus a measurement of the plateau value of R , i.e. the matrix element, from which the quark content f_{T_q} is determined using Eq. (37). We should mention that the excited-state contamination in the sum method is found to be $O(e^{-\Delta_E t})$ [38, 39] where Δ_E is the energy gap between the nucleon ground state and the lowest excited state and t is the sink time of the nucleon propagator (we have taken the source time at $t_0 = 0$). On the other hand, the contamination of the excited state in the plateau method is $O(e^{-\Delta_E t_p} + (e^{-\Delta_E(t_s - t_p)}))$ where t_p is the time when the plateau appears and t_s is the fixed sink time of the nucleon propagator. Since in practice $t \sim t_s > t_p$ and $t \sim t_s > (t_s - t_p)$, the excited-state contamination is less than that of the plateau method and this has been demonstrated in the calculation of the isovector g_A [39].

VI. NUMERICAL RESULTS

This work is carried out using valence overlap fermions on 176 configurations with $2 + 1$ flavors of sea domain-wall fermions. The lattice size is $24^3 \times 64$ with the inverse lattice spacing

$1/a = 1.73(3)$ GeV. The u/d quark mass in the DWF sea is 0.005 which corresponds to a pion mass of 331 MeV. The sea strange-quark mass is 0.04. This pion mass is matched for the overlap fermion with a valence u/d quark mass of 0.016. With the help of the multimass algorithm [40], we compute a set of 26 quark masses for the nucleon and 10 for the loops with an overhead of about 8% [41] of the cost of inverting the lowest quark mass. The summed ratio of the strangeness is shown in Fig. 11(a) for $am_s = 0.063$ and $am_{ud} = 0.016$.

The nucleon propagator in the DI of the three-point function is calculated with a smeared grid source at $t = 0$ and $t = 32$. The time-forward propagator with positive-parity projection and the time-backward propagator with negative-parity projection are averaged. We see that a linear slope develops after $t = 6$ where the nucleon starts to emerge in the nucleon correlator and it is in the middle of the plateau from Fig. 9.

To see the contributions to $R'(t, t_0)$ and the respective slopes from the low modes and the high modes in the loop, we plot them separately in Fig. 11(b). It is interesting to observe that practically all the contributions to $R'(t, t_0)$ are coming from the low modes despite the fact that the low modes saturate only $\sim 15\%$ of the strange quark loop. The high-mode contribution is quite small. Since the low-mode contribution is exact, it only has variance from the gauge ensemble. The variance of the high-mode contribution comes from both the noise and gauge ensembles. Since its contribution is small, there is no need to improve this part of the estimate with more noise vectors.

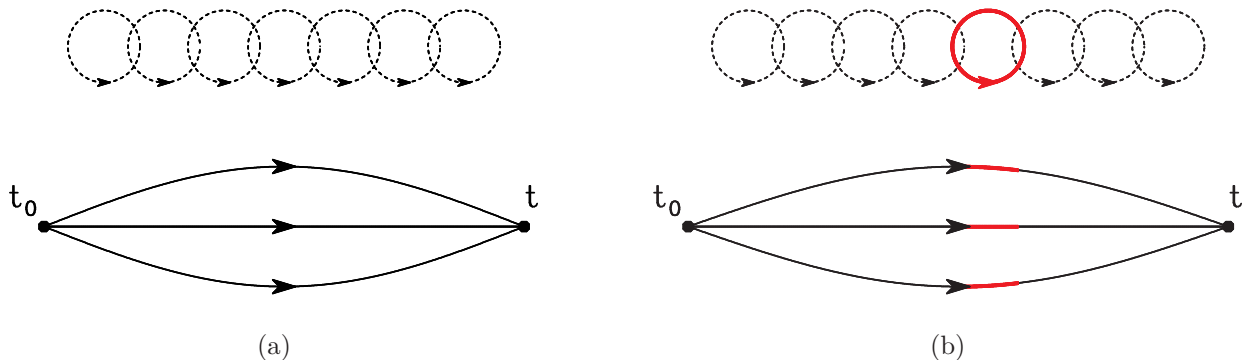


FIG. 10. The cartoon shows the summed ratio method for the disconnected three-point function. The red part is the additional contribution to the sum when the propagator length is increased, which is within the plateau region.

