A CONSTRUCTION OF SLICE KNOTS VIA ANNULUS TWISTS

TETSUYA ABE AND MOTOO TANGE

Abstract. We give a new construction of slice knots via annulus twists. The simplest slice knots obtained by our method are those constructed by Omae. In this paper, we introduce a sufficient condition for given slice knots to be ribbon, and prove that all Omae's knots are ribbon.

1. Introduction

The annulus twist is a certain operation on knots along an annulus embedded in the 3-sphere S^3. Osoinach [Os] found that this operation is useful in the study of 3-manifolds. Using annulus twists, he gave the first example of a 3-manifold admitting infinitely many presentations by 0-framed knots. For more studies, see [AJOT, AJLO, BGL, K, Tak, Te, Om].

Recently, the first author, Jong, Omae and Takeuchi [AJOT] constructed knots related to the slice-ribbon conjecture: Let K be a slice knot admitting an annulus presentation (for the definition, see Section 2) and K_n ($n \in \mathbb{Z}$) the knot obtained from K by the n-fold annulus twist. They proved that K_n bounds a smoothly embedded disk in a certain homotopy 4-ball $W(K_n)$ with $\partial W(K_n) \approx S^3$. A natural question is the following:

Question. Is $W(K_n)$ diffeomorphic to the standard 4-ball B^4?

If $W(K_n)$ is not diffeomorphic to B^4, then the homotopy 4-sphere obtained by capping it off is a counterexample of the smooth 4-dimensional Poincaré conjecture. For related studies, see [A1, A2, FGMW, G1, G2, N, NS, Tan]. Our first result is the following:

Theorem 3.1. *Let K be a ribbon knot admitting an annulus presentation and K_n ($n \in \mathbb{Z}$) the knot obtained from K by the n-fold annulus twist. Then the homotopy 4-ball $W(K_n)$ associated to K_n is diffeomorphic to B^4, that is,*

$$W(K_n) \approx B^4.$$

In particular, K_n is a slice knot.

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Here recall the slice-ribbon conjecture. A knot in $S^3 = \partial B^4$ is called *slice* if it bounds a smoothly embedded disk in B^4. A knot in S^3 is called *ribbon* if it bounds a smoothly immersed disk in S^3 with only ribbon singularities. It is well known that every ribbon knot is slice. The *slice-ribbon conjecture* states that any slice knot is ribbon. There are some affirmative results on the slice-ribbon conjecture, see [CD, GJ, Le, Li]. On the other hand, Gompf, Scharlemann and Thompson [GST] demonstrated slice knots which might not be ribbon. Similarly, there is no apparent reason for the slice knots K_n in Theorem 3.1 to be ribbon.

Let $\mathcal{K}_n$ ($n \geq 0$) be the knot obtained from 8_{20} (with an appropriate annulus presentation) by the n-fold annulus twist. These are the simplest slice knots obtained by our method, and were studied by Omae [Om] in a different viewpoint. We will prove that these slice knots are ribbon. To prove this, we introduce a sufficient condition for given slice knots to be ribbon.

Lemma 5.1. *Let HD be a handle diagram of B^4. Suppose that HD is changed into the empty handle diagram of B^4 by handle slides, adding or canceling 1/2-handle pairs, and isotopies. Then the belt sphere of any 2-handle of HD is a ribbon knot.*

Our second result is the following.

Theorem 5.4. *The slice knot $\mathcal{K}_n$ ($n \geq 0$) is ribbon.*

We outline the proof as follows: By the construction, $\mathcal{K}_n$ ($n \geq 0$) is isotopic to the belt-sphere of a 2-handle of a certain handle diagram HD of B^4 without 3-handles, see the proof of Lemma 2.5. By (rather long) handle calculus, we prove that HD is changed into the empty handle diagram of B^4 by handle slides, canceling 1/2-handle pairs, and isotopies. By Lemma 5.1, $\mathcal{K}_n$ is ribbon.

In Section 6, we propose two conjectures. The first one is the following.

Conjecture 6.1. *Let HD be a handle diagram of B^4 without 3-handles. Then the belt-sphere of any 2-handle of HD is a ribbon knot.*

Note that, if Conjecture 6.1 is true, then slice knots in Theorem 3.1 and Gompf, Scharlemann and Thompson's slice knots in [GST] are ribbon. In this sense, to solve Conjecture 6.1 is the first step toward an affirmative answer to the slice-ribbon conjecture. For the details, see Section 6.

This paper is organized as follows: In Section 2, we recall some definitions which we will use. In Section 3, we prove the main result (Theorem 3.1). First, we give a handle decomposition of $W(K_n)$. After adding a canceling 2/3-handle pair to $W(K_n)$ suitably, we prove $W(K_n) \approx B^4$. In Section 4, we give an alternative proof of Theorem 3.1 in a special case by a log transformation. In Section 5, we give a sufficient condition for given slice knots to be ribbon (Lemma 5.1). As an application, we prove Theorem 5.4. In Section 6, we give two conjectures.

Notations. We denote by $M_K(n)$ the 3-manifold obtained from S^3 by n-surgery on K and by $X_K(n)$ the smooth 4-manifold obtained from B^4 by attaching a 2-handle along K with framing n.w The symbol $\approx$ stands for a diffeomorphism. We denote by $\mathcal{K}$ the knot 8_{20} and by $\mathcal{K}_n$ ($n \in \mathbb{Z}$) the knot obtained from 8_{20} with the annulus presentation in Figure 2. In figures, we denote by $\sim$ an isotopy and by $\to$ a handle slide, a handle canceling or a blow-up.

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2. Preliminary

In this section, we define an annulus twist, annulus presentation and recall the knots constructed by Omae and homotopy 4-balls.

Annulus twist. Let V be the solid torus standardly embedded in S^3 and V' the 3-manifold as in Figure 1. Then the following is known.

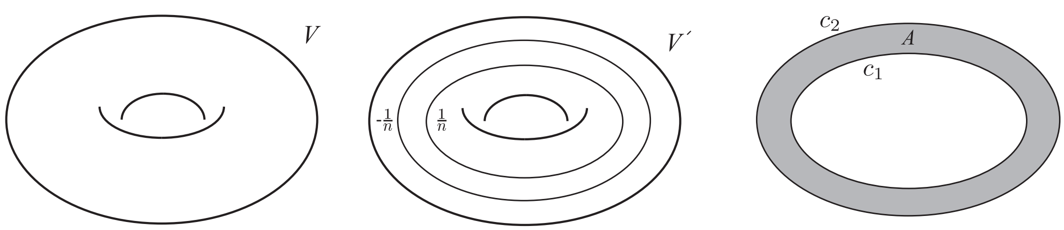

Figure 1. The definitions of V, V', and A, c_1, c_2.

Lemma 2.1 (cf. Theorem 2.1 in [Os]). *There exists a (natural) diffeomorphism*

$$\varphi_n : V' \longrightarrow V$$

such that $\varphi_n|_{\partial V'} = id$.

Remark 2.2. *Osoinach* [Os] *considered the diffeomorphism* φ_n^{-1}.

Let $A \subset \mathbb{R}^2 \cup \{\infty\} \subset S^3$ be an embedded annulus and set $\partial A = c_1 \cup c_2$ as in Figure 1. An *n-fold annulus twist along A* is the following operation:

(1) Regard c_1 as a $\frac{1}{n}$-framed knot and c_2 as a $-\frac{1}{n}$-framed knot for $n \in \mathbb{Z}$, and
(2) take a solid torus V' which is a neighborhood of A, and
(3) apply the diffeomorphism φ_n in Lemma 2.1.

A 1-fold annulus twist along A is called an *annulus twist along A*.

Annulus presentation. The first author, Jong, Omae and Takeuchi [AJOT] introduced the notion of an annulus presentation[1] of a knot for which we can associate an annulus.

We recall the definitions of an annulus presentation of a knot as follows: Let $A \subset \mathbb{R}^2 \cup \{\infty\} \subset S^3$ be a trivially embedded annulus with an ε-framed unknot c in S^3 as shown in the left side of Figure 2, where $\varepsilon = \pm 1$. Take an embedding of a band b: $I \times I \to S^3$ such that

- $b(I \times I) \cap \partial A = b(\partial I \times I)$,
- $b(I \times I) \cap \mathrm{int} A$ consists of ribbon singularities, and
- $b(I \times I) \cap c = \emptyset$,

where $I = [0,1]$. Throughout this paper, we assume that $A \cup b(I \times I)$ is orientable. This means that we deal with only 0-framed knots, see [AJOT]. For simplicity, we also assume that $\varepsilon = -1$. If a knot $K \subset S^3$ is isotopic to the knot $(\partial A \setminus b(\partial I \times I)) \cup b(I \times \partial I)$ in $M_c(-1) \approx S^3$, then we say that K admits an *annulus presentation* (A, b, c). A typical example of an annulus presentation of a knot is given in Figure 2.

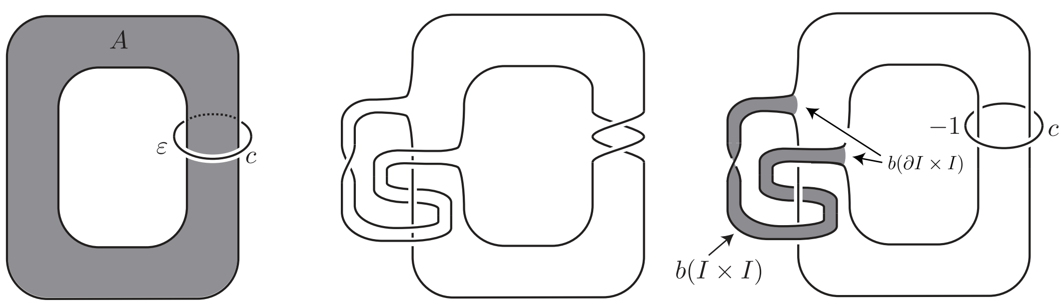

FIGURE 2. The knot 8_{20} depicted in the center admits an annulus presentation as in the right side.

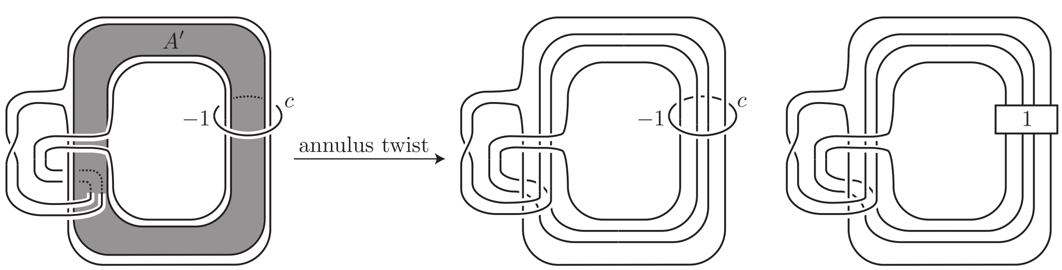

FIGURE 3. The associated annulus A' (left side), an annulus twist along A', and the resulting knot (right side).

