Model-free CPPI

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May 25, 2013

Abstract

We consider Constant Proportion Portfolio Insurance (CPPI) and its dynamic extension, which may be called Dynamic Proportion Portfolio Insurance (DPPI). It is shown that these investment strategies work within the setting of Föllmer's pathwise Itô calculus, which makes no probabilistic assumptions whatsoever. This shows, on the one hand, that CPPI and DPPI are completely independent of any choice of a particular model for the dynamics of asset prices. They even make sense beyond the class of semimartingale sample paths and can be successfully defined for models admitting arbitrage, including some models based on fractional Brownian motion. On the other hand, the result can be seen as a case study for the general issue of robustness in the face of model uncertainty in finance.

1 Introduction

The purpose of this paper is twofold. On the one hand, it deals with Constant Proportion Portfolio Insurance (CPPI) and its dynamic extension, which may be called Dynamic Proportion Portfolio Insurance (DPPI). On the other hand, it deals with the general issues of model uncertainty and model risk in finance by presenting a case study in which a problem of dynamic trading can be solved in a probability-free manner.

Constant Proportion Portfolio Insurance (CPPI) was first studied by Perold [24], Black and Jones [6], and Black and Perold [7]. It provides a strategy that yields superlinear participation in future asset returns while retaining a security guarantee on a part of the invested capital ("the floor"). In the Black & Scholes framework, which is the basis for most academic studies on CPPI, constructing a CPPI strategy is equivalent to hedging a certain power option. Moreover, in this framework, the CPPI strategy has no *gap risk* in the sense that its value stays above the floor with probability one. On the other hand, Cont and Tankov [12], Balder et al. [2], and Paulot and Lacroze [23] show that the CPPI strategy may break through the floor in incomplete market models in which asset prices may jump or in which the portfolio may only be rebalanced at a finite number of trading dates. In this sense, the CPPI strategy may fail in these settings, and one is faced with the question of quantifying the resulting gap risk, which is important in practice [2, 12, 23].

The failure of the CPPI strategy in the incomplete market models of [2, 12, 23] on the one hand, and the absence of gap risk in the complete Black & Scholes framework on the other hand, raise the question whether the completeness of the underlying market model is related to the possible nonexistence of gap risk. More generally, one may ask which model features are crucial for setting up a CPPI strategy:

- Can one choose every general semimartingale model?
- What is the role of arbitrage? In particular, must the underlying market model be arbitrage-free to set up the CPPI strategy?
- If absence of arbitrage is not essential, can one even go beyond the class of general semimartingale models and allow for fractal or fractional models such as those in [4, 10, 25, 27]?
- Are there other sources for gap risk apart from jumps in asset prices or discrete rebalancing times?

In this paper, we address all these questions by considering CPPI in the probability-free setting of Föllmer's pathwise Itô calculus [15]; see also [3, 5, 14, 16, 18, 26, 28]. In this framework, the dynamics of asset prices are simply described by a single trajectory satisfying a few basic assumptions. In particular, this framework does not postulate any probabilistic mechanism that governs the choice of a particular price evolution. All that is required from the price trajectory of a risky asset is that it is continuous and admits a continuous quadratic variation in a pathwise sense. These two conditions are satisfied, in particular, by the typical sample paths of any continuous semimartingale, regardless of whether the semimartingale admits an equivalent martingale measure or not. A continuous quadratic variation exists even for a much larger class of trajectories than the class of semimartingale sample paths. An example are the typical sample paths of fractional Brownian motion with Hurst index $H > \frac{1}{2}$, which have vanishing quadratic variation. It is perhaps interesting to note here that vanishing quadratic variation immediately yields the existence of arbitrage opportunities via a simple application of Föllmer's pathwise Itô formula to the function $f(x) = x^2$; see [16] or [18, Section 5.1].

Our first main result will show that, in this very general context, CPPI can be defined as a self-financing trading strategy and that CPPI has no gap risk in the the sense that its value always stays above the floor. This means in particular that neither the completeness nor the absence of arbitrage play any role in the definition of CPPI and for the possible existence of gap risk. Gap risk is therefore exclusively generated by jumps in the asset price dynamics or by constraints on the rebalancing times of the portfolio.

Our second main result concerns a dynamic extension of the CPPI strategy in which the multiplier level may depend on quantities including time and price evolution. While the possibility of such a Dynamic Proportion Portfolio Insurance (DPPI) has been mentioned several times in the literature, the author was unable to find any corresponding mathematical analysis. Here we treat DPPI in the same strictly pathwise framework as CPPI. We show that in this framework DPPI can always be defined as a self-financing trading strategy and that its value never breaks through the floor.

The beauty of Föllmer's pathwise approach to continuous-time trading lies in the fact that just one single price trajectory is needed. This corresponds to the reality of financial markets, where prices are given only once and the "experiment" of pricing a given asset in a specific state of the world can never be repeated. A proponent of the frequentist interpretation of probability may thus argue that it is therefore anyway impossible to measure the "objective" probability law according to which market scenarios are selected. But even if one does not share such a strong view on the interpretation of probabilistic models of price evolutions, one will still feel compelled to acknowledge that the complexity of economic dynamics will make it practically impossible to accurately describe the probability law of the price evolution. That is, probabilistic models are subject to Knightian uncertainty and the resulting model risk [21]. In recent years, the issue of Knightian uncertainty in finance has received increasing attention; see, e.g., [8, 11, 13, 17, 19, 20, 22] and [18, Section 5]. In this context, the pathwise approach is remarkable as it completely avoids the choice of a probabilistic model. It was known previously that, for example, hedging strategies for variance swaps and related derivatives could be constructed within this pathwise framework [14, 18]. The present paper now adds that also CPPI strategies can be constructed in a purely pathwise manner, so that our result can also be viewed as a case study in model uncertainty.

In the subsequent Section 2 we first recall some basic facts about Föllmer's pathwise Itô calculus and its financial implementation. Our main results on CPPI and DPPI strategies are stated in Theorems 5 and 7, respectively. The proofs of these results are based on the *associativity* of Föllmer's pathwise Itô integral, which is a result of independent interest. It is stated, among some other facts on pathwise Itô calculus, in Section 3. The proofs of Theorems 5 and 7 are given in Section 4.

2 Statement of results

Constant Proportion Portfolio Insurance (CPPI) is a self-financing investment strategy that allows for a superlinear participation in future assert returns while simultaneously retaining a guaranteed capital level. In the academic literature, this strategy has so far been discussed within various probabilistic models for the evolution of the price process. A common feature of these studies is that price processes are assumed to be semimartingales and market models are often taken as complete. Yet, it is a well-known fact that in a financial context the choice of a probabilistic model is typically itself subject to Knightian uncertainty; see, e.g., [18, Section 5]. Our goal in this paper is to show that this restriction to probabilistic semimartingale models is unnecessary in the case of CPPI strategies. We will show that the strategy works in a strictly pathwise setting that not only includes all continuous semimartingales but also applies to the sample paths of many stochastic processes that are not semimartingales such as price processes based on fractional Brownian motion. More precisely, we will work in a probability-free framework that is based on Föllmer's pathwise Itô calculus [15]. In the context of a financial market model, this pathwise Itô calculus has been applied to the hedging of derivatives in [5] and [16]; see also [28] for an introduction and [18, Section 5.1] for a short, recent survey.