The renormalized strangeness content reads

$$\langle N|\bar{s}s|N\rangle_R = Z_S^{\overline{MS}}(2GeV) \langle N|\bar{s}s|N\rangle, \quad (41)$$

where the nonperturbative renormalization constant $Z_S^{\overline{MS}}(2GeV) = 1.121(6)$ (stat) has been calculated in the regularization-independent momentum-subtraction renormalization scheme [42] with input of Z_A from the chiral Ward identity. It is plotted in Fig. 12 as a function of the valence u/d quark mass. The results are obtained from fitting the range from $t = 7$ to 14 in each quark-mass case. The lowest quark mass gives $m_\pi = 250$ MeV.

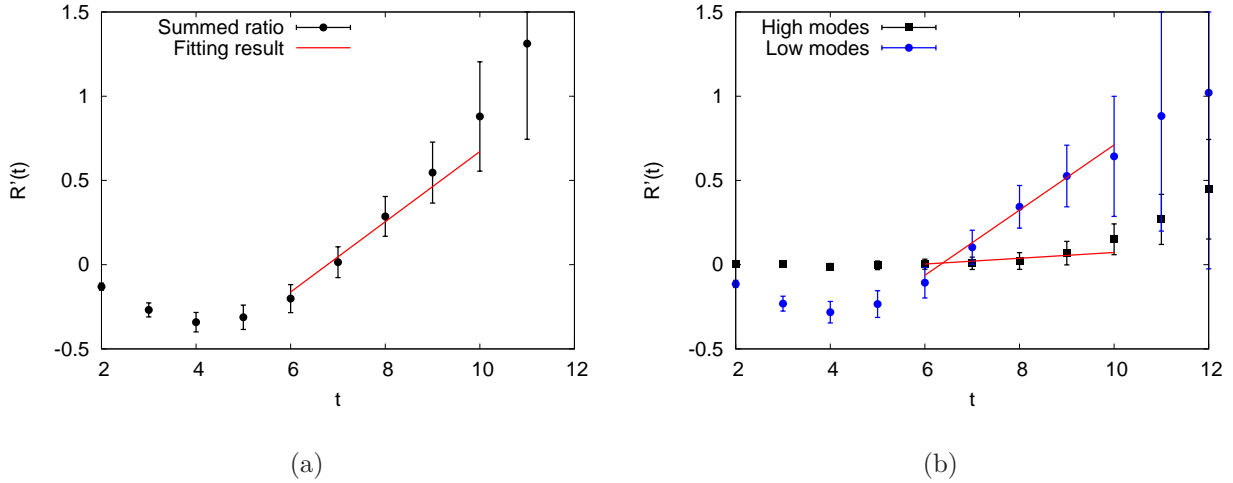


FIG. 11. (a) The plot for the summed ratio of the strangeness. The corresponding pion mass is 330 MeV. (b) The separate contributions from the low modes and the high modes.

To obtain a value at the physical point we must perform a chiral extrapolation. The mixed-action formula can be derived from the partially quenched expression in Ref. [43], using the simple conversion of partially quenched to mixed-action formulas described in Refs. [44, 45]. The next-to-leading-order mixed-action extrapolation formula is [46]

$$\begin{aligned} \langle N|\bar{s}s|N\rangle &= \langle N|\bar{s}s|N\rangle^{LO} + \frac{2g_{\Delta N}^2}{(4\pi f_\pi)^2} \left(\langle N|\bar{s}s|N\rangle^{LO} - \langle \Delta|\bar{s}s|\Delta\rangle^{LO} \right) (J(m_\pi, \Delta, \mu) + J(\tilde{m}_{ju}, \Delta, \mu)) \\ &\quad + E_s(\mu) \frac{m_\pi^2}{(4\pi f_\pi)^2} + E_s^{PQ}(\mu) \frac{m_{ju}^2 - m_\pi^2}{(4\pi f_\pi)^2} + E_s^a(\mu) \frac{a^2 \Delta_{mix}}{(4\pi f_\pi)^2}, \end{aligned} \quad (42)$$