[1]In [AJOT], it was called a band presentation.

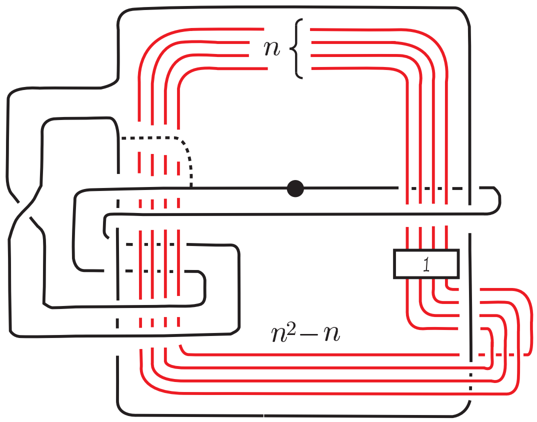

FIGURE 4. A handle decomposition of W_n $(n \geq 0)$.

Let K be a knot admitting an annulus presentation (A, b, c). Shrinking the annulus A slightly, we obtain an annulus $A' \subset A$ as shown in Figure 3. We apply the n-fold $(n \in \mathbb{Z})$ annulus twist along A' and blow down the -1-framed unknot c. Figure 3 illustrates the case $n = 1$. We call the resulting knot *the knot obtained from* K *by the* n*-fold annulus twist* without mentioning A'. The first author, Jong, Omae and Takeuchi proved the following:

Lemma 2.3 ([AJOT])**.** *Let* K *be a knot admitting an annulus presentation and* K_n $(n \in \mathbb{Z})$ *the knot obtained from* K *by the* n*-fold annulus twist. Then*

$$M_K(0) \approx M_{K_n}(0).$$

If K *is a slice knot, then* K_n *bounds a smoothly embedded disk in a homotopy 4-ball* $W(K_n)$ *such that* $\partial W(K_n) \approx S^3$*.*

Remark 2.4. *Under the assumption of Lemma 2.3, we can also prove that* $X_K(0) \approx X_{K_n}(0)$*, see* [AJOT]*. The homotopy 4-ball* $W(K_n)$ *in Lemma 2.3 depends on the choice of a diffeomorphism between* $M_K(0)$ *and* $M_{K_n}(0)$*.*

The knots obtained from 8_{20} and homotopy 4-balls. Let 8_{20} be the knot in the center of Figure 2. Then it admits an annulus presentation, see the right side of Figure 2. Let $\mathcal{K}_n$ be the knot obtained from 8_{20} by the n-fold annulus twist. Omae studies these knots in [Om]. We prove the following two lemmas.

Lemma 2.5. *The above knot* $\mathcal{K}_n$ $(n \geq 0)$ *bounds a smoothly embedded disk in a homotopy 4-ball* W_n *such that* $\partial W_n \approx S^3$ *and it has the handle decomposition as in Figure 4.*

For the dotted circle notation for the complements of ribbon disks, see subsection 1.4 in [A] (see also subsection 6.2 in [GS]). The first half of this lemma follows from Lemma 2.3. For the sake of completeness, we give the proof.

Proof. Let $f_n : M_{\mathcal{K}_0}(0) \to M_{\mathcal{K}_n}(0)$ be the diffeomorphism described in Figure 5 (here we ignore the framed knots colored red). For the detail of this diffeomorphism, see [Te].

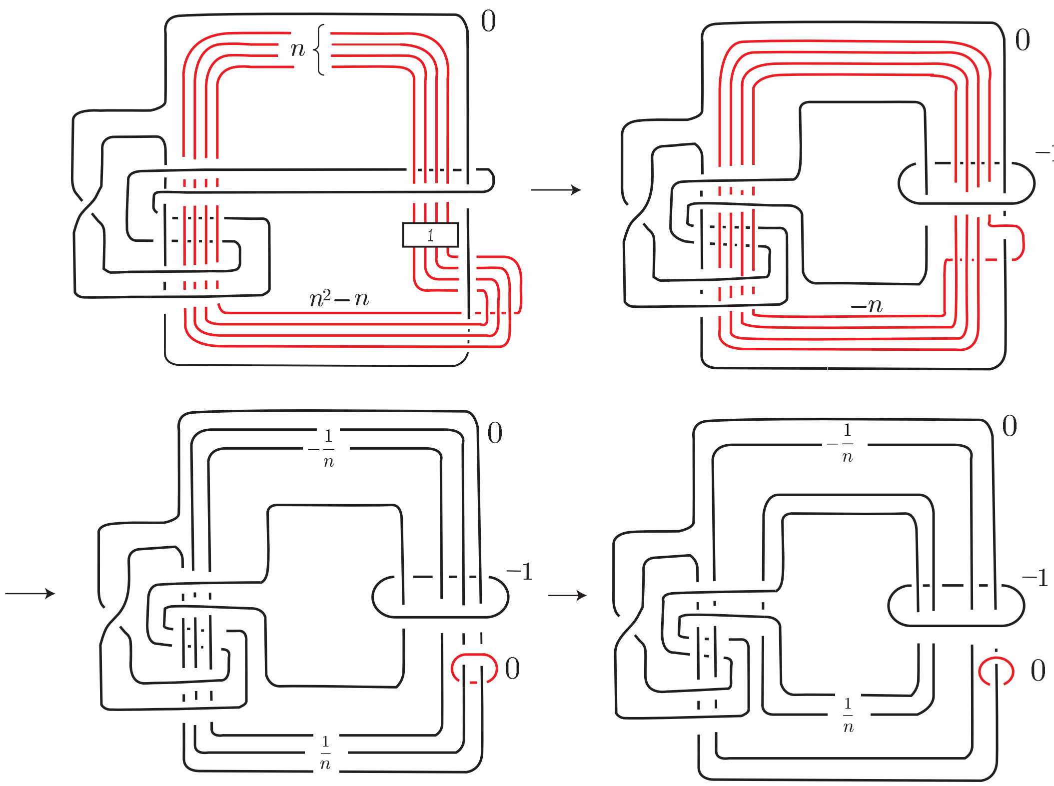

FIGURE 5. A diffeomorphism from $M_{\mathcal{K}_0}(0)$ to $M_{\mathcal{K}_n}(0)$. $M_{\mathcal{K}_0}(0)$ is represented by the first picture. The second picture is obtained by a blow up. The third picture is obtained by applying φ_n^{-1} in Lemma 2.1. The last picture is obtained by a handle slide. Then we obtain $M_{\mathcal{K}_n}(0)$ from the last picture by applying φ_n in Lemma 2.1 and a blow down.

The knot $\mathcal{K}_0$ is ribbon. Indeed, if we add a band along the dashed arc as in the left side of Figure 6, then we obtain the two component unlink. Let D^2 be the corresponding smoothly, properly embedded disk in B^4 such that $\partial D^2 = \mathcal{K}_0$ and X the 4-manifold obtained from B^4 by removing an open tubular neighborhood of D^2 in B^4 (see Figure 7). Note that ∂X is (naturally) diffeomorphic to $M_{\mathcal{K}_0}(0)$. If we attach a 2-handle along the meridian of $\mathcal{K}_0$ in $M_{\mathcal{K}_0}(0) \approx \partial X$ with framing 0, then the resulting 4-manifold is diffeomorphic to B^4. The homotopy 4-ball W_n is obtained from X by attaching a 2-handle along the meridian μ_n of $\mathcal{K}_n$ in $M_{\mathcal{K}_n}(0) \approx \partial X$ with framing 0. Schematic pictures are given in Figure 7. The knot $\mathcal{K}_n$ is isotopic to the boundary of the cocore disk of the 2-handle attached along μ_n. Thus $\mathcal{K}_n$ bounds the cocore disk in W_n, that is, a smoothly embedded disk in W_n.

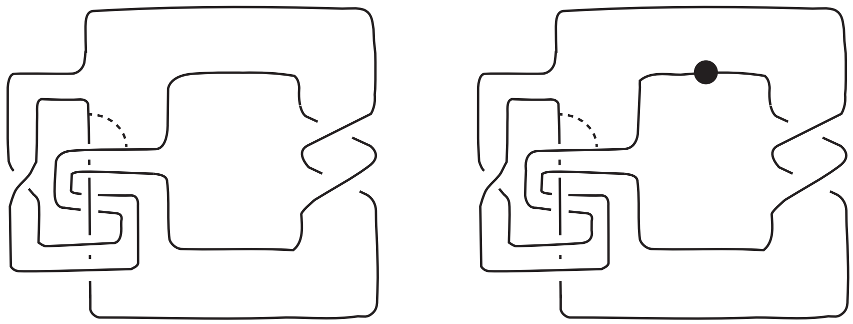

FIGURE 6. $\mathcal{K}$ with a dashed arc and the handle decomposition of X.

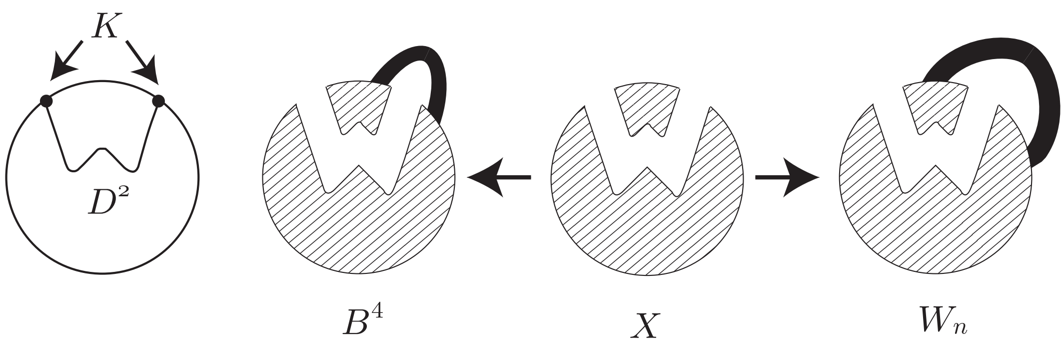

FIGURE 7.