The beauty of the probability-free framework is to assume just one price trajectory as given. We assume that this trajectory includes two assets, a locally riskless bond and a risky asset. Bond prices are described by

$$B_t = \exp\Big(\int_0^t r_s\,ds\Big), \tag{1}$$

where $r : [0, \infty) \to \mathbb{R}$ is measurable and satisfies $\int_0^t |r_s| \, ds < \infty$ for all $t > 0$. Prices of the risky asset are modeled by a single continuous function $S : [0, \infty) \to (0, \infty)$.

In discrete time, trading is possible at time points $0 = t_0 < t_1 < \cdots$, and we assume that $\lim_n t_n = +\infty$. The set $\mathbb{T} = \{t_0, t_1, \dots\}$ is the corresponding *time grid*. Continuous-time trading needs to be defined in terms of an approximation from discrete time. To this end, we fix a sequence $(\mathbb{T}_N)_{N \in \mathbb{N}}$ of time grids satisfying $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \cdots$ and $\lim_N \sup_{t_i \in \mathbb{T}_N} |t_{i+1} - t_i| = 0$. An example of such a sequence is provided by the dyadic time grids, $\mathbb{T}_N = \{k2^{-N} \,|\, k = 0, 1, \dots\}$. Following Föllmer [15], we will say that a continuous trajectory $X : [0, \infty) \to \mathbb{R}$ has *continuous quadratic variation* $[X]$ *along the sequence* $(\mathbb{T}_N)$ if for each $t > 0$ the limit

$$[X]_t := \lim_{N \uparrow \infty} \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \big(X_{t_{i+1}} - X_{t_i} \big)^2 \tag{2}$$

exists, and if $t \mapsto [X]_t$ becomes a continuous function on $[0, \infty)$ for the choice $[X]_0 = 0$. Note that $t \mapsto [X]_t$ is nondecreasing and hence locally of finite variation. The existence of the continuous quadratic variation $[X]$ along $(\mathbb{T}_N)$ guarantees that X can serve as an integrator in Föllmer's pathwise Itô calculus [15]. We state below the corresponding pathwise Itô formula in the form in which it will be needed for the statement of our results on CPPI. Their proofs will require a more general, multidimensional version, which is given in Section 3.

The class $C^{1,2}(\mathbb{R}^n \times \mathbb{R})$ will consist of all functions $f(\boldsymbol{a}, x)$ that are continuously differentiable in $(\boldsymbol{a}, x) \in \mathbb{R}^n \times \mathbb{R}$ and twice continuously differentiable in $x \in \mathbb{R}$. We will write f_{a^k} for the partial derivative of f with respect to the k^{th} coordinate of the vector $\boldsymbol{a} = (a^1, \dots, a^n)$ and f_x and f_{xx} for the first and second partial derivatives with respect to x.

Theorem 1 (Föllmer [15])**.** *Suppose that the continuous trajectory X admits the continuous quadratic variation $[X]$ along $(\mathbb{T}_N)$, that $\boldsymbol{A} : [0, \infty) \to \mathbb{R}^n$ is a continuous function whose components are locally of finite variation, and that $f \in C^{1,2}(\mathbb{R}^n \times \mathbb{R})$. Then*

$$f(\boldsymbol{A}_t, X_t) - f(\boldsymbol{A}_0, X_0) = \sum_{k=1}^n \int_0^t f_{a^k}(\boldsymbol{A}_s, X_s) \, dA_s^k + \int_0^t f_x(\boldsymbol{A}_s, X_s) \, dX_s + \frac{1}{2} \int_0^t f_{xx}(\boldsymbol{A}_s, X_s) \, d[X]_s,$$

where $\int_0^t f_{a^k}(\boldsymbol{A}_s, X_s) \, dA_s^k$ and $\int_0^t f_{xx}(\boldsymbol{A}_s, X_s) \, d[X]_s$ are taken in the usual sense of Riemann–Stieltjes integrals and the pathwise Itô integral $\int_0^t f_x(\boldsymbol{A}_s, X_s) \, dX_s$ *is given by the following limit of nonanticipative Riemann sums:*

$$\int_0^t f_x(\boldsymbol{A}_s, X_s) \, dX_s = \lim_{N \uparrow \infty} \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} f_x(\boldsymbol{A}_{t_i}, X_{t_i})(X_{t_{i+1}} - X_{t_i}). \tag{3}$$

The preceding theorem implies in particular the existence of the pathwise Itô integral (3). We therefore can define a class of admissible integrands:

Definition 2. Suppose that the continuous trajectory X admits the continuous quadratic variation $[X]$ along $(\mathbb{T}_N)$. A real-valued function $t \mapsto \xi_t$ is called an *admissible integrand for* X if for each $T > 0$ there exists $n \in \mathbb{N}$, a function $g \in C^1(\mathbb{R}^{n+1})$, and a continuous function $\boldsymbol{A} : [0, \infty) \to \mathbb{R}^n$ whose components are of finite variation on $[0, T]$ such that $\xi_t = g(\boldsymbol{A}_t, X_t)$ for $0 \le t \le T$.

When ξ is an admissible integrand for X and g and $\boldsymbol{A}$ are as in Definition 2, then $f(\boldsymbol{a}, x) := \int_0^x g(\boldsymbol{a}, y)\,dy$ belongs to $C^{1,2}(\mathbb{R}^n \times \mathbb{R})$, and Theorem 1 implies that the Itô integral

$$\int_0^t f_x(\boldsymbol{A}_s, X_s)\,dX_s = \int_0^t g(\boldsymbol{A}_s, X_s)\,dX_s = \int_0^t \xi_s\,dX_s$$

can be defined through the limit on the right-hand side of (3).

Let us now return to our financial context, in which bond prices are given by (1) and prices of the risky asset are modeled by a continuous path $S : [0, \infty) \to (0, \infty)$. We will assume from now on that S admits the continuous quadratic variation $[S]$ along $(\mathbb{T}_N)$. A trading strategy will be a pair (ξ, η) of of functions $[0, \infty)$, where ξ_t describes the number of shares in the risky asset that are held at time t, while η_t stands for the number of shares in the bond. Using pathwise Itô calculus, we can now define the notion of a self-financing trading strategy:

Definition 3. Let (ξ, η) be a pair of real-valued measurable functions such that ξ is an admissible integrand for S and $\int_0^t |\eta_s r_s|\,ds < \infty$ for all $t \geq 0$. The pair (ξ, η) is called a *self-financing strategy* if the corresponding *portfolio value*,

$$V_t := \xi_t S_t + \eta_t B_t, \quad t \geq 0,$$

satisfies the identity

$$V_t = V_0 + \int_0^t \xi_s\,dS_s + \int_0^t \eta_s\,dB_s, \quad t \geq 0.$$

Remark 4. In the preceding definition, trading strategies are based on the notion of admissible integrands introduced in Definition 2. It is worth pointing out that this notion allows for a large class of integrands, which, for instance, includes the delta hedging strategies for many practically relevant exotic and plain-vanilla options in Markovian market models such as geometric Brownian motion or local volatility; see [26]. Moreover, for $\boldsymbol{A}$ in Definition 2 one can take a continuous function of moving averages, $t \mapsto \int_{(t-\delta)^+}^t S_s\,ds$, or running maxima, $t \mapsto \max_{(t-\delta)^+ \leq s \leq t} S_s$, because these are continuous functions of t with finite variation on every interval $[0, T]$.

