HECKE GRIDS AND CONGRUENCES FOR WEAKLY HOLOMORPHIC MODULAR FORMS

SCOTT AHLGREN AND NICKOLAS ANDERSEN

ABSTRACT. Let U(p) denote the Atkin operator of prime index p. Honda and Kaneko proved infinite families of congruences of the form $f|U(p) \equiv 0 \pmod{p}$ for weakly holomorphic modular forms of low weight and level and primes p in certain residue classes, and conjectured the existence of similar congruences modulo higher powers of p. Partial results on some of these conjectures were proved recently by Guerzhoy. We construct infinite families of weakly holomorphic modular forms on the Fricke groups $\Gamma^*(N)$ for N=1,2,3,4 and describe explicitly the action of the Hecke algebra on these forms. As a corollary, we obtain strengthened versions of all of the congruences conjectured by Honda and Kaneko.

1. Introduction

For a prime number p, let U(p) denote Atkin's operator, which acts on power series via

$$\left(\sum a(n)q^n\right)\big|U(p):=\sum a(pn)q^n.$$

In recent work, Honda and Kaneko [4] generalize a theorem of Garthwaite [2] in order to establish infinite families of congruences of the form

$$f \big| U(p) \equiv 0 \pmod{p}$$

for weakly holomorphic modular forms of low weight and level. For example, it is shown that for any prime $p \equiv 1 \pmod{3}$ and any $k \in \{4, 6, 8, 10, 14\}$ we have

$$\frac{E_k(6z)}{\eta^4(6z)} | U(p) \equiv 0 \pmod{p}. \tag{1.1}$$

For another example, if $p \equiv 1 \pmod{4}$, $k \in \{4, 6\}$, and $f \in M_k(\Gamma_0(2))$ has p-integral Fourier expansion, then it is shown that

$$\frac{f(4z)}{\eta^2(4z)\eta^2(8z)} | U(p) \equiv 0 \pmod{p}. \tag{1.2}$$

Honda and Kaneko conjecture that these extend to congruences modulo higher powers of p. For example, they conjecture that for any $p \equiv 1 \pmod{3}$, the congruence (1.1) can be replaced by

$$\frac{E_k(6z)}{\eta^4(6z)} | U(p^n) \equiv 0 \pmod{p^{n(k-3)}} \text{ for any } n \ge 1.$$
 (1.3)

Date: June 28, 2018.

²⁰¹⁰ Mathematics Subject Classification. Primary 11F33.

The first author was supported by a grant from the Simons Foundation (#208525 to Scott Ahlgren).

In recent work, Guerzhoy [3] studies the conjectures (1.3) using the p-adic theory of weak harmonic Maass forms. In the case when k = 4, he shows that if $p \equiv 1 \pmod{6}$, then there exists an integer A_p such that for all n we have

$$\frac{E_4(6z)}{\eta^4(6z)} | U(p^n) \equiv 0 \pmod{p^{n-A_p}},$$
 (1.4)

and that if $p \equiv 5 \pmod{6}$, then there exists an integer A_p such that for all n we have

$$\frac{E_4(6z)}{\eta^4(6z)} | U(p^n) \equiv 0 \pmod{p^{\lfloor \frac{n}{2} \rfloor - A_p}}. \tag{1.5}$$

In this paper, we show that the congruences conjectured by Honda and Kaneko result from the existence of "Hecke grids" of weakly holomorphic modular forms on Fricke groups. These are infinite families of forms on which the Hecke algebra acts in a systematic way. These are similar to the well-known grid of Zagier [7] which encodes the traces of singular moduli; a similar Hecke action on this grid [1] explains the many congruences among these traces.

Since the congruences are straightforward consequences of identities involving the Hecke operators we will focus here on the identities themselves. As an example of the results, we consider the case related to (1.4) and (1.5). Using Theorem 2 below with k = r = 4, we see that there is an infinite family of forms $F_d(z) \in M_2^!(\Gamma_0(36))$ with p-integral coefficients, and with $F_1(z) = \frac{E_4(6z)}{\eta^4(6z)} = \sum a_1(n)q^n$, such that

$$F_1|T(p^n) = \begin{cases} p^n F_{p^n} & \text{if } p^n \equiv 1 \pmod{6}, \\ p^n F_{p^n} + a_1(p^n)\eta^4(6z) & \text{if } p^n \equiv 5 \pmod{6}. \end{cases}$$
(1.6)

Using relations among the Hecke operators (we sketch the proof in Section 3 below), we conclude that

$$F_1 | U(p^n) \equiv \begin{cases} 0 \pmod{p^n} & \text{if } p \equiv 1 \pmod{6}, \\ 0 \pmod{p^{\lfloor \frac{n}{2} \rfloor}} & \text{if } p \equiv 5 \pmod{6}. \end{cases}$$
 (1.7)

In other words, (1.4) and (1.5) are true with $A_p = 0$ for every n.

In some cases, (1.6) and (1.7) can be strengthened. For example, if $G_1(z) = \frac{E_6(6z)}{\eta^4(6z)}$, Theorem 2 gives a family G_d with the property that $G_1 | T(p^n) = p^{3n} G_{p^n}$ for all $p \geq 5$. We conclude that $G_1 | U(p^n) \equiv 0 \pmod{p^{3n}}$, as shown in [3]. This phenomenon will occur whenever the parameter ℓ in Theorem 2 is non-zero.

In a similar way, we obtain strengthened versions of the other conjectures in [4]. For example, consider the congruence (1.2) in the case k=4. Any form $f \in M_4(\Gamma_0(2))$ can be written uniquely as the sum $f=af^++bf^-$, where $f^+(z)=1+48q+\ldots$ and $f^-(z)=1-80q+\ldots$ are eigenforms for the Fricke involution $f(z)\mapsto 2^{-2}z^{-4}f(-1/2z)$.

Define

$$F_1^+(z) := \frac{f^-(4z)}{\eta^2(4z)\eta^2(8z)}, \qquad F_1^-(z) := \frac{f^+(4z)}{\eta^2(4z)\eta^2(8z)} = \sum a_1^-(n)q^n.$$

Using Theorem 3 below, we conclude that for positive odd d there are p-integral forms $F_d^{\pm} \in M_2^!(\Gamma_0(32))$ with the following properties: For all prime powers p^n we have

$$F_1^+ | T(p^n) = p^n F_{p^n}^+.$$

If $p^n \equiv 1 \pmod{4}$ then

$$F_1^- | T(p^n) = p^n F_{p^n}^-.$$

If $p^n \equiv 3 \pmod{4}$ then

$$F_1^-|T(p^n) = p^n F_{p^n}^- + a_1^-(p^n) \cdot \eta^2(4z)\eta^2(8z).$$

We conclude as above that

$$F_1^{\pm} | U(p^n) \equiv \begin{cases} 0 \pmod{p^n} & \text{if } p \equiv 1 \pmod{4}, \\ 0 \pmod{p^{\lfloor \frac{n}{2} \rfloor}} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$
 (1.8)

For all odd primes p, any $f \in M_4(\Gamma_0(2))$ having p-integral coefficients is a p-integral linear combination of f^+ and f^- . It follows that (1.8) holds for $\frac{f(4z)}{\eta^2(4z)\eta^2(8z)}$; this establishes a stronger version of the conjecture of [4].

