

Colored HOMFLY polynomials as multiple sums over paths or standard Young tableaux

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Abstract

If a knot is represented by an m -strand braid, then HOMFLY polynomial in representation R is a sum over characters in all representations $Q \in R^{\otimes m}$. Coefficients in this sum are traces of products of quantum $\hat{\mathcal{R}}$ -matrices along the braid, but these matrices act in the space of intertwiners, and their size is equal to the multiplicity M_{RQ} of Q in $R^{\otimes m}$. If R is the fundamental representation $R = [1] = \square$, then $M_{\square Q}$ is equal to the number of paths in representation graph, which lead from the fundamental vertex \square to the vertex Q . In the basis of paths the entries of the $m - 1$ relevant $\hat{\mathcal{R}}$ -matrices are associated with the pairs of paths and are non-vanishing only when the two paths either coincide or differ by at most one vertex; as a corollary $\hat{\mathcal{R}}$ -matrices consist of just 1×1 and 2×2 blocks, given by very simple explicit expressions. If cabling method is used to color the knot with the representation R , then the braid has $m|R|$ strands, Q have a bigger size $m|R|$, but only paths passing through the vertex R are included into the sums over paths which define the products and traces of the $m|R| - 1$ relevant $\hat{\mathcal{R}}$ -matrices. In the case of $SU(N)$ this path sum formula can also be interpreted as a multiple sum over the standard Young tableaux. By now it provides the most effective way for evaluation of the colored HOMFLY polynomials, conventional or extended, for arbitrary braids.

1 Introduction

Knot polynomials are currently among the central objects of interest in quantum field theory: they are exactly at the border between the known and unknown. The knot polynomials can be defined as Wilson loop averages

$$H_R^{\mathcal{K}C\mathcal{M}}(q|G) = \left\langle \text{Tr}_R P \exp \left(\oint_{\mathcal{K}} \mathcal{A} \right) \right\rangle_{CS} \quad (1)$$

i.e. generic gauge-invariant observables in the simplest version of the 3-dimensional Yang-Mills theory, the topological Chern-Simons model [1] with the action

$$\frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left(\mathcal{A} d\mathcal{A} + \frac{2}{3} \mathcal{A}^3 \right) \quad (2)$$

and they depend on a closed contour \mathcal{K} in a three dimensional manifold \mathcal{M} , on the representation R of the gauge group $G = SU(N)$ and on the coupling constant $q = e^{2\pi i/(k+N)}$. Since the theory is topological, the dependence is actually only on the topological class of \mathcal{K} , i.e. the contour can be considered as a knot. For the simply connected space $\mathcal{M} = R^3$ or S^3 the average $H(q|G)$ is actually a polynomial in q and $A = q^N$, hence, the name "knot polynomial".

The study of knot polynomials in topology goes back to [2], and they were put into the context of quantum field theory in the seminal works by A.Schwarz [3] and E.Witten [4], further developed in [5, 6]. Since the theory is topological, there are no dynamical phenomena like confinement, instead a close relation exists to the 2-dimensional conformal theories [4, 7], very much in the spirit of AdS/CFT correspondence [8]. In this way the knot polynomials are related to the most difficult part of conformal field theory, to modular transformations,

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which through the AGT relations [9] are connected to the S -duality between the $\mathcal{N} = 2$ supersymmetric Yang-Mills theories [10]. This makes the study of knot polynomials the next task after the structure of conformal blocks themselves is more or less understood in terms of the Dotsenko-Fateev matrix models [11] and other similar representations [12].

An additional interest is induced by existence of non-trivial deformations of the knot polynomials: to superpolynomials [13, 14, 15] and refined Chern-Simons theory [16] (see also [17]), to *extended* knot polynomials [18], which puts them into the class of τ -function like objects, etc. At least naively [19], they belong to the family of Hurwitz partition functions [20], more general than the conventional KP/Toda τ -functions, probably related to the generalized τ -functions of [21].

However, for any kind of generic investigation and application of knot polynomials, they should be first effectively calculated and represented in a theoretically appealing form, allowing evaluation of these polynomials for particular knots and representations. There are different competitive approaches to do this, e.g. [22, 23]. The goal of this letter is to summarize the results of our method [18, 24, 25, 26], which provides a complete, nice and practically efficient solution to this problem.

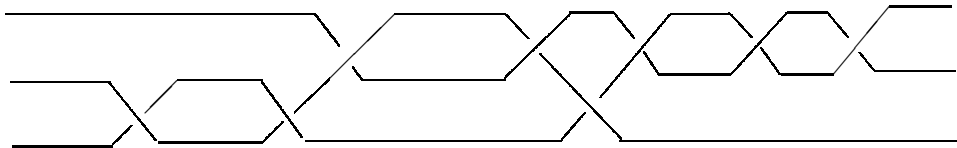
2 HOMFLY polynomials via quantum \mathcal{R} -matrices

The method may begin with choosing the temporal gauge $A_0 = 0$ [27] in Chern-Simons theory, then the theory becomes quadratic with the ultralocal propagator $\theta(t)\delta(\vec{x})$. Then the original knot in 3-dimensions is substituted by a 2-dimensional knot diagram (a 4-valent oriented graph), and the Wilson average reduces to a q -graded trace of the product of quantum \mathcal{R} -matrices, standing at the vertices of the graph [5].