We can now proceed toward defining the CPPI strategy in our model-free setting. At time $t = 0$, one is given the initial capital $V_0 > 0$, a security level $\alpha \in [0, 1]$, and a multiplier $m > 0$. The security level specifies the proportion of the initial capital that one is not willing to risk. That is, the portfolio value should never fall below the *floor* $\alpha V_0 B_t$, which one would have attained by investing the fraction αV_0 of the initial capital into the bond right from the start. Now suppose that the portfolio value V_t of the CPPI strategy at time t is already given. The amount

$$C_t := V_t - \alpha V_0 B_t \tag{4}$$

by which the portfolio value exceeds the floor $\alpha V_0 B_t$ is called the *cushion*. The cushion should always be nonnegative so that the amount V_t is indeed bounded from below by $\alpha V_0 B_t$ at any time. In executing a CPPI strategy we invest a multiple $m > 0$ of the cushion into the risky asset. That means that we should have

$$\xi_t := \frac{mC_t}{S_t}. \tag{5}$$

The remaining capital is invested into the riskless asset, i.e.,

$$\eta_t = \frac{V_t - \xi_t S_t}{B_t} = \frac{V_t - mC_t}{B_t}. \tag{6}$$

Note that we can have $mC_t > V_t$, which means that it is possible that the CPPI strategy does not include any risk-free investment and, instead, is short in cash. The formulas (5) and (6) provide a feedback description of the CPPI strategy. It is, however, not clear *a priori* that this feedback description gives rise to a self-financing strategy. More precisely, the following question arises:

- Does there exist a self-financing strategy (ξ, η) whose portfolio value $V_t = \xi_t S_t + \eta_t B_t$ is such that the identities (4), (5), and (6) hold?

When this question can be answered affirmatively, the following two questions arise:

- Is the CPPI strategy free of gap risk? That is, does the portfolio value V_t of the CPPI strategy always exceed the floor $\alpha V_0 B_t$ or, equivalently, do we have $C_t \geq 0$ for all $t \geq 0$?

- Are CPPI strategies unique in the sense that there can be at most one unique self-financing strategy (ξ, η) such that (4), (5), and (6) hold?

Our first main results yields that all three questions can be answered affirmatively. In view of the generality of our setup, this result implies in particular that notions of market completeness or absence of arbitrage are not needed for CPPI to work. It also follows that gap risk is not caused by issues such as market incompleteness. Gap risk only results when one is not able to instantaneously adjust the portfolio in response to asset price changes, as it occurs in the presence of jumps [12] or under constraints on the available trading dates [2, 23].

Theorem 5. *For given $V_0 \geq 0$, $\alpha \in [0, 1]$, and $m > 0$, we define*

$$C_t = (1 - \alpha) V_0 \Big(\frac{S_t}{S_0}\Big)^m B_t^{1-m} e^{-\frac{1}{2}m(m-1)[\log S]_t} \tag{7}$$

and

$$V_t := C_t + \alpha V_0 B_t. \tag{8}$$

Then the equations (4), (5), and (6) define a self-financing strategy (ξ, η) with associated portfolio value V. In particular the CPPI strategy has no gap risk in the sense that its portfolio value always stays above the floor:

$$V_t \geq \alpha V_0 B_t \qquad \text{for all } t \geq 0.$$

Moreover, (ξ, η) is the unique self-financing trading strategy for which (4), (5), and (6) are satisfied.

Remark 6. It is interesting to analyze the various terms in (7) in regards of their contributions to the return of the CPPI strategy. In a Black–Scholes setting, which provides the framework for most academic studies on CPPI strategies, B_t and $[\log S]_t$ are deterministic quantities and can be treated as constants when t is fixed. Therefore the performance of the CPPI strategy can be described as a constant times the m^{th} power, $(S_t/S_0)^m$, of asset returns. This view, however, conceals some of the downside risks that are associated with volatile model parameters. The impact of volatile interest rates is described by the term B_t^{1-m}, which, for the common case $m > 1$, will decrease returns when interest rates go up. Next, the term $e^{-\frac{1}{2}m(m-1)[\log S]_t}$ describes the influence of volatility on the return of the CPPI strategy, because

$$[\log S]_t = \lim_{N\uparrow\infty} \sum_{\substack{t_i,t_{i+1}\in\mathbb{T}_N \\ t_{i+1}\le t}} \big(\log S_{t_{i+1}} - \log S_{t_i}\big)^2 \tag{9}$$

is often called the realized variance of S. The reason for this terminology is the fact that an approximating sum on the right-hand side of (9) can be regarded as the payoff of a variance swap with maturity t; see [9]. Moreover, it follows from [28, Proposition 2.2.10] that $[\log S]_t = \int_0^t \sigma_s^2\,ds$ when $d[S]_t = \sigma_t^2 S_t^2\,dt$. Formula (7) thus states that an increase in realized variance adversely impacts returns by way of the exponential function $x \mapsto e^{-\frac{1}{2}m(m-1)x}$.

Theorem 5 is in fact a corollary of our following, more general result. It deals with the situation in which the multiplier m is not chosen as a constant but may vary in time. Such an extension to *Dynamic Proportion Portfolio Insurance (DPPI)* is natural, because the multiplier m in the CPPI strategy can be regarded as a measure for the leverage of the CPPI investment strategy, and one may wish to choose varying amounts of leverage over time. For instance for a pension fund with a fixed retirement date it can make sense to start off with a high leverage and to revert to a more conservative, lower leverage factor as retirement approaches. Moreover, leverage should be allowed to depend on the current spot, interest rates, and on performance indicators such as realized variance, moving averages, or running maxima. We model this dynamic adjustment of leverage by a continuous multiplier function $m_t \ge 0$. As before, when a security level $\alpha \in [0,1]$ and the value of the investment strategy V_t at time t are given, we define the cushion C_t by

$$C_t = V_t - \alpha V_0 B_t \ge 0 \tag{10}$$

and make the following respective allocations into risky asset and bond:

$$\xi_t = \frac{m_t C_t}{S_t} \qquad \text{and} \qquad \eta_t = \frac{V_t - \xi_t S_t}{B_t} = \frac{V_t - m_t C_t}{B_t}. \tag{11}$$

Theorem 7. *Suppose that $\alpha \in [0,1]$ and $V_0 \ge 0$ are given and that m_t is an admissible integrand for S. Then m_t/S_t is an admissible integrand for S. If we define*

$$C_t := (1-\alpha)V_0 \exp\left(\int_0^t \frac{m_s}{S_s}\,dS_s - \frac{1}{2}\int_0^t \frac{m_s^2}{S_s^2}\,d[S]_s + \int_0^t (1-m_s) r_s\,ds\right) \tag{12}$$

and

$$V_t := C_t + \alpha V_0 B_t, \tag{13}$$

then (10) *and* (11) *defines a self-financing trading strategy with portfolio value* V. *In particular, the DPPI strategy has no gap risk in the sense that its value never breaks through the floor:*

$$V_t \geq \alpha V_0 B_t \qquad \textit{for all } t \geq 0.$$

Moreover, (ξ, η) *is the unique self-financing trading strategy for which* (10) *and* (11) *are satisfied.*

3 Complements on Föllmer's pathwise Itô calculus

Pathwise Itô calculus goes back to Föllmer [15], where a strictly pathwise Itô formula was proved. This topic was further developed in the lectures of Hans Föllmer, some of which form the basis of the book [28]. The proofs of Theorems 5 and 7 require some techniques in pathwise Itô calculus that go beyond the material in [15, 28]. In particular, we need the so-called associativity of the pathwise Itô integral. This property is stated in Theorem 13 and is of independent interest.

The statements of Theorems 5 and 7 involve only the pathwise Itô formula in the one-dimensional form of Theorem 1; their proofs and Theorem 13 require a d-dimensional integrator $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^d)$. So let us recall the pathwise Itô formula in the multidimensional form in which it will henceforth be needed. To enhance the readability, we will write multidimensional objects in boldface type.