The following strengthened versions of these conjectures for $\Gamma_0(3)$ and $\Gamma_0(4)$ arise from the identities of Theorems 4 and 5 below. Let $p \geq 5$ be prime and let $N \in \{3,4\}$. Suppose that $f \in M_4(\Gamma_0(N))$ has p-integral coefficients and define

$$H_3(z) := \eta^2(3z)\eta^2(9z) = q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} + \cdots,$$

 $H_4(z) := \eta^4(6z) = q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} + \cdots.$

Then we have

$$\frac{f(3z)}{H_N(z)} \equiv \begin{cases} 0 \pmod{p^n} & \text{if } p \equiv 1 \pmod{3}, \\ 0 \pmod{p^{\lfloor \frac{n}{2} \rfloor}} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Finally, we mention that similar results will hold if the initial forms F_1 are replaced by other members of the grid.

2. Preliminaries

We begin with some brief background and a proposition about the action of the Hecke operators on the spaces in question. It will be most natural to work with the Fricke groups $\Gamma^*(N)$ for $N \in \{1, 2, 3, 4\}$ (see [6, Section 1.6] for background). For these levels, the groups are generated by the translation

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the Fricke involution

$$W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Let k be a positive integer. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, define the slash operator $\Big|_k$ by

$$f|_k \gamma := (\det \gamma)^{k/2} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Define $\Gamma_0(M, N) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(1) : M | c \text{ and } N | b \}$. For primes p, define the Hecke operator $T_k(p)$ by

$$f|T_k(p) := f|U(p) + p^{k-1}f(pz) = p^{\frac{k}{2}-1} \left(\sum_{\lambda=0}^{p-1} f|_k \begin{pmatrix} 1 & \lambda \\ 0 & p \end{pmatrix} + f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right).$$
 (2.1)

For (t,p)=1, define the conjugated operator $T_k^{(t)}(p):=A_tT_k(p)A_t^{-1}$, where

$$A_t := \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$f|T_k^{(t)}(p) := p^{\frac{k}{2}-1} \left(\sum_{\lambda=0}^{p-1} f|_k \begin{pmatrix} 1 & t\lambda \\ 0 & p \end{pmatrix} + f|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right), \tag{2.2}$$

and if $f = \sum a_f(n)q^{n/t}$, then

$$f|T_k^{(t)}(p) = \sum (a_f(pn) + p^{k-1}a_f(n/p)) q^{n/t}.$$
 (2.3)

For prime powers p^n we have $T_k^{(t)}(p^n) = A_t T_k(p^n) A_t^{-1}$ and the recurrence relation

$$T_k^{(t)}(p^{n+1}) = T_k^{(t)}(p^n)T_k^{(t)}(p) - p^{k-1}T_k^{(t)}(p^{n-1}).$$
(2.4)

We suppress the subscript k when it is clear from context.

We say that ν is a multiplier system for a group Γ if ν is a character on Γ of absolute value 1 (see [6, Section 1.4] for details). Then $M_k^!(\Gamma, \nu)$ is the space of holomorphic functions f on \mathbb{H} whose poles are supported at the cusps of Γ , and which satisfy

$$f|_{k}\gamma = \nu(\gamma)f\tag{2.5}$$

for all $\gamma \in \Gamma$.

The multiplier system ν_{η} on $\Gamma^*(1)$ for the Dedekind η function

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

is given by

$$\nu_{\eta}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\frac{d}{c}\right)^{*} \exp\left(\frac{2\pi i}{24}\left((a+d)c - bd(c^{2}-1) - 3c\right)\right) & \text{if } c \text{ is odd,} \\ \left(\frac{c}{d}\right)_{*} \exp\left(\frac{2\pi i}{24}\left((a+d)c - bd(c^{2}-1) + 3d - 3 - 3cd\right)\right) & \text{if } c \text{ is even} \end{cases}$$
(2.6)

(see [5, Chapter 4]). The symbols $\left(\frac{d}{c}\right)^*$ and $\left(\frac{c}{d}\right)_*$ denote extensions of the Jacobi symbol to negative integers, and take the values ± 1 .

The following proposition describes the effect of the conjugated Hecke operators $T_k^{(t)}(p^n)$ on these spaces.

Proposition 1. Let $N \in \{1, 2, 3, 4\}$ and suppose that t is a positive integer. Suppose that ν is a multiplier system on $\Gamma^*(N)$ which takes values among the 2t-th roots of unity, and that ν is trivial on $\Gamma_0(Nt, t)$. Then for primes $p \nmid N$ with $p^2 \equiv 1 \pmod{2t}$, we have

$$T_k^{(t)}(p^n): M_k^!(\Gamma^*(N), \nu) \to M_k^!(\Gamma^*(N), \nu^{p^n}).$$

Proof. We proceed by induction on n. For n = 1, it is enough to show that for each of the two generators γ we have

$$f|T^{(t)}(p)|_{k}\gamma = \nu^{p}(\gamma)f|T^{(t)}(p).$$

We begin with the translation T. We have

$$f | T^{(t)}(p) | T = p^{\frac{k}{2} - 1} \left(\sum_{\lambda = 0}^{p - 1} f \Big|_k \begin{pmatrix} 1 & t\lambda \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + f \Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

$$= p^{\frac{k}{2} - 1} \left(\sum_{\lambda = 0}^{p - 1} f \Big|_k \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t\lambda + 1 - p^2 \\ 0 & p \end{pmatrix} + f \Big|_k \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right).$$

Define λ' by $\lambda' \equiv \lambda + (1 - p^2)/t \pmod{p}$ and $0 \leq \lambda' \leq p - 1$. Then

$$f|T^{(t)}(p)|T = p^{\frac{k}{2}-1} \left(\sum_{\lambda'=0}^{p-1} \nu^p(T) f\Big|_k \begin{pmatrix} 1 & t\lambda' \\ 0 & p \end{pmatrix} + \nu^p(T) f\Big|_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right)$$
$$= \nu^p(T) f|T_k^{(t)}(p).$$

Since conjugation by W_N interchanges $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, we have

$$f|T^{(t)}(p)|_{k}W_{N} = p^{\frac{k}{2}-1} \left(\sum_{\lambda=0}^{p-1} f|_{k} \begin{pmatrix} 1 & t\lambda \\ 0 & p \end{pmatrix} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} + f|_{k} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} W_{N} \right)$$

$$= p^{\frac{k}{2}-1} \left(\sum_{\lambda=1}^{p-1} f|_{k} \begin{pmatrix} Nt\lambda & -1 \\ Np & 0 \end{pmatrix} + f|_{k}W_{N} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + f|_{k}W_{N} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right).$$

Define λ' by $Nt^2\lambda\lambda'+1\equiv 0\pmod{p}$ and $1\leq \lambda'\leq p-1$. Then

$$\begin{pmatrix} Nt\lambda & -1 \\ Np & 0 \end{pmatrix} = \begin{pmatrix} \frac{1+Nt^2\lambda\lambda'}{p} & t\lambda \\ Nt\lambda' & p \end{pmatrix} W_N \begin{pmatrix} 1 & t\lambda' \\ 0 & p \end{pmatrix}.$$

By assumption we have

$$\nu \begin{pmatrix} \frac{1+Nt^2\lambda\lambda'}{p} & t\lambda \\ Nt\lambda' & p \end{pmatrix} = 1.$$

Therefore

$$f|T^{(t)}(p)|_{k}W_{N} = \nu(W_{N})p^{\frac{k}{2}-1} \left(\sum_{\lambda'=1}^{p-1} f|_{k} \begin{pmatrix} 1 & t\lambda' \\ 0 & p \end{pmatrix} + f|_{k} \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + f|_{k} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \right)$$
$$= \nu(W_{N})f|T^{(t)}(p) = \nu^{p}(W_{N})f|T^{(t)}(p),$$

since p is odd and W_N is an involution.