Next, we draw a handlebody picture of W_n. Recall that X has the handle decomposition as in the right of Figure 6. The diffeomorphism from ∂X to $M_{\mathcal{K}_0}(0)$, denoted by g, is given by changing the dot to 0. By the construction, W_n is obtained from X by attaching a 2-handle along $(f_n \circ g)^{-1}(\mu_n)$ in ∂X with a suitable framing. By Figure 5, the framing is $n^2 - n$ and W_n has the handle decomposition as in Figure 4. $\square$

Lemma 2.6. *The above knot $\mathcal{K}_n$ $(n < 0)$ bounds a smoothly embedded disk in a homotopy 4-ball W_n such that $\partial W_n \approx S^3$ and it has the handle decomposition as in Figure 8.*

Proof. Set $n = -m$ for some positive integer m. Let $f_{-m} : M_{\mathcal{K}_0}(0) \to M_{\mathcal{K}_{-m}}(0)$ be the diffeomorphism described in Figure 9 (here we ignore the framed knots colored red).

The knot $\mathcal{K}_0$ is ribbon. Indeed, if we add a band along the dashed arc as in the left side of Figure 6, then we obtain the two component unlink. Let D^2 be the corresponding smoothly, properly embedded disk in B^4 such that $\partial D^2 = \mathcal{K}_0$ and X the 4-manifold obtained from B^4 by removing an open tubular neighborhood of D^2 in B^4. Note that ∂X is (naturally) diffeomorphic to $M_{\mathcal{K}_0}(0)$. The homotopy 4-ball W_{-m} is obtained from X by attaching a 2-handle along the meridian μ_{-m} of $\mathcal{K}_{-m}$ in $M_{\mathcal{K}_{-m}}(0) \approx \partial X$ with framing 0. The knot $\mathcal{K}_{-m}$ is isotopic to the boundary of the cocore disk of the 2-handle

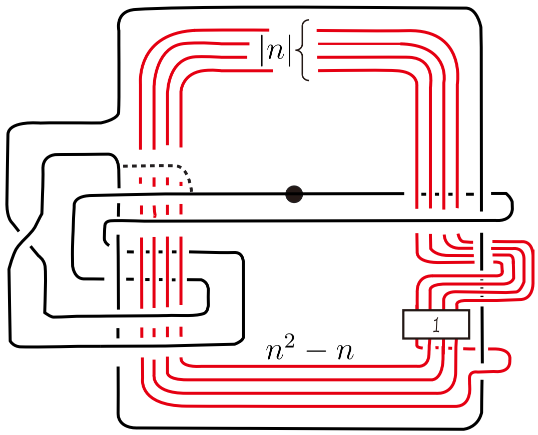

FIGURE 8. A handle decomposition of W_n $(n < 0)$.

attached along μ_{-m}. Thus $\mathcal{K}_{-m}$ bounds the cocore disk in W_{-m}, that is, a smoothly embedded disk in W_{-m}.

Next, we draw a handlebody picture of W_{-m}. Recall that X has the handle decomposition as in the right of Figure 6. The diffeomorphism from ∂X to $M_{\mathcal{K}_0}(0)$, denoted by g, is given by changing the dot to 0. By the construction, W_{-m} is obtained from X by attaching a 2-handle along $(f_{-m} \circ g)^{-1}(\mu_{-m})$ in ∂X with a suitable framing. By Figure 9, the framing is $m^2 + m(= n^2 - n)$. Therefore $W_{-m}(= W_n)$ has the handle decomposition as in Figure 8. □

3. A construction of slice knots via annulus twists.

In this section, we prove the following theorem by introducing a canceling 2/3-handle pair.

Theorem 3.1. *Let K be a ribbon knot admitting an annulus presentation and K_n $(n \in \mathbb{Z})$ the knot obtained from K by the n-fold annulus twist. Then the homotopy 4-ball $W(K_n)$ associated to K_n is diffeomorphic to B^4, that is,*

$$W(K_n) \approx B^4.$$

In particular, K_n is a slice knot.

Proof. First we consider the case $\mathcal{K} = 8_{20}$ with the annulus presentation as the right side of Figure 2 and $n \geq 0$. By Lemma 2.5, $\mathcal{K}_n$ bounds a smoothly embedded disk in the homotopy 4-ball W_n given by the picture in Figure 4. We prove the following claims.

Claim 1. W_n $(n \geq 0)$ also has the handle decomposition given by the picture in Figure 10.

Proof. Inserting a canceling 1/2-handle pair to W_n, we obtain the first picture in Figure 11. Note that, in Figure 11, we ignore the dashed arc because it is disjoint from the handle slides below. By handle slides, we obtain the

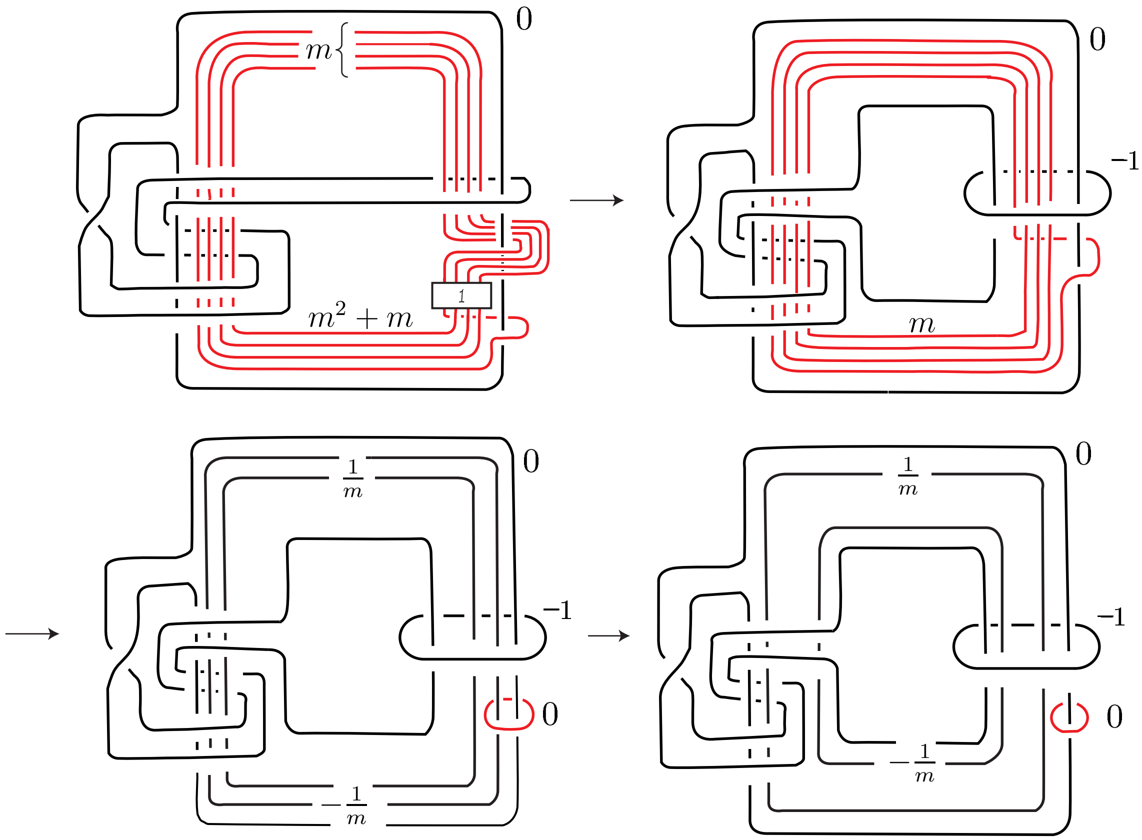

FIGURE 9. A diffeomorphism from $M_{\mathcal{K}_0}(0)$ to $M_{\mathcal{K}_{-m}}(0)$. $M_{\mathcal{K}_0}(0)$ is represented by the first picture. The second picture is obtained by a blow up. The third picture is obtained by applying φ_{-m}^{-1} in Lemma 2.1. The last picture is obtained by a handle slide. Then we obtain $M_{\mathcal{K}_{-m}}(0)$ from the last picture by applying φ_{-m} in Lemma 2.1 and a blow down.

second picture. By inserting a canceling 1/2-handle pair to W_n and handle slides, we obtain the third picture. After a 1-handle slide (and a 2-handle slide, annihilating a canceling 1/2-handle pair and isotopy), we obtain the last picture. Therefore, W_n has the handle decomposition given by the picture in Figure 10. □

Claim 2. $W_n \approx W_{n-1}$.

Proof. We show that $\gamma, \lambda \subset \partial W_n$ described in Figure 12 are isotopic and each curve is the unknot in $\partial W_n = S^3$. By Claim 1, W_n has the handle decomposition given by the first picture in Figure 12. We replace the two dotted circles with the zero-framed circles. Then we obtain the second picture in Figure 12. Handle calculus in Figure 12 illustrates the diffeomorphism from ∂W_n to S^3.

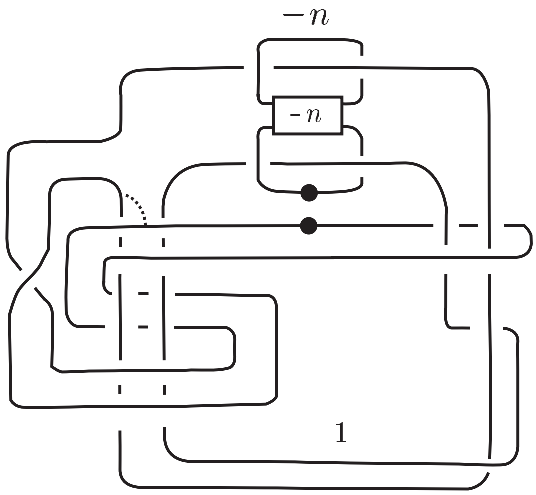

FIGURE 10. A handle decomposition of W_n.

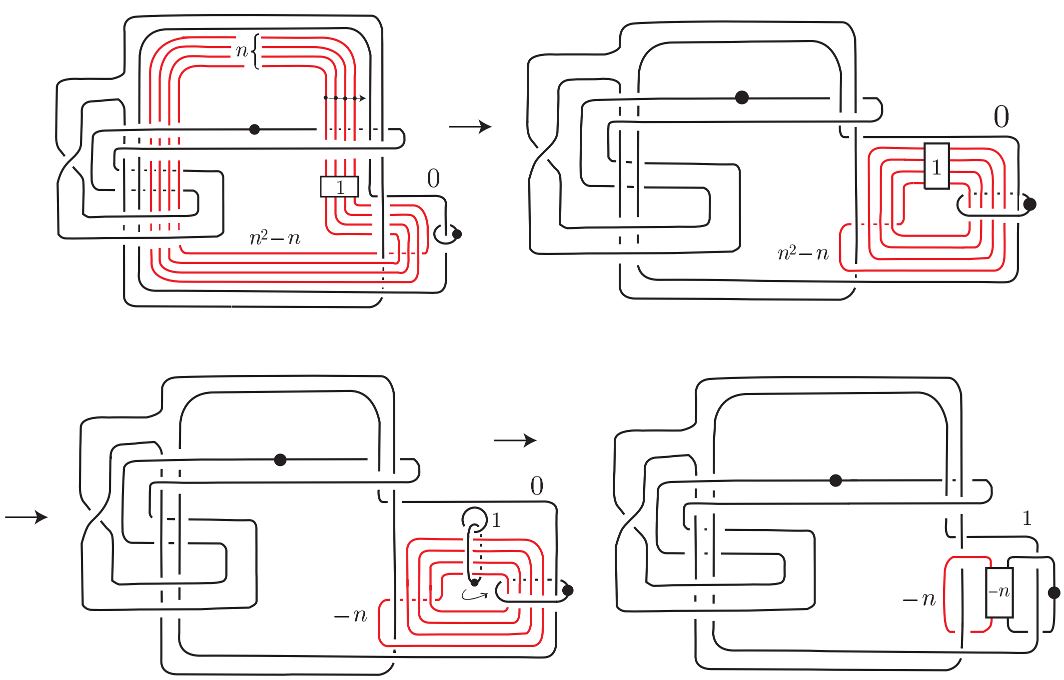

FIGURE 11. Handle decompositions of W_n $(n \geq 0)$.