We fix a sequence $(\mathbb{T}_N)_{N\in\mathbb{N}}$ of time grids satisfying $\mathbb{T}_1 \subset \mathbb{T}_2 \subset \cdots$ and $\lim_N \sup_{t_i \in \mathbb{T}_N} |t_{i+1} - t_i| = 0$. We moreover suppose that $\boldsymbol{X} : [0, \infty) \to \mathbb{R}^d$ is continuous and that for all k and m the real-valued path $X_t^k + X_t^m$ has continuous quadratic variation $[X^k + X^m]$. This assumption is equivalent to the existence of the *covariation of* X^k *and* X^m defined by

$$\begin{aligned} [X^k, X^m]_t &:= \frac{1}{2}\Big([X^k + X^m]_t - [X^k]_t - [X^m]_t\Big) \\ &= \lim_{N\uparrow\infty} \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \leq t}} \big(X_{t_{i+1}}^k - X_{t_i}^k\big)\big(X_{t_{i+1}}^m - X_{t_i}^m\big). \end{aligned} \tag{14}$$

Here the latter identity follows by polarization of the corresponding sums in (2). Clearly, $[X^k]$ exists as $\frac{1}{4}[X^k + X^k]$. Note that $[X^k, X^m]_t$ is locally of finite variation as a function of t, because it is the difference of the nonincreasing functions $[X^k + X^m]_t$ and $[X^k]_t + [X^m]_t$.

Remark 8. We will need the following facts that easily follow from Propositions 2.2.2, 2.2.9, and 2.3.2 in [28]. Suppose that Y is continuous and admits the continuous quadratic variation $[Y]$ along $(\mathbb{T}_N)$ and A is continuous and locally of finite variation. Then both $[A]$ and $[Y + A]$ exist along $(\mathbb{T}_N)$ and are given by $[A] = 0$ and $[Y + A] = [Y]$. By means of the polarization identity (14) we get moreover that $[Y, A] = 0$.

The class $C^{1,2}(\mathbb{R}^n \times \mathbb{R}^d)$ will consist of all functions $f(\boldsymbol{a}, \boldsymbol{x})$ that are continuously differentiable in $(\boldsymbol{a}, \boldsymbol{x}) \in \mathbb{R}^n \times \mathbb{R}^d$ and twice continuously differentiable in $\boldsymbol{x} \in \mathbb{R}^d$. We will write f_{a^k} for the partial derivative of f with respect to the k^{th} component of $\boldsymbol{a} = (a^1, \dots, a^n)$. The gradient of f in direction $\boldsymbol{x} = (x^1, \dots, x^d)$ will be denoted by

$$\nabla_{\boldsymbol{x}} f = \Big(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^d}\Big),$$

and we will write $f_{x^k x^m}$ for the second partial derivatives with respect to the components x^k and x^m of the vector $\boldsymbol{x}$. The Euclidean inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ will be denoted by $\boldsymbol{x} \cdot \boldsymbol{y}$.

Theorem 9 (Föllmer [15])**.** *Suppose that the continuous trajectory $\boldsymbol{X} : [0, \infty) \to \mathbb{R}^d$ admits for all k and m the continuous covariation $[X^k, X^m]$ along $(\mathbb{T}_N)$, that $\boldsymbol{A} : [0, \infty) \to \mathbb{R}^n$ is a continuous function whose components are locally of finite variation, and that $f \in C^{1,2}(\mathbb{R}^n \times \mathbb{R}^d)$. Then*

$$\begin{aligned} f(\boldsymbol{A}_t, \boldsymbol{X}_t) - f(\boldsymbol{A}_0, \boldsymbol{X}_0) &= \int_0^t \nabla_{\boldsymbol{x}} f(\boldsymbol{A}_s, \boldsymbol{X}_s)\, d\boldsymbol{X}_s + \sum_{k=1}^n \int_0^t f_{a^k}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, dA_s^k \\ &\quad + \frac{1}{2} \sum_{k,m=1}^d \int_0^t f_{x^k x^m}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, d[X^k, X^m]_s, \end{aligned}$$

where $\int_0^t f_{a^k}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, dA_s^k$ and $\int_0^t f_{x^k x^m}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, d[X^k, X^m]_s$ are taken in the usual sense of Stieltjes integrals and the pathwise Itô integral *is given by the following limit of nonanticipative Riemann sums:*

$$\int_0^t \nabla_{\boldsymbol{x}} f(\boldsymbol{A}_s, \boldsymbol{X}_s)\, d\boldsymbol{X}_s = \lim_{N \uparrow \infty} \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \nabla_{\boldsymbol{x}} f(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i}) \cdot (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}). \tag{15}$$

Proof. For $f \in C^2(\mathbb{R}^{n+d})$ the result follows from Remarque 1 in [15] and by noting that the quadratic variations $[A^k]$ and covariations $[A^k, A^\ell]$ and $[A^k, X^i]$ $(k, \ell = 1, \dots, n,\ i = 1 \dots, d)$ vanish identically according to Remark 8. The extension to $f \in C^{1,2}(\mathbb{R}^n \times \mathbb{R}^d)$ is obtained just as in the proof of our Theorem 13 below by using Taylor development of $f(\boldsymbol{a}, \boldsymbol{x})$ up to first order in $\boldsymbol{a}$ and up to second order in $\boldsymbol{x}$. ☐

Remark 10. In (15) it is typically *not* possible to write

$$\int_0^t \nabla_{\boldsymbol{x}} f(\boldsymbol{A}_s, \boldsymbol{X}_s)\, d\boldsymbol{X}_s = \sum_{i=1}^d \int_0^t f_{x^i}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, dX_s^i,$$

because the integrals $\int_0^t f_{x^i}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, dX_s^i$ on the right-hand side need not exist individually as the limits of nonanticipative Riemann sums.

Theorem 9 implies in particular that the pathwise Itô integral $\int_0^t \boldsymbol{\xi}_s\, d\boldsymbol{X}_s$ can be defined via (15) when the integrand $\boldsymbol{\xi}$ is of the form $\boldsymbol{\xi}_t = \nabla_{\boldsymbol{x}} f(\boldsymbol{A}_s, \boldsymbol{X}_s)$ for some continuous function $\boldsymbol{A} : [0, \infty) \to \mathbb{R}^n$ whose components are locally of finite variation and for $f \in C^{1,2}(\mathbb{R}^n \times \mathbb{R}^d)$. Since in the case $d > 1$ not every C^1-function $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$ is of the form $\boldsymbol{g} = \nabla_{\boldsymbol{x}} f$ for some $f \in C^{1,2}(\mathbb{R}^n \times \mathbb{R}^d)$, the following definition of d-dimensional admissible integrands needs to be slightly more complicated than its one-dimensional counterpart, Definition 2.

Definition 11. Suppose that the continuous trajectory $\boldsymbol{X} : [0, \infty) \to \mathbb{R}^d$ admits the continuous covariations $[X^k, X^m]$ along $(\mathbb{T}_N)$, $k, m = 1, \ldots, d$. A function $t \mapsto \boldsymbol{\xi}_t \in \mathbb{R}^d$ is called an *admissible integrand for* $\boldsymbol{X}$ if for each $T > 0$ there exists $n \in \mathbb{N}$, a function $f \in C^{1,2}(\mathbb{R}^n \times \mathbb{R}^d)$, and a continuous function $\boldsymbol{A} : [0, \infty) \to \mathbb{R}^n$ whose components are of finite variation on $[0, T]$ such that $\boldsymbol{\xi}_t = \nabla_{\boldsymbol{x}} f(\boldsymbol{A}_t, X_t)$ for $0 \le t \le T$.

