

LOCAL ENERGY DECAY FOR MAXWELL FIELDS PART I: SPHERICALLY SYMMETRIC BLACK-HOLE BACKGROUNDS

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1. INTRODUCTION

Let $(\mathcal{M}, g_{\alpha\beta})$ be a 4 dimensional Lorentzian manifold. In this paper we study the dispersive properties of two-forms $F_{\alpha\beta}$ in terms of their divergences when $(\mathcal{M}, g_{\alpha\beta})$ is both stationary and spherically symmetric, and satisfies additional local and global assumptions which are natural generalizations of the special properties held by asymptotically flat solutions for the Einstein vacuum equations $R_{\alpha\beta} = 0$. Specifically, we make the following definition:

Definition 1.1 (Spherically symmetric stationary “black holes”). *Let $\mathcal{M} \approx \mathbb{R}^2 \times \mathbb{S}^2$ be a spherically symmetric Lorentzian manifold with metric:*

$$(1.1) \quad g = h_{ab}dx^a dx^b + r^2 \delta_{AB} dx^A dx^B, \quad a = 0, 1, \quad A = 2, 3,$$

where δ_{AB} is the standard round metric on \mathbb{S}^2 , and where the 2 dimensional Lorentzian metric h_{ab} depends only on the (x^a, x^b) variables. Denote by $\langle \cdot, \cdot \rangle$ the inner product of g . Then we say that (\mathcal{M}, g) is a (generalized) “spherically symmetric stationary black hole” if the following conditions hold:

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- i) (Stationary asymptotic flatness) There exists a spherically symmetric time function t defined on all of \mathcal{M} , i.e. $t = t(x^a)$ and $\langle dt, dt \rangle < 0$, such that (t, r) forms a system of coordinates in the $x^A = \text{const.}$ plane and one has $h_{ab} = \text{diag}(-1, 1) + O(r^{-1})$ as $r \rightarrow \infty$. Furthermore, in this system of coordinates one has $\partial_t h_{ab} \equiv 0$ and $\partial_r^k h_{ab} = O(r^{-1-k})$ for $k \geq 1$.
- ii) (Non-degenerate forward global hyperbolicity) There exists a value $r_0 > 0$ such that $r = r_0$ is also space-like, and the sign of $g^{rr} = \langle dr, dr \rangle$ changes only once in $r \geq r_0$. We define the value $r = r_M$ to be the unique hypersurface where $g^{rr} = 0$. In addition we assume $\partial_r g^{rr}$ never vanishes.
- iii) (Strictly hyperbolic trapping) There is a one and only one value $r_{\mathcal{T}}$ in the region where $\langle dr, dr \rangle > 0$ such that the time-like surface $r = r_{\mathcal{T}}$ is a trapped set for all null geodesics initially tangent to it. Furthermore, this trapped set is normally hyperbolic in the sense of dynamical systems.¹

A number of remarks about this definition are in order:

- Remark 1.2.** a) The time coordinate of part i) above is regular and should not be confused with the (singular) Schwarzschild t coordinate in the standard metric $ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\omega^2$.
- b) The assumptions i) and ii) above imply that ∂_t (computed in (t, r) coordinates) is timelike on $r > r_M$, null for $r = r_M$, and space-like on $r < r_M$. This can immediately be seen via Cramer's rule which gives $\langle \partial_t, \partial_t \rangle = \det(h)g^{rr}$ where $\det(h) \approx -1$ because it is a regular $(1+1)$ Lorentzian metric in the (t, r) plane. In particular this shows that the phenomena of super-radiance is absent for the case of (general) spherically symmetric black holes.
- c) Since $g^{rr} = 0$ at $r = r_M$ this hypersurface is null, and by the previous remark $\partial_t|_{r=r_M}$ is the null generator. The assumption $\partial_r g^{rr} \neq 0$ implies this null hypersurface enjoys a red shift effect similar to that of Schwarzschild and non-extremal Reissner-Norström, and this is one of the main stability mechanisms we exploit in our work here (we refer the reader to [7] and [10] for further information regarding this issue).
- d) It turns out that condition iii) also implies there are no trapped null geodesics in the region $r > r_M$ other than those tangent to $r = r_{\mathcal{T}}$ (e.g. no null geodesics which oscillate in a band $r_1 < r < r_2$). Furthermore, it turns out this condition has a simple geometric description which is $V'(r_{\mathcal{T}}) = 0$ and $V''(r_{\mathcal{T}}) < 0$, where we have set $V = -r^{-2}\langle \partial_t, \partial_t \rangle_g$.

Now orient $(\mathcal{M}, g_{\alpha\beta})$ so that (dt, dr, dx^2, dx^3) is a positive basis of $T^*(\mathcal{M})$, where $r = \text{const}$ is given the outward (i.e. towards $r \rightarrow \infty$) orientation on each 3-surface $t = \text{const}$. Recall that given such an oriented Lorentzian manifold there is a unique isomorphism $\star : \Lambda^p \rightarrow \Lambda^{4-p}$ such that $\langle \omega, \sigma \rangle_g dV_g = \omega \wedge \star \sigma$ where $dV_g = \sqrt{|g|} dt \wedge dr \wedge dV_{\mathbb{S}^2}$, and $dV_{\mathbb{S}^2}$ is the standard (outward oriented) volume form on \mathbb{S}^2 .

Let $F_{\alpha\beta}$ be an antisymmetric two-tensor on \mathcal{M} , and define $F^\star = \star F$. We label its divergences as follows:

$$(1.2) \quad \nabla^\beta F_{\alpha\beta}^\star = I_\alpha, \quad \nabla^\beta F_{\alpha\beta} = J_\alpha.$$

We call I_α the *magnetic source* and J_α the *electric source* of $F_{\alpha\beta}$. On physical grounds one usually sets $I \equiv 0$, but for mathematical purposes we will not do so here. If both $I \equiv 0$ and $J \equiv 0$, we say $F_{\alpha\beta}$ is a *free Maxwell field*.

It is well known that the first order system (1.2) is hyperbolic, so it makes sense to estimate the values of $F_{\alpha\beta}$ in the wedge $t \geq 0$ and $r \geq r_0$ in terms of the values of (I, J) in $t \geq 0$ and $r \geq r_0$ and the initial restriction of components $F_{\alpha\beta}|_{t=0}$. Based on experience with the scalar wave equation on backgrounds such as $(\mathcal{M}, g_{\alpha\beta})$, see e.g. [16] and [20], one would expect to prove local energy decay estimates which are roughly of the form²

$$(1.3) \quad \| r^{-\frac{1}{2}-\epsilon} F \|_{L^2(dV_g)[0,T]} \leq C_\epsilon \left(\| F|_{t=0} \|_{L^2(dV_g|_{t=0})} + \| r^{\frac{1}{2}+\epsilon} I \|_{L^2(dV_g)[0,T]} + \| r^{\frac{1}{2}+\epsilon} J \|_{L^2(dV_g)[0,T]} \right),$$

where the components of F, I, J are written with respect to regular basis such as $(\partial_a, r^{-1}\partial_A)$. Unfortunately, such a naive estimate is false due to the presence of finite energy bound state solutions to (1.2). These arise as follows:

¹See Lemma 5.21 below for a precise formulation of this condition.

²In the sequel we will prove a scale invariant version which corresponds to $\epsilon = 0$. In addition one needs a logarithmic weight on the trapped set $r = r_{\mathcal{T}}$. The precise version will be given shortly.

Let $\mathcal{S} \subseteq \mathcal{M}$ be a compact space-like two surface homotopic to some sphere $t = \text{const}, r = \text{const}$, and define the quantities (in terms of pull-backs of F):

$$(1.4) \quad Q_{\mathcal{S}} = \int_{\mathcal{S}} F|_{\mathcal{S}}, \quad Q_{\mathcal{S}}^* = - \int_{\mathcal{S}} F^*|_{\mathcal{S}},$$

where we give the integrals the outward orientation induced by dV_g by first restricting t and then r . We call $Q_{\mathcal{S}}^*$ and (resp $Q_{\mathcal{S}}$) the *electric* (resp *magnetic*) charge contained within \mathcal{S} . Let \mathcal{S}' be some other space-like two surface such that there exists a tube $\Sigma(\mathcal{S}', \mathcal{S})$ with boundary $\partial\Sigma(\mathcal{S}', \mathcal{S}) = \mathcal{S}' \sqcup \mathcal{S}$ and the positive outward orientation through \mathcal{S}' . Then by Stokes theorem and the properties of the \star operator (see equation (1.10) below) one has:

$$(1.5) \quad Q_{\mathcal{S}'} - Q_{\mathcal{S}} = - \int_{\Sigma(\mathcal{S}', \mathcal{S})} (\star I)|_{\Sigma}, \quad Q_{\mathcal{S}'}^* - Q_{\mathcal{S}}^* = - \int_{\Sigma(\mathcal{S}', \mathcal{S})} (\star J)|_{\Sigma}.$$

In particular if $F_{\alpha\beta}$ is a free Maxwell field then $Q_{\mathcal{S}}$ and $Q_{\mathcal{S}}^*$ are constants equal to:

$$(1.6) \quad Q_{\infty} = \lim_{r \rightarrow \infty} \int_{S_{t,r}} F_{\hat{A}\hat{B}}(t, r) dV_{\mathbb{S}^2}, \quad Q_{\infty}^* = \lim_{r \rightarrow \infty} \int_{S_{t,r}} F_{tr}(t, r) dV_{\mathbb{S}^2},$$

where $S_{t,r}$ are the spheres $t = \text{const}, r = \text{const}$, and $e_{\hat{A}}, e_{\hat{B}}$ is an orthonormal basis of $S_{t,r}$ with respect to the restriction of g .³ If either of these values is non-zero it represents a finite energy non-dispersing component of $F_{\alpha\beta}$, i.e. an obstruction to estimate (1.3).

1.1. Statement of the main theorem. The main result of the present paper is that in the class of finite energy solutions to (1.2) the (local) charges (1.4) represent the *only* obstruction to local energy decay. To state this result properly, it is necessary to compute the finite energy stationary free Maxwell fields. These turn out to be spanned by the pair:

$$\overline{F}^{\text{electric}} = \frac{1}{4\pi r^2} \sqrt{|h|} dt \wedge dr, \quad \overline{F}^{\text{magnetic}} = \frac{1}{4\pi} \sqrt{|\delta|} dx^2 \wedge dx^3.$$

If F is a general solution to (1.2), we project it dynamically onto the span of $\overline{F}^{\text{electric}}$ and $\overline{F}^{\text{magnetic}}$ as follows:

$$(1.7) \quad \overline{F} = Q^*(t, r) \overline{F}^{\text{electric}} + Q(t, r) \overline{F}^{\text{magnetic}},$$

where $Q^*(t, r)$ and $Q(t, r)$ denote the integrals (1.4) taken over the spheres $S_{t,r}$, and can be viewed as the radial components of F .

To state our main theorem we need one additional bit of notation concerning our choice of norms. Let $\mathcal{R}_j = \{(t, r, x^A, x^B) \mid r \sim 2^j\}$ for $j \in \mathbb{Z}$. Let r_0 be a fixed value of r such that $\langle dr, dr \rangle < 0$ as in Definition 1.1. Then for a value $T > 0$ we set:

$$(1.8) \quad \|F\|_{LE[0,T]} = \sup_j \|r^{-\frac{1}{2}} F\|_{L^2(\mathcal{R}_j)_{[r_0, \infty) \times [0, T]}}, \quad \|J\|_{LE^*[0,T]} = \sum_j \|r^{\frac{1}{2}} J\|_{L^2(\mathcal{R}_j)_{[r_0, \infty) \times [0, T]}},$$

where the integrals are taken with respect to the volume form dV_g . The main result of this paper is the following:

Theorem 1.3 (Local energy decay for Maxwell fields). *Let F be a solution to (1.2) on a space-time (\mathcal{M}, g) which satisfies the axioms of Definition 1.1, and define \overline{F} according to (1.4) and (1.7). Then one has the uniform bound:*

$$(1.9) \quad \|(w_{\ln})^{-1}(F - \overline{F})\|_{LE[0,T]} \lesssim \|F(0)\|_{L^2(dV_{g|_{t=0}})} + \|w_{\ln}(I, J)\|_{LE^*[0,T]},$$

where $w_{\ln}(r) = (1 + |\ln|r - r_{\mathcal{T}}||)/(1 + |\ln(r)|)$. Here the components of F, I, J are taken with respect to any regular coordinate system (t, r) as in Definition 1.1, and any normal frame $e_{\hat{A}}, e_{\hat{B}}$ on the spheres $S_{t,r}$.

³It is not hard to see that under conditions i) and ii) of Definition 1.1 these values do not depend on the choice of t . Here $e_{\hat{A}}, e_{\hat{B}}$ are bounded coefficient linear combinations of $r^{-1}\partial_2, r^{-1}\partial_3$; we will reserve the notation e_A, e_B to denote an orthonormal basis with respect to the (fixed) round metric δ_{AB} on \mathbb{S}^2 , i.e. a bounded coefficient linear combination of ∂_2, ∂_3 .

1.2. The charge equations. A preliminary reduction of the main theorem. In this section we derive some further equations for the charges (1.4). This will allow us to give an alternate version of Theorem 1.3, which is what we will actually prove in the sequel.

Recall that on a 4 dimensional Lorentian manifold with oriented volume form dV_g , the Hodge \star operator can be written in an arbitrary basis as follows:

$$\begin{aligned} \star 1 &= dV_g, & (\star J)_{\alpha\beta\gamma} &= -\epsilon_{\alpha\beta\gamma}{}^\delta J_\delta, & (\star F)_{\alpha\beta} &= \frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}, \\ (\star G)_\alpha &= -\frac{1}{6}\epsilon_\alpha{}^{\beta\gamma\delta} G_{\beta\gamma\delta}, & \star\omega &= \frac{1}{24}\epsilon^{\alpha\beta\gamma\delta}\omega_{\alpha\beta\gamma\delta}, \end{aligned}$$

for 0 through 4 forms, where $\epsilon_{\alpha\beta\gamma\delta}$ are the components of dV_g . One also has $\star^2 = (-1)^{p+1}$ on each class of forms Λ^p (again in the 4 dimensional case).

It will also be useful for us to have 2 dimensional versions of the above formulas in both the Lorentzian and Riemannian cases, in particular for the factor metrics h_{ab} and δ_{AB} of (1.1). We label their volume forms and Hodge star operators by ϵ_{ab} , ϵ_{AB} , \star_h , and \star_δ (resp). In the Riemannian case we have $\star_\delta^2 = (-1)^p$ while in the Lorentzian case $\star_h^2 = (-1)^{p+1}$ as usual. The dual of 0 through 2 in is given by:

$$\star_h 1 = dV_h, \quad (\star_h J)_a = -\epsilon_a{}^b J_b, \quad \star\omega = \frac{1}{2}\epsilon^{ab}\omega_{ab},$$

with identical formulas for \star_δ . In the sequel we will follow the convention that if I or J is some 1 form on \mathcal{M} then \star_h (resp \star_δ) act on it by contraction in the a (resp A) indices.

Next, we have the L^2 adjoint of d with respect to the inner product $\langle\omega, \sigma\rangle_{L^2} = \int_{\mathcal{M}} \omega \wedge \star\sigma$, which in the 4 dimensional case is simply $d^\star = \star d\star$. This operator also has a convenient expression in terms of the connection Levi-Civita connection ∇ of g which is:

$$d^\star J = -\nabla^\alpha J_\alpha, \quad (d^\star F)_\alpha = \nabla^\beta F_{\alpha\beta}, \quad (d^\star G)_{\alpha\beta} = -\nabla^\gamma G_{\alpha\beta\gamma},$$

on 1 through 3 forms.

One also has analogs of the previous formulas in the 2D case (both Riemannian and Lorentzian). To avoid confusion we set \underline{d} to denote the exterior derivative in the (t, r) plane, and \not{d} to denote exterior differentiation on the spheres \mathbb{S}^2 given by $t = \text{const}, r = \text{const}$. Then one also has co-derivative operators \underline{d}^\star and \not{d}^\star , with similar formulas to the last line above, where now the divergences involve the Levi-Civita connections of h and δ .

Using the above formulas one may write the Maxwell equations (1.2) as follows:

$$(1.10) \quad dF^\star = \star J, \quad dF = -\star I.$$

This allows us to give a useful characterization of the charges (1.4).