It is most convenient to choose the knot diagram in the form of a closure of a braid. If the braid has m strands, then the product involves $m - 1$ different \mathcal{R} -matrices: $\mathcal{R}_{(i)}$ stands at the intersection of strands i and $i + 1$, and $i = 1, \dots, m - 1$. For instance, for the 3-strand braid one has

$$H_R^{(a_1, b_1 | a_2, b_2 | \dots)} = \text{Tr}_{R^{\otimes 3}}^{\text{grad}} \mathcal{R}_{(1)}^{a_1} \mathcal{R}_{(2)}^{b_1} \mathcal{R}_{(1)}^{a_2} \mathcal{R}_{(2)}^{b_2} \dots \quad (3)$$

In the pattern picture $a_1 = 0, b_1 = -2, a_2 = 2, b_2 = -1, a_3 = 3$:



Similarly, for arbitrary m

$$H_R^{(a_{11}, \dots, a_{1, m-1} | a_{21}, \dots, a_{2, m-1} | \dots)} = \text{Tr}_{R^{\otimes m}}^{\text{grad}} \mathcal{R}_{(1)}^{a_{11}} \dots \mathcal{R}_{(m-1)}^{a_{1, m-1}} \mathcal{R}_{(1)}^{a_{21}} \dots \mathcal{R}_{(m-1)}^{a_{2, m-1}} \dots \quad (4)$$

The trace here is defined with additionally inserted element $q^{R^{\otimes m}}$ so that

$$\chi_Q = \text{Tr}_Q^{\text{grad}} I = \dim_q^G(Q) \quad (5)$$

are the quantum dimensions of the representation R (the characters of the group G at the special values $p_k = \frac{A^k - A^{-k}}{q^k - q^{-k}}$). In this formula the \mathcal{R} matrices are of the huge size $\dim(R)^2 \times \dim(R)^2$ and this expression can seem absolutely hopeless to evaluate for generic group G and representation R .

However, things are actually much more simple. The product $R^{\otimes m}$ can be expanded into a sum of irreducible representations, generically, with non-trivial multiplicities. The crucial observation is that $\mathcal{R}_{(i)}$ act as unity in each irreducible representation, so that the matrices in (4) can be reduced to $\hat{\mathcal{R}}_{(i)}$ of a much smaller size, equal to just the multiplicities M_{RQ} of Q in $R^{\otimes m}$ [18, 24]:

$$H_R^{\mathcal{K}}(G) = \mathfrak{N}^{w(\mathcal{K})} \sum_{Q \in R^{\otimes m}} C_{RQ}^{\mathcal{K}} \chi_Q(G),$$

$$\boxed{C_{RQ}^{(a_{11}, \dots, a_{1, m-1} | \dots)} = \text{tr}_{M_{RQ}} \hat{\mathcal{R}}_{(1)}^{a_{11}} \dots \hat{\mathcal{R}}_{(m-1)}^{a_{1, m-1}} \hat{\mathcal{R}}_{(1)}^{a_{21}} \dots \hat{\mathcal{R}}_{(m-1)}^{a_{2, m-1}} \dots} \quad (6)$$

Here \mathfrak{N} is a normalization factor which emerges due to our choice of non-standard normalization of \mathcal{R} -matrices and $w(\mathcal{K})$ is the writhe number. The standard normalization of \mathcal{R} -matrix in the vertical framing is restored

with the factor $q^{-2\kappa_R}$, where $\kappa_R = \sum_{i,j \in R} (j - i)$ and the sum runs over the Young diagram corresponding to R . In order to restore the topological invariance, one has to change framing with a factor of $A^{-|R|} q^{-2\kappa_R}$ which totally gives $\mathfrak{N} = A^{-|R|} q^{-4\kappa_R}$.

What is important, in formula (6) the knot and group dependencies are separated and one can consider H_R as a function of $A = q^N$ rather than N : the parameter N enters only the quantum dimensions χ_Q and can be easily substituted by A . Moreover, this formula actually introduces the *extended* HOMFLY polynomial $H_R\{p\}$, if χ_Q are interpreted as characters, which are functions of infinitely many time-variables $\{p_k\}$ instead of N or A [18, 24]. The topological invariance is, however, lost beyond the topological locus $p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}}$.