The following result is a straightforward extension of [28, Proposition 2.3.3], and its proof is left to the reader.

Proposition 12. *Suppose that* $\boldsymbol{X}$ *is as in Theorem 9, that* $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(\nu)}$ *are admissible integrands for* $\boldsymbol{X}$*, and that*

$$Y_t^\ell := \int_0^t \boldsymbol{\xi}_s^{(\ell)} \, d\boldsymbol{X}_s, \qquad \ell = 1, \ldots, \nu.$$

Then $\boldsymbol{Y}_t = (Y_t^1, \ldots, Y_t^\nu)$ *is a continuous trajectory that admits the continuous covariations*

$$[Y^k, Y^\ell]_t = \sum_{i,j=1}^d \int_0^t \xi_s^{(k),i} \xi^{(\ell),j} \, d[X^i, X^j]_s, \qquad k, \ell = 1, \ldots, \nu.$$

The preceding proposition implies in particular that $\boldsymbol{Y}_t = (Y_t^1, \ldots, Y_t^\nu)$ is again an admissible integrator for pathwise Itô calculus. The following *associativity rule* for the pathwise Itô integral shows that one can express a pathwise Itô integral with respect to $\boldsymbol{Y}$ as a pathwise Itô integral with respect to $\boldsymbol{X}$.

Theorem 13 (Associativity of the pathwise Itô integral)**.** *Suppose that* $\boldsymbol{X}$*,* $\boldsymbol{\xi}^{(1)}, \ldots, \boldsymbol{\xi}^{(\nu)}$ *and* $\boldsymbol{Y}$ *are as in Proposition 12, and let* $\boldsymbol{\eta} = (\eta^1, \ldots, \eta^\nu)$ *be an admissible integrand for* $\boldsymbol{Y}$*. Then* $\sum_{\ell=1}^\nu \eta^\ell \boldsymbol{\xi}^{(\ell)}$ *is an admissible integrand for* $\boldsymbol{X}$ *and*

$$\int_0^t \boldsymbol{\eta}_s \, d\boldsymbol{Y}_s = \int_0^t \sum_{\ell=1}^\nu \eta_s^\ell \boldsymbol{\xi}_s^{(\ell)} \, d\boldsymbol{X}_s.$$

Proof of Theorem 13. We fix $T \ge 0$. For $t \le T$, let $\boldsymbol{\xi}^{(\ell)}$ be of the form $\boldsymbol{\xi}_t^{(\ell)} = \nabla_{\boldsymbol{x}} f^\ell(\boldsymbol{A}_t^{(\ell)}, \boldsymbol{X}_t)$ for $n_\ell \in \mathbb{N}$, continuous $\boldsymbol{A}^{(\ell)} : [0, T] \to \mathbb{R}^{n_\ell}$ with components of finite variation, and $f^\ell \in C^{1,2}(\mathbb{R}^{n_\ell} \times \mathbb{R}^d)$. We also define

$$A_t^{(\ell),n_\ell+1} := \sum_{k=1}^{n_\ell} \int_0^t f_{a^k}^\ell(\boldsymbol{A}_s^{(\ell)}, \boldsymbol{X}_s) \, dA_s^k + \frac{1}{2} \sum_{k,m=1}^d \int_0^t f_{x^k x^m}^\ell(\boldsymbol{A}_s^{(\ell)}, \boldsymbol{X}_s) \, d[X^k, X^m]_s. \tag{16}$$

Then $A^{(\ell),n_\ell+1}$ is continuous and of finite variation on $[0, T]$ by standard properties of Stieltjes integrals (see [29, Theorem I.5c]). Moreover, the pathwise Itô formula from Theorem 1 implies that

$$Y_t^\ell = f^\ell(\boldsymbol{A}_t^{(\ell)}, \boldsymbol{X}_t) - f^\ell(\boldsymbol{A}_0^{(\ell)}, \boldsymbol{X}_0) - A_t^{(\ell),n_\ell+1} = F^\ell(\widetilde{\boldsymbol{A}}_t^{(\ell)}, \boldsymbol{X}_t) \tag{17}$$

where

$$\widetilde{\boldsymbol{A}}_t^{(\ell)} := (A_t^{(\ell),1}, \ldots, A_t^{(\ell),n_\ell}, A_t^{(\ell),n_\ell+1})$$

and

$$F^\ell(\widetilde{\boldsymbol{a}},\boldsymbol{x}) := f^\ell(\boldsymbol{a},\boldsymbol{x}) - f^\ell(\boldsymbol{A}_0,\boldsymbol{X}_0) - a^{n_\ell+1} \qquad \text{for } \widetilde{\boldsymbol{a}} = (\boldsymbol{a}, a^{n_\ell+1}) \in \mathbb{R}^{n_\ell}\times\mathbb{R}.$$

Clearly, $\widetilde{\boldsymbol{A}}^{(\ell)} : [0,T] \to \mathbb{R}^{n_\ell+1}$ is continuous and has finite total variation on $[0,T]$, and F^ℓ belongs to $C^{1,2}(\mathbb{R}^{n_\ell+1}\times\mathbb{R}^d)$. Moreover,

$$\nabla_{\boldsymbol{x}} F^\ell(\widetilde{\boldsymbol{a}},\boldsymbol{x}) = \nabla_{\boldsymbol{x}} f^\ell(\boldsymbol{a},\boldsymbol{x}) \qquad \text{for } \widetilde{\boldsymbol{a}} = (\boldsymbol{a}, a^{n_\ell+1}) \in \mathbb{R}^{n_\ell}\times\mathbb{R}. \tag{18}$$

Let us denote

$$\boldsymbol{F}(\boldsymbol{a},\boldsymbol{x}) := \big(F^1(\widetilde{\boldsymbol{a}}^{(1)},\boldsymbol{x}),\dots,F^\nu(\widetilde{\boldsymbol{a}}^{(\nu)},\boldsymbol{x})\big) \qquad \text{for } \boldsymbol{a} = (\widetilde{\boldsymbol{a}}^{(1)},\dots,\widetilde{\boldsymbol{a}}^{(\nu)}) \in \mathbb{R}^{n_1+\cdots+n_\nu+\nu}.$$

By writing $\boldsymbol{A}_t := (\widetilde{\boldsymbol{A}}^{(1)},\dots,\widetilde{\boldsymbol{A}}^{(\nu)})$, the identity (17) becomes

$$\boldsymbol{Y}_t = \boldsymbol{F}(\boldsymbol{A}_t,\boldsymbol{X}_t). \tag{19}$$