Furthermore, if we regard γ (or λ) as a -1-framed knot, then it is isotopic to the 0-framed unknot in S^3. Now we insert a canceling 2/3-handle pair to W_n. Then W_n is diffeomorphic to the first picture in Figure 13. By a handle slide, we obtain the second picture, which is diffeomorphic to W_{n-1}. □

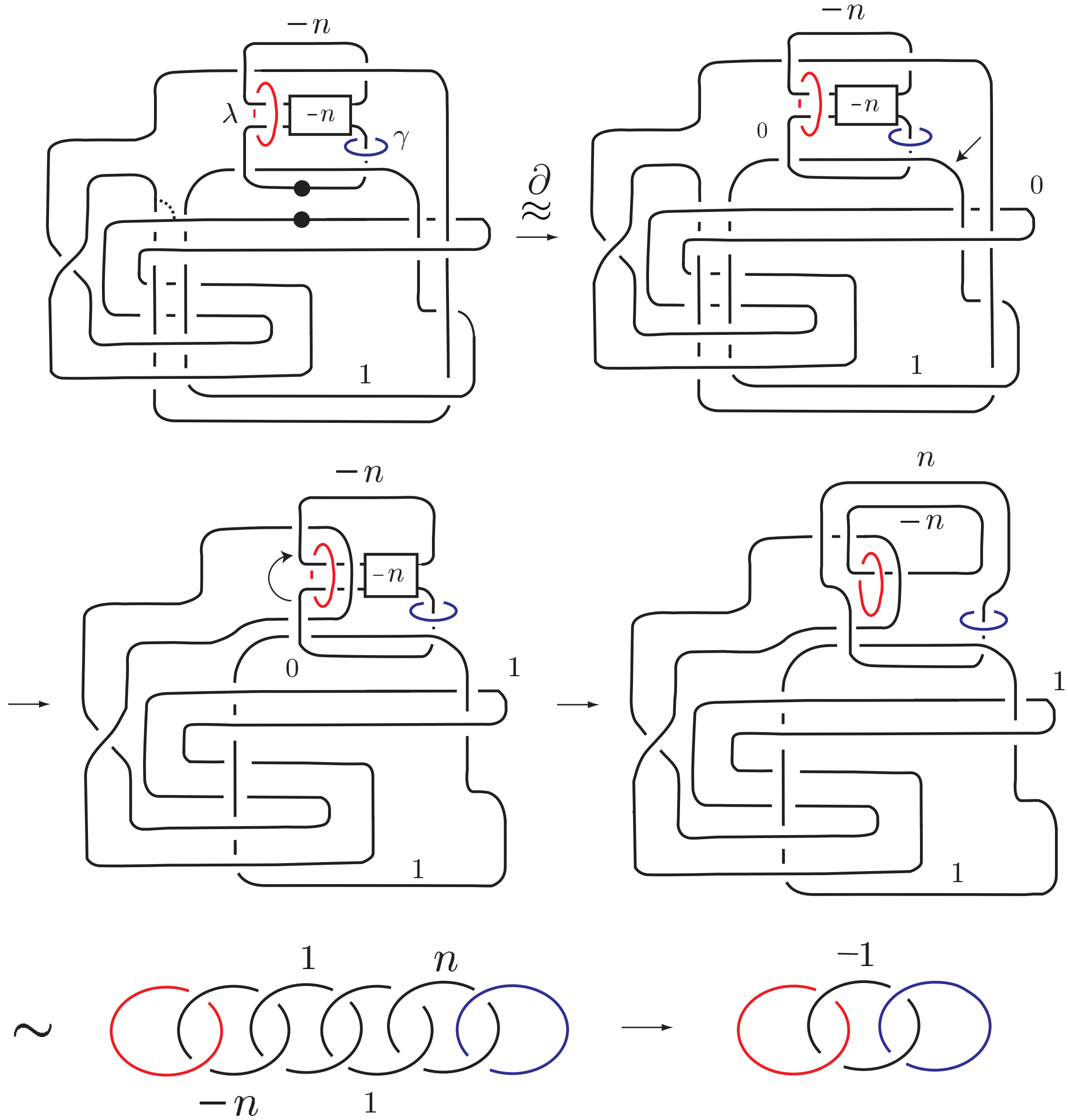

FIGURE 12. A specific diffeomorphism identifying ∂W_n with S^3 which tells us that two curves $\gamma, \lambda \subset \partial W_n$ are isotopic.

By Claim 2, $W_n \approx W_{n-1} \approx \cdots \approx W_1 \approx W_0$. By the construction, $W_0 \approx B^4$. Therefore $W_n \approx B^4$ and $\mathcal{K}_n$ is a slice knot.

Next we consider the case $\mathcal{K} = 8_{20}$ with the annulus presentation as the right side of Figure 2 and $n < 0$. By Lemma 2.6, $\mathcal{K}_n$ bounds a smoothly embedded disk in the homotopy 4-ball W_n given by the picture in Figure 8. We prove the following claim.

Claim 3. W_n $(n < 0)$ also has the handle decomposition given by the picture in Figure 10.

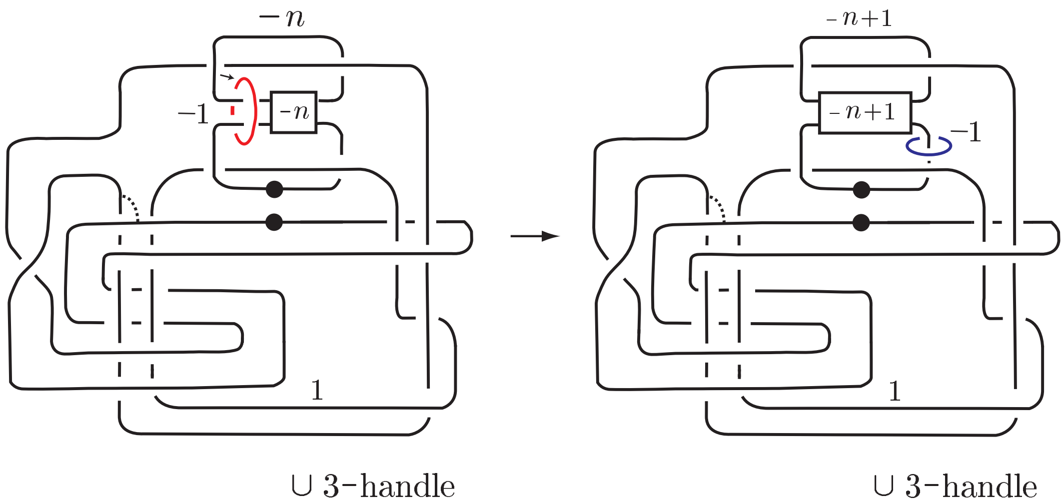

FIGURE 13. A handle slide.

Proof. Inserting a canceling 1/2-handle pair to W_n, we obtain the first picture in Figure 14. Note that, in Figure 14, we ignore the dashed arc because it is disjoint from the handle slides below. By a similar handle calculus to that in Figure 11, we obtain the second picture. Therefore, W_n has the handle decomposition given by the picture in Figure 10. □

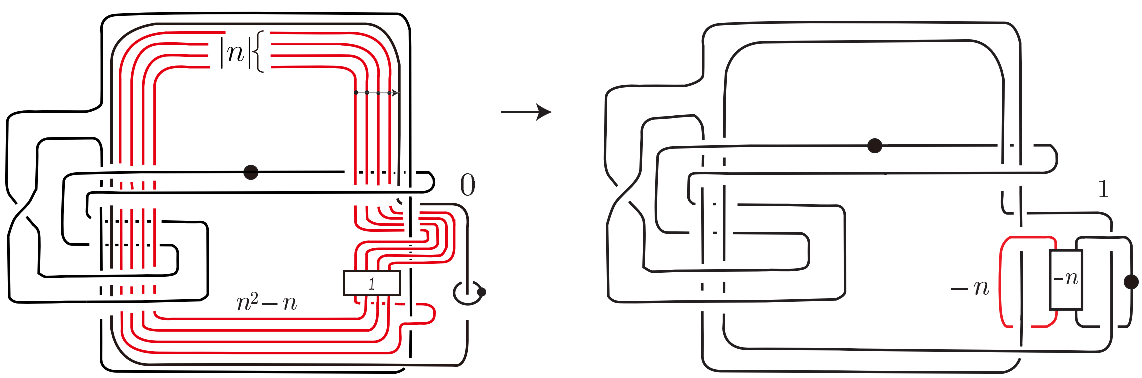

FIGURE 14. Handle decompositions of W_n $(n < 0)$.

By the same argument as that in Claim 2, we can prove that $W_n \approx B^4$ and $\mathcal{K}_n$ is a slice knot.

Now we consider the general case. First suppose that $n \geq 0$. In this case, we can also associate a diffeomorphism $f_n : M_K(0) \to M_{K_n}(0)$ as described in Figure 5. Let μ_n be the meridian of K_n in $M_{K_n}(0)$. Then $f_n^{-1}(\mu_n)$ is as in the first picture in Figure 15 (after ignoring the framing). Since K is ribbon, there exist mutually disjoint bands $B_1, \cdots, B_m$ such that if we surgery along these bands, then we obtain the $(m+1)$-component unlink. Furthermore, (by deforming these bands slightly) we can assume that $B_i \cap f_n^{-1}(\mu_n) = \emptyset$ for each $i \in \{1, 2, \cdots, m\}$. Then, as the proof of Lemma 2.5, we see that K_n bounds a smoothly embedded disk in a homotopy 4-ball $W(K_n)$ which has the handle decomposition as in the second picture in Figure 15. Note that we do not draw dashed arcs in Figure 15. It is proved that $W(K_n)$ also has the handle decomposition as in the third picture in Figure 15 similarly. Then we can prove that $W(K_n) \approx B^4$ by the same argument. Therefore K_n is a slice knot.