Suppose that $n \ge 1$ and recall the recurrence (2.4) satisfied by $T^{(t)}(p^{n+1})$. By induction, the form $f | T^{(t)}(p^n) | T^{(t)}(p)$ has multiplier system $\nu^{p^{n+1}}$ and the form $f | T^{(t)}(p^{n-1})$ has multiplier system $\nu^{p^{n-1}}$. Since the values of ν are 2t-th roots of unity and $p^2 \equiv 1 \pmod{2t}$, these systems are the same. Therefore,

$$T_k^{(t)}(p^{n+1}): M_k^!(\Gamma^*(N), \nu) \to M_k^!(\Gamma^*(N), \nu^{p^{n+1}}).$$

3. Hecke grids on $\Gamma^*(1)$

We construct Hecke grids on $\Gamma^*(1) = \Gamma_0(1)$ which begin with the forms $E_k(z)/\eta^r(z)$ for $k \in \{4, 6, 8, 10, 14\}$ and $r \in \{4, 8, 12, 16, 20\}$ (similar results hold for all positive integers $r \leq 24$, but to state them would require unwieldy notation).

Let ν be the multiplier system for $\eta^4(z)$ on $\Gamma^*(1)$. We compute using (2.6) that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^*(1)$, then

$$\nu(A) = \zeta_6^{(a+d)c - bd(c^2 - 1) - 3c}. (3.1)$$

Here $\zeta_m := e^{2\pi i/m}$.

Theorem 2. Suppose that $k \in \{4, 6, 8, 10, 14\}$ and that $r \in \{4, 8, 12, 16, 20\}$. Define s/t = r/24 in lowest terms and let $\ell \in \{0, 1, 2\}$ be the unique integer satisfying $12\ell + k - r \in \{0, 4, 6, 8, 10, 14\}$.

(a) If d > 0 and $d \equiv s \pmod{t}$ then there exist unique forms

$$f_d = q^{-d/t} + \sum_{\substack{n>0\\n \equiv -s \text{ mod } t}} a_d(n)q^{n/t} \in M_{k-r/2}^!(\Gamma^*(1), \overline{\nu}^{r/4}).$$
(3.2)

(b) There exists a unique form

$$f_{t\ell-s} = q^{s/t-\ell} + \dots \in S_{k-r/2}(\Gamma^*(1), \nu^{r/4}).$$
 (3.3)

Furthermore, if $d > t\ell - s$ and $d \equiv -s \pmod{t}$ then there exist unique forms

$$f_d = q^{-d/t} + \sum_{\substack{n > s - t\ell \\ n \equiv s \bmod t}} a_d(n) q^{n/t} \in M_{k-r/2}^!(\Gamma^*(1), \nu^{r/4}). \tag{3.4}$$

(c) Suppose that p is an odd prime. If $p^n \equiv 1 \pmod{t}$ then we have

$$f_s|T^{(t)}(p^n) = p^{(k-r/2-1)n}f_{p^ns}.$$
 (3.5)

If $p^n \equiv -1 \pmod{t}$ then we have

$$f_s|T^{(t)}(p^n) = \begin{cases} p^{(k-r/2-1)n} f_{p^n s} + a_s(p^n) f_{-s} & \text{if } \ell = 0, \\ p^{(k-r/2-1)n} f_{p^n s} & \text{otherwise.} \end{cases}$$
(3.6)

Remark. An analogue of Theorem 2 with $1 \le r \le 23$ is also true, with the following modifications. When $r \equiv 2 \pmod{4}$ the multiplier system of $\eta^r(z)$ includes the character $\left(\frac{-1}{\bullet}\right)$, and the case k-r=12 needs to be treated separately. When r is odd, one uses the half-integral weight Hecke operators, and there are fewer cases since $p^{2n} \equiv 1 \pmod{t}$ for all n.

Before proving Theorem 2, we sketch the proof of (1.7).

Proof of (1.7). Note that $F_d(z) = f_d(6z)$ in the notation of Theorem 2. We have the relation

$$F_1 | U(p^n) = F_1 | T(p^n) - \sum_{j=1}^n p^j F_1 | U(p^{n-j}) | V(p^j).$$
(3.7)

The case $p \equiv 1 \pmod{3}$ follows in a straightforward way by induction.

Suppose that $p \equiv 2 \pmod{3}$. If n is even then (3.5) gives $F_1 | T(p^n) \equiv 0 \pmod{p^n}$. If n is odd, induction shows that $a_1(p^n) \equiv 0 \pmod{p^{\frac{n-1}{2}}}$, so that $F_1 | T(p^n) \equiv 0 \pmod{p^{\lfloor \frac{n}{2} \rfloor}}$ by (3.6). Using (3.7) we conclude that

$$F_1 | U(p^n) \equiv 0 \pmod{p^{\alpha}}$$

where $\alpha = \min\left\{ \lfloor \frac{n}{2} \rfloor, j + \lfloor \frac{n-j}{2} \rfloor \right\} = \lfloor \frac{n}{2} \rfloor$.

Proof of Theorem 2. Let $\Delta(z) := \eta^{24}(z)$ and let j(z) denote the Hauptmodul on $\Gamma_0(1)$ given by

$$j(z) := \frac{E_4^3}{\Delta(z)} = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots \in M_0^!(\Gamma_0(1)).$$

(a) Set $f_s(z) := E_k(z)/\eta^r(z) = q^{-s/t} + O(q^{1-s/t})$. For d > s with $d \equiv s \pmod{t}$ define

$$f_d(z) := j(z)^{(d-s)/t} f_{d-t}(z) + \sum_{m=2}^{(d-s)/t} c_m f_{d-mt}(z),$$

where the c_m are chosen so that $f_d(z) = q^{-d/t} + O(q^{1-s/t})$. These forms satisfy the requirements in (3.2). For uniqueness, suppose there are two forms f_d and f'_d satisfying (3.2) and define $g(z) := \eta^r(z)(f_d(z) - f'_d(z)) = O(q)$. Then g(z) is in $S_k(\Gamma_0(1))$. Since this space is trivial for $k \in \{4, 6, 8, 10, 14\}$, we conclude that $f_d = f'_d$.