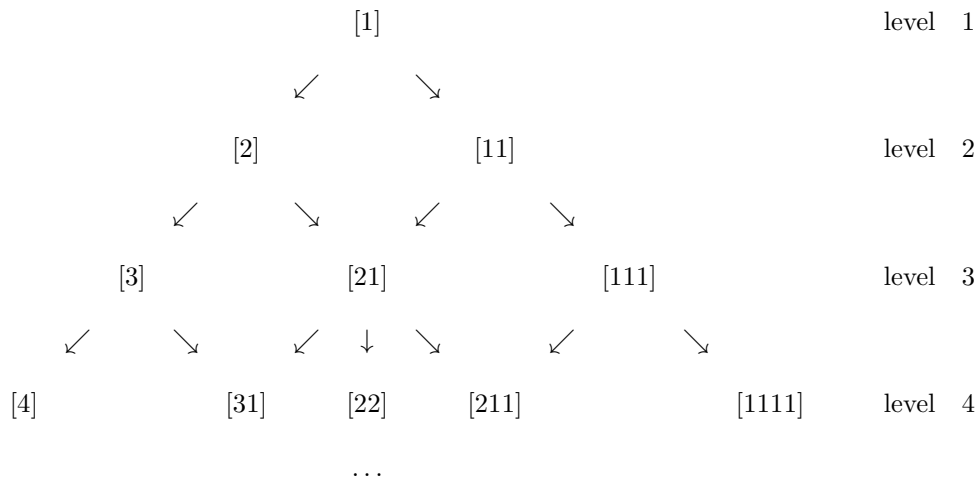
To deal with this formula one needs an explicit expression for the $\hat{\mathcal{R}}$ -matrices, see [24, 28]. Those papers contain many various observations about the structure of $\hat{\mathcal{R}}$ -matrices, and they were used to calculate many non-trivial knot polynomials, still a complete description is not yet found on that way, except for the fundamental representation case of $R = [1] = \square$, fully described in [25], and for the (anti)symmetric representation case [29, 28] (some results in this latter case are also reproduced by alternative methods [30]).

It is therefore natural to attack the case of arbitrary R with the help of the cabling approach [31] and apply the results of [25]. This is successfully done in [32, 26], this letter being a short summary of [25] and [26], purified from all the details and extensive list of examples evaluated there.

Note also that here we restrict our discussion to the knots only. The links can be dealt with similarly, however, they require some technical complications, thus, for the sake of brevity, we skip this extension (see details in [26]).

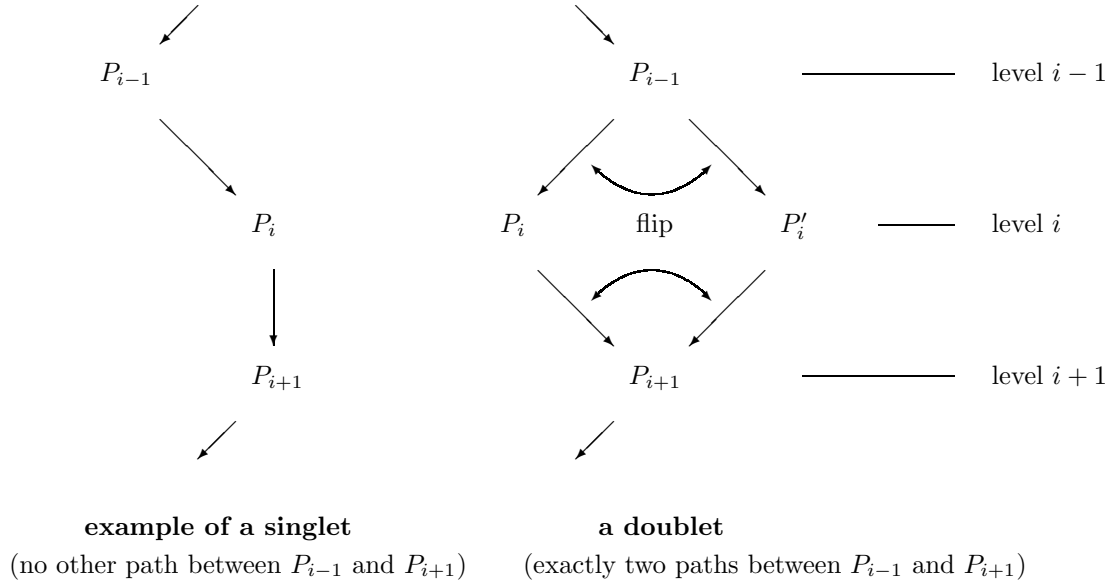
3 $\hat{\mathcal{R}}$ -matrices via paths in the representation graph

Since the cabling reduces the problem from R to the case of the fundamental representation, the results of [25] are directly applicable [26] and we begin from repeating them in a concise and pictorial form.