where $\langle N, \Delta|\bar{s}s|N, \Delta\rangle^{LO}$ are the leading-order contributions to the strange matrix element in the chiral limit and the nonanalytic chiral loop function is [47]

$$J(m, \Delta, \mu) = 2\Delta \sqrt{\Delta^2 - m^2 + i\epsilon} \log \left(\frac{\Delta - \sqrt{\Delta^2 - m^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m^2 + i\epsilon}} \right) + m^2 \log \left(\frac{m^2}{\mu^2} \right) + 2\Delta^2 \log \left(\frac{4\Delta^2}{m^2} \right), \quad (43)$$

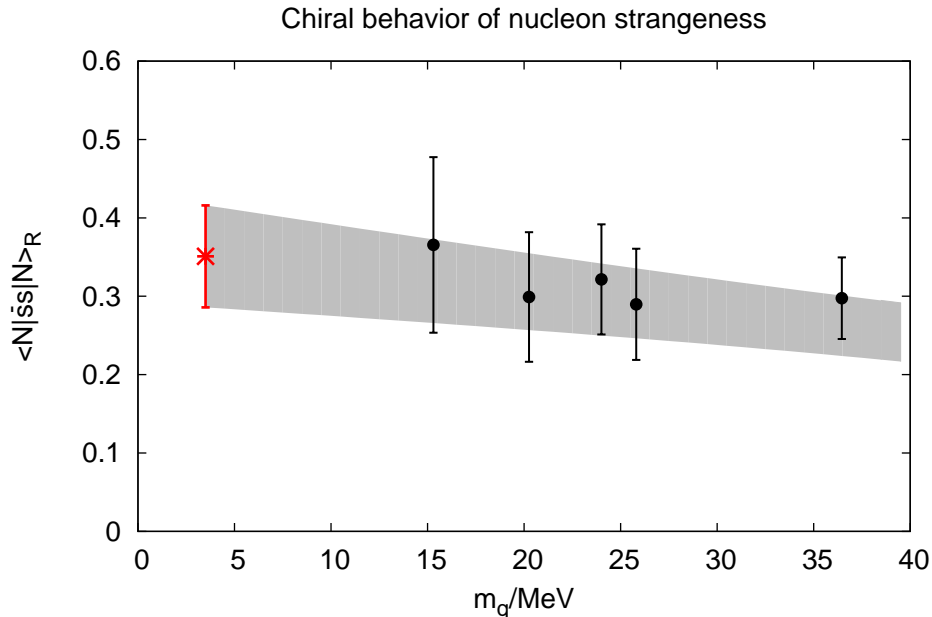


FIG. 12. The dependence of the renormalized strangeness content on the renormalized u/d quark mass in the nucleon propagator. The round dots show the data points with $am_s = 0.063$ for reference and the error band shows the global fitting results with physical strange mass.

where Δ is the mass difference of the Δ baryon and the nucleon.

In the $SU(2)$ case without an explicit delta degree of freedom, Eq. (42) can be simplified to

$$\langle N|\bar{s}s|N\rangle = \langle N|\bar{s}s|N\rangle^{LO} + E_{s,\Delta}(\mu)\frac{m_\pi^2}{(4\pi f_\pi)^2} + E_{s,\Delta}^{PQ}(\mu)\frac{m_{ju}^2 - m_\pi^2}{(4\pi f_\pi)^2} + E_{s,\Delta}^a(\mu)\frac{a^2\Delta_{mix}}{(4\pi f_\pi)^2}. \quad (44)$$

The only mixed action low-energy constant Δ_{mix} [44] between the valence overlap fermion and sea domain-wall fermions is found to be small [48] — it shifts the pion mass at 300 MeV by a mere 16 MeV. We can safely neglect the mixed-action effects since they are quite small compared to the statistical error.