For the case $n < 0$, by a similar argument to that in Claim 3, K_n bounds a smoothly embedded disk in a homotopy 4-ball $W(K_n)$ which has the handle decomposition as in the third picture in Figure 15. Then we can prove that $W(K_n) \approx B^4$ by the same argument again. Therefore K_n is a slice knot.

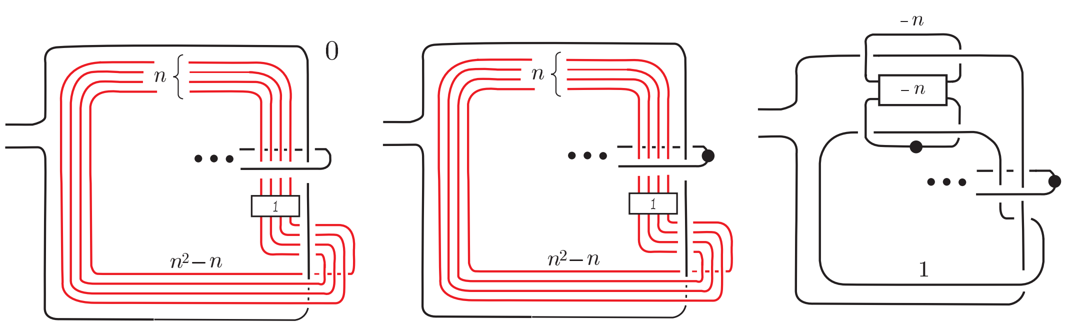

FIGURE 15.

□

4. Log transformation and fishtail neighborhood

In this section, we give an alternative proof of Theorem 3.1 in the case $\mathcal{K} = 8_{20}$. More precisely, we prove that W_n and W_0 are related by a log transformation along a certain torus in W_n, where W_n is the homotopy 4-ball given by the picture in Figure 10. Lemma 4.1 due to Gompf ensures that W_n and W_0 are diffeomorphic, which implies that $W_n \approx B^4$.

Log transformation. Let X be an oriented 4-manifold, T an embedded torus with $T \cdot T = 0$ and $\varphi : T^2 \times \partial D^2 \to \partial\nu(T)$ a diffeomorphism, where $\nu(T)(\approx T^2 \times D^2)$ is a closed neighborhood of T in X. Removing int $\nu(T)$ from X and attaching $T^2 \times D^2$ by φ, we obtain

$$(X - \text{int } \nu(T)) \cup_\varphi T^2 \times D^2.$$

Suppose that

$$\varphi_*([\{\text{pt.}\} \times \partial D^2]) = p[\{\text{pt.}\} \times \partial D^2] + q[\gamma \times \{\text{pt.}\}]$$

for some essential simple closed curve γ in T. Then we call this surgery a *logarithmic transformation with multiplicity p, direction γ and auxiliary multiplicity q*. If $p = 1$, we call this logarithmic transformation a *q-fold Dehn twist along T parallel to γ*.

Fishtail neighborhood. The fishtail neighborhood F is an elliptic surface which has the handle decomposition in Figure 16. It is well known that the -1-framed meridian in Figure 16 is isotopic to the vanishing cycle of F. In [G2] Gompf proved the following assertion.

Lemma 4.1 ([G2])**.** *Let X be a 4-manifold and T be a regular fiber of a fishtail neighborhood F embedded in X. Then the q-fold Dehn twist along T parallel to the vanishing cycle of F does not change the diffeomorphism type of X.*

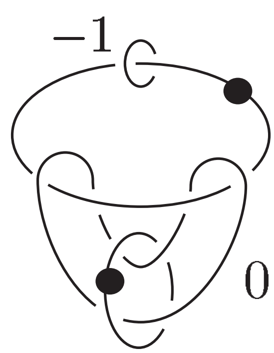

FIGURE 16. A handle decomposition of F.

We prove the following.

Lemma 4.2. *The homotopy 4-ball W_n also has the handle decomposition given by the first picture in Figure 17.*

Proof. We fix a diffeomorphism identifying ∂W_n with S^3. We use the diffeomorphism described in Figure 12 again. Recall that this diffeomorphism tells us that the -1-framed γ is isotopic to the 0-framed unknot in S^3 (for the detail, see the proof of Theorem 3.1). Therefore, by inserting a canceling 2/3-handle pair to W_n, we obtain

$$W_n \approx W_n + \gamma^{-1} \cup (\text{3-handle}),$$

where $W_n + \gamma^{-1}$ is the handlebody given by the second picture in Figure 17.

Next we fix a diffeomorphism identifying $\partial(W_n+\gamma^{-1})$ with $S^1\times S^2$ described in Figure 18 (for a while, we ignore the curve μ). This diffeomorphism tells us that $\mu \subset \partial(W_n + \gamma^{-1})$ is the unknot in $S^1 \times S^2$. Furthermore, if we regard μ as a 0-framed knot, then it is isotopic to the 0-framed unknot in $S^1 \times S^2$. Therefore, by inserting a canceling 2/3-handle pair to W_n, we obtain the first picture in Figure 17. □

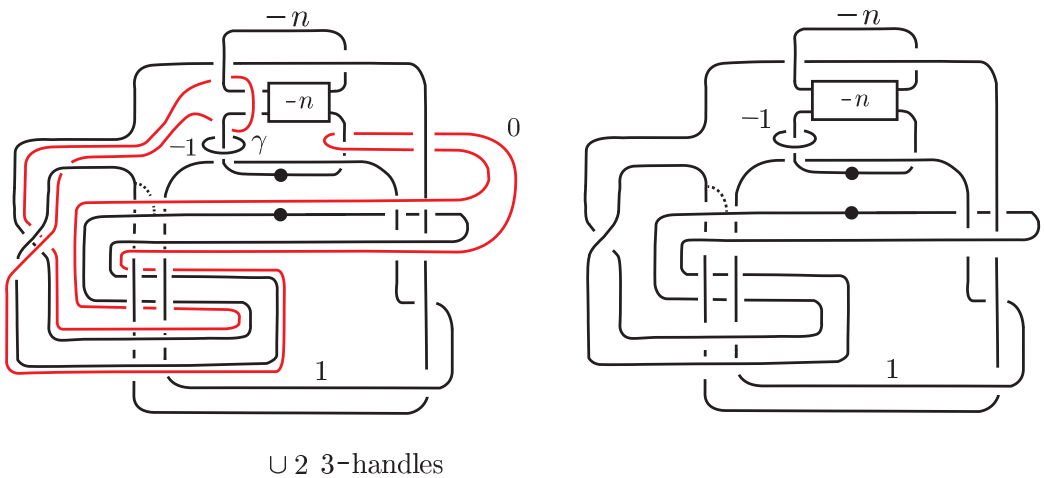

Figure 17. A handle decomposition of W_n and the handlebody picture of $W_n + \gamma^{-1}$.

Now we prove the main result in this section.

Proof of Theorem 3.1 in the case $\mathcal{K} = 8_{20}$. The second picture of Figure 19 is a sub-handlebody of W_n. By isotopy, we see that it is diffeomorphic to $F \cup$ (1-handle), where F is the fishtail neighborhood. Therefore, by removing the 1-handle, we can find F as a submanifold of W_n.

Let T be a regular fiber of F embedded in W_n. The 1-fold Dehn twist along T parallel to γ is 1-untwisting along γ. For the detail, see [AY] or [GS]. Thus the local deformation is as in Figure 20. As a result, performing the n-fold Dehn twist along T parallel to γ and removing the canceling 2/3-handle pairs, we obtain W_0 which is diffeomorphic to B^4. By Lemma 4.1, $W_n \approx W_0$. Therefore $\mathcal{K}_n$ (obtained from 8_{20}) is a slice knot. □

5. A sufficient condition to be ribbon

In this section, we give a sufficient condition for a slice knot to be ribbon (Lemma 5.1) and prove that all the knots obtained from 8_{20} by annulus twists are ribbon (Theorem 5.4).

Lemma 5.1. *Let HD be a handle diagram of B^4. Suppose that HD is changed into the empty handle diagram of B^4 by handle slides, adding or canceling 1/2-handle pairs, and isotopies. Then the belt sphere of any 2-handle of HD is a ribbon knot.*

Proof. Let

$$HD = HD_0 \to HD_1 \to \cdots \to HD_n = \text{(empty handle diagram)}$$

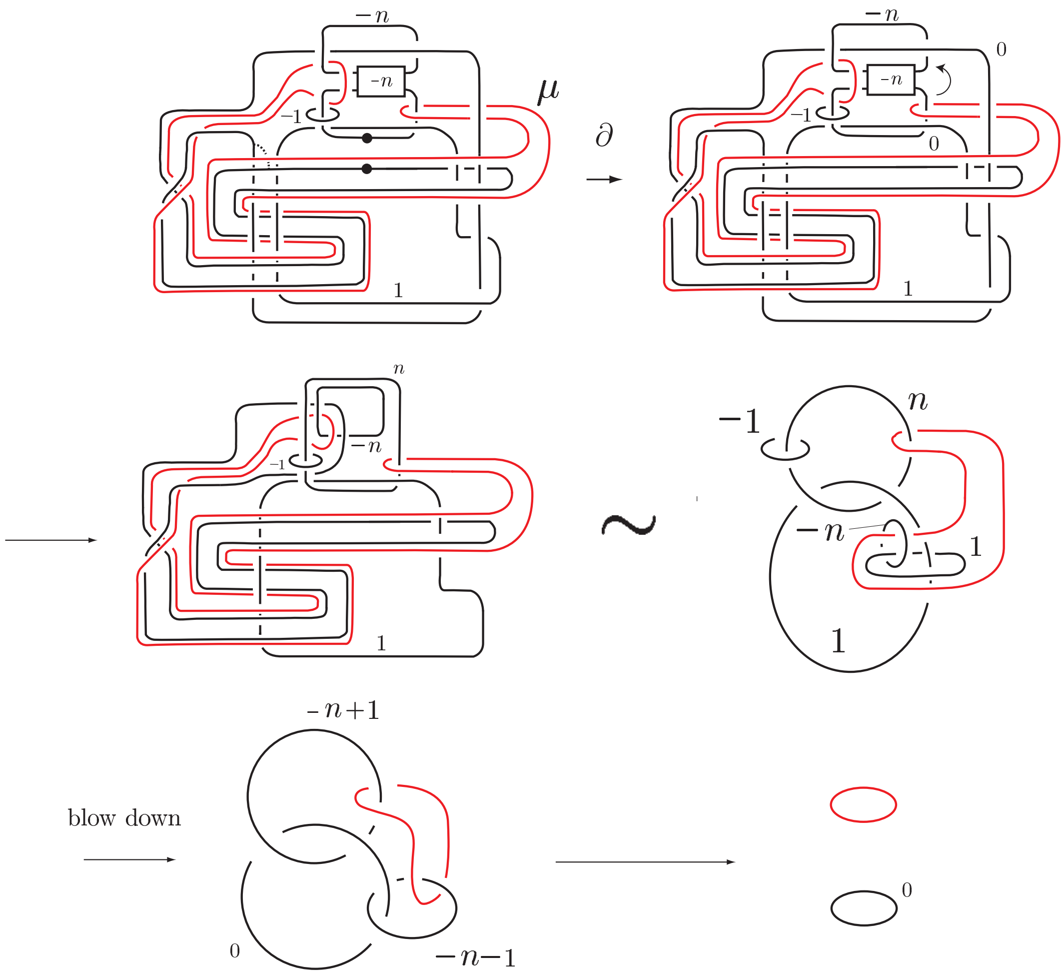

FIGURE 18. A diffeomorphism identifying $\partial(W_n + \gamma^{-1})$ with $S^1 \times S^2$ which tells us that the curve μ is the unknot in $S^1 \times S^2$.