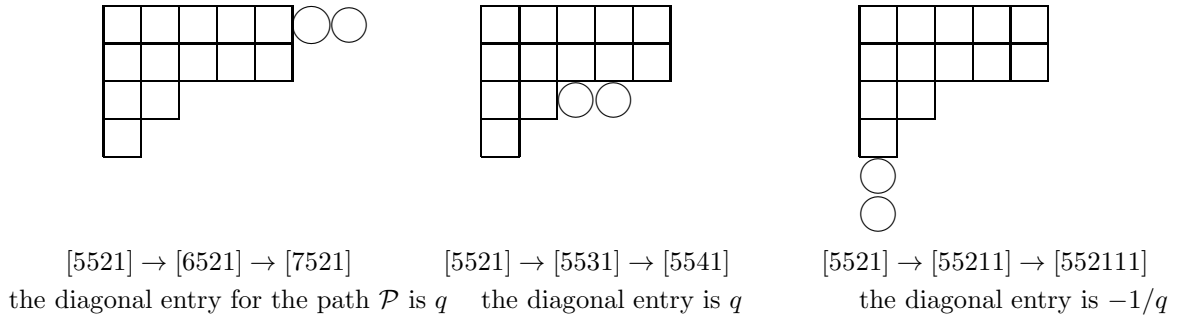


Implicit in [25] is representation of the coefficients $C_{\square Q}$ in the form of a sum over paths in the representation graph [26]. The first four levels of the representation graph of [24] are shown in the Figure: in an obvious way it describes the multiplication of fundamental representations [1]. The multiplicity $M_{\square Q}$ of the representation Q in $\square^{\otimes |Q|}$ is obviously equal to the number of directed paths in the representation graph, connecting \square and Q . More generally, M_{RQ} is equal to the number of directed paths between R and Q . The matrices $\hat{\mathcal{R}}_{(i)}$, $i = 1, \dots, m - 1$ can be represented in the basis of paths between Q and \square and according to [25, 26] they have extremely simple form in this basis.

First of all, with each index i of the matrix $\hat{\mathcal{R}}_{(i)}$ one associates a level i in the graph. A given path \mathcal{P} is passing through exactly one vertex P_i at level i and through some two adjacent vertices P_{i-1} and P_{i+1} at levels $i - 1$ and $i + 1$. The structure of the representation graph is such that these P_{i-1} and P_{i+1} are connected either by a single two-segment path (singlet) (then it is a fragment of our P) or by two such paths (doublet), the segments of our path \mathcal{P} and another path \mathcal{P}' . We call the transformations $\mathcal{P} \leftrightarrow \mathcal{P}'$ a *flip*.



In the former case (singlet) our path \mathcal{P} provides a diagonal element in $\hat{\mathcal{R}}_{(i)}$ and it is equal to either q or $-1/q$. In the language of Young diagrams the singlet appears when the two boxes added to the diagram P_{i-1} in order to form P_{i+1} lie either in the same row, then we put q at the diagonal of $\hat{\mathcal{R}}_i$; or in the same column, then we put $-1/q$.



examples of singlets

In the latter case (the doublet) the two boxes are neither in the same row nor in the same column, and the two paths \mathcal{P} and \mathcal{P}' form a 2×2 block in $\hat{\mathcal{R}}_{(i)}$. This block is described as follows. First, that of the paths \mathcal{P} and \mathcal{P}' which lies to the left of the other, corresponds to the left column and to the first row of the 2×2 block. Second, the Young diagrams P_{i+1} is obtained by adding two boxes to the diagram P_{i-1} , and the two paths correspond to doing this in two different orders, thus providing at the intermediate stage the two adjacent vertices P_i and P'_i . The two added boxes are connected by a hook in the Young diagram, which has length n (measured between the *centers* of the two boxes). Then the 2×2 block is equal to

$$\begin{pmatrix} -q^{-n}c_n & s_n \\ s_n & q^n c_n \end{pmatrix}, \quad c_n = \frac{1}{[n]_q} = \frac{q - q^{-1}}{q^n - q^{-n}}, \quad s_n = \sqrt{1 - c_n^2} = \frac{\sqrt{[n-1]_q [n+1]_q}}{[n]_q} \quad (7)$$

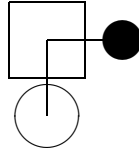
The following picture shows that for $P_{i-1} = [5521]$, $P_{i+1} = [6522]$, $P_i = [6521]$ and $P'_i = [5522]$ the parameter $n = 7$:

m=3:

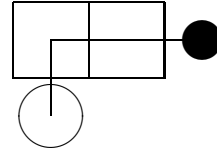
For $Q = [3]$ and $[111]$ there are unique paths from \square and the corresponding matrices $\hat{\mathcal{R}}_{(1)} = \hat{\mathcal{R}}_{(2)}$ are again 1×1 and are equal to q and $-1/q$ respectively.