Combining Eq. (40) and Eq. (44), we obtain a global fit using different time slices and quark masses. The fitting model for strangeness reads

$$R'(t, t_0)_{t \gg t_0} \rightarrow [\langle N|\bar{s}s|N\rangle + A(m_l - m_l^0) + B(m_s - m_s^0)] t + C_{m_l, m_s}, \quad (45)$$

where m_l^0 and m_s^0 are the physical quark masses corresponding to the correct π and K masses, and C_{m_l, m_s} is a set of constants.

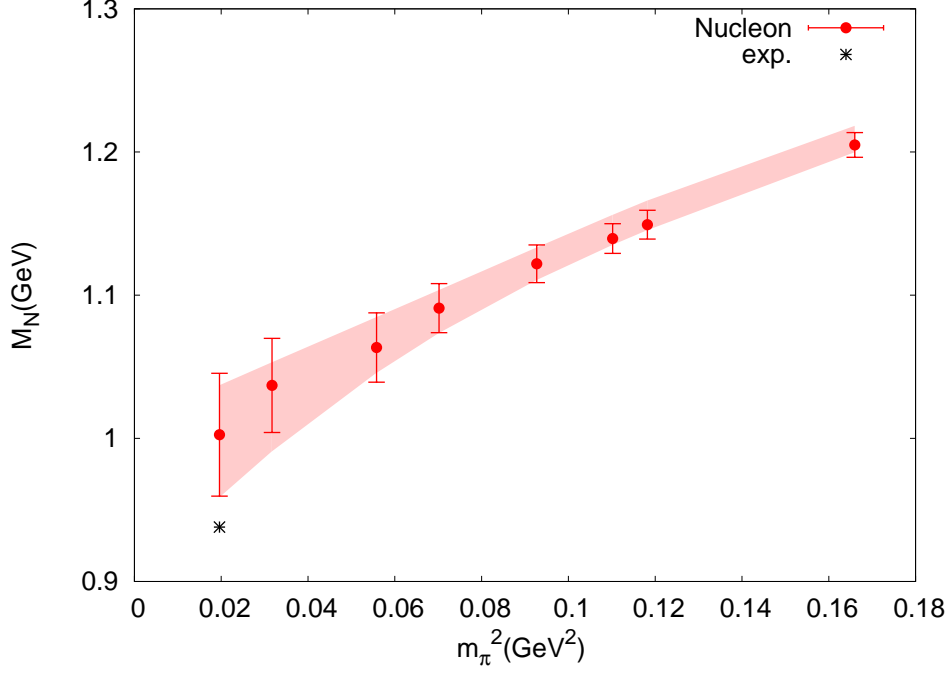


FIG. 13. The chiral extrapolation of the nucleon mass.

After the chiral extrapolation in the valence quark mass and the interpolation in the loop quark mass, we get the renormalized strangeness matrix element $\langle N|\bar{s}s|N\rangle_R = 0.341(63)$ and $m_s\langle N|\bar{s}s|N\rangle = 33.3(6.2)$ MeV with a fitting range from $t = 7$ to 14 and $am_s^0 = 0.063$. We also calculate the nucleon mass and extrapolate it to the chiral limit with $m_N(m_\pi) = m_N(0) + C_1m_\pi^2 + C_2m_\pi^3$ as shown in Fig. 13 and we obtain $m_N(0) = 0.998(39)$ GeV at the physical pion mass. Using this number, we obtain $f_{T_s} = 0.0334(62)$. We plot the recent results of $m_s\langle N|\bar{s}s|N\rangle$ in Fig. 14 from calculations with dynamical fermions with $N_f = 2$ and $N_f = 2 + 1$ (the ETM collaboration's calculation is with $N_f = 2 + 1 + 1$). We see that our result has a small statistical error and its 5σ relative error is comparable to those of Refs. [13, 15, 16].