(b) Set

$$f_{t\ell-s} := \frac{E_{12\ell+k-r}(z)}{\Delta^{\ell}(z)} \eta^r(z) = q^{s/t-\ell} + O(q^{1+s/t-\ell}),$$

where $E_0(z) := 1$, and set $f_{-s} = 0$ if $\ell \neq 0$. For $d > t\ell - s$ with $d \equiv -s \pmod{t}$, define

$$f_d(z) := j(z)^{(d+s)/t} f_{d-t}(z) + \sum_{m=2}^{(d+s)/t-\ell} c_m f_{d-mt}(z),$$

where the c_m are chosen so that $f_d(z) = q^{-d/t} + O(q^{1+s/t-\ell})$. If there are two forms f_d and f'_d which each satisfy (3.3) or (3.4) then the form

$$g(z) := \Delta^{\ell}(z) \frac{f_d(z) - f'_d(z)}{\eta^r(z)} = O(q)$$

has trivial multiplier system, so it is an element of $S_{12\ell+k-r}(\Gamma_0(1))$. This space is trivial since $12\ell+k-r \in \{0,4,6,8,10,14\}$, so $f_d = f'_d$.

(c) Since $rt/24 = s \in \mathbb{Z}$ we see from (3.1) that the multiplier system $\nu^{r/4}$ is trivial on $\Gamma_0(t,t)$ and takes values which are t-th roots of unity. Therefore Proposition 1 gives

$$f_s | T_{k-r/2}^{(t)}(p^n) \in M_{k-r/2}^!(\Gamma^*(1), \overline{\nu}^{p^n r/4}).$$

It follows from this and (2.3) that if $p^n \equiv 1 \pmod{t}$ then

$$f_s|T^{(t)}(p^n) = p^{(k-r/2-1)n}q^{-p^ns/t} + O(q^{1-s/t}) \in M^!_{k-r/2}(\Gamma^*(1), \overline{\nu}^{r/4}),$$

while if $p^n \equiv -1 \pmod{t}$ then

$$f_s | T^{(t)}(p^n) - a_s(p^n) f_{-s} = p^{(k-r/2-1)n} q^{-p^n s/t} + O(q^{1+s/t-\ell}) \in M^!_{k-r/2}(\Gamma^*(1), \nu^{r/4}).$$

By uniqueness we obtain (3.5) and (3.6).

Example 1. Computing as described in the proof above with k=6 and r=4, we obtain

$$f_{1} = q^{-\frac{1}{6}} - 500q^{\frac{5}{6}} - 18634q^{\frac{11}{6}} - 196520q^{\frac{17}{6}} - 1277535q^{\frac{23}{6}} - 6146028q^{\frac{29}{6}} + \cdots$$

$$f_{7} = q^{-\frac{7}{6}} - 71750q^{\frac{5}{6}} - 86461760q^{\frac{11}{6}} - 13650854021q^{\frac{17}{6}} - 851755409792q^{\frac{23}{6}} + \cdots$$

$$f_{13} = q^{-\frac{13}{6}} - 2401000q^{\frac{5}{6}} - 24581234095q^{\frac{11}{6}} - 19372032655696q^{\frac{17}{6}} + \cdots$$

$$f_{19} = q^{-\frac{19}{6}} - 44127125q^{\frac{5}{6}} - 2445793637760q^{\frac{11}{6}} - 6837455343912760q^{\frac{17}{6}} + \cdots$$

and

$$\begin{split} f_5 &= q^{-\frac{5}{6}} - 4q^{\frac{1}{6}} - 196882q^{\frac{7}{6}} - 42199976q^{\frac{13}{6}} - 2421343603q^{\frac{19}{6}} + \cdots \\ f_{11} &= q^{-\frac{11}{6}} - 14q^{\frac{1}{6}} - 22281280q^{\frac{7}{6}} - 40574734265q^{\frac{13}{6}} - 12603830624640q^{\frac{19}{6}} + \cdots \\ f_{17} &= q^{-\frac{17}{6}} - 40q^{\frac{1}{6}} - 953031331q^{\frac{7}{6}} - 8662803937424q^{\frac{13}{6}} - 9545716711560680q^{\frac{19}{6}} + \cdots \\ f_{23} &= q^{-\frac{23}{6}} - 105q^{\frac{1}{6}} - 24011843968q^{\frac{7}{6}} - 837470540062104q^{\frac{13}{6}} - 2657886912184060160q^{\frac{19}{6}} + \cdots \end{split}$$

4. Hecke grids on $\Gamma^*(2)$

In this section we construct grids on $\Gamma^*(2)$ which lead to the congruences (1.8). Let

$$h_2(z) := \eta^2(z)\eta^2(2z) = q^{\frac{1}{4}} - 2q^{\frac{5}{4}} - 3q^{\frac{9}{4}} + 6q^{\frac{13}{4}} + \cdots$$

The grids begin with forms f/h_2 , where $f \in M_4(\Gamma_0(2))$. This space is two-dimensional and is spanned by the forms

$$F_2^+(z) := \frac{1}{5} \left(4E_4(2z) + E_4(z) \right) = 1 + 48q + 624q^2 + 1344q^3 + \cdots,$$

$$F_2^-(z) := \frac{1}{3} \left(4E_4(2z) - E_4(z) \right) = 1 - 80q - 400q^2 - 2240q^3 + \cdots.$$

Here F_2^+ and F_2^- are eigenforms of W_2 with eigenvalues ± 1 , respectively. Since $M_6(\Gamma_0(2))$ is also two-dimensional, the results in this section have analogues for k=6, using the eigenforms

$$G_2^{\pm}(z) := \frac{8E_6(2z) \pm E_6(z)}{8 \pm 1} \in M_6(\Gamma_0(2)).$$

The details are similar, and are omitted.

Let ν_{\pm} denote the multiplier system for $h_2(z)$ on $\Gamma_0(2)$, extended to $\Gamma^*(2)$ via $\nu_{\pm}(W_2) = \pm 1$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, then a computation involving (2.6) gives

$$\nu_{\pm}(\gamma) = i^{d(b-c/2)},\tag{4.1}$$

which is trivial on $\Gamma_0(8,4)$. We have

$$h_2 \in S_2(\Gamma^*(2), \nu_-).$$

Theorem 3. (a) If d > 0 and $d \equiv 1 \pmod{4}$, then there exist unique forms

$$f_d^{\pm} = q^{-d/4} + \sum_{\substack{n>0\\n\equiv 3 \bmod 4}} a_d^{\pm}(n)q^{n/4} \in M_2^!(\Gamma^*(2), \overline{\nu}_{\pm}). \tag{4.2}$$