However, for $Q = [21]$ the situation is already different. There are two paths between \square and $[21]$, and the left one, $[1] \rightarrow [2] \rightarrow [21]$ contains the segment $([1], [2])$, while the second path $[1] \rightarrow [11] \rightarrow [21]$ contains the segment $([1], [11])$. This means that the matrix $\hat{\mathcal{R}}_{(1)}$ in the sector $Q = [21]$ is 2×2 , it is diagonal with the entries q and $-1/q$. The matrix $\hat{\mathcal{R}}_{(2)}$ is again 2×2 , but it is not diagonal, because the two paths are connected exactly by the *flip*. Since the length of the hook in this case is $n = 2$, our rules imply that $\hat{\mathcal{R}}_{(2)} = \frac{1}{[2]_q} \begin{pmatrix} -1/q^2 & \sqrt{[3]_q} \\ \sqrt{[3]_q} & q^2 \end{pmatrix}$ and the formula for the arbitrary 3-strand braid is [24]

$$A^{a_1+b_1+a_2+b_2+\dots} H_{\square}^{(a_1 b_1 | a_2 b_2 | \dots)} = q^{a_1+b_1+a_2+b_2+\dots} \chi_{[3]}(G) + (-1/q)^{a_1+b_1+a_2+b_2+\dots} \chi_{[111]}(G) + \text{tr}_{2 \times 2} \left\{ \begin{pmatrix} q & 0 \\ 0 & -1/q \end{pmatrix}^{a_1} \begin{pmatrix} -\frac{1}{q^2 [2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & q^2 \end{pmatrix}^{b_1} \begin{pmatrix} q & 0 \\ 0 & -1/q \end{pmatrix}^{a_2} \begin{pmatrix} -\frac{1}{q^2 [2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & q^2 \end{pmatrix}^{b_2} \dots \right\} \quad (10)$$



$[1] \rightarrow [21]$
 $n = 2$



$[2] \rightarrow [31]$
 $n = 3$

m=4:

For $Q = [4]$ and $[1111]$ all the three matrices $\hat{\mathcal{R}}_{1,2,3}$ are 1×1 and equal to q and $-1/q$ respectively.

For $Q = [31]$ there are three paths. Ordered from the left to the right they are:

$$\alpha = [1] \rightarrow [2] \rightarrow [3] \rightarrow [31], \quad \beta = [1] \rightarrow [2] \rightarrow [21] \rightarrow [31], \quad \gamma = [1] \rightarrow [11] \rightarrow [21] \rightarrow [31]$$

At level 2 the flip relates β and γ , at level 3 relates α and β . This implies that in sector $[31]$ one has, [24]

$$\hat{\mathcal{R}}_{(1)} = \begin{pmatrix} q & & \\ & q & \\ & & -1/q \end{pmatrix}, \quad \hat{\mathcal{R}}_{(2)} = \begin{pmatrix} q & 0 & 0 \\ 0 & -\frac{1}{q^2 [2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ 0 & \frac{\sqrt{[3]_q}}{[2]_q} & q^2 \end{pmatrix}, \quad \hat{\mathcal{R}}_{(3)} = \begin{pmatrix} -\frac{1}{q^3 [3]_q} & \frac{\sqrt{[2]_q [4]_q}}{[3]_q} & 0 \\ \frac{\sqrt{[2]_q [4]_q}}{[3]_q} & \frac{q^3}{[3]_q} & 0 \\ 0 & 0 & q \end{pmatrix} \quad (11)$$

For $Q = [211]$ the answer is similar: the three paths are now

$$\bar{\alpha} = [1] \rightarrow [2] \rightarrow [21] \rightarrow [211], \quad \bar{\beta} = [1] \rightarrow [11] \rightarrow [21] \rightarrow [211], \quad \bar{\gamma} = [1] \rightarrow [11] \rightarrow [111] \rightarrow [211]$$

and

$$\hat{\mathcal{R}}_{(1)} = \begin{pmatrix} q & & \\ & -1/q & \\ & & -1/q \end{pmatrix}, \quad \hat{\mathcal{R}}_{(2)} = \begin{pmatrix} -\frac{1}{q^2 [2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} & 0 \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} & 0 \\ 0 & 0 & -1/q \end{pmatrix}, \quad \hat{\mathcal{R}}_{(3)} = \begin{pmatrix} -1/q & 0 & 0 \\ 0 & -\frac{1}{q^3 [3]_q} & \frac{\sqrt{[2]_q [4]_q}}{[3]_q} \\ 0 & \frac{\sqrt{[2]_q [4]_q}}{[3]_q} & \frac{q^3}{[3]_q} \end{pmatrix} \quad (12)$$