Similarly, we can compute the charmness content $m_c\langle N|\bar{c}c|N\rangle$ using the same method. The summed ratio of the charmness is shown in Fig. 15(a) and the low/high separation is given in Fig. 15(b). We see that the low-mode part still plays an important role, but in contrast to the strangeness case, the high-mode contribution is no longer small.

The chiral extrapolation is shown in Fig. 16. We take $am_c^0 = 0.67$ which corresponds to the physical charm mass in the global analysis of the charmonium spectrum with three

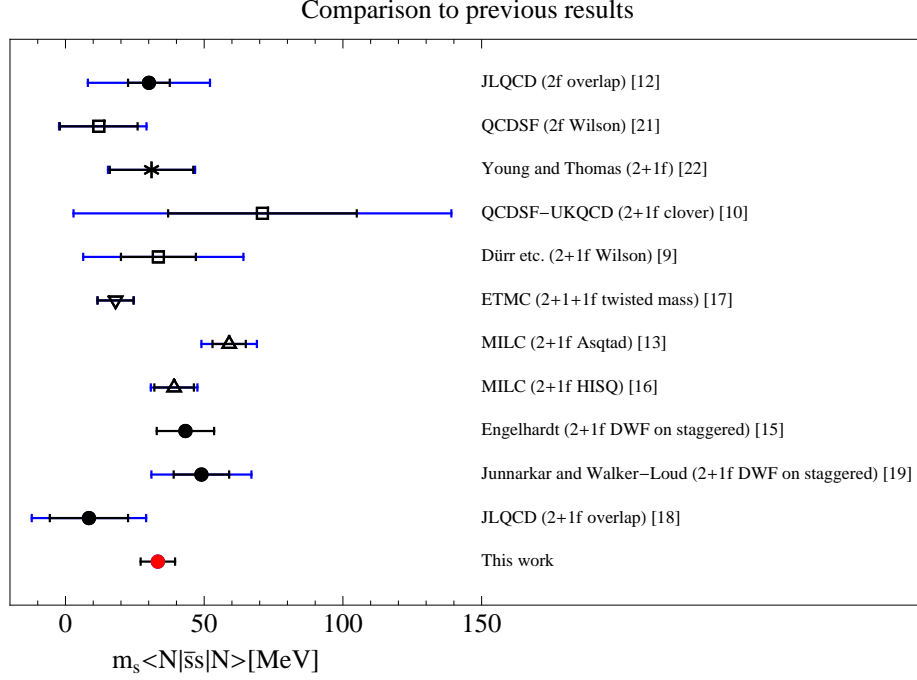


FIG. 14. A comparison of the result of our calculation of the strangeness σ term $\sigma_s = m_s \langle N | \bar{s}s | N \rangle$ with those of other groups. The statistical errors are denoted by black error bars and the total errors are denoted by blue error bars.