Since $\boldsymbol{\eta}$ is an admissible integrand for $\boldsymbol{Y}$, there are $m \in \mathbb{N}$, $h \in C^{1,2}(\mathbb{R}^m\times\mathbb{R}^\nu)$, and continuous $\boldsymbol{D} : [0,T] \to \mathbb{R}^m$ with finite variation such that $\boldsymbol{\eta}_t = \nabla_{\boldsymbol{y}} h(\boldsymbol{D}_t,\boldsymbol{Y}_t)$ for $0 \le t \le T$. Using (19), (18), and the notation $\nabla_{\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{a},\boldsymbol{x})$ for the Jacobi matrix of $\boldsymbol{x} \mapsto \boldsymbol{F}(\boldsymbol{a},\boldsymbol{x})$, we get

$$\begin{aligned}\sum_{\ell=1}^{\nu} \eta_t^\ell \boldsymbol{\xi}_t^{(\ell)} &= \sum_{\ell=1}^{\nu} h_{y^\ell}(\boldsymbol{D}_t,\boldsymbol{Y}_t)\nabla_{\boldsymbol{x}} f^\ell(\boldsymbol{A}_t^{(\ell)},\boldsymbol{X}_t) = \nabla_{\boldsymbol{y}} h(\boldsymbol{D}_t,\boldsymbol{F}(\boldsymbol{A}_t,\boldsymbol{X}_t))\cdot\nabla_{\boldsymbol{x}}\boldsymbol{F}(\boldsymbol{A}_t,\boldsymbol{X}_t)\\ &= \nabla_{\boldsymbol{x}}\widetilde{h}(\widetilde{\boldsymbol{D}}_t,\boldsymbol{X}_t),\end{aligned}$$

where $\widetilde{\boldsymbol{D}}_t = (\boldsymbol{D}_t,\boldsymbol{A}_t)$, and $\widetilde{h}((\boldsymbol{d},\boldsymbol{a}),\boldsymbol{x}) := h(\boldsymbol{d},\boldsymbol{F}(\boldsymbol{a},\boldsymbol{x}))$ belongs to $C^{1,2}(\mathbb{R}^K\times\mathbb{R}^d)$ for $K = m + n_1 + \cdots + n_\nu + \nu$. It follows in particular that $\sum_{i=1}^\nu \eta^\ell\boldsymbol{\xi}^{(\ell)}$ is an admissible integrand for $\boldsymbol{X}$.

The definition (15) of the Itô integral and (19) imply that

$$\begin{aligned}\int_0^t \boldsymbol{\eta}_s\,d\boldsymbol{Y}_s &= \lim_{N\uparrow\infty}\sum_{\substack{t_i,t_{i+1}\in\mathbb{T}_N\\ t_{i+1}\le t}} \boldsymbol{\eta}_{t_i}\cdot\big(\boldsymbol{F}(\boldsymbol{A}_{t_{i+1}},\boldsymbol{X}_{t_{i+1}}) - \boldsymbol{F}(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})\big)\\ &= \lim_{N\uparrow\infty}\sum_{\substack{t_i,t_{i+1}\in\mathbb{T}_N\\ t_{i+1}\le t}}\sum_{\ell=1}^{\nu}\eta_{t_i}^\ell\left(F^\ell(\boldsymbol{A}_{t_{i+1}},\boldsymbol{X}_{t_{i+1}}) - F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})\right),\end{aligned} \tag{20}$$

where, by abuse of notation, we write $F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})$ instead of $F^\ell(\widetilde{\boldsymbol{A}}_{t_i}^{(\ell)},\boldsymbol{X}_{t_i})$. Using multidimensional Taylor development up to first order in $\boldsymbol{a}$ and up to second order in $\boldsymbol{x}$, we get

$$\begin{aligned}&F^\ell(\boldsymbol{A}_{t_{i+1}},\boldsymbol{X}_{t_{i+1}}) - F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})\\ &\quad= F^\ell(\boldsymbol{A}_{t_{i+1}},\boldsymbol{X}_{t_{i+1}}) - F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_{i+1}}) + F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_{i+1}}) - F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})\\ &\quad= \nabla_{\boldsymbol{a}}F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})\cdot(\boldsymbol{A}_{t_{i+1}}-\boldsymbol{A}_{t_i}) + \boldsymbol{\delta}_i^\ell\cdot(\boldsymbol{A}_{t_{i+1}}-\boldsymbol{A}_{t_i})\\ &\qquad+\nabla_{\boldsymbol{x}}F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})\cdot(\boldsymbol{X}_{t_{i+1}}-\boldsymbol{X}_{t_i})\\ &\qquad+\frac{1}{2}(\boldsymbol{X}_{t_{i+1}}-\boldsymbol{X}_{t_i})\cdot\nabla_{\boldsymbol{x}}^2F^\ell(\boldsymbol{A}_{t_i},\boldsymbol{X}_{t_i})(\boldsymbol{X}_{t_{i+1}}-\boldsymbol{X}_{t_i}) + (\boldsymbol{X}_{t_{i+1}}-\boldsymbol{X}_{t_i})\cdot\boldsymbol{\varepsilon}_i^\ell(\boldsymbol{X}_{t_{i+1}}-\boldsymbol{X}_{t_i}),\end{aligned}$$

where

$$\boldsymbol{\delta}_i^\ell = \int_0^1 \nabla_{\boldsymbol{a}} F^\ell\big(\boldsymbol{A}_{t_i} + s(\boldsymbol{A}_{t_{i+1}} - \boldsymbol{A}_{t_i}), \boldsymbol{X}_{t_{i+1}}\big)\, ds - \nabla_{\boldsymbol{a}} F^\ell(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i}),$$

$\nabla^2_{\boldsymbol{x}} F^\ell$ is the Hessian of F^ℓ with respect to $\boldsymbol{x}$, and, for some $\theta \in [0,1]$,

$$\boldsymbol{\varepsilon}_i^\ell = \frac{1}{2}\Big(\nabla^2_{\boldsymbol{x}} F^\ell\big(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i} + \theta(\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i})\big) - \nabla^2_{\boldsymbol{x}} F^\ell(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i})\Big).$$

The continuity of $\nabla_{\boldsymbol{a}} F^\ell$ and $\nabla^2_{\boldsymbol{x}} F^\ell$ implies that

$$\max_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le T}} \big(|\boldsymbol{\delta}_i^\ell| + \|\boldsymbol{\varepsilon}_i^\ell\|\big) \longrightarrow 0 \qquad \text{as } N \uparrow \infty,$$

where $|\cdot|$ denotes the Euclidean norm and $\|\boldsymbol{\varepsilon}_i^\ell\|^2 := \max_{|\boldsymbol{x}|=1} \boldsymbol{x} \cdot \boldsymbol{\varepsilon}_i^\ell \boldsymbol{x}$. When denoting the total variation of A^k over the interval $[0,T]$ by $\|A^k\|_{\mathrm{var}}$, we thus get

$$\Big| \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \boldsymbol{\delta}_i^\ell \cdot (\boldsymbol{A}_{t_{i+1}} - \boldsymbol{A}_{t_i}) \Big| \le \max_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le T}} |\boldsymbol{\delta}_i^\ell| \sum_{k=1}^K \|A^k\|_{\mathrm{var}} \longrightarrow 0,$$

as $N \uparrow \infty$. Furthermore,

$$\Big| \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \cdot \boldsymbol{\varepsilon}_i^\ell (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \Big| \le \max_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le T}} \|\boldsymbol{\varepsilon}_i^\ell\|^2 \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \cdot (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}).$$

Since the rightmost sum converges to the finite limit $[X^1]_t + \cdots + [X^d]_t$, the right-hand side above tends to zero as $N \uparrow \infty$.

Next, the standard existence result for Stieltjes integrals (e.g., [29, Theorem I.4a]) implies that

$$\begin{aligned}
&\lim_{N \uparrow \infty} \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \eta^\ell_{t_i} \nabla_{\boldsymbol{a}} F^\ell(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i}) \cdot (\boldsymbol{A}_{t_{i+1}} - \boldsymbol{A}_{t_i}) \\
&= \sum_{k=1}^K \int_0^t \eta_s^\ell F^\ell_{a^k}(\boldsymbol{A}_s, \boldsymbol{X}_s)\, dA^k_s \\
&= \sum_{k=1}^{n_\ell} \int_0^t \eta_s^\ell f^\ell_{a^k}(\boldsymbol{A}^{(\ell)}_s, \boldsymbol{X}_s)\, dA^{(\ell),k}_s - \int_0^t \eta_s^\ell\, dA^{(\ell),n_\ell+1}_s \\
&= -\frac{1}{2} \sum_{k,m=1}^d \int_0^t f^\ell_{x^k x^m}(\boldsymbol{A}^{(\ell)}_s, \boldsymbol{X}_s)\, d[X^k, X^m]_s,
\end{aligned}$$

where we have used (16) and the associativity of the Stieltjes integral [29, Theorem I.6b] in the final step.