For $Q = [22]$ there are just two paths,

$$\delta = [1] \rightarrow [2] \rightarrow [21] \rightarrow [22], \quad \bar{\delta} = [1] \rightarrow [11] \rightarrow [21] \rightarrow [22]$$

and they are related by the flip at level 2. This means that both $\hat{\mathcal{R}}_{(1)}$ and $\hat{\mathcal{R}}_{(3)}$ are 2×2 , and diagonal, while $\hat{\mathcal{R}}_{(2)}$ is not:

$$\hat{\mathcal{R}}_{(1)} = \hat{\mathcal{R}}_{(3)} = \begin{pmatrix} q & 0 \\ 0 & -1/q \end{pmatrix}, \quad \hat{\mathcal{R}}_{(2)} = \begin{pmatrix} -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{pmatrix} \quad (13)$$

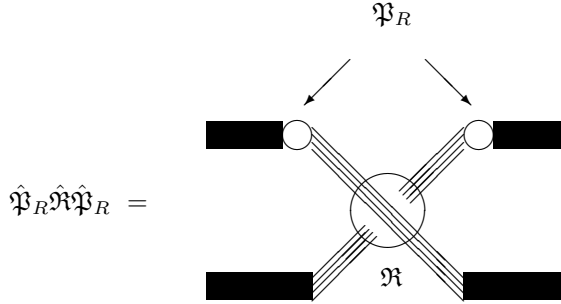
The counterpart of (9) and (10) is now obvious, but rather lengthy, so we do not write it down here.

5 Cabling method

Cabling is based on the fact that the representation R appears in the product of $|R|$ fundamental representations, $R \in \square^{|R|}$, thus, the answer for $H_R^{\mathcal{K}}$ can be extracted from the answer for $H_{\square^{|R|}}^{\mathcal{K}^{|R|}}$ by the projection:

$$H_R^{\mathcal{K}}(G) = \mathfrak{N}_c \sum_{Q \in \square^{\otimes m|R|}} \mathfrak{C}_{\square Q}^{\mathcal{K}^{|R|}} \chi_Q(G) \quad (14)$$

where the normalization factor is slightly corrected as compared with (6), see [26] for details. Cabling is widely used not only in the knot theory [31], but also in the theory of \mathcal{R} -matrices and integrable systems [33] (where it is called the fusion procedure).



Here $\mathcal{K}^{|R|}$ is an $m|R|$ -strand braid, obtained from \mathcal{K} by substituting each line (strand) by a bunch (cable) of $|R|$ strands so that the intersection of two strands is now a peculiar combination \mathfrak{R} of $|R|^2$ original \mathcal{R} -matrices, but in the fundamental representations. In other words,

$$\mathfrak{C}_{\square Q}^{\mathcal{K}^{|R|}} = \text{tr}_{M_{\square Q}} \hat{\mathfrak{P}}_R \hat{\mathfrak{R}}_{(1)}^{a_{11}} \dots \hat{\mathfrak{R}}_{(m-1)}^{a_{1,m-1}} \hat{\mathfrak{R}}_{(1)}^{a_{21}} \dots \hat{\mathfrak{R}}_{(m-1)}^{a_{2,m-1}} \dots \quad (15)$$

$\hat{\mathfrak{P}}_R$ is the projection from the reducible representation $\square^{\otimes |R|}$ onto R . It can actually be inserted everywhere in between the \mathfrak{R} matrices, but for the case of *knots* (not links) a single insertion is sufficient.

Now, (15) can be written explicitly and calculated with the help of sec.3, but in terms of the $m|R|$ -strand braid and the corresponding $\hat{\mathcal{R}}$ matrices replaced with $\hat{\mathfrak{R}}$. This is a rather cumbersome expression even for the simplest knots (therefore we do not rewrite (15) in this way) but absolutely straightforward and adequate for practical computations.

Moreover, here comes a bonus of the path sum representation: **the projector $\hat{\mathfrak{P}}_R$ is nearly trivial**, at least in the case of knots: one should include only the directed paths from Q to \square , which pass through the vertex R , this effectively decreases the size of the matrices $\hat{\mathfrak{R}}_{(I)}$, $I = 1, \dots, m-1$ from $M_{\square Q}$ to $M_{\square R} \cdot M_{RQ}$. **NB:** The constituent matrices $\hat{\mathcal{R}}_{(i)}$, $i = 1, \dots, m|R| - 1$ can *not* be reduced in this way: one can *not* insert the projector *inside* $\hat{\mathfrak{R}}$.

6 Cabling for $m=2$ and $|\mathbf{R}|=2$:

Now we can use the last example in s.4 to demonstrate how the cabling works in our formulas. The 4-strand braids are enough to describe only the 2-strand (torus) knots in representations [2] and [11]. In this case there is just a single combined matrix

$$\hat{\mathfrak{R}}_{(1)} = \hat{\mathcal{R}}_{(2)} \hat{\mathcal{R}}_{(1)} \hat{\mathcal{R}}_{(3)} \hat{\mathcal{R}}_{(2)} \quad (16)$$

According to our rules, if we consider, for example, $H_{[2]}^{(n)}$ for a 2-strand knot, we should leave in $\hat{\mathfrak{R}}_{(1)}$ only the paths passing through the vertex [2].