(b) If d > 0 and $d \equiv 3 \pmod{4}$, then there exist unique forms

$$f_d^+ = q^{-d/4} + \sum_{\substack{n>0\\n\equiv 1 \text{ mod } 4}} a_d^+(n)q^{n/4} \in M_2^!(\Gamma^*(2), \nu_+)$$
(4.3)

$$f_d^- = q^{-d/4} + \sum_{\substack{n \ge 5\\ n \equiv 1 \bmod 4}} a_d^-(n) q^{n/4} \in M_2^!(\Gamma^*(2), \nu_-). \tag{4.4}$$

(c) For all odd prime powers p^n we have

$$f_1^+ | T^{(4)}(p^n) = p^n f_{p^n}^+.$$

If $p^n \equiv 1 \pmod{4}$ then

$$f_1^-|T^{(4)}(p^n)=p^nf_{p^n}^-.$$

If $p^n \equiv 3 \pmod{4}$ then

$$f_1^- | T^{(4)}(p^n) = p^n f_{p^n}^- + a_1^-(p^n) \cdot h_2.$$

Proof. Let $j_2(z)$ denote the Hauptmodul on $\Gamma^*(2)$ given by

$$j_2(z) := \frac{\Delta(z)}{\Delta(2z)} + 24 + 2^{12} \frac{\Delta(2z)}{\Delta(z)} = \frac{1}{q} + 4372q + 96256q^2 + \dots \in M_0^!(\Gamma^*(2)).$$

Since h_2 has eigenvalue -1 under W_2 , we define

$$f_1^+ := \frac{F_2^-}{h_2} = q^{-\frac{1}{4}} - 78q^{\frac{3}{4}} - 553q^{\frac{7}{4}} - 3586q^{\frac{11}{4}} - 11325q^{\frac{15}{4}} + \dots \in M_2^!(\Gamma^*(2), \overline{\nu}_+),$$

$$f_1^- := \frac{F_2^+}{h_2} = q^{-\frac{1}{4}} + 50q^{\frac{3}{4}} + 727q^{\frac{7}{4}} + 2942q^{\frac{11}{4}} + 12995q^{\frac{15}{4}} + \dots \in M_2^!(\Gamma^*(2), \overline{\nu}_-).$$

For $d \equiv 1 \pmod{4}$ we can construct f_d^+ satisfying (4.2) as a linear combination of $f_{d-4} \cdot j_2$ and f_{d-4}, \ldots, f_1 . To prove uniqueness, suppose that f_d^+ and g_d^+ are two forms with these properties. Let ω_- be the multiplier system on $\Gamma^*(2)$ which maps T to 1 and W_2 to -1. Then

$$h_2(z)(f_d^+(z) - g_d^+(z)) = O(q) \in M_4^!(\Gamma^*(2), \omega_-).$$

Since there is only one cusp, this is in fact a cusp form, and is therefore equal to zero.

The remaining forms are constructed in similar fashion. When $d \equiv 3 \pmod{4}$, we begin with the forms

$$f_3^+ := \frac{F_2^+ F_2^-}{h_2^3} = q^{-\frac{3}{4}} - 26q^{\frac{1}{4}} - 3775q^{\frac{5}{4}} - 92634q^{\frac{9}{4}} + \dots \in M_2^!(\Gamma^*(2), \nu_+),$$

$$f_{-1}^- := h_2 = q^{\frac{1}{4}} - 2q^{\frac{5}{4}} - 3q^{\frac{9}{4}} + 6q^{\frac{13}{4}} + \dots \in M_2^!(\Gamma^*(2), \nu_-).$$

We conclude the proof by applying (2.3) and Proposition 1 to the forms f_1^{\pm} to obtain the equalities listed in (c).

Example 2. We have

$$f_1^+ = q^{-\frac{1}{4}} - 78q^{\frac{3}{4}} - 553q^{\frac{7}{4}} - 3586q^{\frac{11}{4}} - 11325q^{\frac{15}{4}} + \cdots$$

$$f_5^+ = q^{-\frac{5}{4}} - 2265q^{\frac{3}{4}} - 291480q^{\frac{7}{4}} - 8976715q^{\frac{11}{4}} - 155852328q^{\frac{15}{4}} + \cdots$$

$$f_9^+ = q^{-\frac{9}{4}} - 30878q^{\frac{3}{4}} - 16474122q^{\frac{7}{4}} - 1629968274q^{\frac{11}{4}} - 71856917725q^{\frac{15}{4}} + \cdots$$

$$f_{13}^+ = q^{-\frac{13}{4}} - 232056q^{\frac{3}{4}} - 443763544q^{\frac{7}{4}} - 107298900269q^{\frac{11}{4}} - 10015296762600q^{\frac{15}{4}} + \cdots$$

$$\begin{split} f_3^+ &= q^{-\frac{3}{4}} - 26q^{\frac{1}{4}} - 3775q^{\frac{5}{4}} - 92634q^{\frac{9}{4}} - 1005576q^{\frac{13}{4}} - 8083772q^{\frac{17}{4}} + \cdots \\ f_7^+ &= q^{-\frac{7}{4}} - 79q^{\frac{1}{4}} - 208200q^{\frac{5}{4}} - 21181014q^{\frac{9}{4}} - 824132296q^{\frac{13}{4}} + \cdots \\ f_{11}^+ &= q^{-\frac{11}{4}} - 326q^{\frac{1}{4}} - 4080325q^{\frac{5}{4}} - 1333610406q^{\frac{9}{4}} - 126807791227q^{\frac{13}{4}} + \cdots \\ f_{15}^+ &= q^{-\frac{15}{4}} - 755q^{\frac{1}{4}} - 51950776q^{\frac{5}{4}} - 43114150635q^{\frac{9}{4}} - 8679923860920q^{\frac{13}{4}} + \cdots , \end{split}$$

as well as

$$\begin{split} f_1^- &= q^{-\frac{1}{4}} + 50q^{\frac{3}{4}} + 727q^{\frac{7}{4}} + 2942q^{\frac{11}{4}} + 12995q^{\frac{15}{4}} + \cdots \\ f_5^- &= q^{-\frac{5}{4}} + 2599q^{\frac{3}{4}} + 281448q^{\frac{7}{4}} + 9097141q^{\frac{11}{4}} + 154926040q^{\frac{15}{4}} + \cdots \\ f_9^- &= q^{-\frac{9}{4}} + 29154q^{\frac{3}{4}} + 16632054q^{\frac{7}{4}} + 1625776110q^{\frac{11}{4}} + 71919500835q^{\frac{15}{4}} + \cdots \\ f_{13}^- &= q^{-\frac{13}{4}} + 238728q^{\frac{3}{4}} + 442272424q^{\frac{7}{4}} + 107373859795q^{\frac{11}{4}} + 10013399068440q^{\frac{15}{4}} + \cdots \end{split}$$

and

$$\begin{split} f_{-1}^- &= q^{\frac{1}{4}} - 2q^{\frac{5}{4}} - 3q^{\frac{9}{4}} + 6q^{\frac{13}{4}} + 2q^{\frac{17}{4}} + \cdots \\ f_3^- &= q^{-\frac{3}{4}} + 4365q^{\frac{5}{4}} + 87512q^{\frac{9}{4}} + 1034388q^{\frac{13}{4}} + 7956216q^{\frac{17}{4}} + \cdots \\ f_7^- &= q^{-\frac{7}{4}} + 201242q^{\frac{5}{4}} + 21384381q^{\frac{9}{4}} + 821362450q^{\frac{13}{4}} + 18482815673q^{\frac{17}{4}} + \cdots \\ f_{11}^- &= q^{-\frac{11}{4}} + 4135599q^{\frac{5}{4}} + 1330181256q^{\frac{9}{4}} + 126896378153q^{\frac{13}{4}} + 6154813925224q^{\frac{17}{4}} + \cdots \end{split}$$