Next, as observed in [15], taking $X = X^k$ in (2), the convergence in (2) can be interpreted as vague convergence of the point measures

$$\sum_{t_i, t_{i+1} \in \mathbb{T}_N} (X^k_{t_{i+1}} - X^k_{t_i})^2 \delta_{t_i}$$

toward the continuous and nonnegative Radon measure $d[X^k]_t$. Therefore,

$$\sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \varphi(t_i)(X^k_{t_{i+1}} - X^k_{t_i})^2 \longrightarrow \int_0^t \varphi(s)\, d[X^k]_s$$

holds for any continuous function φ due to the portmanteau theorem (e.g., [1, Theorem 14.3]). Via the polarization identity in (14), we get the analogous result for the covariation $[X^k, X^m]$ replacing $[X^k]$. This implies

$$\begin{aligned}
&\sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \eta^\ell_{t_i} (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \cdot \nabla^2_{\boldsymbol{x}} F^\ell(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i})(\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \\
&= \sum_{k,m=1}^d \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \eta^\ell_{t_i} F^\ell_{x^k x^m}(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i})(X^k_{t_{i+1}} - X^k_{t_i})(X^m_{t_{i+1}} - X^m_{t_i}) \\
&\longrightarrow \sum_{k,m=1}^d \int_0^t \eta^\ell_s F^\ell_{x^k x^m}(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i})\, d[X^k, X^m]_s \\
&= \sum_{k,m=1}^d \int_0^t \eta^\ell_s f^\ell_{x^k x^m}(\boldsymbol{A}^{(\ell)}_{t_i}, \boldsymbol{X}_{t_i})\, d[X^k, X^m]_s.
\end{aligned}$$

Moreover, $\nabla_{\boldsymbol{x}} F^\ell(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i}) = \nabla_{\boldsymbol{x}} f^\ell(\boldsymbol{A}^{(\ell)}_{t_i}, \boldsymbol{X}_{t_i}) = \boldsymbol{\xi}^{(\ell)}_t$, and so

$$\begin{aligned}
\sum_{\ell=1}^\nu \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \eta^\ell_{t_i} \nabla_{\boldsymbol{x}} F^\ell(\boldsymbol{A}_{t_i}, \boldsymbol{X}_{t_i}) \cdot (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \quad &= \sum_{\substack{t_i, t_{i+1} \in \mathbb{T}_N \\ t_{i+1} \le t}} \sum_{\ell=1}^\nu \eta^\ell_{t_i} \boldsymbol{\xi}^{(\ell)}_{t_i} \cdot (\boldsymbol{X}_{t_{i+1}} - \boldsymbol{X}_{t_i}) \\
&\longrightarrow \int_0^t \sum_{\ell=1}^\nu \eta^\ell_s \boldsymbol{\xi}^{(\ell)}_s \, d\boldsymbol{X}_s.
\end{aligned}$$

Putting everything together, we see that the limit on the right-hand side of (20) is given by

$$\begin{aligned}
-\frac{1}{2} \sum_{k,m=1}^d \int_0^t f^\ell_{x^k x^m}(\boldsymbol{A}^{(\ell)}_s, \boldsymbol{X}_s)\, d[X^k, X^m]_s + \frac{1}{2} \sum_{k,m=1}^d \int_0^t f^\ell_{x^k x^m}(\boldsymbol{A}^{(\ell)}_s, \boldsymbol{X}_s)\, d[X^k, X^m]_s \\
+ \int_0^t \sum_{\ell=1}^\nu \eta^\ell_s \boldsymbol{\xi}^{(\ell)}_s \, d\boldsymbol{X}_s = \int_0^t \sum_{\ell=1}^\nu \eta^\ell_s \boldsymbol{\xi}^{(\ell)}_s \, d\boldsymbol{X}_s.
\end{aligned}$$

This concludes the proof. □

4 Proofs of Theorems 7 and 5

Proof of Theorem 7. We note first that m_t/S_t is an admissible integrand for S, because m_t is an admissible integrand, and $1/S_t$ can locally for $t \in [0, T]$ be written as $f(S_t)$ for some function

$f \in C^1(\mathbb{R})$ since S_t is bounded away from zero for $0 \leq t \leq T$. In particular, formula (12) is well-defined. Let us write

$$C_t = X_t A_t,$$

where

$$X_t := \exp\left(\int_0^t \frac{m_s}{S_s}\, dS_s - \frac{1}{2}\int_0^t \frac{m_s^2}{S_s^2}\, d[S]_s\right)$$

and

$$A_t := (1-\alpha)V_0 \exp\left(\int_0^t (1-m_s) r_s\, ds\right).$$

Using the function $f(a,x) := ax$ in Theorem 1 yields the integration by parts formula

$$C_t - C_0 = X_t A_t - X_0 A_0 = \int_0^t A_s\, dX_s + \int_0^t X_s\, dA_s. \tag{21}$$

We now define

$$Y_t := \int_0^t \frac{m_s}{S_s}\, dS_s \qquad \text{and} \qquad L_t := \frac{1}{2}\int_0^t \frac{m_s^2}{S_s^2}\, d[S]_s.$$

Then L is continuous and of locally finite variation by standard properties of the Stieltjes integral (see [29, Theorem I.5c]), and Y has the continuous quadratic variation $[Y]_t = 2L_t$ by Proposition 12. Moreover, applying Theorem 1 to the function $g(a,y) = e^{y-a}$ yields

$$\begin{aligned} X_t - X_0 &= g(L_t, Y_t) - g(L_0, Y_0) \\ &= \int_0^t g_y(L_s, Y_s)\, dY_s + \int_0^t g_a(L_s, Y_s)\, dL_s + \frac{1}{2}\int_0^t g_{yy}(L_s, Y_s)\, d[Y]_s \\ &= \int_0^t X_s\, dY_s, \end{aligned}$$

where we have applied Theorem 13 to the Stieltjes integral $\int_0^t g_\ell(L_s, Y_s)\, dL_s$ (instead of Theorem 13 one can here also apply [29, Theorem I.6b]). We also have

$$A_t - A_0 = \int_0^t A_s(1-m_s) r_s\, ds = \int_0^t \frac{A_s(1-m_s)}{B_s}\, dB_s.$$

Plugging these results into (21) and applying Theorem 13 several times yields that

$$\frac{X_s A_s m_s}{S_s} = \frac{m_s C_s}{S_s} = \xi_s$$

is an admissible integrand for S and that

$$\begin{aligned} C_t - C_0 &= \int_0^t A_s X_s\, dY_s + \int_0^t \frac{X_s A_s(1-m_s)}{B_s}\, dB_s \\ &= \int_0^t \frac{m_s C_s}{S_s}\, dS_s + \int_0^t \frac{C_s(1-m_s)}{B_s}\, dB_s. \end{aligned} \tag{22}$$

It follows that $V_t := C_t + \alpha V_0 B_t$ satisfies

$$V_t - V_0 = \int_0^t \frac{m_s C_s}{S_s}\,dS_s + \int_0^t \frac{C_s(1-m_s) + \alpha V_0 B_s}{B_s}\,dB_s = \int_0^t \xi_s\,dS_s + \int_0^t \eta_s\,dB_s,$$

where ξ and η are as in (11). Finally, we clearly have

$$V_t = \xi_t S_t + \eta_t B_t,$$

which shows that (ξ, η) is indeed a self-financing strategy with portfolio value V.