be a sequence of handle diagrams satisfying the condition of Lemma 5.1. By rearranging the sequence, we can assume the following.

$$HD_0 \to HD_1 \to \cdots \to HD_k \quad \text{(adding canceling 1/2-handle pairs)},$$

$$HD_k \to HD_{k+1} \to \cdots \to HD_l \quad \text{(handle slides)},$$

$$HD_l \to HD_{l+1} \to \cdots \to HD_n \quad \text{(annihilating canceling 1/2-handle pairs)}.$$

Let β be the belt sphere of any 2-handle of HD. Then it is the unknot in HD and we denote by β_i $(i = 1, 2, \cdots, l)$ the corresponding knot in HD_i. We see that β_l is also the unknot in HD_l. Furthermore, we can find a smoothly embedded disk D in HD_l such that $\partial D = \beta_l$, the disk D does not intersect any dotted 1-handles[2], and D intersects transversely with some attaching spheres

[2] We can choose D in this way since the link which consists of dotted circles (representing 1-handles) and β_l is the unlink.

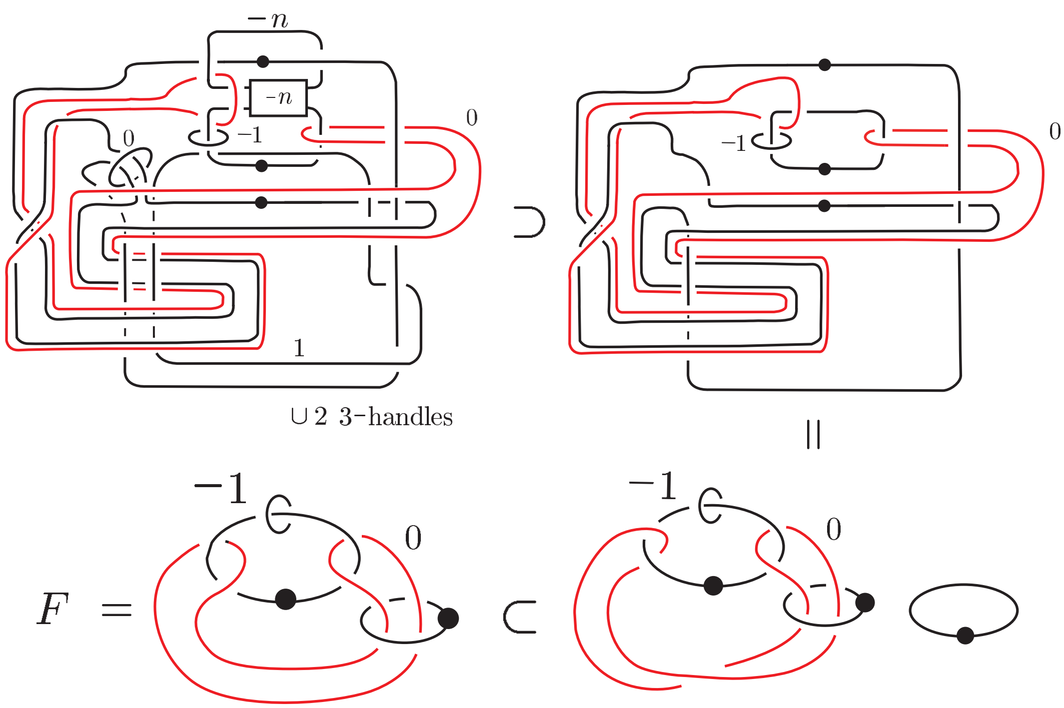

FIGURE 19. An embedding of the fishtail neighborhood F.

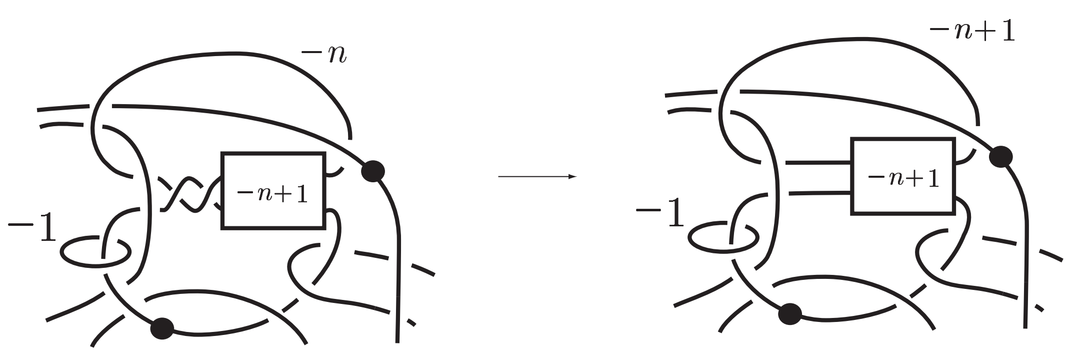

FIGURE 20. The 1-fold Dehn twist along T parallel to γ.

of 2-handles as the left in Figure 21. Let m be the number of intersections between D and the attaching spheres of 2-handles of HD_l. By band surgeries along mutually disjoint bands $B_1, B_2, \cdots, B_{m-1}$ as the middle picture in Figure 21, we obtain an m-component link L such that each component is the meridian of the attaching sphere of a 2-handle of HD_l.

Finally we consider the sequence $HD_l \to \cdots \to HD_n$. Let L' be the link in HD_n which is corresponding to L. Then it is the m-component unlink in S^3. In other words, the knot β is deformed into the m-component unlink by band surgeries along $m-1$ bands. This means that β is a ribbon knot. □

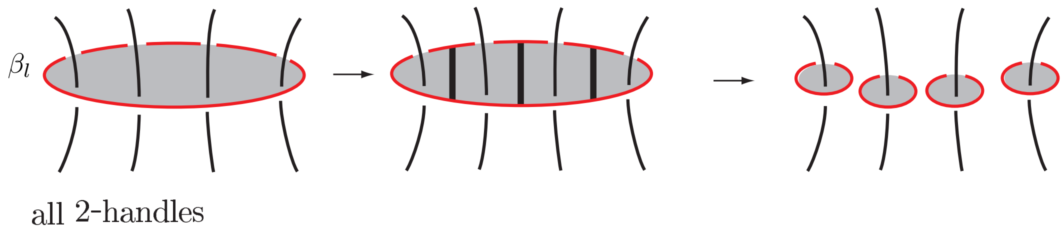

FIGURE 21. Band surgeries along mutually disjoint bands ($m = 4$).

Let 8_{20} be the knot with the annulus presentation as in the right side of Figure 2 and $\mathcal{K}_n$ ($n \geq 0$) the knot obtained from 8_{20} by the n-fold annulus twist. By Theorem 3.1, $\mathcal{K}_n$ is a slice knot. There is no apparent reason for $\mathcal{K}_n$ to be ribbon. Our result is that, indeed, $\mathcal{K}_n$ is a ribbon knot. To prove this, we first observe the following.

Lemma 5.2. *The slice knot $\mathcal{K}_n$ is located as in Figure 22.*

Proof. By the proofs of Lemma 2.5 and Theorem 3.1, we obtain this lemma immediately. □

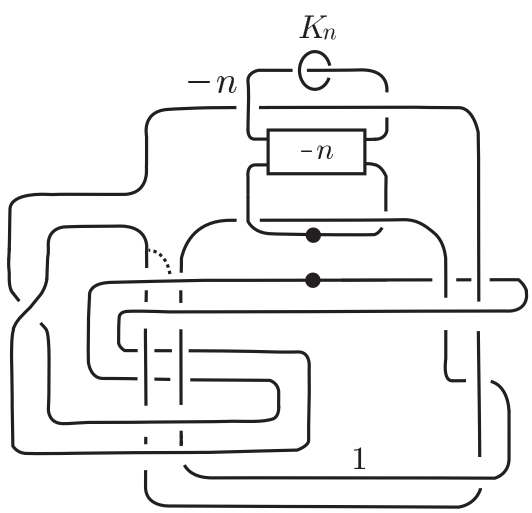

FIGURE 22. The slice knot $\mathcal{K}_n$ in ∂W_n.