5. Hecke grids on $\Gamma^*(3)$

Let

$$h_3(z) := \eta^2(z)\eta^2(3z) = q^{\frac{1}{3}} - 2q^{\frac{4}{3}} - q^{\frac{7}{3}} + 5q^{\frac{13}{3}} + \cdots$$

We construct grids on $\Gamma^*(3)$ starting with the forms f/h_3 , where $f \in M_4(\Gamma_0(3))$. This space is two-dimensional, spanned by the W_3 -eigenforms

$$F_3^+(z) := \frac{1}{10}(9E_4(3z) + E_4(z)) = 1 + 24q + 216q^2 + 888q^3 + 1752q^4 + \cdots,$$

$$F_3^-(z) := \frac{1}{8}(9E_4(3z) - E_4(z)) = 1 - 30q - 270q^2 - 570q^3 - 2190q^4 + \cdots.$$

Let ν_{\pm} denote the multiplier system of $h_3(z)$ on $\Gamma_0(3)$, extended to $\Gamma^*(3)$ via $\nu_{\pm}(W_3) = \pm 1$. Using (2.6), we see that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(3)$, we have

$$\nu_{\pm}(\gamma) = \zeta_3^{\frac{c}{3}(a+d)+bd},\tag{5.1}$$

which is trivial on $\Gamma_0(9,3)$.

Theorem 4. (a) If d > 0 and $d \equiv 1 \pmod{3}$, then there exist unique forms

$$f_d^{\pm} = q^{-d/3} + \sum_{\substack{n > 0 \\ n \equiv 2 \bmod 3}} a_d^{\pm}(n) q^{n/3} \in M_2^!(\Gamma^*(3), \overline{\nu}_{\pm}).$$

(b) If d > 0 and $d \equiv 2 \pmod{3}$, then there exist unique forms

$$f_d^+ = q^{-d/3} + \sum_{\substack{n>0 \\ \equiv 1 \bmod 3}} a_d^+(n) q^{n/3} \in M_2^!(\Gamma^*(3), \nu_+)$$

$$f_d^- = q^{-d/3} + \sum_{\substack{n \ge 4 \\ n \equiv 1 \bmod 3}} a_d^-(n) q^{n/3} \in M_2^!(\Gamma^*(3), \nu_-).$$

(c) Suppose $p \geq 5$ is prime. We have

$$f_1^+|T^{(3)}(p^n)=p^nf_{p^n}^+$$

If $p^n \equiv 1 \pmod{3}$ then

$$f_1^-|T^{(3)}(p^n)=p^nf_{p^n}^-$$

If $p^n \equiv 2 \pmod{3}$ then

$$f_1^- | T^{(3)}(p^n) = p^n f_{p^n}^- + a_1^-(p^n) \cdot h_3.$$

Proof. Let ω_{-} be the multiplier which maps W_3 to -1, and define

$$G_3^-(z) := \frac{1}{2} (E_2(3z) - E_2(z)) = 1 + 12q + 36q^2 + 12q^3 + 84q^4 + \dots \in M_2(\Gamma^*(3), \omega_-).$$

The four grids are constructed beginning with the forms

$$f_{1}^{+} := \frac{F_{3}^{-}}{h_{3}} = q^{-\frac{1}{3}} - 28q^{\frac{2}{3}} - 325q^{\frac{5}{3}} - 1248q^{\frac{8}{3}} - 5016q^{\frac{11}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(3), \overline{\nu}_{+}),$$

$$f_{1}^{-} := \frac{F_{3}^{+}}{h_{3}} = q^{-\frac{1}{3}} + 26q^{\frac{2}{3}} + 269q^{\frac{5}{3}} + 1452q^{\frac{8}{3}} + 4920q^{\frac{11}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(3), \overline{\nu}_{-}),$$

$$f_{2}^{+} := \frac{F_{3}^{-}G_{3}^{-}}{h_{3}^{2}} = q^{-\frac{2}{3}} - 14q^{\frac{1}{3}} - 652q^{\frac{4}{3}} - 7462q^{\frac{7}{3}} - 47525q^{\frac{10}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(3), \nu_{+}),$$

$$f_{-1}^{-} := h_{3} = q^{\frac{1}{3}} - 2q^{\frac{4}{3}} - q^{\frac{7}{3}} + 5q^{\frac{13}{3}} + 4q^{\frac{16}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(3), \nu_{-}).$$

The remaining forms f_d^{\pm} are constructed using the Hauptmodul $j_3(z)$ on $\Gamma^*(3)$ given by

$$j_3(z) = \frac{\eta^{12}(z)}{\eta^{12}(3z)} + 12 + 3^6 \frac{\eta^{12}(3z)}{\eta^{12}(z)} = q^{-1} + 783q + 8672q^2 + \dots \in M_0^!(\Gamma^*(3)).$$

Example 3. We have

$$f_1^+ = q^{-\frac{1}{3}} - 28q^{\frac{2}{3}} - 325q^{\frac{5}{3}} - 1248q^{\frac{8}{3}} - 5016q^{\frac{11}{3}} + \cdots$$

$$f_4^+ = q^{-\frac{4}{3}} - 326q^{\frac{2}{3}} - 23600q^{\frac{5}{3}} - 471884q^{\frac{8}{3}} - 5409712q^{\frac{11}{3}} + \cdots$$

$$f_7^+ = q^{-\frac{7}{3}} - 2132q^{\frac{2}{3}} - 513250q^{\frac{5}{3}} - 25773728q^{\frac{8}{3}} - 636531533q^{\frac{11}{3}} + \cdots$$

$$f_{10}^+ = q^{-\frac{10}{3}} - 9505q^{\frac{2}{3}} - 6467264q^{\frac{5}{3}} - 677506240q^{\frac{8}{3}} - 30773378240q^{\frac{11}{3}} + \cdots$$

and

$$f_{2}^{+} = q^{-\frac{2}{3}} - 14q^{\frac{1}{3}} - 652q^{\frac{4}{3}} - 7462q^{\frac{7}{3}} - 47525q^{\frac{10}{3}} + \cdots$$

$$f_{5}^{+} = q^{-\frac{5}{3}} - 65q^{\frac{1}{3}} - 18880q^{\frac{4}{3}} - 718550q^{\frac{7}{3}} - 12934528q^{\frac{10}{3}} + \cdots$$

$$f_{8}^{+} = q^{-\frac{8}{3}} - 156q^{\frac{1}{3}} - 235942q^{\frac{4}{3}} - 22552012q^{\frac{7}{3}} - 846882800q^{\frac{10}{3}} + \cdots$$