Proposition 1.4 (The charge equations). *Let $F_{\alpha\beta}$ solve (1.10), and define $\bar{F}_{\alpha\beta}$ as on lines (1.4) and (1.7). Then one has:*

$$(1.11) \quad \nabla^\beta \bar{F}_{\alpha\beta}^\star = \bar{I}_\alpha, \quad \nabla^\beta \bar{F}_{\alpha\beta} = \bar{J}_\alpha,$$

where $\bar{I}_A = \bar{J}_A \equiv 0$ for $A = 2, 3$, and for $a = 0, 1$:

$$(1.12) \quad \bar{I}_a = \frac{1}{4\pi} \int_{\mathbb{S}^2} I_a dV_{\mathbb{S}^2}, \quad \bar{J}_a = \frac{1}{4\pi} \int_{\mathbb{S}^2} J_a dV_{\mathbb{S}^2}.$$

Furthermore, if the source components I_a and J_a , for $a = 0, 1$ have sufficiently fast decay as $r \rightarrow \infty$ we have:

$$(1.13) \quad Q(t, r) = Q_\infty + \int_r^\infty \int_{\mathbb{S}^2} (\star_h I)_r(t, s) s^2 dV_{\mathbb{S}^2}, \quad Q^\star(t, r) = Q_\infty^\star + \int_r^\infty \int_{\mathbb{S}^2} (\star_h J)_r(t, s) s^2 dV_{\mathbb{S}^2},$$

where Q_∞, Q_∞^\star are given on line (1.6).

An immediate consequence of the last proposition is that Theorem 1.3 is reduced to the following result. Furthermore, to understand arbitrary $F_{\alpha\beta}$ one only needs to add to this the formulas (1.13).

Theorem 1.5 (Local energy decay estimates for chargeless fields). *Let $F_{\alpha\beta}$ be a solution to (1.2) in the region $r \geq r_0$ and $0 \leq t \leq T$ with uniformly vanishing charges $Q(t, r) = Q^*(t, r) \equiv 0$. Then one has:*

$$\int_{\mathbb{S}^2} I_a dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} J_a dV_{\mathbb{S}^2} \equiv 0, \quad a = 0, 1,$$

in $r \geq r_0$ and $0 \leq t \leq T$ as well, and in addition there is a local energy decay estimate:

$$(1.14) \quad \|(w_{ln})^{-1}F\|_{LE[0,T]} \lesssim \|F(0)\|_{L^2(dV_{g|_{t=0}})} + \|w_{ln}(I, J)\|_{LE^*[0,T]},$$

with the same notation as in (1.9) above.

Remark 1.6. *The equations (1.13) show that if the charges Q_∞ and Q_∞^* vanish (initially) then one may obtain a local energy decay estimate for the full field $F_{\alpha\beta}$ in terms of initial data $\|F(0)\|_{L^2(dV_{g|_{t=0}})}$, and certain integrals of I and J . However, to estimate the local energy decay contribution of the averages \bar{I}, \bar{J} requires a RHS norm with a different scaling. Precisely, we have the estimate*

$$(1.15) \quad \|(w_{ln})^{-1}F\|_{LE[0,T]} + \|r\bar{F}\|_{LE[0,T]} \lesssim \|F(0)\|_{L^2(dV_{g|_{t=0}})} + \|w_{ln}(I, J)\|_{LE^*[0,T]} + \|r(\bar{I}, \bar{J})\|_{LE^*[0,T]}.$$

1.3. Background and further remarks. The motivation for Theorem 1.3 is its relation to stability problems in general relativity, in particular the problem of proving non-linear stability of the Kerr family of metrics (see [13] and [10] for a survey). It is generally accepted wisdom that the proof of such a stability theorem will be strongly predicated on the theory of L^∞ decay estimates for first order hyperbolic systems. It is also well known that for hyperbolic (and even other dispersive) PDE the assumption of local energy decay estimates of the form (1.9) allows one to prove much more refined decay estimates in higher L^p and weighted L^2 spaces (e.g. Strichartz, conformal energy, and uniform L^∞ estimates).

While there has been great progress lately towards understanding the local energy decay and higher L^p behavior of solutions to the *scalar* wave equation $\square_g = \nabla^\alpha \nabla_\alpha$ on black-hole backgrounds (e.g. see [5], [7], [8], [9], [14], [15], [16], [17], [20], [21], [23]) relatively little has been done for the case of higher spin equations such as Maxwell fields. The main estimates we are aware of to date are of conformal energy and uniform L^∞ type in the case of Schwarzschild space due to work of P. Blue [4], and similar estimates in the case of Kerr with $|a| \ll M$ more recently due to Andersson-Blue [1]. In a companion to our paper [18] it is also shown that local energy decay estimates of the form (1.9) imply much more refined L^∞ estimates of the form studied in [21] and [17] for general solutions to (1.2) on a broad class of non-symmetric backgrounds. This puts additional emphasis on generalizing Theorem 1.3 to a wider class of space-times.

We remark here that the problem of showing local energy decay for higher spin fields such as Maxwell's equations is fundamentally different from the case of scalar fields due to the presence of finite energy bound state solutions (i.e. charges). A somewhat better model for this type of behavior would be the scalar wave equation \square_g on a black-hole background plus a possibly negative potential; a problem which is completely open. The method of the present paper is simplified to a certain extent by the happy coincidence that for Maxwell fields on spherical backgrounds the only finite energy bound state solutions can be eliminated by subtraction of the spherical average from the sources (I, J) and then removing the charge from $r = \infty$ (this latter process essentially “removes the charge from initial data”). In a sequel to this paper we plan to show that an analog of Theorem 1.3 holds for the case of arbitrary small and sufficiently well localized perturbations of the metric given by Definition 1.1. The proof of this more general result involves a number of complications not present here.

Finally, we remark that it is not hard to check the Schwarzschild and family of non-extremal⁴ Reissner-Nordström metrics all obey the general requirements of Definition 1.1. Furthermore, if $g_{\alpha\beta}$ satisfies such assumptions, so does any sufficiently small and well localized stationary spherically symmetric perturbation of it. In general we believe that small (not necessarily symmetric or stationary) perturbations of such metrics give rise to a general class of space-times with “good dispersive properties” not only for \square_g , but for a variety of first and second order wave equations. We plan to address this in future works.

⁴For recent work on local energy decay type estimates for scalar fields in the extremal case we refer the reader to work of Aretakis [2] and [3].

1.4. Organization of the paper. This paper is organized as follows. In the remainder of this section we give a quick proof of Proposition 1.4, followed by some basic notation that will be useful in the sequel.

In the Section 2 we recall the basic energy estimates for Maxwell fields. These will allow us to reduce Theorem 1.5 to a spatially localized form. However, beyond this simple localization, energy estimates at the level of $F_{\alpha\beta}$ are not used in the remainder of the proof.

Section 3 reduces the proof of estimate (1.14) to the spin-zero components of $F_{\alpha\beta}$. There are two of these, which represent the two dynamical degrees of freedom in an electro-magnetic field. The proof of this reduction follows from elliptic Hodge theory and does not rely on any time dependent analysis.

In Section 4 we introduce the main estimate of the paper which is contained in Theorem 4.12. This is a local energy decay estimate for a certain second order wave equation which is satisfied by the spin-zero components of $F_{\alpha\beta}$. Reduction of first order relativistic systems to such second order wave equations is standard in the literature (e.g. see [22], [4], and [12] for the case of the full Einstein equations). However, our work is the first to provide sharp local energy decay estimates for general inhomogeneous solutions to such equations (in the case of Maxwell), with weights that correspond to local energy decay at the level of the physical energy of the original fields. Specifically, our estimates have a sharp L^2 scaling as opposed to an \dot{H}^1 scaling, so in particular they contain more information at low frequencies. To accomplish this one must keep track of a remnant of the tensorial character of the original first order system which resides in the (inhomogeneous) second order equation. It turns out this feature of the second order equation is intimately connected to the issue of estimating “dynamic charges” which come from the source terms (I, J) on RHS(1.2).

Sections 5 and 6 are the technical heart of the paper. In Section 5 we prove a preliminary local energy decay estimate for our spin-zero Teukolsky equation. This estimate is a direct analog of the local energy decay estimate in [16]. Our proof combines ideas from [5], [7], and [16]. More specifically, we use red-shift estimates similar to those of [7] along with certain “energy estimates” to obtain a local energy decay estimate close to $|r - r_M| \ll 1$ with a source error at $r \approx r_M + \epsilon$. We then introduce an analog of Regge-Wheeler coordinates in $r > r_M$ and prove a local energy decay estimate in the exterior along the lines of [5], but with sharp norms as introduced in [16]. Interestingly, we are able to show that one can directly glue together the exterior and horizon estimates because the conservation of (degenerate) energy close to $r = r_M$ controls the worst part of the error coming from truncation of the Regge-Wheeler estimate. In effect this allows one to work in two convenient coordinate systems simultaneously, one of them singular, and to still obtain a regular global estimate.

In Section 6 we upgrade the preliminary scalar local energy decay estimate of Section 5 to an estimate which has a non-local character and takes into account the structure of the source terms (I, J) to the original first order system (1.2). To accomplish this we use a certain “gauge transformation” of the inhomogeneous spin-zero wave equation which is analogous to a space-time Coulomb gauge for Maxwell fields. This allows us to directly estimate the portion of the spin-zero components of $F_{\alpha\beta}$ which depend elliptically on the sources (I, J) , the remainder being estimated by the dispersive local energy decay estimate explained above.

In an Appendix we list some auxiliary estimates that come up in the bulk of the proof. These are isolated for the convenience of the reader, and so they don’t interrupt the flow of the main argument.

1.5. Proof of Proposition 1.4. Here we give a quick demonstration. More general formulas which also imply this proposition will be given in Lemma 3.10 below.

Step 1:(*Proof of formulas (1.13)*) Using the identities (1.5) we have

$$Q_\infty - Q(t, r) = - \int_{\Sigma_{t,r}} (\star I)|_{\Sigma_{t,r}} , \quad Q_\infty^* - Q^*(t, r) = - \int_{\Sigma_{t,r}} (\star J)|_{\Sigma_{t,r}} ,$$

where $\Sigma_{t,r} = \{(x^0, x^1, x^2, x^3) \mid x^0 = t, x^1 > r\}$. One also has that with the positive orientation:

$$(\star I)|_{\Sigma_{t,r}} = -\frac{1}{2}\epsilon_{rAB}{}^a I_a dr \wedge dx^A \wedge dx^B = -r^2(\epsilon_h)_a{}^b I_b dr \wedge dV_{\mathbb{S}^2} ,$$

with a similar formula for $(\star J)|_{\Sigma_{t,r}}$. The result follows.

Step 2:(*Variation of the charge*) From equations (1.13) we compute the exterior derivatives of Q, Q^* in the (t, r) variables. Since the formulas (1.13) are identical, it suffices to compute things for the magnetic charge

Q . We claim that:

$$(1.16) \quad dQ = -r^2 \int_{\mathbb{S}^2} (\star_h I)_a dV_{\mathbb{S}^2} dx^a .$$

For the ∂_r derivative this is immediate from (1.13).

On the other hand, to compute the derivative with respect to t we use the continuity equation $d \star I = 0$, which follows immediately from (1.10). In terms of covariant derivatives one has:

$$\nabla^a (r^2 I)_a + \nabla^A I_A = -\underline{d}^\star (r^2 I) - \not{d}^\star I = 0 ,$$

where ∇_a (resp ∇_A) is the Levi-Civita connection of h_{ab} (resp δ_{AB}). Using the identity $\underline{d}^\star = \star_h \underline{d} \star_h$ we may write the continuity equation in the mixed form:

$$(1.17) \quad \star_h \underline{d} (r^2 \star_h I) = -\not{d}^\star I .$$

In term of (t, r) derivatives, and with our choice of orientation, this gives:

$$\partial_t (r^2 \star_h I)_r = \partial_r (r^2 \star_h I)_t - \not{d}^\star (\sqrt{|h|} \star_\delta I) .$$

Since an angular divergence integrates to zero on \mathbb{S}^2 , after applying the fundamental theorem of calculus in the radial variable we are left with:

$$\partial_t Q = -r^2 \int_{\mathbb{S}^2} (\star_h I)_t dV_{\mathbb{S}^2} ,$$

which proves (1.16).

Step 3: (*Computing $d\bar{F}$*) First note that $d(Q^\star(t, r)\bar{F}^{electric}) = 0$ for any function $Q^\star(t, r)$. On the other hand, by line (1.16) we have:

$$d(Q(t, r)\bar{F}^{magnetic}) = -\frac{1}{4\pi} r^2 (\star_h \int_{\mathbb{S}^2} IdV_{\mathbb{S}^2})_a \cdot dx^a \wedge dV_{\mathbb{S}^2} = -\star \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} I_a dV_{\mathbb{S}^2} \cdot dx^a \right) .$$

Adding these identities, and noting the second identity of line (1.10), gives the second equation on line (1.11).

The first identity on line (1.11) follows from the previous calculation and duality.

1.6. Some notation. We list here several notational conventions that will be used in the sequel.

As usual we denote $A \lesssim B$ (resp $A \ll B$; $A \approx B$) if $A \leq CB$ for some fixed $C > 0$ which may change from line to line (resp $A \leq \epsilon B$ for a small $\epsilon > 0$; both $A \lesssim B$ and $B \lesssim A$).

For a collection of non-commutative operators we often denote their ordered product without parenthesis. A typical example of this would be a mixture of pseudodifferential operators such as $\partial_r \Delta^{-1} f \partial_r$, where f is a function of (r, x^A) .

If $\phi(t, r, x^A)$ is a space-time function we denote by $\phi[t_0]$ the restriction of the gradient $d\phi$ to the hypersurface $t = t_0$. This notation is context sensitive as the components of $d\phi$ will be given with respect to different frames depending on the application. In practice there are only two frames used here, either $(\partial_t, \partial_r, e_{\hat{A}})$ or $(\partial_t, \partial_r, e_A)$. The choice (i.e. normalization of angular derivatives) will be clear from context.

A basic estimate that will be used many times in the sequel is the following: If f is a sufficiently smooth function on \mathbb{S}^2 we denote by $\bar{f} = (4\pi)^{-1} \int_{\mathbb{S}^2} f dV_{\mathbb{S}^2}$ its spherical average. Then Poincaré's estimate says that:

$$(1.18) \quad \|f - \bar{f}\|_{L^2(dV_{\mathbb{S}^2})} \lesssim \|\not{d}f\|_{L^2(dV_{\mathbb{S}^2})} .$$

2. ENERGY ESTIMATES FOR MAXWELL FIELDS

In this section we recall the basic energy estimates for Maxwell fields on a Lorentzian background. These will be used in a auxiliary manner in the sequel, but are not necessary for the core part of the proof of local energy decay.⁵ For any antisymmetric two-form on a $(3+1)$ space-time we define:

$$Q_{\alpha\beta}[F] = F_{\alpha\gamma} F_\beta{}^\gamma - \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} = \frac{1}{2} (F_{\alpha\gamma} F_\beta{}^\gamma + F_{\alpha\gamma}^\star F_\beta^{\star\gamma}) .$$

⁵Our proof of local energy decay uses energy conservation in an essential way, but only for certain components of $F_{\alpha\beta}$. For these specific components we will only need to show certain spatially localized versions of the conservation of energy.

Then with the notation of line (1.2) we have the divergence identity:

$$(2.19) \quad \nabla^\alpha Q_{\alpha\beta}[F] = \frac{1}{2}(F_{\alpha\beta}J^\alpha + F_{\alpha\beta}^*I^\alpha).$$

To measure energy contributions we use the following definition:

Definition 2.7 (Regular space-like hypersurfaces). *Fix (t, r) coordinates as in Definition 1.1. We say a space-like hypersurface \mathcal{S} in $(\mathcal{M}, g_{\alpha\beta})$ is “regular” if there exists a constant $c > 0$ such that:*

$$T = b_+\ell + b_-\underline{\ell}, \quad b_\pm > c,$$

where T is the future directed (with respect to $-\nabla t$) normal to \mathcal{S} and $\ell, \underline{\ell}$, are future directed null vector-fields such that:

$$\ell = -\nabla t + a\partial_r, \quad \underline{\ell} = -\nabla t - a\partial_r, \quad a > 0.$$

In the usual way we have the following:

Proposition 2.8 (Energy estimates for Maxwell fields). *Let $\mathcal{R} \subseteq \mathcal{M}$ be an open set bounded by two regular space-like hypersurfaces \mathcal{C}_1 and \mathcal{C}_2 , where \mathcal{C}_2 is to the future of \mathcal{C}_1 (with respect to $-\nabla t$). If $F_{\alpha\beta}$ solves (1.2) on \mathcal{R} then one has the following uniform bound in $\epsilon > 0$:*

$$(2.20) \quad \|F|_{\mathcal{C}_2}\|_{L^2(dV_{g|_{\mathcal{C}_2}})} \lesssim \epsilon \| (w_{ln})^{-1}F \|_{LE(\mathcal{R})} + \|F|_{\mathcal{C}_1}\|_{L^2(dV_{g|_{\mathcal{C}_1}})} + \epsilon^{-1} \|w_{ln}(I, J)\|_{LE^*(\mathcal{R})}.$$

Here the components of F, I, J are computed in any regular frame such as $\{\partial_t, \partial_r, e_{\hat{A}}, e_{\hat{B}}\}$, and the notation $F|_{\mathcal{C}_i}$ denotes the restriction of such scalar components to \mathcal{C}_i (not the pull-back of forms).