Now we turn toward the proof of the uniqueness of the DPPI strategy. To this end, let (ξ, η) be the self-financing strategy constructed above, with portfolio value $V_t = \xi_t S_t + \eta_t B_t$ and cushion $C_t = V_t - \alpha V_0 B_t$. Suppose moreover that $(\widetilde{\xi}, \widetilde{\eta})$ is another self-financing strategy with portfolio value $\widetilde{V}_t = \widetilde{\xi}_t S_t + \widetilde{\eta}_t B_t$ and cushion $\widetilde{C}_t = \widetilde{V}_t - \alpha V_0 B_t$ such that $\widetilde{V}_0 = V_0$ and $\widetilde{\xi}_t = m_t \widetilde{C}_t / S_t$.

From the self-financing condition we necessarily have that

$$\widetilde{\eta}_t = \frac{\widetilde{V}_t - \widetilde{\xi}_t S_t}{B_t} = \frac{\widetilde{C}_t + \alpha V_0 B_t - m_t \widetilde{C}_t}{B_t}$$

and hence

$$\begin{aligned}
\widetilde{C}_t - \widetilde{C}_0 &= \widetilde{V}_t - V_0 - \alpha V_0 (B_t - B_0) \\
&= \int_0^t \frac{m_s \widetilde{C}_s}{S_s}\,dS_s + \int_0^t \frac{\widetilde{C}_s + \alpha V_0 B_s - m_s \widetilde{C}_s}{B_s}\,dB_s - \int_0^t \alpha V_0\,dB_s \\
&= \int_0^t \frac{m_s \widetilde{C}_s}{S_s}\,dS_s + \int_0^t (1-m_s)\widetilde{C}_s r_s\,ds
\end{aligned}$$

In the preceding part of the proof we showed that C_t satisfies the same Itô integral equation; see (22). When letting

$$Y_t^{(1)} := e^{-\int_0^t (1-m_s) r_s\,ds}\widetilde{C}_t \qquad \text{and} \qquad Y_t^{(2)} := e^{-\int_0^t (1-m_s) r_s\,ds} C_t,$$

one easily checks via (21) that $Y^{(i)}$ satisfies

$$Y_t^{(i)} = Y_0^{(i)} + \int_0^t \frac{m_s Y_s^{(i)}}{S_s}\,dS_s, \qquad 0 \le t \le T,\ i = 1, 2. \tag{23}$$

Here, the pathwise Itô integral exists since, e.g., $m_t Y_t^{(1)} / S_t = e^{-\int_0^t (1-m_s) r_s\,ds}\widetilde{\xi}_t$ is clearly an admissible integrand for S.

It follows from equation (23), Remark 8, and Proposition 12 that the quadratic variations $[Y^{(i)}]$ and the covariation $[Y^{(1)}, Y^{(2)}]$ exist and are given by

$$[Y^{(i)}]_t = \int_0^t \frac{m_t^2 (Y_t^{(i)})^2}{S_t^2}\,d[S]_t \qquad \text{and} \qquad [Y^{(1)}, Y^{(2)}]_t = \int_0^t \frac{m_t^2 Y_t^{(1)} Y_t^{(2)}}{S_t^2}\,d[S]_t. \tag{24}$$

In particular, $\boldsymbol{Y}_t = (Y_t^{(1)}, Y_t^{(2)})$ can be used as integrator in the pathwise Itô formula.

Now let $T > 0$ be given. Then by (12) there exists $\varepsilon > 0$ such that $Y_t^{(2)} \geq \varepsilon$ for $0 \leq t \leq T$. Let $f \in C^2(\mathbb{R}^2)$ be a function such that $f(y^1, y^2) = y^1/y^2$ for $y^2 \geq \varepsilon/2$. Theorem 9 then yields that

$$\begin{aligned} f(\boldsymbol{Y}_t) - f(\boldsymbol{Y}_0) = \int_0^t \nabla f(\boldsymbol{Y}_s)\, d\boldsymbol{Y}_s + \frac{1}{2}\int_0^t f_{y^1y^1}(\boldsymbol{Y}_s)\, d[Y^{(1)}]_s \\ + \frac{1}{2}\int_0^t f_{y^2y^2}(\boldsymbol{Y}_s)\, d[Y^{(2)}]_s + \int_0^t f_{y^1y^2}(\boldsymbol{Y}_s)\, d[Y^{(1)}, Y^{(2)}]_s. \end{aligned} \tag{25}$$

Applying Theorem 13 with $\nu = 2$, $\boldsymbol{\eta}_t = \nabla f(\boldsymbol{Y}_s)$, $\xi_t^{(\ell)} := m_s Y_s^{(\ell)}/S_s$, $d = 1$, and $X = S$ yields that the Itô integral above is given by

$$\int_0^t \nabla f(\boldsymbol{Y}_s)\, d\boldsymbol{Y}_s \;=\; \int_0^t \left(f_{y^1}(\boldsymbol{Y}_s) \frac{m_s Y_s^{(1)}}{S_s} + f_{y^2}(\boldsymbol{Y}_s) \frac{m_s Y_s^{(2)}}{S_s} \right) dS_s.$$

Since $f_{y^1}(\boldsymbol{Y}_s) = 1/Y_t^{(2)}$ and $f_{y^2}(\boldsymbol{Y}_s) = -Y_t^{(1)}/(Y_t^{(2)})^2$, we see that the integrand of the right-hand integral vanishes. Hence $\int_0^t \nabla f(\boldsymbol{Y}_s)\, d\boldsymbol{Y}_s = 0$ for $0 \leq t \leq T$. Moreover, $f_{y^1y^1} = 0$ and so also the second integral on the right-hand side of (25) vanishes. Finally, one easily shows with (24) and the associativity of the Stieltjes integral that the remaining two integrals on the right-hand side of (25) add up to zero. Thus, $Y_t^{(1)}/Y_t^{(2)} = f(\boldsymbol{Y}_t) = f(\boldsymbol{Y}_0) = Y_0^{(1)}/Y_0^{(2)} = 1$ and so $\widetilde{C}_t = C_t$ for all $t \in [0, T]$. Therefore the uniqueness of the DPPI strategy follows. □

Proof of Theorem 5. Take $T > 0$ and let $\varepsilon > 0$ be such that $S_t \geq \varepsilon$ for $0 \leq t \leq T$. Then we take $f \in C^2(\mathbb{R})$ such that $f(x) = \log x$ for $x \geq \varepsilon/2$. When m is constant, an application of the pathwise Itô formula to $mf(S_t)$ yields that

$$\int_0^t \frac{m}{S_s}\, dS_s = m \log S_t - m \log S_0 + \frac{m}{2}\int_0^t \frac{1}{S_s^2}\, d[S]_s.$$

Moreover, [28, Proposition 2.2.10] yields that

$$[\log S]_t = \int_0^t \frac{1}{S_s^2}\, d[S]_s.$$

Hence, formula (12) becomes

$$\begin{aligned} C_t &= (1-\alpha)V_0 \exp\left(m \log S_t - m \log S_0 - \frac{m(m-1)}{2}\int_0^t \frac{1}{S_s^2}\, d[S]_s + (1-m)\int_0^t r_s\, ds \right) \\ &= (1-\alpha)V_0 \left(\frac{S_t}{S_0}\right)^m e^{-\frac{1}{2}m(m-1)[\log S]_t} B_t^{1-m}. \end{aligned}$$

This concludes the proof. □

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