$$f_{11}^{+} = q^{-\frac{11}{3}} - 456q^{\frac{1}{3}} - 1967168q^{\frac{4}{3}} - 405065521q^{\frac{7}{3}} - 27975798400q^{\frac{10}{3}} + \cdots$$

as well as

$$\begin{split} f_1^- &= q^{-\frac{1}{3}} + 26q^{\frac{2}{3}} + 269q^{\frac{5}{3}} + 1452q^{\frac{8}{3}} + 4920q^{\frac{11}{3}} + \cdots \\ f_4^- &= q^{-\frac{4}{3}} + 376q^{\frac{2}{3}} + 23488q^{\frac{5}{3}} + 468634q^{\frac{8}{3}} + 5427008q^{\frac{11}{3}} + \cdots \\ f_7^- &= q^{-\frac{7}{3}} + 2026q^{\frac{2}{3}} + 516638q^{\frac{5}{3}} + 25767436q^{\frac{8}{3}} + 636345829q^{\frac{11}{3}} + \cdots \\ f_{10}^- &= q^{-\frac{10}{3}} + 9449q^{\frac{2}{3}} + 6456448q^{\frac{5}{3}} + 677710592q^{\frac{8}{3}} + 30773024128q^{\frac{11}{3}} + \cdots \end{split}$$

and

$$\begin{split} f_{-1}^- &= q^{\frac{1}{3}} - 2q^{\frac{4}{3}} - q^{\frac{7}{3}} + 5q^{\frac{13}{3}} + 4q^{\frac{16}{3}} + \cdots \\ f_2^- &= q^{-\frac{2}{3}} + 778q^{\frac{4}{3}} + 7104q^{\frac{7}{3}} + 47245q^{\frac{10}{3}} + 232128q^{\frac{13}{3}} + \cdots \\ f_5^- &= q^{-\frac{5}{3}} + 18898q^{\frac{4}{3}} + 723347q^{\frac{7}{3}} + 12912896q^{\frac{10}{3}} + 152125263q^{\frac{13}{3}} + \cdots \\ f_8^- &= q^{-\frac{8}{3}} + 234680q^{\frac{4}{3}} + 22546688q^{\frac{7}{3}} + 847138240q^{\frac{10}{3}} + 18799619328q^{\frac{13}{3}} + \cdots \end{split}$$

6. Hecke grids on $\Gamma^*(4)$

The three-dimensional space $M_4(\Gamma_0(4))$ is spanned by $\{E_4(2z), F_4^+(z), F_4^-(z)\}$, where

$$F_4^+(z) := \frac{1}{15} (16E_4(4z) + E_4(z) - 2E_4(2z)),$$

$$F_4^-(z) := \frac{1}{15} (16E_4(4z) - E_4(z)).$$

The forms $E_4(2z)$ and $F_4^+(z)$ have eigenvalue +1 under the Fricke involution W_4 , while the form F_4^- has eigenvalue -1. Let

$$h_4(z) := \eta^4(2z) = q^{\frac{1}{3}} - 4q^{\frac{7}{3}} + 2q^{\frac{13}{3}} + 8q^{\frac{19}{3}} - 5q^{\frac{25}{3}} + \cdots$$

We construct grids on $\Gamma^*(4)$ starting with forms $f(z)/h_4$, where $f(z) \in M_4(\Gamma_0(4))$.

Recall that $E_4(z)/\eta^4(z)$ is the first member of one of the $\Gamma^*(1)$ grids. So we need concern ourselves only with the subspace spanned by $\{F_4^+, F_4^-\}$. The distinguishing feature of F_4^+ is the fact that it vanishes to order 2 at the cusp 1/2.

Let ν_{\pm} denote the multiplier system for $\eta^4(2z)$ on $\Gamma_0(4)$, extended to $\Gamma^*(4)$ by $\nu_{\pm}(W_4) = \pm 1$. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, then by applying (3.1) to the matrix $\begin{pmatrix} a & 2b \\ c/2 & d \end{pmatrix} = A_2 \gamma A_2^{-1}$ we obtain

$$\nu_{+}(\gamma) = \zeta_3^{bd(1-(c/2)^2) + \frac{c}{4}(a+d)}.$$
(6.1)

Note that ν_{\pm} is trivial on $\Gamma_0(12,3)$. Since $\eta^4(2z)\big|_2W_4=-\eta^4(2z)$, we have $h_4\in S_2(\Gamma^*(4),\nu_-)$.

Theorem 5. (a) If d > 0 and $d \equiv 1 \pmod{3}$, then there exist unique forms

$$f_d^+ = q^{-d/3} + \sum_{\substack{n>0\\n\equiv 2 \bmod 3}} a_d^+(n)q^{n/3} \in M_2^!(\Gamma^*(4), \overline{\nu}_+).$$
 (6.2)

Furthermore, there exist unique forms

$$f_d^- = q^{-d/3} + \sum_{\substack{n>0\\n=2 \text{ mod } 3}} a_d^-(n)q^{n/3} \in M_2^!(\Gamma^*(4), \overline{\nu}_-)$$
(6.3)

which vanish at the cusp 1/2.

(b) If $0 < d \equiv 2 \pmod{3}$, then there exist unique forms

$$f_d^+ = q^{-d/3} + \sum_{\substack{n>0\\n\equiv 1 \bmod 3}} a_d^+(n)q^{n/3} \in M_2^!(\Gamma^*(4), \nu_+)$$
(6.4)

and

$$f_d^- = q^{-d/3} + \sum_{\substack{n \ge 4 \\ n = 1 \text{ mod } 3}} a_d^-(n) q^{n/3} \in M_2^!(\Gamma^*(4), \nu_-).$$

$$(6.5)$$

(c) Suppose $p \geq 5$ is prime. We have

$$f_1^+ | T^{(3)}(p^n) = p^n f_{p^n}^+.$$

If $p^n \equiv 1 \pmod{3}$ then

$$f_1^-|T^{(3)}(p^n)=p^nf_{p^n}^-$$

If $p^n \equiv 2 \pmod{3}$ then

$$f_1^-|T^{(3)}(p^n) = p^n f_{p^n}^- + a_1^-(p^n) \cdot h_4.$$

Proof. Let

$$G_4^-(z) := \frac{1}{3} \left(4E_2(4z) - E_2(z) \right) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + \dots \in M_2(\Gamma^*(2), \omega_-)$$