Remark 5.3. *Let K be any ribbon knot in ∂B^4. Then it is not difficult to see that B^4 admits a handle decomposition*

$$h^0 \cup h^1_1 \cup \cdots \cup h^1_n \cup h^2_1 \cup \cdots \cup h^2_n$$

such that the belt sphere of some 2-handle is isotopic to K*, where* h^0 *is a* 0-*handle,* $h^1_i (i = 1, \cdots, n)$ *is a 1-handle and* $h^2_j (j = 1, \cdots, n)$ *is a 2-handle. For the converse, see Conjecture 6.1.*

Now we prove the following:

Theorem 5.4. *The slice knot* $\mathcal{K}_n$ $(n \geq 0)$ *is ribbon.*

Proof. Let HD be the handle diagram given by the picture in Figure 22. By Lemma 5.2, $\mathcal{K}_n$ is isotopic to the belt sphere of a 2-handle of HD. By Lemma 5.1, if HD is changed into the empty handle diagram by handle slides, adding or canceling 1/2-handle pairs, and isotopies, then $\mathcal{K}_n$ is a ribbon knot. Such operations are realized in Figures 23, 24, 25 and 26. As a result, $\mathcal{K}_n$ is a ribbon knot. □

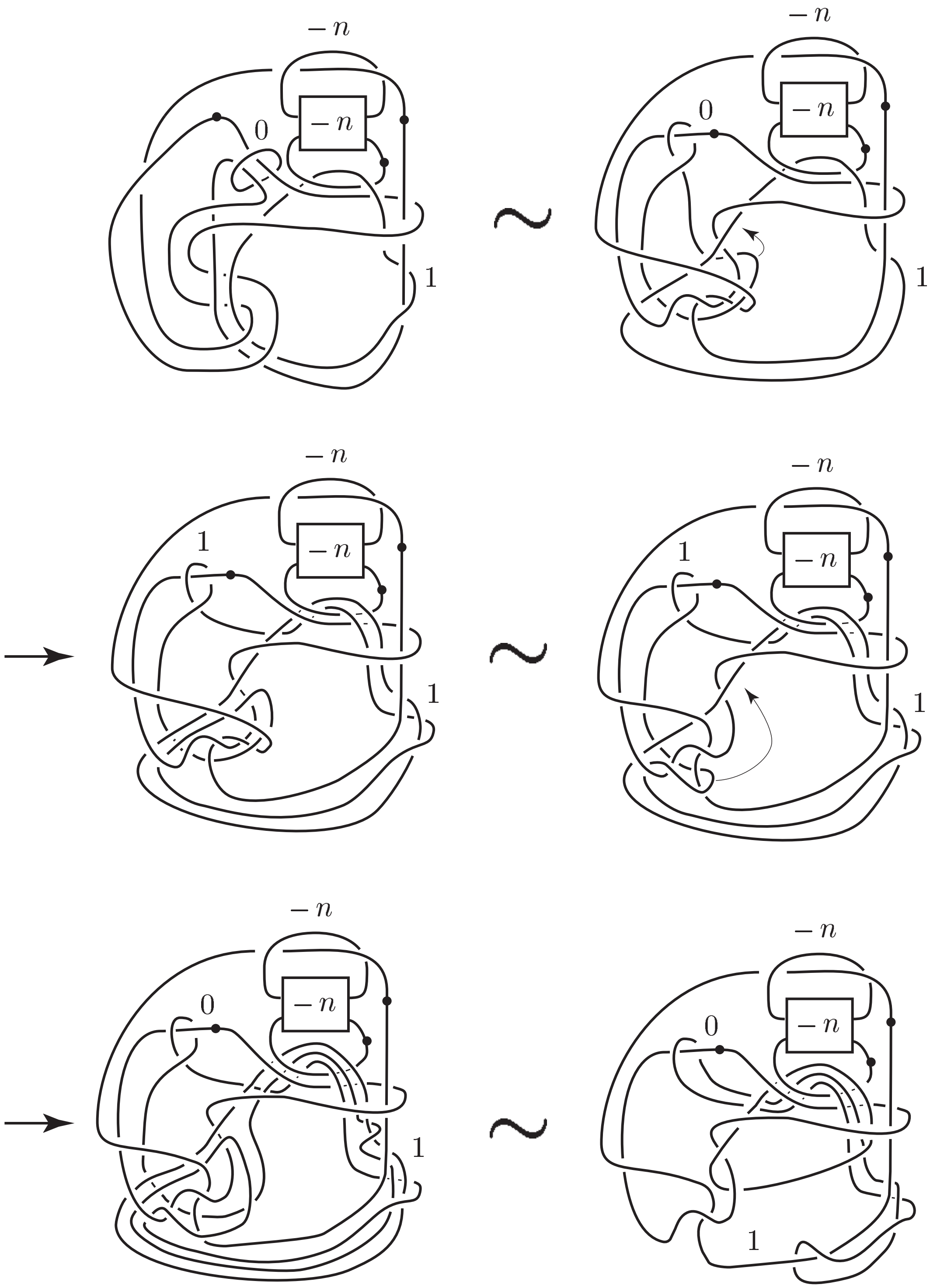

FIGURE 23. Handle calculus without adding canceling 2/3-handle pairs.

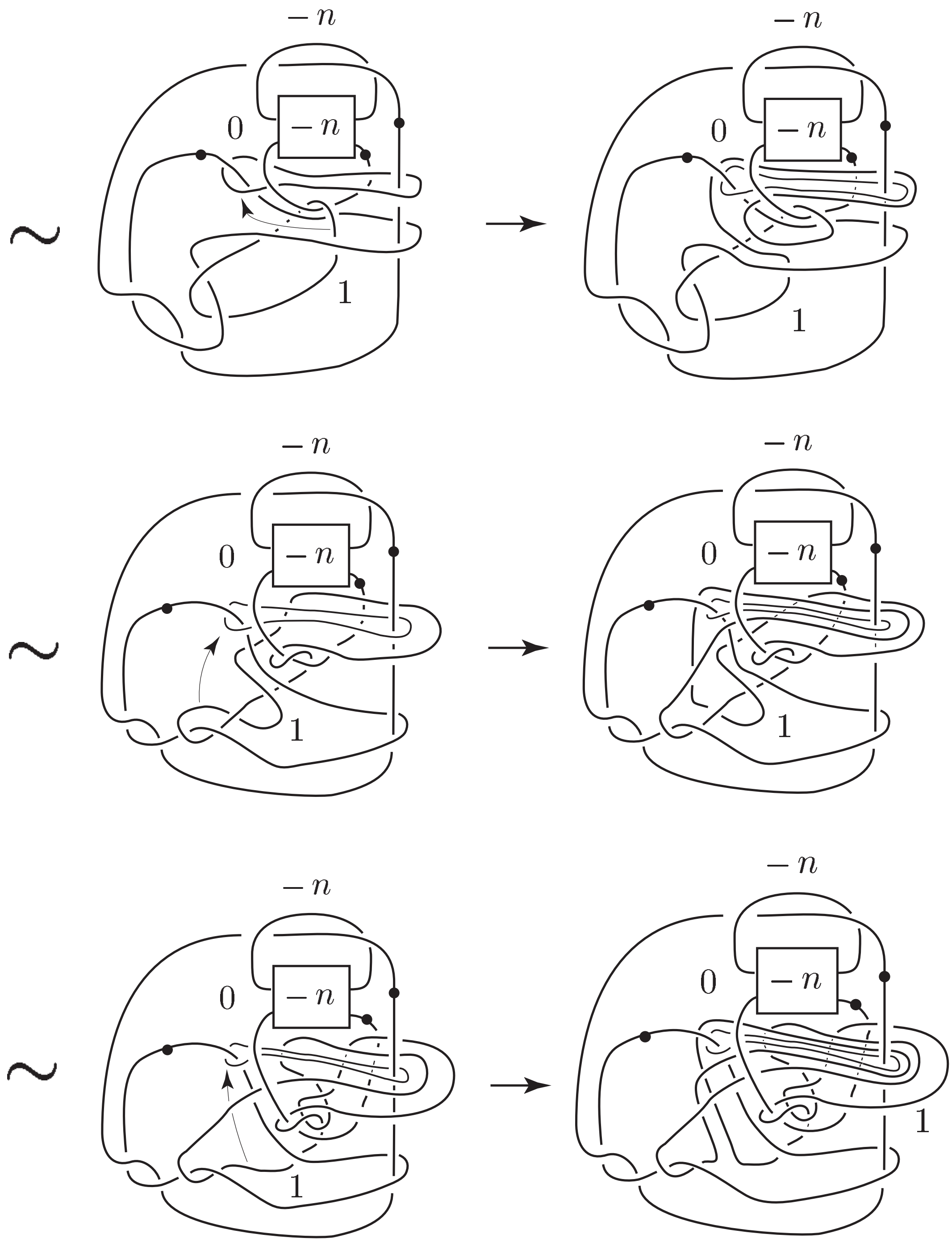

Figure 24. Handle calculus without adding canceling 2/3-handle pairs.

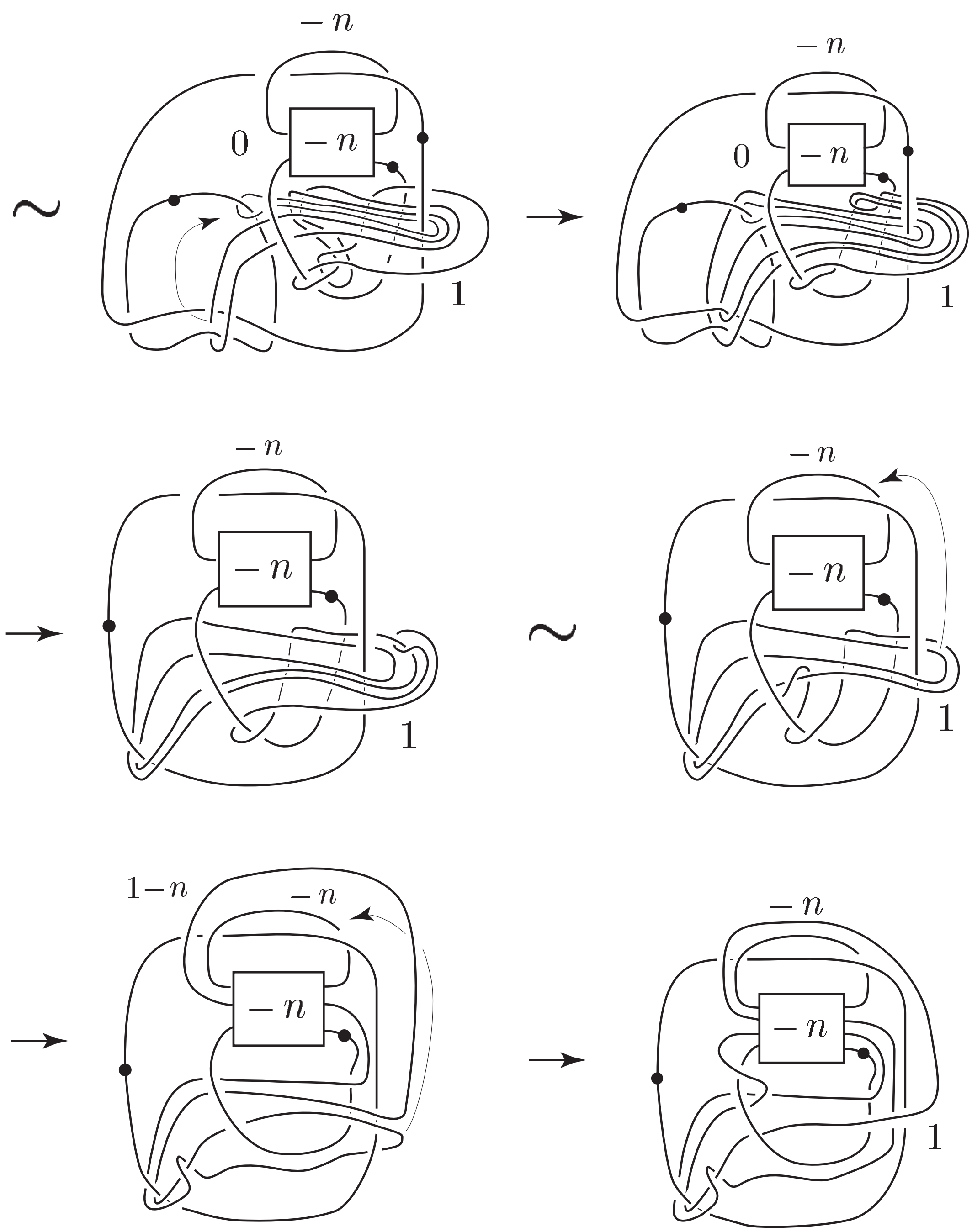

FIGURE 25. Handle calculus without adding canceling 2/3-handle pairs.