Proof. The proof is standard. First contract $Q_{\alpha\beta}$ above with the vector-field $-\nabla t$. Then using the divergence identity (2.19) one employs Stokes theorem and Cauchy-Schwartz. The key observation is that:

$$-Q(T, \nabla t) \approx \sum_{\alpha < \beta} |F_{\alpha\beta}|^2,$$

where T is the (future) normal to \mathcal{C}_i and the RHS sum is taken over the frame $\{\partial_t, \partial_r, e_{\hat{A}}, e_{\hat{B}}\}$. See [11] for more details. \square

3. FIRST ORDER FORMULATION OF THE EQUATIONS. REDUCTION OF THE MAIN THEOREM TO A SPIN-ZERO LOCAL ENERGY DECAY ESTIMATE

We begin with some first order equations for components of $F_{\alpha\beta}$ when (1.2) holds. First we introduce some notation:

$$(3.21) \quad \phi = \frac{1}{2}\epsilon^{AB}F_{AB}, \quad \phi^* = -\frac{1}{2}\epsilon^{AB}F_{AB}^*, \quad \mathbb{F}_{aA} = F_{aA} \quad \mathbb{F}_{aA}^* = F_{aA}^*.$$

Note that the quantities ϕ, ϕ^* are scalars. Here we think of \mathbb{F}, \mathbb{F}^* as sections of $T^*(\mathbb{R}^2) \otimes T^*(\mathbb{S}^2)$, in other words \mathbb{F} is a tensor with four components $F_{02}, F_{03}, F_{12}, F_{13}$ and similarly for \mathbb{F}^* . We let $\star_h, \underline{d}, \underline{d}^*$ (resp $\star_\delta, \underline{d}, \underline{d}^*$) act on \mathbb{F}, \mathbb{F}^* in the obvious way, by touching only the first (resp second) set of components. It turns out that by Hodge duality the second tensor \mathbb{F}^* is redundant, specifically:

$$(3.22) \quad \mathbb{F}_{aA}^* = \frac{1}{2}\epsilon_{aA}{}^{\gamma\delta}F_{\gamma\delta} = -(\epsilon_h)_a{}^b(\epsilon_\delta)_A{}^B F_{bB} = -(\star_h \star_\delta \mathbb{F})_{aA}.$$

In particular notice that the 6 scalar quantities ϕ, ϕ^* and \mathbb{F}_{aA} for $a = 0, 1, A = 2, 3$ span the values of $F_{\alpha\beta}$.

Remark 3.9. *One may compare the quantities listed on line (3.21) to the standard null decomposition of an electromagnetic field on Minkowski space as defined by Christodoulou-Klainerman [6]. Then we have $\phi = r^2\sigma$, $\phi^* = r^2\rho$, and \mathbb{F}_{aA} is a linear combination of $r\alpha_A$ and $r\underline{\alpha}_A$. We remark that since all components are on an equal footing with respect to the natural L^2 energy (2.20), there is no need to further decompose \mathbb{F}_{aA} in proving our estimates.*

Our main result here is to relate the (t, r) derivatives of ϕ, ϕ^* to the angular derivatives of \mathbb{F} and the sources I, J . This will generalize (1.16) above.

Lemma 3.10 (Gradient identities for ϕ, ϕ^*). *Let $F_{\alpha\beta}$ satisfy the equations (1.2), and define the quantities on line (3.21) from it. Then one has:*

$$(3.23) \quad \underline{d}\phi = \star_\delta \not{d}\not{F} - r^2 \star_h I ,$$

$$(3.24) \quad \underline{d}\phi^* = -\star_\delta \not{d}\not{F}^* - r^2 \star_h J ,$$

$$(3.25) \quad \star_h \underline{d}\phi^* = \not{d}^* \not{F} - r^2 J .$$

Proof. Notice that line (3.24) follows from (3.23) and Hodge duality which sends $\phi \rightarrow -\phi^*$, $\not{F} \rightarrow \not{F}^*$, and $I \rightarrow -J$.

Next, we see that line (3.25) follows by applying \star_h to line (3.24), using $[\star_h, \star_\delta \not{d}] = 0$, and then using the identity (3.22) with $\not{d}^* = -\star_\delta \not{d} \star_\delta$.

To show (3.23) we expand the second member of line (1.10) in the basis $(\partial_a, \partial_A, \partial_B)$ which gives:

$$\partial_a F_{AB} - \nabla_A F_{aB} + \nabla_B F_{aA} = -(\star I)_{aAB} ,$$

where ∇ is the standard round connection on \mathbb{S}^2 . Tracing the last line with $\frac{1}{2}\epsilon^{AB}$ and using the identity:

$$\frac{1}{2}\epsilon^{AB}(\star I)_{aAB} = -\frac{1}{2}\epsilon^{AB}r^2\epsilon_a{}^b\epsilon_{AB}I_b = -r^2\epsilon_a{}^bI_b = r^2(\star_h I)_a ,$$

we have the desired result. \square

3.1. Reduction of the main theorem. The equations (3.23) and (3.25) completely determine the values of \not{F} in terms of ϕ, ϕ^* . This may be quantified as follows:

Proposition 3.11 (Reduction to spin-zero components). *Let \not{F}, ϕ, ϕ^* , and I, J solve the system (3.23) and (3.25). Suppose in addition that:*

$$(3.26) \quad \int_{\mathbb{S}^2} \phi dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} \phi^* dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} I_a dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} J_a dV_{\mathbb{S}^2} \equiv 0 , \quad a = 0, 1.$$

Then one has the L^2 estimate:

$$(3.27) \quad \|(w_{ln})^{-1}\not{F}\|_{LE[0,T]} \lesssim \|(w_{ln})^{-1}r^{-1}(-\not{\Delta})^{-\frac{1}{2}}(\underline{d}\phi, \underline{d}\phi^*)\|_{LE[0,T]} + \|(I, J)\|_{LE^*[0,T]} ,$$

where $-\not{\Delta} = \not{d}^* \not{d}$ is the scalar Laplace-Beltrami operator on \mathbb{S}^2 . Here the components of \not{F} are taken in a normalized basis $(\partial_a, r^{-1}\partial_A)$.

Proof. This follows immediately from L^2 Hodge estimates on \mathbb{S}^2 . Recall that the operators $\not{d}\not{v} = -\not{d}^*$ and $\text{c}\not{v}\text{r}\text{l} = \star_\delta \not{d}$ on $T^*(\mathbb{S}^2)$ give rise to a double sided singular integral L^2 estimate:

$$(3.28) \quad \|\omega\|_{L^2(\mathbb{S}^2)} \approx \|(-\not{\Delta})^{-\frac{1}{2}}\not{d}\not{v}\omega\|_{L^2(\mathbb{S}^2)} + \|(-\not{\Delta})^{-\frac{1}{2}}\text{c}\not{v}\text{r}\text{l}\omega\|_{L^2(\mathbb{S}^2)} .$$

Setting $(\omega_a)_A = \not{F}_{a\hat{A}}$ to be a one form on \mathbb{S}^2 indexed by $a = 0, 1$ with values the normalized components of \not{F} , we may write (3.23) and (3.25) as:

$$\not{d}\not{v}\omega = -r^{-1}\star_h \underline{d}\phi^* - rJ , \quad \text{c}\not{v}\text{r}\text{l}\omega = r^{-1}\underline{d}\phi + r\star_h I .$$

The estimate (3.27) follows at once from this, the boundedness of $(-\not{\Delta})^{-\frac{1}{2}}$ on $L^2(\mathbb{S}^2)$ functions with zero average, and the inclusion $r^{-1}LE^* \subseteq LE$. \square

4. SECOND ORDER WAVE EQUATIONS AND LOCAL ENERGY DECAY ESTIMATES. PROOF OF THE FIRST ORDER DECAY ESTIMATE

In this section we reduce the estimation of the first term on RHS (3.27) to the following Theorem, which one may view as the main technical result of the paper. To state it we introduce the following operator, which we call the *spin-zero wave equation*:

$$(4.29) \quad \square^0 = \square_h + r^{-2}\not{\Delta} = \nabla^a \nabla_a + r^{-2}\nabla^A \nabla_A .$$

For this operator we have:

Theorem 4.12 (Inverse angular gradient local energy decay estimates). *Let $\phi(x^a, x^A)$ be a scalar function defined on the slab $[0, T] \times [r_0, \infty) \times \mathbb{S}^2$. In addition let $G_a(x^a, x^A)$ be a one form in the x^a variables, whose coefficients also depending on $x^A \in \mathbb{S}^2$, such that $\star_h \underline{d}G = K$. Finally let $H(x^a, x^A)$ be another scalar function. Suppose that all of these objects obey the moment condition:*

$$(4.30) \quad \int_{\mathbb{S}^2} \phi(x^a) dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} G_b(x^a) dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} H(x^a) dV_{\mathbb{S}^2} \equiv 0, \quad b = 0, 1,$$

throughout $[0, T] \times [r_0, \infty)$. If ϕ, G, H are all supported in $\{r \leq CT\}$ and

$$(4.31) \quad \square^0 \phi = \nabla^a G_a + H,$$

then one has the local energy decay type estimate:

$$(4.32) \quad \begin{aligned} \|(w_{ln})^{-1} r^{-1} (\underline{d}(-\underline{\Delta})^{-\frac{1}{2}} \phi, r^{-1} \phi)\|_{LE[0, T]} &\lesssim \|(-\underline{\Delta})^{-\frac{1}{2}} r^{-1} (\underline{d}\phi(0) - G(0))\|_{L^2(dV_{g|_{t=0}})} + \|r^{-2} \phi(0)\|_{L^2(dV_{g|_{t=0}})} \\ &+ \|w_{ln} r^{-1} (r^{-1} G, (-\underline{\Delta})^{-\frac{1}{2}} K, (-\underline{\Delta})^{-\frac{1}{2}} H)\|_{LE^*[0, T]}. \end{aligned}$$

Here $w_{ln}(r) = (1 + |\ln|r - r_{\mathcal{T}}||)/(1 + |\ln(r)|)$ as usual.

The proof of this theorem will occupy the second portion of the paper. First, we use it to prove Theorem 1.5. In light of the energy estimates (2.20) the main step is to show the following:

Proposition 4.13 (Local energy decay estimates for spin-zero components). *Let \mathbb{F}, ϕ, ϕ^* , and I, J satisfy the assumptions of Proposition 3.11, and in addition suppose each of these quantities is supported in the region $r < CT$ for C a sufficiently large fixed constant. Then one has the uniform (in T) bound:*

$$(4.33) \quad \begin{aligned} \|(w_{ln})^{-1} r^{-1} (-\underline{\Delta})^{-\frac{1}{2}} (\underline{d}\phi, \underline{d}\phi^*)\|_{LE[0, T]} + \|(w_{ln})^{-1} r^{-2} (\phi, \phi^*)\|_{LE[0, T]} &\lesssim \|F|_{t=0}\|_{L^2(dV_{g|_{t=0}})} \\ &+ \|w_{ln}(I, J)\|_{LE^*[0, T]} + \|(\star_h I_0(r_0), \star_h J_0(r_0))\|_{L^2(dV_{\mathbb{S}^2 dt}[0, T])}. \end{aligned}$$

Proof that Theorem 4.12 implies Proposition 4.13. This boils down to a direct algebraic calculation, whose point is to derive a second order equation of the form (4.31) for ϕ, ϕ^* , where G_a and H can be estimated in terms of I and J .

Step 1: (Derivation of the second order equation) All of our computations here are more easily done with respect to a conformal metric. Let Ω be a weight function and set $\tilde{g} = \Omega^2 g$. We denote by $\tilde{\star}$ the corresponding Hodge operator, which obeys the identity $\tilde{\star} = \Omega^{4-2p} \star$ on each Λ^p . In particular notice that the quantity F^* is conformally invariant. With respect to the new metric the Maxwell system can be written as (alternatively):

$$\begin{aligned} dF &= -\Omega^{-2} \tilde{\star} I, & dF^* &= \Omega^{-2} \tilde{\star} J, \\ d^{\tilde{\star}} F &= \Omega^{-2} J, & d^{\tilde{\star}} F^* &= \Omega^{-2} I. \end{aligned}$$

Combining these formulas we have:

$$\square^{hodge} F = d^{\tilde{\star}}(\Omega^{-2} \tilde{\star} I) - d(\Omega^{-2} J), \quad \square^{hodge} F^* = -d^{\tilde{\star}}(\Omega^{-2} \tilde{\star} J) - d(\Omega^{-2} I),$$

where $\square^{hodge} = -(d d^{\tilde{\star}} + d^{\tilde{\star}} d)$ is the Hodge Laplacian of \tilde{g} .

In terms of the connection $\tilde{\nabla}$ of \tilde{g} we may write \square^{hodge} as follows:

$$\begin{aligned} \square^{hodge} F_{\alpha\beta} &= \tilde{\nabla}^\gamma (\tilde{\nabla}_\alpha F_{\beta\gamma} + \tilde{\nabla}_\gamma F_{\alpha\beta} + \tilde{\nabla}_\beta F_{\gamma\alpha}) - \tilde{\nabla}_\alpha \tilde{\nabla}^\gamma F_{\beta\gamma} + \tilde{\nabla}_\beta \tilde{\nabla}^\gamma F_{\alpha\gamma}, \\ &= \tilde{\nabla}^\gamma \tilde{\nabla}_\gamma F_{\alpha\beta} + [\tilde{\nabla}_\alpha, \tilde{\nabla}_\gamma] F_{\beta\gamma} + [\tilde{\nabla}_\beta, \tilde{\nabla}_\gamma] F_{\alpha\gamma}, \\ &= \square_{\tilde{g}} F_{\alpha\beta} - \tilde{R}_\alpha{}^\gamma F_{\gamma\beta} - \tilde{R}_\beta{}^\gamma F_{\alpha\gamma} - \tilde{R}_{\alpha\beta}{}^{\gamma\delta} F_{\gamma\delta}, \end{aligned}$$

where $\square_{\tilde{g}}$ is the covariant wave equation acting on two-forms, $\tilde{R}_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor of \tilde{g} , and $\tilde{R}_{\alpha\beta} = \tilde{g}^{\gamma\delta} \tilde{R}_{\alpha\gamma\delta\beta}$ is its Ricci curvature. Here our curvature convention is $[\nabla_\alpha, \nabla_\beta] \omega_\gamma = R_{\alpha\beta}{}^\delta{}_\gamma \omega_\delta$ for one-forms.

We now choose Ω in order to simplify the form of \square^{hodge} above. This is done by choosing $\Omega = r^{-1}$ so \tilde{g} becomes a pure direct sum metric instead of a warped product. For this new metric the curvatures

diagonalize into pure a, b and A, B components. It is the latter which is important for us here, which is simply the Riemann and Ricci curvature of \mathbb{S}^2 , that is:

$$\tilde{R}_{ABCD} = \delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}, \quad \tilde{R}_{AB} = \delta_{AB}.$$

For the D'Alembertian this gives the simple formula:

$$\square^{hodge} F_{AB} = \square_{\tilde{g}} F_{AB}.$$

Tracing this with respect to $\frac{1}{2}\epsilon^{AB}$ gives us:

$$\square_{\tilde{g}}\phi = \frac{1}{2}\epsilon^{AB}d\tilde{\star}(r^2\tilde{\star}I)_{AB} - \frac{1}{2}\epsilon^{AB}d(r^2J)_{AB}, \quad \square_{\tilde{g}}\phi^* = \frac{1}{2}\epsilon^{AB}d\tilde{\star}(r^2\tilde{\star}J)_{AB} + \frac{1}{2}\epsilon^{AB}d(r^2I)_{AB},$$

where $\square_{\tilde{g}} = \tilde{\nabla}^a\tilde{\nabla}_a + \mathbb{A} = r^2\square^0$ is now the scalar covariant wave equation of \tilde{g} .

It remains to compute the traces on the two RHS of the last line above. For the exterior derivative terms we immediately have:

$$\frac{1}{2}\epsilon^{AB}d(r^2I)_{AB} = r^2\star_\delta\phi I, \quad \frac{1}{2}\epsilon^{AB}d(r^2J)_{AB} = r^2\star_\delta\phi J.$$

For the co-derivative expressions we compute:

$$d\tilde{\star}(r^2\tilde{\star}I)_{AB} = \star d(r^2I)_{AB} = r^2\epsilon_{AB} \cdot \frac{1}{2}\epsilon^{ab}\underline{d}(r^2I)_{ab} = r^2\epsilon_{AB}\star_h\underline{d}(r^2I) = -r^2\epsilon_{AB}\nabla^a(r^2\star_h I)_a,$$

with an identical formula for $d\tilde{\star}(r^2\tilde{\star}J)_{AB}$. All together this yields:

$$(4.34) \quad \square^0\phi = -\nabla^a(r^2\star_h I)_a - \star_\delta\phi J, \quad \square^0\phi^* = -\nabla^a(r^2\star_h J)_a + \star_\delta\phi I.$$

Step 2: (*Application of estimate (4.32)*) It suffices to treat the first equation on line (4.34) as the second is of the same form. To set up for estimate (4.32) we define:

$$G := -r^2\star_h I, \quad H := -\star_\delta\phi J, \quad K := \phi^* I.$$

Using the continuity equation (1.17) we have $\star_h dG = K$ as required. Notice also the the moment conditions (4.30) are all satisfied thanks to (3.26).