The four grids are constructed beginning with the forms

$$f_{1}^{+} := \frac{F_{4}^{-}}{h_{4}} = q^{-\frac{1}{3}} - 16q^{\frac{2}{3}} - 140q^{\frac{5}{3}} - 512q^{\frac{8}{3}} - 1474q^{\frac{11}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(4), \overline{\nu}_{+}),$$

$$f_{1}^{-} := \frac{F_{4}^{+}}{h_{4}} = q^{-\frac{1}{3}} + 16q^{\frac{2}{3}} + 116q^{\frac{5}{3}} + 512q^{\frac{8}{3}} + 1598q^{\frac{11}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(4), \overline{\nu}_{-}),$$

$$f_{2}^{+} := \frac{F_{4}^{-}G_{4}^{-}}{h_{4}^{2}} = q^{-\frac{2}{3}} + 8q^{\frac{1}{3}} - 240q^{\frac{4}{3}} - 2016q^{\frac{7}{3}} - 10380q^{\frac{10}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(4), \nu_{+}),$$

$$f_{-1}^{-} := h_{4} = q^{\frac{1}{3}} - 4q^{\frac{7}{3}} + 2q^{\frac{13}{3}} + 8q^{\frac{19}{3}} - 5q^{\frac{25}{3}} + \dots \in M_{2}^{!}(\Gamma^{*}(4), \nu_{-}).$$

The remaining forms f_d^{\pm} are constructed using the Hauptmodul $j_4(z)$ on $\Gamma^*(4)$ given by

$$j_4(z) := \frac{\eta^8(z)}{\eta^8(4z)} + 8 + \frac{\eta^8(4z)}{\eta^8(z)} = \frac{1}{q} + 276q + 2048q^2 + \dots \in M_0^!(\Gamma^*(4)).$$

For $d \equiv 1 \pmod{3}$, the forms f_d^- are constructed so that they vanish at 1/2. This property is necessary to establish uniqueness, for if f_d^- and g_d^- satisfy (6.3) then

$$h_4 \cdot (f_d^- - g_d^-) = O(q)$$

vanishes at ∞ and vanishes to order 2 at 1/2. But nonzero weight 4 forms on $\Gamma_0(4)$ can have at most 2 zeros, so $f_d^- = g_d^-$.

Example 4. We have

$$f_1^+ = q^{-\frac{1}{3}} - 16q^{\frac{2}{3}} - 140q^{\frac{5}{3}} - 512q^{\frac{8}{3}} - 1474q^{\frac{11}{3}} + \cdots$$

$$f_4^+ = q^{-\frac{4}{3}} - 120q^{\frac{2}{3}} - 5120q^{\frac{5}{3}} - 69872q^{\frac{8}{3}} - 585728q^{\frac{11}{3}} + \cdots$$

$$f_7^+ = q^{-\frac{7}{3}} - 576q^{\frac{2}{3}} - 69950q^{\frac{5}{3}} - 2115584q^{\frac{8}{3}} - 34400960q^{\frac{11}{3}} + \cdots$$

$$f_{10}^+ = q^{-\frac{10}{3}} - 2076q^{\frac{2}{3}} - 606208q^{\frac{5}{3}} - 34664448q^{\frac{8}{3}} - 955187200q^{\frac{11}{3}} + \cdots$$

$$f_{2}^{+} = q^{-\frac{2}{3}} - 8q^{\frac{1}{3}} - 240q^{\frac{4}{3}} - 2016q^{\frac{7}{3}} - 10380q^{\frac{10}{3}} + \cdots$$

$$f_{5}^{+} = q^{-\frac{5}{3}} - 28q^{\frac{1}{3}} - 4096q^{\frac{4}{3}} - 97930q^{\frac{7}{3}} - 1212416q^{\frac{10}{3}} + \cdots$$

$$f_{8}^{+} = q^{-\frac{8}{3}} - 64q^{\frac{1}{3}} - 34936q^{\frac{4}{3}} - 1851136q^{\frac{7}{3}} - 43330560q^{\frac{10}{3}} + \cdots$$

$$f_{11}^{+} = q^{-\frac{11}{3}} - 134q^{\frac{1}{3}} - 212992q^{\frac{4}{3}} - 21891520q^{\frac{7}{3}} - 868352000q^{\frac{10}{3}} + \cdots$$

as well as

$$\begin{split} f_1^- &= q^{-\frac{1}{3}} + 16q^{\frac{2}{3}} + 116q^{\frac{5}{3}} + 512q^{\frac{8}{3}} + 1598q^{\frac{11}{3}} + \cdots \\ f_4^- &= q^{-\frac{4}{3}} + 136q^{\frac{2}{3}} + 5120q^{\frac{5}{3}} + 69392q^{\frac{8}{3}} + 585728q^{\frac{11}{3}} + \cdots \\ f_7^- &= q^{-\frac{7}{3}} + 576q^{\frac{2}{3}} + 70338q^{\frac{5}{3}} + 2115584q^{\frac{8}{3}} + 34391360q^{\frac{11}{3}} + \cdots \\ f_{10}^- &= q^{-\frac{10}{3}} + 2020q^{\frac{2}{3}} + 606208q^{\frac{5}{3}} + 34672640q^{\frac{8}{3}} + 955187200q^{\frac{11}{3}} + \cdots \end{split}$$

and

$$\begin{split} f_{-1}^{-} &= q^{\frac{1}{3}} - 4q^{\frac{7}{3}} + 2q^{\frac{13}{3}} + 8q^{\frac{19}{3}} - 5q^{\frac{25}{3}} - 4q^{\frac{31}{3}} + \cdots \\ f_{2}^{-} &= q^{-\frac{2}{3}} + 272q^{\frac{4}{3}} + 2048q^{\frac{7}{3}} + 10100q^{\frac{10}{3}} + 40960q^{\frac{13}{3}} + \cdots \\ f_{5}^{-} &= q^{-\frac{5}{3}} + 4096q^{\frac{4}{3}} + 98566q^{\frac{7}{3}} + 1212416q^{\frac{10}{3}} + 10351552q^{\frac{13}{3}} + \cdots \\ f_{8}^{-} &= q^{-\frac{8}{3}} + 34696q^{\frac{4}{3}} + 1851392q^{\frac{7}{3}} + 43340800q^{\frac{10}{3}} + 641007616q^{\frac{13}{3}} + \cdots \end{split}$$

References

- [1] Scott Ahlgren. Hecke relations for traces of singular moduli. Bull. Lond. Math. Soc., 44(1):99–105, 2012.
- [2] Sharon Anne Garthwaite. Convolution congruences for the partition function. *Proc. Amer. Math. Soc.*, 135(1):13–20, 2007.
- [3] P. Guerzhoy. On the Honda-Kaneko congruences. In From Fourier analysis and number theory to radon transforms and geometry, volume 28 of Dev. Math., pages 293–302. Springer, New York, 2013.
- [4] Yutaro Honda and Masanobu Kaneko. On Fourier coefficients of some meromorphic modular forms. *Bull. Korean Math. Soc.*, 49(6):1349–1357, 2012.
- [5] Marvin I. Knopp. Modular functions in analytic number theory. Markham Publishing Co., Chicago, Ill., 1970.
- [6] Günter Köhler. Eta products and theta series identities. Springer Monographs in Mathematics. Springer, Heidelberg, 2011.
- [7] Don Zagier. Traces of singular moduli. In Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), volume 3 of Int. Press Lect. Ser., pages 211–244. Int. Press, Somerville, MA, 2002.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801 *E-mail address*: sahlgren@illinois.edu

Department of Mathematics, University of Illinois, Urbana, IL 61801 $E\text{-}mail\ address:}$ nandrsn4@illinois.edu