This means that the single path leading to $Q = [4]$ remains intact, while the one to [1111] is now eliminated; this means that there will be no contribution of $Q = [1111]$ to $H_{[2]}^{(n)}$.

From the three paths which led to $Q = [31]$ only two remain, α and β , thus the third line and the third column should be omitted from the matrix $\hat{\mathfrak{R}}$:

$$\begin{aligned} \hat{\mathfrak{R}} &= \begin{pmatrix} q & 0 & 0 \\ 0 & -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ 0 & \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{pmatrix} \begin{pmatrix} q & & \\ & q & \\ & & -1/q \end{pmatrix} \begin{pmatrix} -\frac{1}{q^3[3]_q} & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & 0 \\ \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & \frac{q^3}{[3]_q} & 0 \\ 0 & 0 & q \end{pmatrix} \begin{pmatrix} q & 0 & 0 \\ 0 & -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ 0 & \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{pmatrix} = \\ &= \begin{pmatrix} -\frac{1}{[3]_q} & -\frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} & \frac{q^2\sqrt{[2]_q[3]_q[4]_q}}{q^2[2]_q[3]_q} \\ -\frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} & -\frac{[4]_q}{[2]_q[3]_q} & -\frac{q^2}{\sqrt{[3]_q}} \\ \frac{q^2\sqrt{[2]_q[3]_q[4]_q}}{[2]_q[3]_q} & -\frac{q^2}{\sqrt{[3]_q}} & 0 \end{pmatrix} \rightarrow \\ &\rightarrow \hat{\mathfrak{P}}_{[2]} \hat{\mathfrak{R}} \hat{\mathfrak{P}}_{[2]} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \hat{\mathfrak{R}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{[3]_q} & -\frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} \\ -\frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} & -\frac{[4]_q}{[2]_q[3]_q} \end{pmatrix} \quad (17) \end{aligned}$$

This 2×2 matrix has two eigenvalues: 0 and -1 .

Similarly from the three paths to $Q = [211]$ only one survives, $\bar{\alpha}$, and $\hat{\mathfrak{R}}$ matrix is reduced to 1×1 :

$$\begin{aligned} \hat{\mathfrak{R}} &= \begin{pmatrix} -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} & 0 \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} & 0 \\ 0 & 0 & -1/q \end{pmatrix} \begin{pmatrix} q & & \\ & -1/q & \\ & & -1/q \end{pmatrix} \begin{pmatrix} -1/q & 0 & 0 \\ 0 & -\frac{1}{q^3[3]_q} & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} \\ 0 & \frac{\sqrt{[2]_q[4]_q}}{[3]_q} & \frac{q^3}{[3]_q} \end{pmatrix} \begin{pmatrix} -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} & 0 \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} & 0 \\ 0 & 0 & -1/q \end{pmatrix} = \\ &= \begin{pmatrix} 0 & \frac{1}{q^2\sqrt{[3]_q}} & \frac{\sqrt{[2]_q[3]_q[4]_q}}{q^2[2]_q[3]_q} \\ \frac{1}{q^2\sqrt{[3]_q}} & -\frac{[4]_q}{[2]_q[3]_q} & \frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} \\ \frac{\sqrt{[2]_q[3]_q[4]_q}}{q^2[2]_q[3]_q} & \frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} & -\frac{1}{[3]_q} \end{pmatrix} \rightarrow \\ &\rightarrow \hat{\mathfrak{P}}_{[2]} \hat{\mathfrak{R}} \hat{\mathfrak{P}}_{[2]} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \hat{\mathfrak{R}} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \rightarrow (0) \quad (18) \end{aligned}$$

Thus, despite a path survives in the $[211]$ sector, its contribution is actually vanishing, as it should be, because $[211] \notin [2] \otimes [2]$.

Also for $Q = [22]$ only one path of the two remains, δ , and

$$\begin{aligned} & \begin{pmatrix} -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & -1/q \end{pmatrix}^2 \begin{pmatrix} -\frac{1}{q^2[2]_q} & \frac{\sqrt{[3]_q}}{[2]_q} \\ \frac{\sqrt{[3]_q}}{[2]_q} & \frac{q^2}{[2]_q} \end{pmatrix} = \begin{pmatrix} \frac{1}{q^2} & 0 \\ 0 & q^2 \end{pmatrix} \\ & \longrightarrow \hat{\mathfrak{P}}_{[2]} \hat{\mathfrak{R}} \hat{\mathfrak{P}}_{[2]} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \hat{\mathfrak{R}} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 \\ q^2 \end{pmatrix} \end{aligned} \quad (19)$$