It remains to bound the terms occurring on the RHS of (4.32) for these choices in terms of RHS (4.33). For the terms H and K this is an immediate consequence of (3.28). Notice that the extra factor r^{-1} disappears when one switches from the \mathbb{S}^2 basis e_A to the regular basis $e_{\hat{A}} = r^{-1}e_A$.

Next, the estimate for the one form G is also immediate due to the truncation condition which implies $(\sqrt{rT})^{-1} \lesssim r^{-1}$. Likewise we have $r^{-2}\phi = F_{\hat{A}\hat{B}}$ in an ordered g orthonormal basis $e_{\hat{A}}, e_{\hat{B}}$ of $S_{t,r}$, which gives the desired bound for the undifferentiated initial data on RHS (4.32).

Finally, we dispense with the gradient terms in the initial data on RHS (4.32). Using equations (3.23) above and the definition of G we have:

$$\underline{d}\phi(0) - G(0) = \star_\delta\phi F(0),$$

and the desired L^2 bound follows again from (3.28) and writing the components of F in a regular basis. \square

4.1. Proof of the main theorem. We are now ready to show Theorem 1.5, which follows easily by combining estimates (2.20), (3.27), and (4.33).

In order to use (4.33) we must first dispense with estimate (1.14) in the region $r > CT$ for some sufficiently large $C > 0$. To do this we will use (2.20) to prove (1.5) in a slightly larger region, namely $r > Ct$. The desired estimate for $0 \leq t \leq 1$ follows easily from local considerations. For larger values of t we can integrate the LHS of estimate (2.20) over a family of uniformly space-like hyper-surfaces such as $r = Ct$ which upon bootstrapping the first term on RHS (2.20) gives:

$$(4.35) \quad \|F\|_{LE(r>Ct)[0,T]} \lesssim \|F(0)\|_{L^2(dV_{g|_{t=0}})} + \|(I, J)\|_{LE^*(r>Ct)[0,T]}.$$

Note that this estimate only uses the simple bound $\|F\|_{LE(r>Ct)[1,T]} \lesssim \sup_{1 \leq t \leq T} \|F|_{C_t}\|_{L^2(dV_{g|_{C_t}})}$ where C_t is the family of hypersurfaces given by $r = Ct$.

It remains to estimate F in the region $r < Ct$ which we enlarge to the region $r < CT$. After truncating $F_{\alpha\beta}$ by a smooth cutoff of the form $\chi(r/T)$ we can immediately reduce matters to proving (1.14) for fields

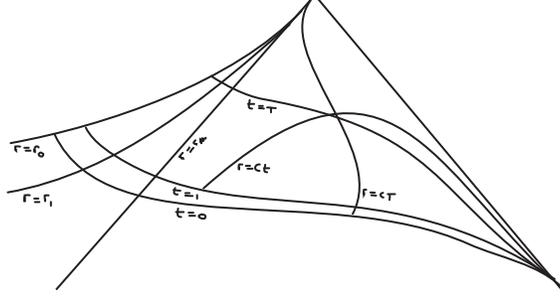


FIGURE 1. Penrose diagram of the regions described in the proof.

supported in $r < 2CT$, as the error generated by differentiation of the cutoff χ on the RHS of (1.14) is handled by $T^{-1} \|\chi'(r/T)F\|_{LE^*[0,T]} \lesssim \|F\|_{LE(r>Ct)[0,T]}$ and then using (4.35).

To proceed further we average estimates (3.27) and (4.33) on the slabs $[0, T] \times \{r > r_*\}$ for r_* in the band $r_0 < r_* < r_1 < r_M$. The purpose of this averaging is simply to trade the resulting error terms $\|(\star_h I_0(r_*), \star_h J_0(r_*))\|_{L^2(dV_{g_2} dt)[0,T]}$ on RHS (4.33) for $\|(I, J)\|_{LE^*(r>r_0)[0,T]}$. Combining the result with estimate (2.20) yields (again for F supported where $r < 2CT$):

$$\|F(r_1)\|_{L^2(dV_{g_1|_{r=r_1}})} + \|(w_{ln})^{-1}F\|_{LE(r>r_1)[0,T]} \lesssim \|F(0)\|_{L^2(dV_{g_1|_{t=0}})} + \|w_{ln}(I, J)\|_{LE^*(r>r_0)[0,T]}.$$

Finally, to fill in the remaining quantity $\|(w_{ln})^{-1}F\|_{LE(r_0 < r < r_1)[0,T]}$ it suffices to integrate the LHS of estimate (2.20) over the family of uniformly space-like hypersurfaces $r = r_*$ for r_* in the band $r_0 \leq r_* \leq r_1$.

5. LOCAL ENERGY DECAY FOR SPIN-ZERO FIELDS PART I: A PRELIMINARY ESTIMATE

In this section we begin our estimates for the spin-zero wave equation (4.29). It is best to think of this as a $(1+1)$ wave equation plus a potential, which is literally true after decomposition into spherical harmonics. There is a natural volume form $dV = dV_h \wedge dV_{\mathbb{S}^2}$ for which \square^0 becomes self adjoint. Here $dV_h = \sqrt{|h|} dx^0 \wedge dx^1$ and $dV_{\mathbb{S}^2}$ is the standard volume on \mathbb{S}^2 . There is also a natural (but not exactly conserved) energy type norm:

$$E(\phi[t]) = \int_{[r_0, \infty] \times \mathbb{S}^2} (\phi_t^2 + \phi_r^2 + r^{-2} |\not{d}\phi(t)|^2) dr dV_{\mathbb{S}^2}, \quad |\not{d}\phi|^2 = g^{AB} \nabla_A \phi \nabla_B \phi,$$

where ∇ is the gradient on \mathbb{S}^2 . Here we take (t, r) to be any regular set of coordinates as in Definition 1.1. We also set up local energy decay norms as follows:

$$\|\phi\|_{LE_0} = \sup_{j \geq 0} 2^{-\frac{j}{2}} \|\chi_j \phi\|_{L^2(dV)}, \quad \|G\|_{LE_0^*} = \sum_{j \geq 0} 2^{\frac{j}{2}} \|\chi_j G\|_{L^2(dV)},$$

for some overlapping set dyadic cutoffs $\chi_j(r)$. Here the integrals are implicitly constrained to the region $r \geq r_0$ which we assume is covered by $j \geq 0$. The main result of this section can now be written as:

Theorem 5.14 (Local energy decay for \square^0). *Let $\square^0 \phi = G$, then one has:*

$$(5.36) \quad \|\partial_r \phi\|_{LE_0[0,T]} + \|(w_{ln})^{-1}(\partial_t \phi, r^{-1} \not{d}\phi)\|_{LE_0[0,T]} \lesssim E^{\frac{1}{2}}(\phi[0]) + \|w_{ln} G\|_{LE_0^*[0,T]},$$

where $w_{ln}(r) = (1 + |\ln|r - r_{\mathcal{T}}||)/(1 + |\ln(r)|)$, and where we assume $\langle \partial_r, \partial_t \rangle_h = 0$ at $r = r_{\mathcal{T}}$.

Remark 5.15. *By Duhamel's principle (note $g^{00} \approx -1$ in regular coordinates) one may replace the norm for G on the RHS of (5.36) with the following:*

$$(5.37) \quad \|G\|_{N_0[0,T]} = \inf_{G_1 + G_2 = G} (\|w_{ln} G_1\|_{LE_0^*[0,T]} + \|G_2\|_{L^1(L^2)[0,T]}),$$

where $L^1(L^2)$ is with respect to dt and $dr dV_{\mathbb{S}^2}$ as above.

Remark 5.16. *The estimate (5.36) is true for a general ϕ even if it is spherically symmetric. On the other hand, if $\int_{\mathbb{S}^2} \phi(t, r) dV_{\mathbb{S}^2} \equiv 0$ then by Poincare's estimate one can also add an LE estimate for $r^{-1}\phi$. However, a bound for $r^{-1}\phi$ in the spherical symmetric case is impossible due to constant solutions which have finite energy even after truncation outside a large dyadic set. Note that this is in stark contrast to the LE estimates for the scalar wave equation \square_g for which one can estimate $r^{-1}\phi$ even in the spherical case (see [16]).*

Remark 5.17. *The condition that $\langle \partial_r, \partial_t \rangle = 0$ at $r = r_{\mathcal{T}}$ is essential if one wishes to gain an unweighted estimate for $\partial_r \phi$. There is an alternate geometric description of this requirement which is that the unweighted derivative must correspond to quantizations of the defining functions for the stable and unstable manifolds of the trapped set.*

The remainder of this section will be devoted to the proof of Theorem 5.14. The method we employ is to estimate separately the contributions coming from the region $\langle dr, dr \rangle \approx 0$ (event horizon), and the region $\langle dr, dr \rangle > 0$ (domain of outer communication). This allows us to build estimates in a modular way, which we believe should also be useful for further applications.

Close to $r = r_M$ our estimates are purely multiplier based, and boil down to the construction of well chosen null frames. There are two key estimates here: (5.51) and (5.52) below. The first captures the contribution of waves parallel to $r = r_M$, and is simply a version of the “red shift” estimates first introduced in [7] (see also [10] for a more general exposition). The second bounds the contribution of waves transverse to $r = r_M$, and turns out to be a version of the conservation of energy.

The region $r > r_M$ involves the most work, and is handled by estimate (5.53) below. To prove it we introduce a generalized “Regge-Wheeler” type coordinate condition to put the radial part h_{ab} of the metric in conformal form. After a little bit of work to show that the truncated estimate (5.53) follows from an untruncated version in Regge-Wheeler coordinates, the analysis follows along the lines of [5] and [16] which is essentially microlocal in nature.

5.1. Description of the geometry. In this section we compute some special coordinate systems in the regions where $\langle dr, dr \rangle \approx 0$ and where $\langle dr, dr \rangle > 0$.

First, recall that according to part ii) of Definition 1.1 there exists a unique $r_M > r_0$ with the property that $g^{rr} = \langle dr, dr \rangle|_{r=r_M} = 0$. Furthermore, the assumption that $\partial_r g^{rr} \neq 0$ throughout $r \geq r_0$ implies $\partial_r g^{rr}|_{r=r_M} > 0$. This is the key non-degeneracy condition which leads to good estimates in the region $\langle dr, dr \rangle \approx 0$. To capture it we make the following definition:

Definition 5.18 (Negatively N-boosted null pairs). *Let $\langle \cdot, \cdot \rangle$ denote the (1 + 1) Lorenzian inner product of h . For a fixed number $N > 0$ we call a pair of vector-fields L, \underline{L} , defined over a time independent neighborhood \mathcal{H} of $r = r_M$, a “negatively N-boosted null pair” if there exists a fixed $c = c(N) > 0$ such that the following hold:*

- i) (Basic relations) *One has the relations $\langle L, L \rangle = \langle \underline{L}, \underline{L} \rangle = 0$ and $\langle L, \underline{L} \rangle = -2$. Both L and \underline{L} are future directed in the sense that $Lt, \underline{L}t > 0$. In addition \underline{L} is incoming in the sense that $\underline{L}r < -c < 0$.*
- ii) (Stationarity) *The frame is stationary, that is $[\partial_t, L] = [\partial_t, \underline{L}] = 0$.*
- iii) (Boosting condition) *One has $\nabla_{\underline{L}} \underline{L} = 2N \underline{\chi} \underline{L}$, where $N > 0$ is as above and $\underline{\chi} = \underline{L}r$.*

Remark 5.19. *Condition iii) above is malleable in the sense that one can construct pairs of vector-fields L, \underline{L} which satisfy i) and ii), and which have $\nabla_{\underline{L}} \underline{L} = \underline{\sigma} \underline{L}$, where $\underline{\sigma}$ is an arbitrary function of r . Our choice of constant in iii) is for later convenience when constructing multipliers.*

In terms of such null pairs close to $r = r_M$ we have the following result, which follows closely the presentation of Dafermos-Rodnianski in [10]:

Lemma 5.20 (Description of the geometry close to $r = r_M$). *Fix a number $N > 0$. Then under the assumptions of Definition 1.1 there exists a (time independent) neighborhood \mathcal{H} of $r = r_M$ and a negatively N-boosted null pair defined over \mathcal{H} as in Definition 5.18 above. Furthermore, for any such null pair there exists a constant $c = c(N) > 0$ such that:*

- i) (Relation to ∂_t) *One may write $\partial_t = q_+ L + q_- \underline{L}$ where $q_+ > c$, and where $q_- = \lambda g^{rr}$ for some weight function $\lambda(r)$ with $\lambda > c$ as well.*
- ii) (Red Shift) *One has the identity $\nabla_{\underline{L}} L = \sigma L$ for some smooth function $\sigma > c$.*

iii) (Area variations) Define $\underline{\chi} = \underline{L}r$ and $\chi = Lr$. Then in addition to $\underline{\chi} < -c$, one also has $\chi = \gamma g^{rr}$ for some smooth function $\gamma > c$.

It remains to discuss the geometry in the region $r > r_M$. For this we have:

Lemma 5.21 (Conformal coordinates in $r > r_M$). Let $h_{ab} = g_{ab}$ denote the radial part of the Lorentzian metric from Definition 1.1. For the statement of this lemma we also set $|h| = |\det(h)|$, where the determinant is computed in regular (t, r) coordinates. Then there exist two functions $s = t + b(r)$ and $r_* = r_*(r)$ defined in the region $r > r_M$, such that $r_*(r_M) = 0$, and such that (s, r_*) are smooth coordinates in $r > r_M$ with:

$$r_* \rightarrow -\infty \text{ as } r \rightarrow r_M, \quad r_* \rightarrow \infty \text{ as } r \rightarrow \infty.$$

Furthermore the following properties hold:

i) (Asymptotics in $r_M < r < C$) One has (note $g^{rr} = h^{rr}$):

$$\partial_r s = -|h|^{-\frac{1}{2}}(g^{rr})^{-1} + \tilde{s}(r), \quad \partial_r r_* = |h|^{-\frac{1}{2}}(g^{rr})^{-1},$$

where \tilde{s} is uniformly bounded with all of its derivatives on $r > r_M$. Here all derivatives are taken with respect to regular (t, r) coordinates as in Definition 1.1.

ii) (Asymptotics as $r \rightarrow \infty$) As $r \rightarrow \infty$ one has the asymptotics:

$$\partial_r^k s = O(r^{-k}), \quad k \geq 1,$$

as well as:

$$\partial_r r_* = 1 + O(r^{-1}), \quad \partial_r^k r_* = O(r^{-k}), \quad k \geq 2.$$

Again everything is computed with respect to regular (t, r) coordinates.

iii) (Conformal form) With respect to (s, r_*) the $(1+1)$ dimensional Lorentzian metric h can be written as:

$$h = \Omega^2(-ds^2 + dr_*^2), \quad \Omega^2 = -g_{00} = |h|g^{rr}.$$

iv) (Trapped set) The unique trapped set of part iii) of Definition 1.1 is at $r_* = 0$. At $r_* = 0$ one has $V'(0) = 0$ and $V''(0) < 0$, where $V = V(r_*)$ is the effective potential $V = r^{-2}\Omega^2$. Moreover, $V' \neq 0$ for $r_* \neq 0$.

v) (Asymptotics of $V(r_*)$) Finally, $V(r_*)$ has the following asymptotic formulas some fixed $c > 0$:

$$\partial_{r_*}^k V(r_*) \approx (r - r_M) \approx e^{cr_*} \text{ as } r_* \rightarrow -\infty, \quad (-1)^k \partial_{r_*}^k V(r_*) \approx r^{-2-k} \approx r_*^{-2-k} \text{ as } r_* \rightarrow \infty.$$

Proof of Lemma 5.20. We proceed in a series of steps:

Step 1: (Construction of the null pair with boosting condition) First we show the existence of L, \underline{L} according to Definition 5.18. We denote by (t, r) any regular system of coordinates as in Definition 1.1, and by $h_{ab} = g_{ab}$ the components of the radial part of g . Set $T = -\nabla t$ the future directed time function gradient. Then we have:

$$\langle T, T \rangle = h^{00} < 0, \quad \langle T, \partial_r \rangle = 0, \quad \langle \partial_r, \partial_r \rangle = h_{rr} > 0.$$

From these one can form a pair of null vectors:

$$\ell = T + \sqrt{-h^{00}/h_{rr}}\partial_r, \quad \underline{\ell} = T - \sqrt{-h^{00}/h_{rr}}\partial_r.$$

Immediately we have $\ell t = \underline{\ell} t = -h^{00} > 0$. In addition note that $\langle \ell, \partial_r \rangle > 0$, thus ℓ is the outgoing null direction and in particular $\ell r|_{r=r_M} = 0$. This also shows $Tr|_{r=r_M} < 0$. Choosing $L = \lambda\ell$ and $\underline{L} = \underline{\ell}$ for an appropriate positive weight function $\lambda(r)$ we immediately have the inner product and future direction properties and of part i) of Definition 5.18, as well as the stationarity condition ii). Furthermore, one computes $\underline{L}r = Tr - \sqrt{-h^{00}/h_{rr}} < 0$ where the inequality holds in a neighborhood of $r = r_M$.