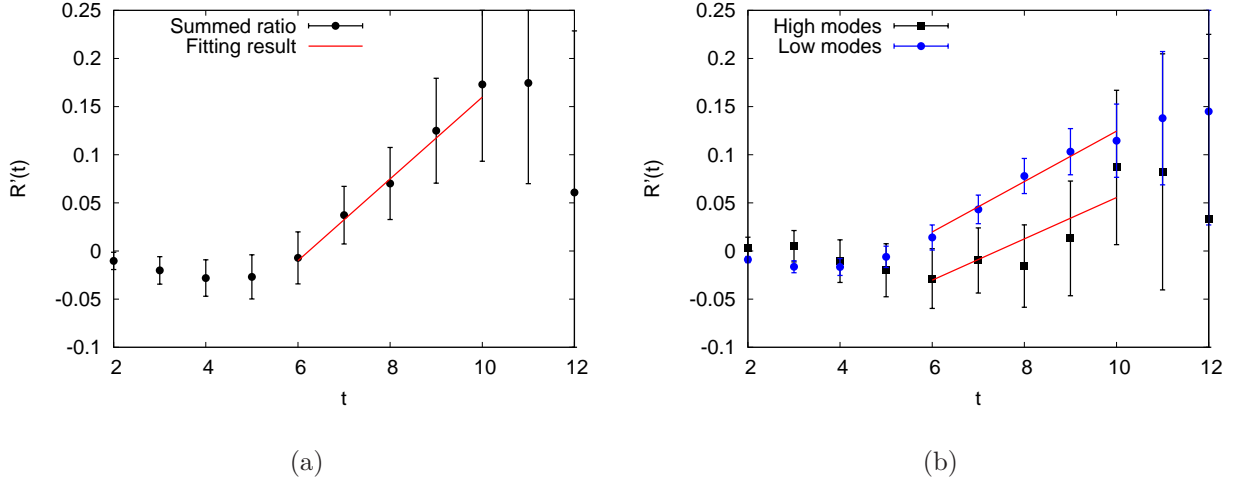


FIG. 15. The summed ratio of the charmness is plotted in the left panel. The separate high-mode and low-mode parts are plotted in the right panel. These results are computed with a valence light quark mass which corresponds to $m_\pi = 330$ MeV.

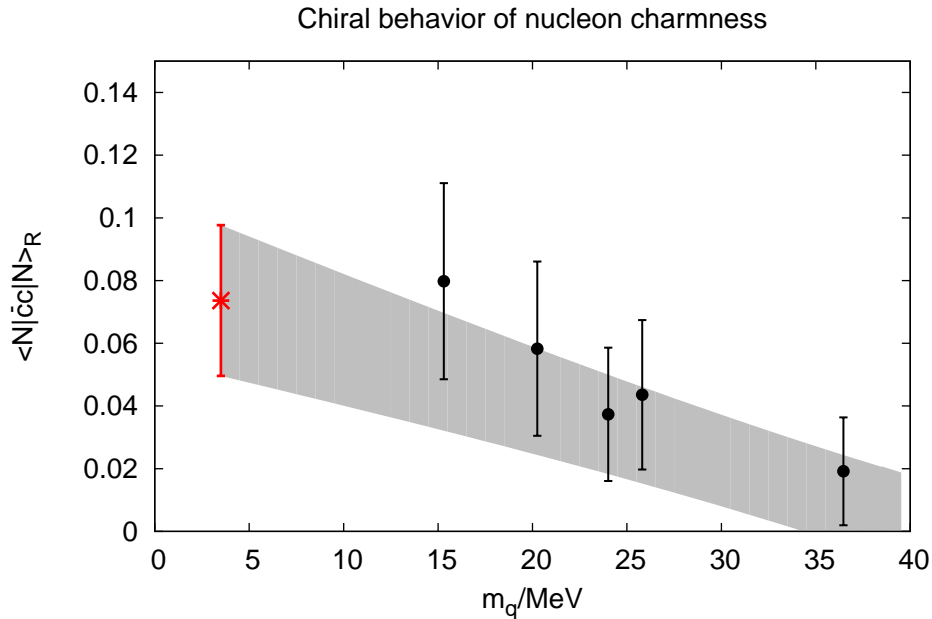


FIG. 16. The dependence of the charmness content on the u/d quark mass in the nucleon propagator. The round dots show the data points with $am_c = 0.67$ for reference and the error band shows the global fitting results with physical charm mass.

sea masses and two lattice spacings [49]. We find the renormalized charmness content $\langle N|\bar{c}c|N\rangle_R = 0.072(23)$ and $f_{T_c} = 0.094(31)$ after the chiral extrapolation and charm-quark-mass interpolation. This number is consistent with the result from the MILC collaboration at $\langle N|\bar{c}c|N\rangle = 0.056(27)$ [16], but with a smaller relative error at 33%. It is interesting to note that $m_c\langle N|\bar{c}c|N\rangle$ at 94(31) MeV is larger than $m_s\langle N|\bar{s}s|N\rangle = 33.3(6.2)$ MeV and agrees with the prediction of 70 MeV in Eq. (10) from the heavy-quark expansion [5].