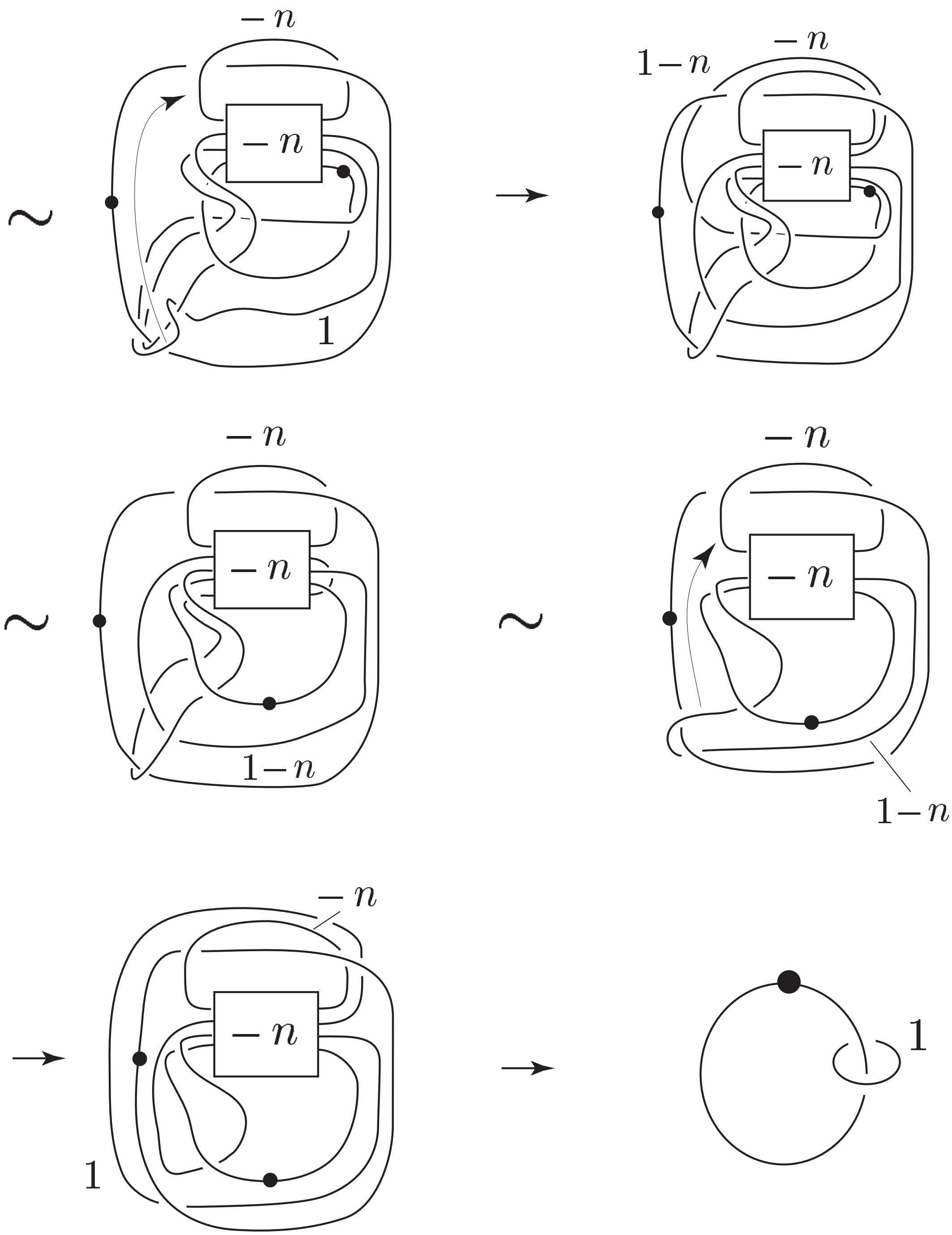

Figure 26. Handle calculus without adding canceling 2/3-handle pairs.

Now we draw a ribbon presentation of $\mathcal{K}_n$. Keeping track of $\mathcal{K}_n$ through the handle calculus, though it is rather troublesome, we can obtain a ribbon presentation of $\mathcal{K}_n$ as in Figure 27.

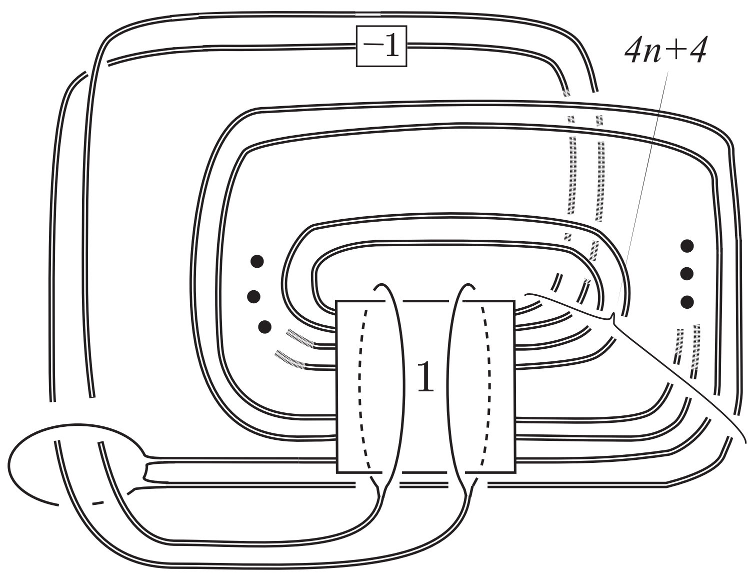

FIGURE 27. A ribbon presentation of $\mathcal{K}_n$ ($n \geq 1$).

6. Two conjectures

In this section, we propose two conjectures. The first one is the following.

Conjecture 6.1. *Let HD be a handle diagram of B^4 without 3-handles. Then the belt-sphere of any 2-handle of HD is a ribbon knot.*

Recall that each slice knot in Theorem 3.1 is isotopic to the belt-sphere of a 2-handle of a certain handle diagram of B^4 without 3-handles, see the proofs of Lemma 2.5 and Theorem 3.1. Therefore, if Conjecture 6.1 is true, then all slice knots in Theorem 3.1 are ribbon.

A partial answer to Conjecture 6.1 has already given in Lemma 5.1. The difficulty to solve this conjecture is explained by yet another following conjecture.

Conjecture 6.2. *There exists a handle diagram HD of B^4 without 3-handles such that we always have to add canceling 2/3-handle pairs when we change HD into the empty handle diagram B^4 by a sequence of handle slides, adding or canceling handle pairs, and isotopies*

A promising candidate to Conjecture 6.2 is the handle diagram $H_{n,k}$ of B^4 given by Gompf [G1] (see the left half of Figure 28), where $n \geq 3$ and $k \neq 0$.

Finally, we observe that, if Conjecture 6.1 is true, then Gompf, Scharlemann and Thompson's slice knots in [GST] are ribbon as follows: Let $L_{n,k}$ be the

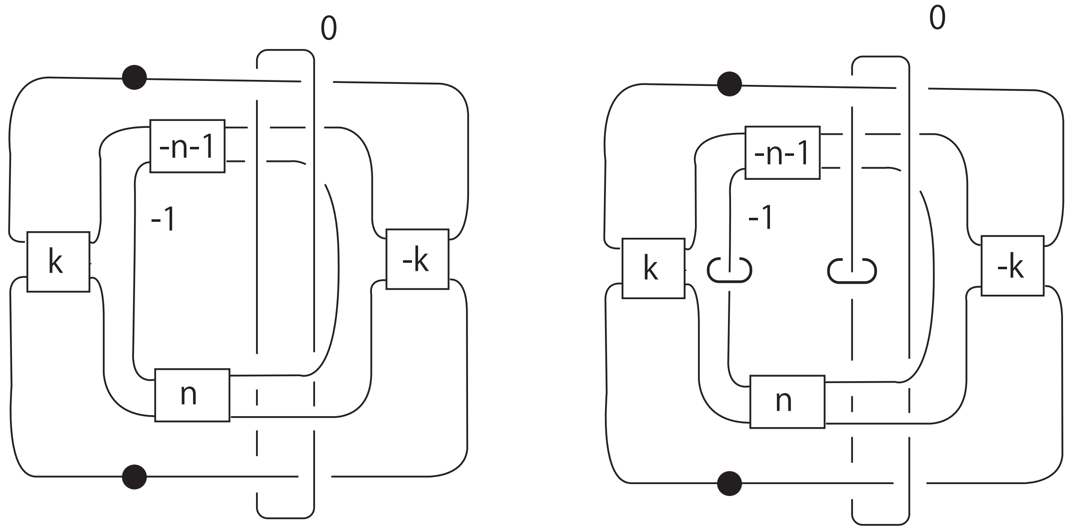

FIGURE 28. The handle diagram $H_{n,k}$ of B^4 (left) and the 2-component link $L_{n,k}$ in $S^3 = \partial B^4$ (right).

2-component link in S^3 which consists of the two belt-spheres of the two 2-handles of $H_{n,k}$, see the right half of Figure 28. By the definition, $L_{n,k}$ is a slice link, that is, it bounds two smoothly embedded disjoint disks in B^4. Each Gompf, Scharlemann and Thompson's slice knot is obtained from $L_{n,k}$ by attaching an arbitrary band. After a single 2-handle slide (along the band), it turns out that the slice knot is isotopic to the belt-sphere of a 2-handle of a certain handle diagram of B^4 without 3-handles. Therefore, if Conjecture 6.1 is true, these slice knots are also ribbon. In this sense, to solve Conjecture 6.1 is the first step toward an affirmative answer to the slice-ribbon conjecture.

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Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan
E-mail address: abe.t.av@m.titech.ac.jp

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail address: tange@math.tsukuba.ac.jp