To establish part iii) of Definition 5.18 we remark that any null pair obeying parts i)-ii) is invariant with respect to boosting $L \rightarrow q^{-1}L$ and $\underline{L} \rightarrow q\underline{L}$ for a smooth positive weight function q . There exists a function $\underline{\sigma}$ such that $\nabla_{\underline{L}}\underline{L} = \underline{\sigma}\underline{L}$. Set $\tilde{\underline{L}} = q\underline{L}$ and define $\tilde{\underline{\sigma}}$ accordingly. Then a quick computation shows that:

$$\tilde{\underline{\sigma}} = \underline{\chi}q' + \underline{\sigma}q, \quad \text{where } \underline{\chi} = \underline{L}r.$$

Setting $\tilde{\underline{\sigma}} = 2N\tilde{\underline{L}}r = 2Nq\underline{\chi}$ gives the following ODE for $q(r)$:

$$q' = (2N - (\underline{\chi})^{-1}\underline{\sigma})q,$$

which is well defined on account of the condition $\underline{\chi} \neq 0$. Such an equation can always be solved on \mathcal{H} subject to the constraint $q > 0$.

Step 2:(*Relation to ∂_t*) To prove part i) of Lemma 5.20 we set q_{\pm} to be the coefficients of ∂_t in the basis $\{L, \underline{L}\}$, and note that $q_{\pm} > 0$ where ∂_t is time-like. Then one has:

$$(5.38) \quad q_+ q_- = -\frac{1}{4} h_{00} = -\frac{1}{4} \det(h) g^{rr} .$$

By assumption ii) of Definition 1.1 the RHS is a function with simple zero at $r = r_M$. Thus only one of q_{\pm} can vanish, and it does so with a simple zero at $r = r_M$. But $r = r_M$ is a null hypersurface with generator proportional to L and on this hypersurface ∂_t is also null (see part b) of Remark 1.2). Thus $\partial_t|_{r=r_M} = cL|_{r=r_M}$ for some $c > 0$, which shows that q_- must have a simple zero at $r = r_M$ and $q_+ > 0$ throughout \mathcal{H} . Referring back to (5.38) gives the formula for q_- in terms of g^{rr} and a non-vanishing weight.

Step 3:(*The “red shift”*) Using the results of the previous steps we have:

$$\sigma = -\frac{1}{2} \langle \nabla_L L, \underline{L} \rangle = \frac{1}{2} \langle \nabla_L \underline{L}, L \rangle = \frac{1}{2} q_+^2 \langle \nabla_{\partial_t} \underline{L}, \partial_t \rangle + O(r - r_M) = \frac{1}{4} q_+^2 \underline{L} h_{00} + O(r - r_M) .$$

For the first term on the RHS we further compute:

$$\underline{L} h_{00} = \underline{L}(r) \partial_r (\det(h)^{-1} h^{rr}) = \underline{L}(r) \det(h)^{-1} \partial_r g^{rr} + O(r - r_M) > 0 , \quad \text{if } |r - r_M| \ll 1 .$$

Step 4:(*Area variation*) In the construction of L, \underline{L} we have already shown that $\underline{L}r < -c$. Using **Step 2** above we compute:

$$0 = \partial_t r = q_+ Lr + \lambda g^{rr} \underline{L}r .$$

The desired result follows by solving for Lr and using the already established properties of $\lambda, q_+, \underline{L}r$. \square

Next, we construct conformal coordinates in the region $r > r_M$.

Proof of Lemma 5.21. Recall that our (1 + 1) metric h as the form $h = g_{ab} dx^a dx^b$ with the conditions of Definition 1.1. We let g_{00}, g_{0r}, g_{rr} denote the components of h in the original (non-singular) (t, r) coordinates. As in the statement of Lemma 5.21 we also set $|h| = |\det(h)|$ computed with respect to (t, r) .

Step 1:(*Normalized coordinates*) To normalize things in the region $g^{rr} > 0$ we first introduce a singular time function whose level sets are perpendicular to the Killing field ∂_t . Setting $t = s - b(r)$ we have in the (s, r) coordinates:

$$h = g_{00} ds^2 + 2(g_{0r} - g_{00} b') ds dr + (g_{rr} + g_{00} (b')^2 - 2g_{0r} b') dr^2 .$$

Now choose $b(r)$ so that $g_{00} b' = g_{0r}$ and $b(r_{\mathcal{T}}) = 0$, so our metric takes the diagonal form:

$$(5.39) \quad h = h_{ss} ds^2 + h_{rr} dr^2 ,$$

where:

$$(5.40) \quad h_{ss} = g_{00} , \quad h_{rr} = g_{rr} - (g_{0r})^2 / g_{00} ,$$

which is now only defined in the region $g_{00} < 0 \Leftrightarrow r > r_M$. We remark that this change of coordinates does not affect the determinant of the metric, and in either case we have:

$$(5.41) \quad |h| = -h_{ss} h_{rr} = (g_{0r})^2 - g_{00} g_{rr} .$$

We refer to (5.39) as the “Schwarzschild form” of the metric h .

Next, we make an additional transformation to put the metric in conformal form by defining $r_* = r_*(r)$ according to the ODE:

$$(5.42) \quad \frac{dr_*}{dr} = \sqrt{-h_{rr}/h_{ss}} , \quad r_*(r_{\mathcal{T}}) = 0 .$$

Using the identity $h_{ss} = -|h|g^{rr}$ and the fact that $h_{rr} = 1/g^{rr}$ the first identity on this last line becomes $\partial_r r_* = |h|^{-\frac{1}{2}} (g^{rr})^{-1}$, which gives the second formula in part i) of the Lemma and immediately yields the rough asymptotics:

$$\frac{dr_*}{dr} \approx 1/g^{rr} \approx 1/(r - r_M) , \quad r \rightarrow r_M , \quad \frac{dr_*}{dr} \approx 1 , \quad r \rightarrow \infty .$$

Thus $r_* \rightarrow -\infty$ as $r \rightarrow r_M$ and $r_* \rightarrow \infty$ as $r \rightarrow \infty$, and our metric takes the final form:

$$(5.43) \quad h = \Omega^2(-ds^2 + dr_*^2), \quad \Omega^2 = -\langle \partial_t, \partial_t \rangle_g, \quad -\infty < r_* < \infty \iff r > r_M.$$

We call this the ‘‘Regge-Wheeler form’’ of h .

Step 2: (*Asymptotics of s as $r \rightarrow r_M$*) Here we derive the first formula in i) of the Lemma. From the formulas of the previous step we have:

$$\partial_r s = g_{0r}/g_{00}, \quad \text{where } g_{00} = -|h|g^{rr}, \quad \text{and } g_{0r} = \sqrt{|h| + g_{00}g_{rr}}.$$

This allows us to write:

$$\partial_r s = -|h|^{-\frac{1}{2}}(g^{rr})^{-1} \cdot (1 - f(r, g^{rr})), \quad f(r, \theta) = 1 - \sqrt{1 - \theta g_{rr}}.$$

Thus $f(r, \theta) = \theta \tilde{f}(r, \theta)$ when $\theta \approx 0$, for some other smooth \tilde{f} , and the desired result follows.

Step 3: (*Asymptotics of (s, r_*) as $r \rightarrow \infty$*) The result for s follows immediately from $\partial_r^k(g_{ab}) = O(r^{-k})$ and $(g_{ab} - \eta_{ab}) = O(r^{-1})$, where $\eta = \text{diag}(-1, 1)$. The result for r_* also follows from these asymptotics for g_{ab} and the formulas on lines (5.40) and (5.42).

Step 4: (*Description of the trapped set at $r = r_{\mathcal{T}}$*) The Regge-Wheeler form (5.43) is particularly convenient for discussing the trapped null geodesics of g in the region $r > r_M$. Since conformal metrics have the same null geodesic flow, it suffices to analyze the direct sum metric:

$$\tilde{g} = -ds^2 + dr_*^2 + \Omega^{-2}r^2 d\omega^2, \quad d\omega^2 = \delta_{AB} dx^A dx^B.$$

The projections of unit speed null geodesics for this metric onto the space parametrized by (r_*, x^A) , where $A = 2, 3$ denote coordinates on \mathbb{S}^2 , are exactly the unit speed geodesics of the three dimensional Riemannian surface of rotation $dl^2 = dr_*^2 + \Omega^{-2}r^2 d\omega^2$. The Hamiltonian for the corresponding geodesic flow is $p(r_*, x^A, \xi_{r_*}, \xi_A) = \frac{1}{2}(\xi_{r_*}^2 + \Omega^2 r^{-2} |\xi|^2)$, where $|\xi|^2 = \delta^{AB} \xi_A \xi_B$ is the Hamiltonian of the standard geodesic flow on \mathbb{S}^2 .

The equations for a trapped sphere of unit speed geodesics at $r = r_0$ now take the form:

$$\dot{r}_*|_{r=r_0} = \{p, r_*\}|_{r=r_0} = 0,$$

where $\{p, f\} = \partial_\xi p \partial_x f - \partial_x p \partial_\xi f$ is the Poisson bracket. In general we have:

$$\dot{r}_* = \xi_{r_*}, \quad \dot{\xi}_{r_*} = -\frac{1}{2} \partial_{r_*}(V) |\xi|^2, \quad \text{where } V = \Omega^2 r^{-2},$$

and thus $r = r_0$ is trapped iff $\xi_{r_*} = \dot{\xi}_{r_*} = 0$ at $r = r_0$, which can happen iff $V'(r_0) = 0$. Therefore, condition iii) of Definition 1.1 implies that $\partial_{r_*} V = 0$ iff $r_* = 0$ in the region $-\infty < r_* < \infty$. It is also manifestly clear that $V > 0$ in that region with $V \rightarrow 0$ as $r_* \rightarrow \pm\infty$. In other words $V(r_*)$ is a positive repulsive potential with unique maximum at $r_* = 0$.⁶ Finally, the non-degeneracy condition in iii) of Definition 1.1 means that $V''(0) \neq 0$, so in fact $V''(0) < 0$. Summarizing this, we may write the normal part of the geodesic flow at $r = r_{\mathcal{T}}$ in the form of a standard planar hyperbolic⁷ fixed point:

$$\dot{r}_* = \xi_r, \quad \dot{\xi}_r = \lambda_0 |\xi|^2 r_* + O(|\xi|^2 r_*^2), \quad \lambda_0 = -\frac{1}{2} V''(0) > 0.$$

Step 5: (*Asymptotics of V*) Finally the asymptotic formulas of part v) for $V(r_*)$ as $r_* \rightarrow \pm\infty$ follows from the explicit formula above and the fact that $r_* = A \ln(r - r_M) + O(1)$ as $r_* \rightarrow -\infty$, and $r_* = r + B \ln(r) + O(r^{-1})$ as $r_* \rightarrow \infty$, where $A > 0$ and B are constant. Further details are left to the reader. \square

5.2. The multiplier method for \square^0 . Before embarking on the proof of Theorem 5.14 we pause to introduce some notation and identities that will be useful for the remainder of the section. For the wave equation \square^0 one has a $(1 + 1)$ energy momentum tensor:

$$Q_{ab}[\phi] = \int_{\mathbb{S}^2} [\partial_a \phi \partial_b \phi - \frac{1}{2} h_{ab} (\partial^c \phi \partial_c \phi + r^{-2} |\not\partial \phi|^2)] dV_{\mathbb{S}^2}.$$

In terms of any null pair basis L, \underline{L} one can write its components as:

$$(5.44) \quad Q_{LL}[\phi] = \int_{\mathbb{S}^2} (L\phi)^2 dV_{\mathbb{S}^2}, \quad Q_{\underline{L}\underline{L}}[\phi] = \int_{\mathbb{S}^2} (\underline{L}\phi)^2 dV_{\mathbb{S}^2}, \quad Q_{L\underline{L}}[\phi] = \int_{\mathbb{S}^2} |\not\partial \phi|^2 dV_{\mathbb{S}^2}.$$

⁶This also immediately implies there are no other trapped geodesics except at $r = r_{\mathcal{T}}$.

⁷In particular the assumption of a unique non-degenerate trapped set implies such a trapped set must be normally hyperbolic.

This object behaves similarly to the energy momentum tensor for $\square_h + V$ where $V = -k^2 r^{-2}$. Specifically one has:

$$\nabla^a Q_{ab}[\phi] = \int_{\mathbb{S}^2} \square^0 \phi \partial_b \phi dV_{\mathbb{S}^2} + r^{-3} \partial_b(r) \int_{\mathbb{S}^2} |\not\partial \phi|^2 dV_{\mathbb{S}^2} .$$

In the usual way this leads to momentum identities for any (spherically symmetric) vector-field $X = X^a \partial_a$:

$$(5.45) \quad \nabla^a ({}^X P_a[\phi]) = \frac{1}{2} \int_{\mathbb{S}^2} Q_{ab}[\phi] ({}^X \pi^{ab}) dV_{\mathbb{S}^2} + r^{-3} X(r) \int_{\mathbb{S}^2} |\not\partial \phi|^2 dV_{\mathbb{S}^2} + \int_{\mathbb{S}^2} \square^0 \phi X \phi dV_{\mathbb{S}^2} , \quad ({}^X P_a[\phi]) = Q_{ab}[\phi] X^b ,$$

where $({}^X \pi_{ab}) = \mathcal{L}_X h_{ab} = \nabla_a X_b + \nabla_b X_a$ is the $(1+1)$ deformation tensor of X .

There are two types of multipliers we will work with:

$$(5.46) \quad X = X^- \underline{L} , \quad Y = Y^0 \partial_t ,$$

for given weight functions $X^-(r), Y^0(r)$. We record their raised deformation tensors here:

Lemma 5.22 (Deformation tensors). *Let X, Y be the multipliers defined on line (5.46). Then with the notation of Definition 5.18 and Lemma 5.20 we have:*

$$(5.47) \quad ({}^X \pi^{LL}) = 0 , \quad ({}^X \pi^{\underline{L}\underline{L}}) = -\chi \cdot (X^-)' + \sigma X^- \quad ({}^X \pi^{L\underline{L}}) = -\frac{1}{2} \underline{\chi} \cdot (X^-)' - N \underline{\chi} X^-$$

$$(5.48) \quad ({}^Y \pi^{LL}) = -\underline{\chi} \cdot q_+(Y^0)' , \quad ({}^Y \pi^{\underline{L}\underline{L}}) = -\chi \cdot q_-(Y^0)' , \quad ({}^Y \pi^{L\underline{L}}) = 0 .$$

and where we have set $\chi = Lr$ and $\underline{\chi} = \underline{L}r$. Close to $r = r_M$ one has the sign relations:

$$(5.49) \quad \chi \approx (r - r_M) , \quad \underline{\chi} \approx -1 , \quad \sigma \approx 1 , \quad q_+ \approx 1 , \quad q_- \approx (r - r_M) .$$

Proof of Lemma 5.22. By raising indices we have $({}^X \pi^{LL}) = \frac{1}{4} ({}^X \pi_{\underline{L}\underline{L}})$, $({}^X \pi^{L\underline{L}}) = \frac{1}{4} ({}^X \pi_{LL})$, and $({}^X \pi^{\underline{L}\underline{L}}) = \frac{1}{4} ({}^X \pi_{L\underline{L}})$. Similarly for $({}^Y \pi^{ab})$. For $X = X^- \underline{L}$ we have:

$$({}^X \pi_{ab}) = e_a(X^-) \langle \underline{L}, e_b \rangle + e_b(X^-) \langle \underline{L}, e_a \rangle + X^- \langle \nabla_{e_a} \underline{L}, e_b \rangle + X^- \langle \nabla_{e_b} \underline{L}, e_a \rangle .$$

Using the orthogonality of \underline{L} to itself we immediately have $({}^X \pi_{\underline{L}\underline{L}}) = 0$, while:

$$(5.50) \quad ({}^X \pi_{LL}) = -4L(X^-) - 2X^- \langle \nabla_L L, \underline{L} \rangle , \quad ({}^X \pi_{L\underline{L}}) = -2\underline{L}(X^-) + X^- \langle \nabla_{\underline{L}} \underline{L}, L \rangle .$$

This immediately implies (5.47).

Now let $Y = Y^0 \partial_t$, for which we have:

$$({}^Y \pi_{ab}) = e_a(Y^0) \langle \partial_t, e_b \rangle + e_b(Y^0) \langle \partial_t, e_a \rangle ,$$

where we have used $({}^{\partial_t} \pi_{ab}) = \langle \nabla_a \partial_t, e_b \rangle + \langle \nabla_b \partial_t, e_a \rangle = 0$. Using the decomposition $\partial_t = q_+ L + q_- \underline{L}$ this immediately gives the first two identities on (5.48). For the last term on that line we have:

$$L(Y^0) \langle \partial_t, \underline{L} \rangle + \underline{L}(Y^0) \langle \partial_t, L \rangle = -2(q_+ L(Y^0) + q_- \underline{L}(Y^0)) = -2\partial_t Y^0 = 0 .$$

□

5.3. Local energy decay component estimates for \square^0 . Proof of Theorem 5.14. We break our proof of Theorem 5.14 down into the three constituents in the next proposition. For this purpose we consider thresholds

$$r_0 \leq r_1 < r_M < r_2 < r_3 < r_{\mathcal{T}}$$

We think of r_3 as fixed, away from r_M and $r_{\mathcal{T}}$. On the other hand r_1 and r_2 are free to float, and we reserve the right to choose them arbitrarily close to r_M . For the statement of our main technical result we let $\chi_{[a,b]}(r)$ denote the indicator function (sharp cutoff) of the interval $a \leq r \leq b$.