By varying the quark mass in the loop with a fixed light quark mass in the nucleon, we can check the quark-mass dependence of f_{T_q} and f_{T_Q} . Figures 17 and 18 display the quark-mass dependence of $\langle N|\bar{q}q|N\rangle_R$ and $f_{T_{q,Q}}$ respectively as a function of the quark mass. We see that the matrix element seems to go down as $1/m_q$ at reasonably large m_q and f_{T_q} appears to be flat beyond $m_q \sim 500$ MeV. This behavior will be checked with higher precision on the $32^3 \times 64$ lattice with $a^{-1} = 2.35$ GeV which can accommodate heavier quark masses than on the present $24^3 \times 64$ lattice.

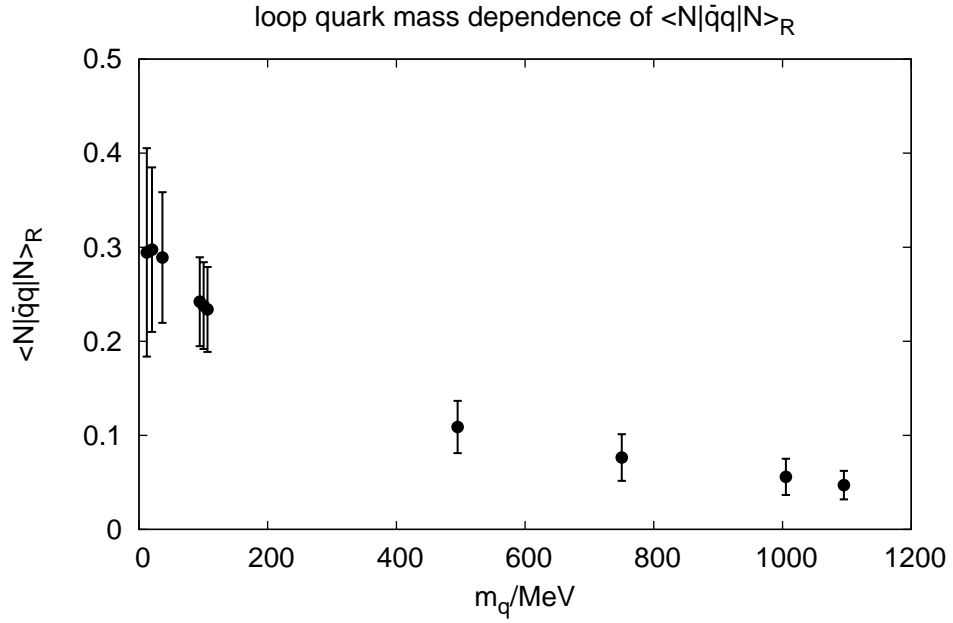


FIG. 17. The dependence of $\langle N|\bar{q}q|N\rangle_R$ on the loop quark mass, with the mass of the nucleon valence quarks fixed at a value corresponding to $m_\pi = 330$ MeV.