Putting everything together one gets for odd n (i.e. for the knot):

$$\begin{aligned} q^{2n} A^{2n} H_{[2]}^{(n)} &= \text{Tr}_{[1] \otimes 4}^{\text{grad}} \hat{\mathfrak{P}}_{[2]} \hat{\mathfrak{R}}^n = \text{Tr}_{[1] \otimes 4}^{\text{grad}} \left(\hat{\mathfrak{P}}_{[2]} \hat{\mathfrak{R}} \hat{\mathfrak{P}}_{[2]} \right)^n = \\ &= q^{4n} \chi_{[4]}(G) + \text{tr}_{2 \times 2} \left\{ \begin{pmatrix} -\frac{1}{[3]_q} & -\frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} \\ -\frac{\sqrt{[2]_q[4]_q}}{[2]_q[3]_q} & -\frac{[4]_q}{[2]_q[3]_q} \end{pmatrix} \right\}^n \chi_{[31]}(G) + q^{-2n} \chi_{[22]}(G) = \\ &= q^{-2n} \left(q^{6n} \chi_{[4]}(G) - q^{2n} \chi_{[31]}(G) + \chi_{[22]}(G) \right) \end{aligned} \quad (20)$$

The formula at the last line is the standard Rosso-Jones expression [34] for the colored HOMFLY polynomial in the case of a torus knot.

NB: Note that the matrices $\hat{\mathfrak{R}}$ do *not* commute with $\hat{\mathfrak{P}}_R$ and that the projector is as simple as described above only for the first cable in the braid.

Note also that two of the five $Q \in [1]^4$ do not contribute to the middle line for somewhat different reasons: $\chi_{[1111]}$ does not appear, because there are no paths, going from \square to $[1111]$ via $[2]$, while for $\chi_{[211]}$ such path exists, just its contribution to the relevant element of $\hat{\mathfrak{R}}$ is zero.

These examples illustrate all the peculiarities of our formulas. Once they are understood, the use of the path sum formalism is straightforward.

7 Open questions

The path sum formula provides a complete solution for the evaluation problem of arbitrary colored HOMFLY polynomials. It is nice looking and theoretically attractive, and it is algorithmic and very effective for practical computations (a vast list of examples is provided in [32, 26]). It represents the knot polynomials in the form of a character expansion [18, 24], and therefore expresses them directly in terms of N , A , or the time-variables p_κ , whatever one prefers. Still there are a few obvious directions to study.

First, our final formula for the colored knots heavily relies on the cabling approach and therefore is in a certain sense more involved than (6). In particular, when $R \neq \square$, the sizes of \mathcal{R} -matrices are considerably bigger than theoretically possible (though the entries are simpler). Of course, **the path sum representation exists for the arbitrary R** , but it is in terms of another representation graph, Γ_R describing powers of the representation R , and the corresponding \mathcal{R} -matrices are more involved (their constituents are no longer just 1×1 and 2×2 blocks). At the same time that graph is the subset of the full one, Γ_\square , which we considered in the present letter, and the sum over the paths from Q to \square , passing through the vertex R , can be also considered as a sum over just the paths from Q to R , more in the spirit of eq.(6). However, there are many such paths in Γ_\square with the same image in Γ_R , and they contribute to the matrix elements of the composite $\hat{\mathfrak{R}}$. The quantities (15) contain also multiple sums over the paths from R to \square , but in fact they do not depend on this, and one can simply fix one such path arbitrarily (as a kind of a gauge fixing). It would be very interesting to find a relevant modification of the path sum formula which would not refer to the cabling procedure, and [28] strongly implies that this can be possible and that such a formula should possess its own beauties.

Second, the representation graph exists for arbitrary Lie algebras, not only for $SU(N)$, and the sum path formula should be straightforwardly generalized to arbitrary groups, in particular, from the HOMFLY (for $SU(N)$) to Kauffman (for $SO(N)$) polynomials.

Third, it is now obvious that the Khovanov-Rozhansky approach [13] and the superpolynomials [14] should have a similar representations in terms of the multiple sums over paths (i.e. over the standard Young tableaux). There are certain advances in this direction for the torus knots [35], particularly close should be the results by [36]. We emphasize once again that **the sums over Young tableaux provide the answers for arbitrary knots**, not only torus: this is a theorem for the HOMFLY polynomials and a plausible conjecture for the superpolynomials.

To this list one should of course add a massive calculation of the colored HOMFLY polynomials for non-torus knots, especially in representations with many lines and rows in the Young diagram, i.e. not just (anti)symmetric representations. Such examples are crucially important for understanding the structure of generic knot polynomials, e.g. *a la* [29, 30, 32, 37] or [19, 38] or [39], and various relations between them [40, 41].

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