Proposition 5.23 (Three modular local energy decay estimates). *Let r_1, r_2, r_3 be three parameters such that $r_0 \leq r_1 < r_M < r_2 < r_3 < r_{\mathcal{T}}$. Then there exists a fixed value of r_3 such that for every $k > 1$ one has*

the following three estimates for test functions ϕ , where the implicit constants are uniform in r_1, r_2 (but may depend on r_3, k):

$$(5.51) \quad \|\chi_{[r_1, r_3]}(\underline{L}\phi, \not{d}\phi)\|_{LE_0[0, T]} \lesssim \|\chi_{[r_3, r_{\mathcal{T}}]}(r - r_{\mathcal{T}})\not{d}\phi\|_{LE_0[0, T]} + E_{\geq r_1}^{\frac{1}{2}}(\phi[0]) + \|\chi_{[r_1, r_{\mathcal{T}}]}\square^0\phi\|_{LE_0^*[0, T]},$$

$$(5.52) \quad \|\chi_{[r_1, r_2]}w_k L\phi\|_{LE_0[0, T]} \lesssim \epsilon \|\chi_{[r_1, \infty)}(w_{ln})^{-1}\partial_t\phi\|_{LE_0[0, T]} + \|\chi_{[r_1, r_2]}w_k(r - r_M)\underline{L}\phi\|_{LE_0[0, T]} \\ + (r_M - r_1)^{\frac{1}{2}}\|\chi_{[r_3, r_{\mathcal{T}}]}(r - r_{\mathcal{T}})\not{d}\phi\|_{LE_0[0, T]} + E_{\geq r_1}^{\frac{1}{2}}(\phi[0]) + \epsilon^{-1}\|\chi_{[r_1, \infty)}w_{ln}\square^0\phi\|_{LE_0^*[0, T]}.$$

In addition there exists a $C_{r_2} > 0$ such that (again with implicit constant uniform in r_1, r_2):

$$(5.53) \quad \|\chi_{[r_2, r_3]}w_k(L\phi, (r - r_M)\underline{L}\phi)\|_{LE_0[0, T]} + \|\chi_{[r_2, r_3]}\not{d}\phi\|_{LE_0[0, T]} \\ + \|\chi_{[r_3, \infty)}\partial_r\phi\|_{LE_0[0, T]} + \|\chi_{[r_3, \infty)}(w_{ln})^{-1}(\partial_t\phi, r^{-1}\not{d}\phi)\|_{LE_0[0, T]} \\ \lesssim |\ln(r_2 - r_M)|^k \|\chi_{[r_M, r_2]}w_k(L\phi, (r - r_M)\underline{L}\phi)\|_{LE_0[0, T]} + C_{r_2} \left(\|\chi_{[r_M, \infty)}w_{ln}\square^0\phi\|_{LE_0^*[0, T]} + E_{\geq r_M}^{\frac{1}{2}}(\phi[0]) \right).$$

The weight functions w_k and w_{ln} in estimates (5.52) and (5.53) are:

$$w_{ln} = (1 + |\ln|r - r_{\mathcal{T}}||)/(1 + |\ln(r)|), \quad w_k(r) = (1 + |\ln|r - r_M||)^{-\frac{1}{2}k} |r - r_M|^{-\frac{1}{2}}.$$

In estimate (5.53) one must also take $\langle \partial_t, \partial_r \rangle = 0$ when $r = r_{\mathcal{T}}$.

Remark 5.24. The purpose of the weight w_{ln} is to account for the degeneracy of local energy decay near the trapped set $r = r_{\mathcal{T}}$. On the other hand the weight w_k is only used in a neighborhood of the horizon $r = r_M$ and accounts for the fact that the ‘‘good component’’ $L\phi$ has a non-degenerate energy there.

Before giving a proof of the above three estimates, we use them to give a short demonstration of Theorem 5.14.

Proof that Proposition 5.23 implies estimate (5.36). First note that while the LE_0 estimate in (5.36) takes place over the region $r \geq r_0$, one can easily reduce it to showing the same bound for $r \geq r_1$, as the region $r_0 \leq r \leq r_1$ is easily covered by known LE_0 bounds on $r \geq r_1$ and energy estimates.

For constants $c, C > 0$ we add together:

$$c \times (5.51) + C |\ln(r_2 - r_M)|^k \times (5.52) + (5.53).$$

Then choose c, C, ϵ, r_1, r_2 so that:

$$c + C^{-1} \ll 1, \quad \text{followed by} \quad |r_1 - r_2|^{\frac{1}{2}} + \epsilon \ll cC^{-1} |\ln(r_2 - r_M)|^{-k}.$$

This allows one to move all local energy decay errors for ϕ to the LHS of the resulting inequality. \square

We now give a proof of the first two estimates in Proposition 5.23. These are based directly on the multiplier calculations of the previous subsection. The last estimate (5.53) is more involved and will be the topic of the next subsection.

Proof of estimate (5.51). For the purposes of showing estimate (5.52) next, it will help to prove a slightly more informative version of (5.51). Let L, \underline{L} be a negatively N-boosted null pair (according to Definition 5.18), and r_1, r_3 sufficiently close to r_M so that Lemma 5.20 holds on a time independent neighborhood \mathcal{H} of $\mathbb{R}_t \times [r_1, r_3]$ with $r = r_{\mathcal{T}}$ to the exterior of \mathcal{H} .

Next, we choose the X multiplier from line (5.46) with $X^-(r) = \varphi(r)$ where $\varphi(r)$ is a smooth non-negative function with $\varphi \equiv 1$ on $[r_1, r_3]$, $\varphi' \leq 0$, and $\varphi \equiv 0$ for r outside of \mathcal{H} . Denote by \mathcal{S} the space-time slab $\{r \geq r_1\} \times \{0 \leq t \leq T\}$, and by $\mathcal{S}(t_0)$ the section $\{t = t_0\} \cap \mathcal{S}$ and similarly for $\mathcal{S}(r_0)$. Then forming the momentum density $^{(X)}P$ as on line (5.45), and integrating over the \mathcal{S} , we have from Stokes’ theorem the identity:

$$\int_{\mathcal{S}(t_0)} \varphi Q_{T\underline{L}} \sqrt{|h|} dr dV_{S^2} + \int_{\mathcal{S}(r_1)} Q_{R\underline{L}} \sqrt{|h|} dV_{S^2} dt + \frac{1}{2} \int_{\mathcal{S}} \varphi^{(X)} \pi^{\underline{L}\underline{L}} (\underline{L}\phi)^2 \sqrt{|h|} dr dV_{S^2} dt \\ + \int_{\mathcal{S}} \varphi r^{-2} (-\underline{\chi})(N - r^{-1}) |\not{d}\phi|^2 \sqrt{|h|} dr dV_{S^2} dt = \int_{\mathcal{S}(0)} \varphi Q_{T\underline{L}} \sqrt{|h|} dr dV_{S^2} + \frac{1}{2} \int_{\mathcal{S}} r^{-2} \underline{\chi} \varphi' |\not{d}\phi|^2 \sqrt{|h|} dr dV_{S^2} dt \\ - \int_{\mathcal{S}} \varphi \square^0 \phi \underline{L}\phi \sqrt{|h|} dr dV_{S^2} dt.$$

where we have set $T = -\nabla t$ and $R = \nabla r = -\frac{1}{2}\chi\underline{L} - \frac{1}{2}\chi L$, which are the future pointing normals to $t = 0, t_0$ and $r = r_1$. Because the level sets $t = \text{const}$ are uniformly space-like we also know that $T = T^+L + T^-\underline{L}$ where $T^\pm \approx 1$ on the support of φ . Thus, for $N > 0$ sufficiently large, the previous identity yields the following uniform estimate where the implicit constant does not depend r_1, r_2 :

$$(5.54) \quad \begin{aligned} & \|\chi_{[r_1, r_3]}(\underline{L}\phi, \not\partial\phi)|_{t=T}\|_{L^2(dr dV_{\mathbb{S}^2})} + \|((r_M - r_1)^{\frac{1}{2}}\underline{L}\phi, \not\partial\phi)|_{r=r_1}\|_{L^2(dV_{\mathbb{S}^2} dt)[0, T]} \\ & + \|\chi_{[r_1, r_3]}(\underline{L}\phi, \not\partial\phi)\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T]} \lesssim \|\chi_{[r_3, r_{\mathcal{T}}]}(r - r_{\mathcal{T}})\not\partial\phi\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T]} \\ & + \|\chi_{[r_1, r_{\mathcal{T}}]}(\underline{L}\phi, \not\partial\phi)|_{t=0}\|_{L^2(dr dV_{\mathbb{S}^2})} + \|\chi_{[r_1, r_{\mathcal{T}}]}\square^0\phi\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T]} . \end{aligned}$$

Disregarding the first two terms on the LHS yields the desired result. \square

Proof of estimate (5.52). Here we use the same setup as in the first paragraph of the previous proof. This time we use the Y multiplier from line (5.46) with $Y^- = q(r)$ where q is a non-negative function such that $q' = w_k^2 \chi_{[r_1, r_2]}$. Thus we have the global bound $C > q > c > 0$ for some set of constants c, C . Applying (5.45) with $(Y)P$ yields:

$$\begin{aligned} & \int_{S(t_0)} q Q_{T0} \sqrt{|h|} dr dV_{\mathbb{S}^2} + \int_{S(r_1)} q Q_{R0} \sqrt{|h|} dV_{\mathbb{S}^2} dt + \frac{1}{2} \int_S (-\chi) q_+ w_k^2 \chi_{[r_1, r_2]} (L\phi)^2 \sqrt{|h|} dV_{\mathbb{S}^2} dr dt \\ & = \frac{1}{2} \int_S \chi q_- w_k^2 \chi_{[r_1, r_2]} |\underline{L}\phi|^2 \sqrt{|h|} dr dV_{\mathbb{S}^2} dt + \int_S q Q_{T0} \sqrt{|h|} dr dV_{\mathbb{S}^2} - \int_S q \square^0 \phi \partial_t \phi \sqrt{|h|} dr dV_{\mathbb{S}^2} dt . \end{aligned}$$

Expanding the boundary terms using part i) of Lemma 5.20 we get:

$$\begin{aligned} & \|\chi_{[r_1, r_2]} w_k L\phi\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T]} \lesssim \|\chi_{[r_1, r_2]} w_k (r - r_M) \underline{L}\phi\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T]} \\ & + (r_M - r_1)^{\frac{1}{2}} \|\chi_{[r_1, r_M]} \underline{L}\phi|_{t=T}\|_{L^2(dr dV_{\mathbb{S}^2})} + (r_M - r_1)^{\frac{1}{2}} \|((r_M - r_1)^{\frac{1}{2}} \underline{L}\phi, \not\partial\phi)|_{r=r_1}\|_{L^2(dV_{\mathbb{S}^2} dt)[0, T]} \\ & + \|(\underline{d}\phi, r^{-1} \not\partial\phi)|_{t=0}\|_{L^2(dr dV_{\mathbb{S}^2})} + \epsilon^{-1} \|\chi_{[r_1, \infty)} w_{ln} \square^0 \phi\|_{LE_0^*[0, T]} + \epsilon \|\chi_{[r_1, \infty)} (w_{ln})^{-1} \partial_t \phi\|_{LE_0[0, T]} . \end{aligned}$$

By itself this estimate is not yet of the form (5.52). However, upon combination with $(r_M - r_1)^{\frac{1}{2}} \times (5.54)$ the future boundary terms at $t = T$ and $r = r_1$ on the RHS of the last line above can be traded for the space-time error $(r_M - r_1)^{\frac{1}{2}} \|\chi_{[r_3, r_{\mathcal{T}}]}(r - r_{\mathcal{T}})\not\partial\phi\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T]}$ which then yields the desired result. \square

5.4. Proof of the main component estimate (5.53). This estimate is most easily recast in terms of the Regge-Wheeler type coordinates of Lemma 5.21. We define the space LE_{RW} and E_{RW} by:

$$\begin{aligned} \|\phi\|_{LE_{RW}[0, T]} & = \sup_{j \geq 0} \|2^{-\frac{1}{2}j} \chi_j \phi\|_{L^2(dr_* dV_{\mathbb{S}^2} ds)[0, T]} , \\ E_{RW}(\phi[s_0]) & = \|(\partial_s \phi, \partial_{r_*} \phi, r^{-\frac{3}{2}}(r - r_M)^{\frac{1}{2}} \not\partial\phi)|_{s=s_0}\|_{L^2(dr_* dV_{\mathbb{S}^2})} , \end{aligned}$$

where χ_j for $j \geq 1$ cuts off smoothly where $|r_*| \approx 2^j$, and where χ_0 is supported where $|r_*| \lesssim 1$. For this section all estimates will be in terms of a rescaled version of \square^0 , which in Regge-Wheeler coordinates is:

$$\square_{RW}^0 = -\partial_s^2 + \partial_{r_*}^2 + V(r_*) \not\Delta = -\langle \partial_t, \partial_t \rangle_h \square^0 , \quad V = -r^{-2} \langle \partial_t, \partial_t \rangle_h .$$

Then our main estimate here reads:

Proposition 5.25 (Local energy decay estimates for \square_{RW}^0). *Let ϕ be a smooth function which vanishes for $r_* \rightarrow \pm\infty$. Then one has the uniform bound:*

$$(5.55) \quad \begin{aligned} & \|(\partial_{r_*} \phi, (w_{ln})^{-1} \partial_s \phi)\|_{LE_{RW}[0, T]} + \|(w_{ln})^{-1} r^{-2} (r - r_M)^{\frac{1}{2}} \not\partial\phi\|_{L^2[0, T]} \\ & \lesssim E_{RW}^{\frac{1}{2}}(\phi[0]) + \|w_{ln} \square_{RW}^0 \phi\|_{LE_{RW}^*[0, T]} , \end{aligned}$$

where $w_{ln} = (1 + |\ln|r_*|)/|\ln(2 + |r_*|)|$.

In addition, if $\bar{\phi}$ is any spherically symmetric function, one has the following estimate uniform in $r_*^0 < -1$:

$$(5.56) \quad \begin{aligned} & \|\chi_{[r_*^0, \infty)}(\partial_s \bar{\phi}, \partial_{r_*} \bar{\phi})\|_{LE_{RW}[0, T]} \lesssim \|\chi_{[2r_*^0, r_*^0]}(\partial_s \bar{\phi}, \partial_{r_*} \bar{\phi})\|_{LE_{RW}[0, T]} \\ & + E_{RW, r_* > 2r_*^0}^{\frac{1}{2}}(\bar{\phi}[0]) + \|\chi_{[2r_*^0, \infty)} \square_{RW}^0 \bar{\phi}\|_{LE_{RW}^*[0, T]} , \end{aligned}$$

Before giving a proof we use these estimate to establish (5.53).

Proof that estimates (5.55) and (5.56) implies (5.53). Given a solution of $\square^0 \phi = G$, will treat the spherically symmetric part $\bar{\phi}$ separately from the non-spherical part $\phi - \bar{\phi}$.

Step 1:(*Control of the spherical part*) That (5.56) implies (5.53) for $\bar{\phi}$ is an immediate consequence of the coordinate asymptotics in parts i) and ii) of Lemma 5.21. First notice that by the asymptotic formulas of part i) of Lemma 5.21 the null vector-field $\partial_s + \partial_{r_*}$ is regular and non-vanishing up to $r = r_M$ in the original (t, r) coordinates. Therefore we have that:

$$L = \partial_s + \partial_{r_*}, \quad \underline{L} = \Omega^{-2}(\partial_s - \partial_{r_*}) = O(r(r - r_M)^{-1})(\partial_s - \partial_{r_*}),$$

is a regular null pair throughout $r > r_M$. Next, note that one has:

$$dsdr_* = \frac{dr_*}{dr} dt dr = O(r(r - r_M)^{-1}) dt dr = O(\Omega^{-2}) dt dr.$$

Combining this with the asymptotics $r_* = A \ln(r - r_M) + O(1)$ as $r_* \rightarrow -\infty$ and $r_* = r + B \ln(r) + O(r^{-1})$ as $r_* \rightarrow \infty$, we see that:

$$\begin{aligned} \|w_k \chi_{[r_M, r_T]} \psi\|_{LE_0} &\lesssim \|\chi_{(-\infty, 0]} \psi\|_{LE_{RW}} \lesssim \|w_{-k} \chi_{[r_M, r_T]} \psi\|_{LE_0}, \\ \|\chi_{[r_T, \infty)} \psi\|_{(LE_0, LE_0^*)} &\approx \|\chi_{[0, \infty)} \psi\|_{(LE_{RW}, LE_{RW}^*)}, \quad \|\chi_{(-\infty, 0]} \psi\|_{LE_{RW}^*} \lesssim \|w_{-k} \chi_{[r_M, r_T]} \psi\|_{LE_0^*}, \end{aligned}$$

where the implicit constants depend on $k > 1$. In addition note that the logarithmic part of w_k is essentially constant when $r_* \approx -2^j$, so it may be removed from the integral in the first term on the RHS of (5.56) above. Finally, note that by standard local estimates one can control $E_{RW, r_* > 2r_*^0}(\phi[0])$ in terms of the standard energy $E_{r > r_M}(\phi[0])$.