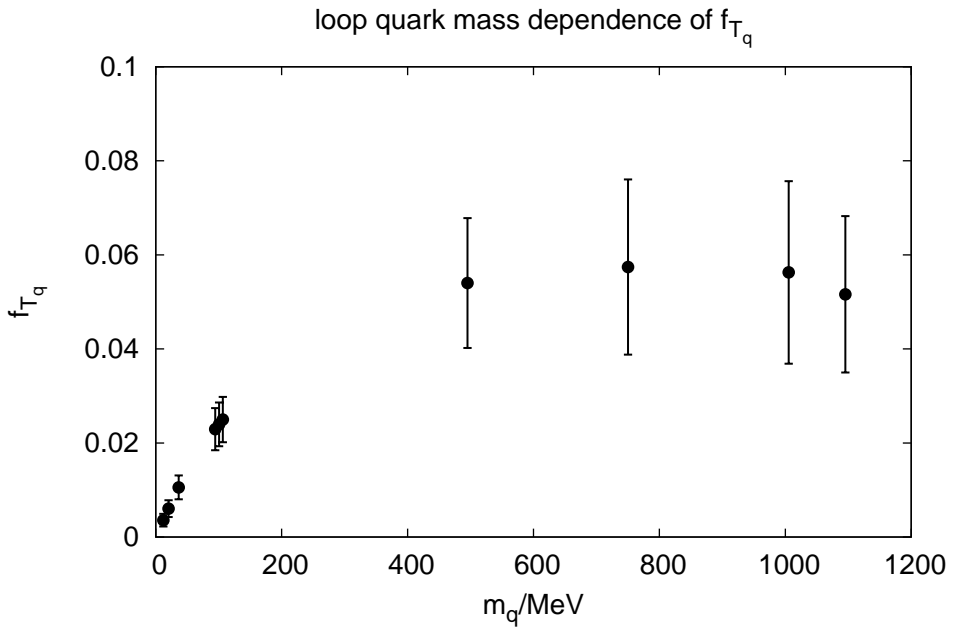


FIG. 18. The same as the last figure for the dependence of $f_{T_{q,Q}}$ of the nucleon on the quark mass.

VII. CONCLUSION

In this work, we have computed both the strangeness and the charmness content of the nucleon with overlap valence fermions on $2 + 1$ -flavor dynamical DWF configurations on the $24^3 \times 64$ lattice with $a^{-1} = 1.73$ GeV and a sea pion mass of 331 MeV. We have employed a smeared Z_3 noise-grid source with low-mode substitution to calculate the nucleon two-point functions, which reduces the error on the nucleon mass by a factor of 7 compared to the calculation with a point source. For the loop part of the three-point disconnected insertion calculation, we used low-mode averaging to compute the low-mode part exactly and used the Z_4 noise on a grid to estimate the high-mode part. It turns out that the high-mode part contributes negligibly to the strangeness content. With the highly improved nucleon propagators and the quark loops, we extrapolated to the physical pion mass and obtained the precise value $f_{T_s} = 0.0334(62)$ with a better than 5σ signal. The statistical error is quite small compared to those of other lattice calculations. The renormalized matrix element is $\langle N | \bar{s}s | N \rangle_R = 0.341(63)$ and the strange-quark sigma term σ_s is $m_s \langle N | \bar{s}s | N \rangle = 33.3(6.2)$ MeV. Similarly, we obtain $f_{T_c} = 0.094(31)$ with a 3σ signal which is the first time such a result has been obtained for the charm beyond a precision of two sigma. The renormalized matrix element is $\langle N | \bar{c}c | N \rangle_R = 0.072(23)$ and the charm-quark sigma term is $m_c \langle N | \bar{c}c | N \rangle = 94(31)$ MeV which is consistent with the prediction based on the heavy quark expansion [5, 50].

Even though our present work has a high precision, a more meaningful comparison would be with the number of inversions one has to do in order to achieve the same precision. To this end, we shall compare our results with the calculations by Engelhardt [15] and JLQCD [18]. They used the direct DI calculation with DWF and overlap which should have an inversion time comparable to the overlap fermion we use. In our case, we used 176 configurations and 48 noise vectors to calculate the high-mode part of the quark loop. In the case of Engelhardt [15], ~ 468 configurations were used each with 1200 noise vectors, giving a relative error of 24% at sink time $T = 10$, comparable to our result. However, this approach requires ~ 66 times as many inversions as ours. As for the JLQCD calculation [18], 288 noise vectors were used on 50 configurations and the error is 2.54 times larger than ours. Thus, to reach the same error as ours, it would take this approach ~ 11 times as many inversions. We attribute the efficiency of our approach to the improvement of both the nucleon propagator

and the quark loop.

In the present work we have considered the statistical error only. We will continue this work on the $32^3 \times 64$ dislocation-suppressing determinant ratio lattices with $a^{-1} = 1.37$ GeV and $m_\pi = 170$ and 250 MeV, as well as the finer $32^3 \times 64$ lattices with $a^{-1} = 2.31$ GeV and $m_\pi = 290$ MeV, in order to extrapolate to the continuum limit and the physical sea pion mass in order to address the systematic errors.

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