Step 2:(*Control of the non-spherical part*) For this we use exactly the same estimates as above with a slight twist. We claim that for $C > 0$ sufficiently large one has the analogous bound to (5.56):

$$(5.57) \quad \begin{aligned} &\|\chi_{[r_*^0, \infty)}(\partial_{r_*}(\phi - \bar{\phi}), (w_{ln})^{-1} \partial_s(\phi - \bar{\phi}))\|_{LE_{RW}[0, T]} + \|\chi_{[r_*^0, \infty)}(w_{ln})^{-1} r^{-2} (r - r_M)^{\frac{1}{2}} \not{d}(\phi - \bar{\phi})\|_{L^2[0, T]} \\ &\lesssim \|\chi_{[2Cr_*^0, r_*^0]} \partial_{r_*}(\phi - \bar{\phi})\|_{LE_{RW}[0, T]} + E_{RW, r_* > 2Cr_*^0}^{\frac{1}{2}}((\phi - \bar{\phi})[0]) + \|\chi_{[2Cr_*^0, \infty)} \square_{RW}^0(\phi - \bar{\phi})\|_{LE_{RW}^*[0, T]}. \end{aligned}$$

To prove it first apply (5.55) to the quantity $\chi_{r_* > 2Cr_*^0}(\phi - \bar{\phi})$, where $\chi_{r_* > 2Cr_*^0} \equiv 1$ for $r_* > Cr_0^*$ with standard derivative bounds. This generates the above estimate modulo two errors:

$$(5.58) \quad Error_1 = \|\chi_{[2Cr_*^0, Cr_*^0]} \langle r_* \rangle^{-1} (\phi - \bar{\phi})(0)\|_{L^2}, \quad Error_2 = \|\chi_{[2Cr_*^0, Cr_*^0]} \langle r_* \rangle^{-1} (\phi - \bar{\phi})\|_{LE_{RW}[0, T]}.$$

To handle these terms we use the sharp Hardy type estimates (7.72) and (7.73) of the Appendix respectively.

By an immediate application of (7.72) with $s = 1$ and $p = 2$, followed by the spherical Poincare estimate (1.18) to control the undifferentiated term on the RHS, we have:

$$Error_1 \lesssim E_{RW, r_* > 2Cr_*^0}^{\frac{1}{2}}((\phi - \bar{\phi})[0]).$$

To handle $Error_2$ apply (7.73) with $C = 2^{j-i}$ and $s = 1$ which yields:

$$Error_2 \lesssim C^{\frac{1}{2}} \|\chi_{[-2Cr_*^0, -r_*^0]} \partial_{r_*}(\phi - \bar{\phi})\|_{LE[0, T]} + C^{-\frac{1}{2}} (\|\chi_0 \not{d}\phi\|_{LE[0, T]} + \|\chi_{[-r_*^0, -1]} \partial_{r_*}(\phi - \bar{\phi})\|_{LE[0, T]}).$$

For $C \gg 1$ the second term of the RHS above may safely be absorbed into the LHS of estimate (5.57). \square

We now give the proofs for Proposition 5.25.

Proof of estimate (5.55). In light of the second estimate (5.56), which is independently proved below, we may assume that $\bar{\phi} = 0$. The method is to combine the analysis of [5] with [16].

Step 1:(*Proof of a degenerate local energy decay estimate*) First we show that for all $k > 1$:

$$(5.59) \quad \begin{aligned} &\|\langle r_* \rangle^{-\frac{1}{2}k} \partial_{r_*} \phi\|_{L^2[0, T]} + \|(w_1)^{-1} r^{-2} (r - r_M)^{\frac{1}{2}} \not{d}\phi\|_{L^2[0, T]} \\ &\lesssim C^{-1} \|\partial_{r_*} \phi\|_{LE_{RW}[0, T]} + \sup_{0 \leq s \leq T} E_{RW}^{\frac{1}{2}}(\phi[s]) + C \|\square_{RW}^0 \phi\|_{LE_{RW}^*[0, T]}, \end{aligned}$$

where the implicit constant depends on k but is uniform in $C > 0$, and $w_1 = r(r - r_T)^{-1}$.

For a $C^1(\mathbb{R})$ (but not necessarily smooth) function $a(r_*)$ we form the operator $A = op^w(ia\xi_{r_*}) = a\partial_{r_*} + \frac{1}{2}a'$. Then for test functions ϕ we integrate the quantity $\square_{RW}^0\phi \cdot A\phi$ over the space-time slab $0 \leq t \leq T$. After some integration by parts we arrive at the Virial type identity (see Section 3 of [5] for further details):

$$\int_0^T \int_{\mathbb{R} \times \mathbb{S}^2} \left(a' \phi_{r_*}^2 - \frac{1}{4} a''' \phi^2 - \frac{1}{2} a V' |\not{d}\phi|^2 \right) dr_* dV_{\mathbb{S}^2} ds = - \int_{\mathbb{R} \times \mathbb{S}^2} \partial_s \phi A \phi dr_* dV_{\mathbb{S}^2} \Big|_0^T - \int_0^T \int_{\mathbb{R} \times \mathbb{S}^2} \square_{RW}^0 \phi A \phi dr_* dV_{\mathbb{S}^2} ds .$$

Here a''' is interpreted in the sense of distributions if necessary. If we choose a so that:

$$\text{(Assumption 1)} \quad a(r_*) \approx \frac{r_*}{1 + |r_*|}, \quad a'(r_*) \lesssim \frac{1}{1 + |r_*|},$$

then using the Hardy/Poincare estimate (7.72) with $s = 1$ and $p = 2$ the boundary terms at $t = 0, T$ are estimated by the energy. In addition, the non-negative weight $-aV'$ is then consistent with estimate (5.59). Thus, by using Young's inequality to split the last term in the previous identity above, and then estimate (7.72) with $s = \frac{3}{2}$ and $p = \infty$ to bound $a'\phi$ in terms of $\partial_{r_*}\phi$ and a local $\not{d}\phi$ contribution (which may be absorbed to the LHS), we have:

$$\begin{aligned} (5.60) \quad & \int_0^T \int_{\mathbb{R} \times \mathbb{S}^2} \left(a' \phi_{r_*}^2 - \frac{1}{4} a''' \phi^2 \right) dr_* dV_{\mathbb{S}^2} ds + \| r^{-3} (r - r_{\mathcal{T}}) (r - r_M)^{\frac{1}{2}} \not{d}\phi \|_{L^2[0, T]}^2 \\ & \lesssim C^{-2} \| \partial_{r_*} \phi \|_{LE_{RW}[0, T]}^2 + \sup_{0 \leq s \leq T} E(\phi[s]) + C^2 \| \square_{RW}^0 \phi \|_{LE_{RW}^*[0, T]}^2 . \end{aligned}$$

To deal with the $\partial_{r_*}\phi$ terms on the LHS of (5.60) we need to further constrain a . To see what is the correct choice, first note that one has the following Hardy type inequality for smooth a and fixed $\lambda > 0$:

$$(5.61) \quad \int \left(a''' - \lambda \frac{(a'')^2}{a'} \right) \phi^2 dr_* \leq \lambda^{-1} \int a' \phi_{r_*}^2 dr_* .$$

This follows at once from integrating $\partial_{r_*}(a''\phi^2)$ and then applying Young's inequality to the term containing ϕ_{r_*} . From this we see that an ideal set of assumptions would be:

$$\text{(Assumption 2)} \quad a' \approx \langle r_* \rangle^{-k} ,$$

$$\text{(Assumption 3)} \quad \frac{(a'')^2}{a'} \leq \gamma a''' , \quad 0 < \gamma < 1 .$$

Taken together, and setting $\lambda^{-1} = 2\gamma$ in (5.61), these assumptions would give the lower bound:

$$(5.62) \quad c_\gamma \| \langle r_* \rangle^{-\frac{1}{2}k} \partial_{r_*} \phi \|_{L^2(dr_*)}^2 \leq (1 - \gamma) \int a' \phi_{r_*}^2 dr_* \leq \int \left(a' \phi_{r_*}^2 - \frac{1}{4} a''' \phi^2 \right) dr_* ,$$

which would be sufficient for our purposes.

Unfortunately it turns out that Assumptions 2 and 3 as stated are inconsistent with each other due to inflections of a' . A natural way around this is to eliminate inflection points by working with a singular weight $a(r_*)$. In other words we trade Assumption 3 for the modification:

$$\text{(Assumption 3')} \quad a'(r_*) = \frac{1}{(1 + |\epsilon r_*|)^k} ,$$

in which case the original Assumption 3 holds for $r_* \neq 0$ with $\gamma = \frac{k}{k+1}$, and (5.61) still holds in the sense of distributions. The difference now is that a''' contains a delta function, so (5.62) must be replaced by:

$$(1 - \gamma) \int a' \phi_{r_*}^2 dr_* \leq \int \left(a' \phi_{r_*}^2 - \frac{1}{4} a''' \phi^2 \right) dr_* + \epsilon k \phi^2|_{r_*=0} .$$

The last term on the RHS is easily eliminated by the assumption that $\bar{\phi} = 0$ which gives:

$$\int_{\mathbb{S}^2} \phi^2|_{r_*=0} dV_{\mathbb{S}^2} \lesssim \| \chi_0(\partial_{r_*}\phi, r_*\not{d}\phi) \|_{L^2(dr_* dV_{\mathbb{S}^2})}^2 .$$

For $\epsilon \ll 1$ sufficiently small this allows us to absorb $\phi^2|_{r_*=0}$ and we arrive at (5.59).

Step 2: (*Improvement of the weights on $\partial_{r_*}\phi$ as $r_* \rightarrow \pm\infty$*) Next, we improve estimate (5.59) to the following:

$$(5.63) \quad \| \partial_{r_*} \phi \|_{LE_{RW}[0, T]} + \| (w_1)^{-1} r^{-2} (r - r_M)^{\frac{1}{2}} \not{d}\phi \|_{L^2[0, T]} \lesssim \sup_{0 \leq s \leq T} E_{RW}^{\frac{1}{2}}(\phi[s]) + \| \square_{RW}^0 \phi \|_{LE_{RW}^*[0, T]} .$$

To prove this we integrate the quantities $\square_{RW}^0 \phi \cdot a_j^\pm \partial_{r_*} \phi$ over the slabs $0 \leq s \leq T$, where we choose $(a_j^\pm)' = \pm 2^{-j} \chi_j \chi_{\pm r_* > 1}$ for $j \geq 1$ and $a_j^\pm \equiv 0$ in a neighborhood of $r_* = 0$. This gives the bound:

$$\begin{aligned} \|\chi_{|r_*| > 2} \partial_{r_*} \phi\|_{LE_{RW}[0, T]} &\lesssim \|\chi_{|r_*| > 1} r^{-2} (r - r_M)^{\frac{1}{2}} \not\partial \phi\|_{L^2[0, T]} \\ &\quad + \|\chi_{|r_*| < 1} \partial_{r_*} \phi\|_{L^2[0, T]} + \sup_{0 \leq s \leq T} E^{\frac{1}{2}}(\phi[s]) + \|\square_{RW}^0 \phi\|_{LE_{RW}^*[0, T]}, \end{aligned}$$

which can be added safely into (5.59) in order to produce (5.63).

Step 3: (*Removal of the future energy and addition of $\partial_s \phi$*) Our goal here is the more complete estimate:

$$(5.64) \quad \|\partial_{r_*} \phi\|_{LE_{RW}[0, T]} + \|(w_1)^{-1} \partial_s \phi\|_{LE_{RW}[0, T]} + \|(w_1)^{-1} r^{-2} (r - r_M)^{\frac{1}{2}} \not\partial \phi\|_{L^2[0, T]} \\ \lesssim E_{RW}^{\frac{1}{2}}(\phi[0]) + \|w_1 \square_{RW}^0 \phi\|_{LE_{RW}^*[0, T]}.$$

To prove this bound first note that by the usual calculations used to produce the conservation of energy one has:

$$\sup_{0 \leq s \leq T} E_{RW}(\phi[s]) \lesssim \|w_1 \square_{RW}^0 \phi\|_{LE_{RW}^*[0, T]} \|(w_1)^{-1} \partial_s \phi\|_{LE_{RW}[0, T]}.$$

On the other hand, to control the quantity $\partial_s \phi$ one integrates $2^{-j} (w_1)^{-2} \chi_j \phi \square_{RW}^0 \phi$ over the slabs $0 \leq s \leq T$. After some integration by parts, using Poincare/Hardy type estimates to bound undifferentiated terms and boundary contributions, and then supping over $j \geq 0$, one has:

$$\begin{aligned} \|(w_1)^{-1} \partial_s \phi\|_{LE_{RW}[0, T]} &\lesssim \|\partial_{r_*} \phi\|_{LE_{RW}[0, T]} + \|(w_1)^{-1} r^{-2} (r - r_M)^{\frac{1}{2}} \not\partial \phi\|_{L^2[0, T]} \\ &\quad + \sup_{0 \leq s \leq T} E_{RW}^{\frac{1}{2}}(\phi[s]) + \|\square_{RW}^0 \phi\|_{LE_{RW}^*[0, T]}. \end{aligned}$$

Adding the last two lines together with (5.63) yields (5.64).

Step 4: (*Improvement of the local weight*) Our final task is to establish (5.64) with the local weight w_1 replaced by the weight w_{ln} as written in estimate (5.55) above. To do this we recall the main local estimate of [16] which can be stated in the following form:

Proposition 5.26. *The following two statements hold:*

a) *Let ϕ be a space-time function which is supported in the region $|r_*| \leq C$ for some fixed $C > 0$. Then one has the bound:*

$$\|\partial_{r_*} \phi\|_{L^2[0, T]} + \|(w_{ln})^{-1} (\partial_s \phi, \not\partial \phi)\|_{L^2[0, T]} \lesssim E^{\frac{1}{2}}(\phi[0]) + \|w_{ln} \square_{RW}^0 \phi\|_{L^2[0, T]}.$$

b) *Let f be a space-time supported in the region $|r_*| \leq C$ for some fixed $C > 0$ with the property that $w_{ln} f \in L^2[0, T]$. Then there exists a function ϕ defined for $s \in [0, T]$ and supported in the region $|r_*| \leq 2C$ such that $\square_{RW}^0 \phi - f$ is supported away from $|r_*| \leq \frac{1}{2}C$ and such that:*

$$(5.65) \quad \sup_{0 \leq s \leq T} E^{\frac{1}{2}}(\phi[s]) + \|\partial_{r_*} \phi\|_{L^2[0, T]} + \|(w_{ln})^{-1} (\partial_s \phi, \not\partial \phi)\|_{L^2[0, T]} + \|\square_{RW}^0 \phi - f\|_{L^2[0, T]} \lesssim \|w_{ln} f\|_{L^2[0, T]}.$$

For a proof of these bounds refer to parts a) and b) of Proposition 3.3 in [16]. Note that in both cases the analysis is purely local and relies only on the structural assumption that the potential $V(r_*)$ in the definition of \square_{RW}^0 has a unique positive non-degenerate local maximum at $r_* = 0$.

To show that parts a) and b) of Proposition 5.26 imply (5.59) with local weight w_{ln} involves a series of steps. First, one employs part b) to produce a $\tilde{\phi}$ such that $\square_{RW}^0 \tilde{\phi} - \chi_0 \square_{RW}^0 \phi$ is supported away from $r_* = 0$ with good L^2 bounds, and such that (5.64) holds for $\tilde{\phi}$ with w_1 replaced by w_{ln} . Then one may apply (5.64) to the remainder $\phi - \tilde{\phi}$ to show that (5.64) holds for ϕ with RHS norm containing w_{ln} instead of w_1 . Finally, one replaces the weight $(w_1)^{-1}$ on the LHS of this last bound with $(w_{ln})^{-1}$ by applying part a) above to $\chi_0 \phi$.

This concludes the proof of (5.55). \square

Proof of estimate (5.56). Note that the spherical part of \square_{RW}^0 is simply the 1D wave equation $\square = -\partial_s^2 + \partial_{r_*}^2$, for which it is easy to produce estimates of the form (5.56) via the multipliers $\chi_{(r_*^0, \pm 2^j)} \partial_{r_*}$, where $r_*^0 \leq -2^j$. This is analogous to **step 2** and **step 3** of the previous proof. Further details are left to the reader. \square

6. LOCAL ENERGY DECAY FOR SPIN-ZERO FIELDS PART II: PROOF OF THE MAIN ESTIMATE

Our main goal here is to prove Theorem 4.12 which we state again here for the convenience of the reader.

Theorem 6.27 (Inverse angular gradient local energy decay estimates). *Let $\phi(t, r, x^A)$ and $H(t, r, x^A)$ be scalars, and let $G_a(t, r, x^A)$ be a one form in (t, r) variables which also depends on the \mathbb{S}^2 coordinates. Suppose all of these quantities have no radial component::*

$$\int_{\mathbb{S}^2} \phi(t, r) dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} H(t, r) dV_{\mathbb{S}^2} = \int_{\mathbb{S}^2} G(t, r) dV_{\mathbb{S}^2} \equiv 0 ,$$

throughout the region $[0, T] \times [r_0, \infty)$. Finally, set $\star_h dG = K$. If ϕ, G, H are all supported in $\{r \leq CT\}$ and

$$(6.66) \quad \square^0 \phi = \nabla^a G_a + H ,$$

then one has the estimate:

$$(6.67) \quad \begin{aligned} \|(w_{ln})^{-1}(\underline{d}(-\underline{\Delta})^{-\frac{1}{2}}\phi, r^{-1}\phi)\|_{LE_0[0, T]} &\lesssim \|(-\underline{\Delta})^{-\frac{1}{2}}(\underline{d}\phi(0) - G(0))\|_{L^2(dr dV_{\mathbb{S}^2})} + \|r^{-1}\phi(0)\|_{L^2(dr dV_{\mathbb{S}^2})} \\ &+ \|w_{ln}(r^{-1}G, (-\underline{\Delta})^{-\frac{1}{2}}K, (-\underline{\Delta})^{-\frac{1}{2}}H)\|_{LE_0^*[0, T]} . \end{aligned}$$

The main step in the proof is the following elliptic result:

Proposition 6.28. *Let G be as in the theorem. Then there exists a function $\tilde{\phi}$ in $[0, T] \times [r_0, \infty)$, with no radial component, satisfying the following estimates:*

$$(6.68) \quad \|r^{-1}(\underline{d}\tilde{\phi}, r^{-1}\tilde{\phi}, r^{-1}\underline{d}\tilde{\phi})\|_{LS_0^*[0, T]} + \|(-\underline{\Delta})^{-\frac{1}{2}}\nabla_{r,t}(G - \underline{d}\tilde{\phi})\|_{LS_0^*[0, T]} \lesssim \|(r^{-1}G, (-\underline{\Delta})^{-\frac{1}{2}}K)\|_{LS_0^*[0, T]},$$

$$(6.69) \quad \|w_{ln}r^{-2}\underline{d}\tilde{\phi}\|_{LS_0^*[0, T]} + \|w_{ln}(-\underline{\Delta})^{-\frac{1}{2}}\nabla_{r,t}(G - \underline{d}\tilde{\phi})\|_{LS_0^*[0, T]} \lesssim \|w_{ln}(r^{-1}G, (-\underline{\Delta})^{-\frac{1}{2}}K)\|_{LS_0^*[0, T]}.$$

We first use the proposition to prove Theorem 6.27. This is achieved by applying the bound (5.36) to the function $\psi = \Delta^{-\frac{1}{2}}(\phi - \tilde{\phi})$. The left hand side of (6.67) is controlled by the left hand side of (5.36) applied to ψ and the first term on the left hand side of (6.68). It remains to estimate the right hand side of (5.36) applied to ψ .

For the energy part $E^{\frac{1}{2}}(\psi[0])$ we use the trace estimate (7.74) of Lemma 7.30 below. Note it is precisely here that truncation to $r \leq CT$ is used. Estimating $\tilde{\phi}$ and $\underline{d}\tilde{\phi} - G$ in this way we obtain from (6.68) the following bound:

$$\|(-\underline{\Delta})^{-\frac{1}{2}}(\underline{d}\tilde{\phi}(0) - G(0))\|_{L^2(dr dV_{\mathbb{S}^2})} + \|r^{-1}\tilde{\phi}(0)\|_{L^2(dr dV_{\mathbb{S}^2})} \lesssim \|(r^{-1}G, (-\underline{\Delta})^{-\frac{1}{2}}K)\|_{LE_0^*[0, T]} ,$$

which is combined with the energy component of the right hand side of (6.67). It remains to estimate $\|\square^0\psi\|_{LE_0^*}$, for which we compute:

$$\begin{aligned} \square^0\psi &= (-\underline{\Delta})^{-\frac{1}{2}}(\nabla^a G_a + H) - (-\underline{\Delta})^{-\frac{1}{2}}(\nabla^a \nabla_a + r^{-2}\underline{\Delta})\tilde{\phi} , \\ &= (-\underline{\Delta})^{-\frac{1}{2}}\nabla^a(G_a - \nabla_a\tilde{\phi}) + r^{-2}(-\underline{\Delta})^{\frac{1}{2}}\tilde{\phi} + (-\underline{\Delta})^{-\frac{1}{2}}H . \end{aligned}$$

and use the bound (6.69) for the first two terms.

Proof of Proposition 6.28. We observe that the proposition has no reference to the metric h . We begin with several simplifications.

Step 1: Using a radial partition of unit, we reduce the problem to the case when we know that G is supported in a dyadic annulus $\{r \approx R\}$, $R \gtrsim 1$, and we seek $\tilde{\psi}$ with support in a slightly enlarged annulus.

Step 2: Secondly, given $R \gtrsim 1$, we use a partition of unit in time on the R scale to reduce the problem to the case when G is supported in a dyadic annulus $\{r \approx R\}$ and a time interval of size $\delta T \approx R$ and we seek $\tilde{\psi}$ with support in a similar but slightly enlarged region.

Step 3: We rescale the problem to the unit scale in t and r . Then matters are reduced to the next lemma. \square

Lemma 6.29. *Let G be a one form in (t, r) variables in $[0, \infty) \times [0, \infty) \times \mathbb{S}^2$ which is supported in $[0, 1] \times [0, 1] \times \mathbb{S}^2$. Then there exists a function $\tilde{\psi}$ in $[0, \infty) \times [0, \infty) \times \mathbb{S}^2$, supported in $[0, 2] \times [0, 2] \times \mathbb{S}^2$, so that the following estimates hold:*

$$(6.70) \quad \|\tilde{\phi}\|_{H^1} + \|(-\mathbb{A})^{-\frac{1}{2}}\nabla_{r,t}(G - \underline{d}\tilde{\phi})\|_{L^2} \lesssim \|G\|_{L^2} + \|(-\mathbb{A})^{-\frac{1}{2}}\underline{d}G\|_{L^2} ,$$

$$(6.71) \quad \|w_{ln}(-\mathbb{A})^{\frac{1}{2}}\tilde{\phi}\|_{L^2} + \|w_{ln}(-\mathbb{A})^{-\frac{1}{2}}\nabla_{r,t}(G - \underline{d}\tilde{\phi})\|_{L^2} \lesssim \|w_{ln}G\|_{L^2} + \|w_{ln}(-\mathbb{A})^{-\frac{1}{2}}\underline{d}G\|_{L^2} .$$

where the logarithmic weight w_{ln} is now centered at $r = \frac{1}{2}$.

We remark that the lemma includes both the case when G is compactly supported, and the cases when we have either a $t = \text{const}$ boundary or an $r = \text{const}$ boundary or both.

Proof. In all cases, we reduce the problem to the boundaryless case by taking suitable even/odd extensions of the components of G to the other three quadrants in the (r, t) plane. Now we have the (t, r) 1-form G with variables $\mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$ supported in $[-1, 1] \times [-1, 1] \times \mathbb{S}^2$.

Using the euclidean metric in (t, r) we define $\tilde{\phi}$ by:

$$\tilde{\phi} = \chi(r, t)\tilde{\phi}_0 , \quad \tilde{\phi}_0 = (-\Delta_{r,t} - \mathbb{A})^{-1}\underline{d}^*G ,$$

where χ is a smooth cutoff supported in $[-2, 2] \times [-2, 2]$ and which equals 1 in $[-1, 1] \times [-1, 1]$.

Then:

$$G - \underline{d}\tilde{\phi} = \chi G_0 - \tilde{\phi}d\chi ,$$

where:

$$\begin{aligned} \tilde{G} &= G - \underline{d}\tilde{\phi}_0 = (1 + \Delta_{r,t}(-\Delta_{r,t} - \mathbb{A})^{-1})G + (-\Delta_{r,t} - \mathbb{A})^{-1}\underline{d}^*\underline{d}G , \\ &= -\mathbb{A}(-\Delta_{r,t} - \mathbb{A})^{-1}G + (-\Delta_{r,t} - \mathbb{A})^{-1}\underline{d}^*\underline{d}G . \end{aligned}$$

Then we have the straightforward bounds:

$$\|\tilde{\phi}_0\|_{H^1} \lesssim \|G\|_{L^2} ,$$

$$\|(-\mathbb{A})^{-\frac{1}{2}}\tilde{G}\|_{H^1} \lesssim \|G\|_{L^2} + \|(-\mathbb{A})^{-\frac{1}{2}}\underline{d}G\|_{L^2} .$$

and the estimate (6.70) follows after truncation.

For the bound (6.71) we need the following weighted versions of the above estimates:

$$\|w_{ln}(-\mathbb{A})^{\frac{1}{2}}\tilde{\phi}_0\|_{L^2} \lesssim \|w_{ln}G\|_{L^2} ,$$

$$\|w_{ln}(-\mathbb{A})^{-\frac{1}{2}}\nabla_{r,t}\tilde{G}\|_{L^2} \lesssim \|w_{ln}G\|_{L^2} + \|w_{ln}(-\mathbb{A})^{-\frac{1}{2}}\underline{d}G\|_{L^2} .$$

These in turn correspond to the $L^2_{w_{ln}}$ boundedness of the singular integral operators $\mathbb{A}(-\Delta_{r,t} - \mathbb{A})^{-1}$, respectively $(-\mathbb{A})^{\frac{1}{2}}\nabla_{r,t}(-\Delta_{r,t} - \mathbb{A})^{-1}$. But this can be easily handled within the context of A_p weights, see for example Chapter V of [19]. \square

7. APPENDIX: SOME HARDY AND POINCARÉ TYPE ESTIMATES

This section contains some auxiliary estimates needed for some of the proofs in the main body of the paper. They are collected here for the convenience of the reader.

To state the estimates denote by χ_j the indicator function (sharp cutoff) of $2^j \leq |x| \leq 2^{j+1}$.

Lemma 7.30 (Hardy and Poincaré estimates). *Let ψ be a function on $\mathbb{R} \times \mathbb{S}^2$. Then one has the pair of uniform bounds for $1 \leq p \leq \infty$ and $0 \leq i \leq j$, and implicit constant depending on $s > \frac{1}{2}$:*

$$(7.72) \quad \|\langle x \rangle^{-s}\psi\|_{\ell^p L^2(|x| \leq 2^j)} \lesssim \|\chi_0\psi\|_{L^2} + \|\langle x \rangle^{1-s}\partial_x\psi\|_{\ell^p L^2(|x| \leq 2^j)} ,$$

$$(7.73) \quad \begin{aligned} \|\chi_j\langle x \rangle^{-s}\psi\|_{L^2} &\lesssim 2^{(\frac{1}{2}-s)(j-i)} (\|\chi_0\psi\|_{L^2} + \|\langle x \rangle^{1-s}\partial_x\psi\|_{\ell^\infty L^2(|x| \leq 2^i)}) \\ &\quad + 2^{(s-\frac{1}{2})(j-i)} \|\langle x \rangle^{1-s}\partial_x\psi\|_{\ell^\infty L^2(2^i \leq |x| \leq 2^{j+1})} . \end{aligned}$$

Here $\|\psi\|_{\ell^p L^2}^p = \|\psi\|_{L^2(|x| < 1)}^p + \sum_{j \geq 0} \|\chi_j\psi\|_{L^2}^p$ (with standard modification for $p = \infty$), where the integration is with respect to $dx dV_{\mathbb{S}^2}$. Similar estimates hold when ψ is also a function of t and the integrals are with respect to $dx dV_{\mathbb{S}^2} dt$.

In addition, let G be a space-time function (variables (t, r, x^A)) defined on the slab $r \geq r_0$ and $0 \leq t \leq T$. Then one has the uniform bound:

$$(7.74) \quad \sup_{t \in [0, T]} \|G(t)\|_{L^2(dr dV_{\mathbb{S}^2})[r_0, \infty)} \lesssim \|(T^{-\frac{1}{2}}G, r^{-\frac{1}{2}}G, r^{\frac{1}{2}}\partial_t G)\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T] \times [r_0, \infty)}.$$

Proof of estimates (7.72) and (7.73). Set χ according to $\chi' = -2^{-k}\chi_k + \chi_0$ and vanishing at infinity. Then $\chi \equiv 0$ for $|x| \geq 2^{k+1}$ and $|x| \leq 1$, and $|\chi| \leq 1$ throughout \mathbb{R} . Integrating the quantity $\partial_x(\chi\psi^2)$ with respect to $dx dV_{\mathbb{S}^2}$ or $dx dV_{\mathbb{S}^2} dt$, multiplying the result by $2^{(1-2s)k}$, and taking the square root we have the uniform bound:

$$(7.75) \quad \|\chi_k \langle x \rangle^{-s} \psi\|_{L^2} \lesssim 2^{(\frac{1}{2}-s)k} \|\chi_0 \psi\|_{L^2} + 2^{(\frac{1}{2}-s)k} \|\psi \partial_x \psi\|_{L^1(1 \leq |x| \leq 2^{k+1})}^{\frac{1}{2}}, \quad s \in \mathbb{R}, \quad k \geq 0.$$

For the second term on the RHS above we have:

$$(7.76) \quad 2^{(\frac{1}{2}-s)k} \|\psi \partial_x \psi\|_{L^1(1 \leq |x| \leq 2^{k+1})}^{\frac{1}{2}} \lesssim \sum_{0 \leq k' \leq k} 2^{(\frac{1}{2}-s)(k-k')} \|\chi_{k'} \langle x \rangle^{-s} \psi\|_{L^2}^{\frac{1}{2}} \|\chi_{k'} \langle x \rangle^{1-s} \partial_x \psi\|_{L^2}^{\frac{1}{2}}.$$

The proof of (7.72) follows at once by summing (7.75) over $0 \leq k \leq j$ and using then using (7.76). To bound the RHS of (7.76) summed over k we use Young's convolution inequality $2^{(\frac{1}{2}-s)|k-k'|} * \ell_{k'}^p \subseteq \ell_k^p$ for $s > \frac{1}{2}$ and then $\|\sqrt{ab}\|_{\ell^p} \leq \epsilon \|a\|_{\ell^p} + \epsilon^{-1} \|b\|_{\ell^p}$ for sequences $a = (a_k)$ and $b = (b_k)$. This produces an estimate for $\|\langle x \rangle^{-s} \psi\|_{\ell^p L^2(1 \leq |x| \leq 2^j)}$ with upper bound RHS(7.72). To fill in $|x| \leq 1$ one uses a standard Sobolev embedding.

It remains to show (7.73). Starting with (7.75) for $k = j$ followed by (7.76), we break the sum of RHS(7.76) into $0 \leq k' \leq i$ and $i \leq k' \leq j$. Following the same procedure of the previous paragraph and using (7.72) to estimate the resulting undifferentiated term, the sum in the first range is bounded by the first term on RHS(7.73). For the second range we use the dyadic weight to sum in ℓ^1 followed by:

$$(7.77) \quad \|\langle x \rangle^{-s} \psi\|_{\ell^\infty L^2(2^i \leq |x| \leq 2^{j+1})}^{\frac{1}{2}} \|\langle x \rangle^{1-s} \partial_x \psi\|_{\ell^\infty L^2(2^i \leq |x| \leq 2^{j+1})}^{\frac{1}{2}} \\ \leq 2^{(\frac{1}{2}-s)(j-i)} \|\langle x \rangle^{-s} \psi\|_{\ell^\infty L^2(2^i \leq |x| \leq 2^{j+1})} + 2^{(s-\frac{1}{2})(j-i)} \|\langle x \rangle^{1-s} \partial_x \psi\|_{\ell^\infty L^2(2^i \leq |x| \leq 2^{j+1})}.$$

For the first term on the RHS above we again use (7.72). □

Proof of estimate (7.74). By the pigeon-hole principle there exists $t_0 \in [0, T]$ such that:

$$\|G(t_0)\|_{L^2(dr dV_{\mathbb{S}^2})[r_0, \infty)} \leq T^{-\frac{1}{2}} \|G\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T] \times [r_0, \infty)}.$$

Then using the fundamental theorem of calculus and integrating on $[t_0, t]$ we have:

$$\|G(t)\|_{L^2(dr dV_{\mathbb{S}^2})[r_0, \infty)}^2 \leq T^{-1} \|G\|_{L^2(dr dV_{\mathbb{S}^2} dt)[0, T] \times [r_0, \infty)}^2 + 2 \left| \int_{t_0}^t \int_{[r_0, \infty) \times \mathbb{S}^2} G \partial_t G \, dr dV_{\mathbb{S}^2} dt \right|.$$

Estimate (7.74) follows immediately from Young's inequality applied to the last line. □

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