# Density of states for GUE, GOE, and interpolating ensembles through supersymmetric approach

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#### Abstract

We use the supersymmetric formalism to derive an integral formula for the density of states of the Gaussian Orthogonal Ensemble, and then apply saddle-point analysis to give a new derivation of the  $1/N$ correction to Wigner's law. This extends the work of Disertori on the Gaussian Unitary Ensemble. We also apply our method to the interpolating ensembles of Mehta–Pandey.

#### 1 Introduction

In this note we study the density of states for the Gaussian Unitary Ensemble (GUE) and the Gaussian Orthogonal Ensemble (GOE). Both are ensembles of  $N \times N$  Hermitian random matrices H, so that the joint distribution of the entries is centered Gaussian, and the covariance of the entries is given by

<span id="page-0-0"></span>
$$
\langle H_{ij} H_{kl} \rangle = \begin{cases} \frac{1}{N} \delta(jk) \delta(il) , & \text{GUE} \\ \frac{1}{N} \left( \delta(jk) \delta(il) + \delta(ik) \delta(jl) \right) , & \text{GOE} \end{cases} . \tag{1}
$$

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Here  $\langle \cdot \rangle$  denotes the average (expectation), and  $\delta$  denotes the Kronecker delta,

$$
\delta(ij) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}
$$

.

In particular, the GUE entries are complex (the diagonal elements are real), whereas the GOE entries are real.

The density of states  $\rho(E)$  is defined by

$$
\rho(E) = \frac{d}{dE} \left\langle \frac{1}{N} \# \{\text{eigenvalues} \le E\} \right\rangle .
$$

Let

$$
\rho_{\rm sc}(E) = \frac{1}{\pi} \sqrt{(1 - E^2/4)_+}
$$

be the (Wigner) semicircle density. We give new proofs for the following two theorems (see below for the history of these and related results):

<span id="page-1-0"></span>Theorem 1. For GUE,

$$
\rho(E) = \rho_{sc}(E) - \frac{(-1)^N}{4\pi^3 N \rho_{sc}(E)^2} \cos\left[N\left(E\sqrt{1 - \frac{E^2}{4}} + 2\arcsin\frac{E}{2}\right)\right] + O(N^{-3/2})
$$

for  $|E| < 2 - \delta$ , and the implicit constant in the O-notation depends only on  $\delta > 0$ .

<span id="page-1-1"></span>Theorem 2. For GOE,

$$
\rho(E) = \rho_{sc}(E) - \frac{1}{4\pi^2 N \rho_{sc}(E)} + O(N^{-3/2})
$$

for  $|E| < 2 - \delta$ , and the implicit constant in the O-notation depends only on  $\delta > 0$ .

Remark. The oscillatory term in the expansion corresponding to GOE is of order  $N^{-2}$ , see Kalisch and Braak [\[9\]](#page-20-0). It can also be derived by our methods.

We also consider the interpolating ensembles of Mehta and Pandey [\[12\]](#page-21-0), which are given by

$$
\sqrt{r} \,\text{GUE} + \sqrt{1-r} \,\text{GOE} \;, \quad 0 \le r \le 1 \; .
$$

The case  $r = 0$  corresponds to the GOE, whereas  $r = 1$  corresponds to the GUE (in the notation of [\[12\]](#page-21-0),  $r = \alpha^2$ ). More explicitly,

<span id="page-2-0"></span>
$$
\langle H_{ij} H_{kl} \rangle = \frac{1}{N} \left( \delta(jk) \delta(il) + (1 - r) \delta(ik) \delta(jl) \right) . \tag{2}
$$

We prove:

<span id="page-2-1"></span>Theorem 3. For the interpolating ensemble [\(2\)](#page-2-0),

$$
\rho(E) = \rho_{sc}(E) - \frac{(-1)^N r}{4\pi^3 N \rho_{sc}(E)^2} \Re \left\{ \frac{e^{-N\left( iE\sqrt{1-E^2/4} + 2\ln(iE/2 + \sqrt{1-E^2/4}) \right)}}{(1-r)(1-E^2/2 - iE\sqrt{1-E^2/4}) + 1} \right\}
$$

$$
- \frac{1-r}{2\pi^2 N \rho_{sc}(E)} \Re \left\{ (2\sqrt{1-E^2/4} - r(-iE/2 + \sqrt{1-E^2/4}))
$$

$$
\times \frac{(-iE/2 + \sqrt{1-E^2/4})^3}{((1-r)(1-E^2/2 - iE\sqrt{1-E^2/4}) + 1)^2} \right\} + O(N^{-3/2})
$$

for  $|E| < 2 - \delta$ , and the implicit constant in the O-notation depends only on  $\delta > 0$ .

These results are based on a saddle-point analysis of the exact integral formulaæ for  $\rho(E)$ , which we prove (in Lemmata [1.1,](#page-4-0) [1.2,](#page-5-0) and [1.3](#page-5-1) below) using the supersymmetric formalism. The supersymmetric formalism, put forth by Berezin (see [\[1\]](#page-20-1) for an early application to Wigner matrices) and developed in the works of Wegner and Efetov, is a very general method to derive dual integral representations for expressions such as an average product of several matrix elements of the resolvent. While widely applied in the physical literature, only a fraction of these applications have been put on rigorous mathematical basis.

On the other hand, the supersymmetric method is potentially applicable to a wide range of problems pertaining to the spectral properties of random matrices and random operators; see the review of Spencer [\[14\]](#page-21-1).

Two of the alternative groups of methods to study the eigenvalue distribution of random matrices are perturbative methods (such as the moment method), and the method of orthogonal polynomials. The moment method was applied by Wigner in the 1950's to prove the weak convergence of the spectral distribution to the semicircle law  $\rho_{\rm sc}$ . A major disadvantage of all perturbative methods is that they typically allow to control the density of

states at some scale  $\epsilon \sim N^{-\kappa}$ , i.e. they do not allow to take  $\epsilon \to +0$  while keeping N fixed (moreover, usually  $\kappa < 1$ , so the perturbative methods are unable to see the oscillatory corrections to  $\rho_{\rm sc}$ ). The supersymmetric method allows to derive exact formulæ for fixed N and  $\epsilon \to +0$ .

The method of orthogonal polynomials, developed in the 1960's by Dyson, Gaudin, and Mehta (see the book of Mehta [\[10\]](#page-20-2)), allows to compute the asymptotics of the density of states in the strong sense and to arbitrary precision. For example, the asymptotic expansions of Theorems [1](#page-1-0) and [2](#page-1-1) (as well as analogous expansions for several other ensembles) were derived by Forrester, Frankel, and Garoni [\[8,](#page-20-3) [6\]](#page-20-4). Theorem [3](#page-2-1) can probably be extracted via asymptotic analysis from formula (4.52) of Mehta and Pandey [\[12\]](#page-21-0); see also  $\S$  5 there.

A vast generalization of Theorems [1](#page-1-0) and [2](#page-1-1) was obtained by Desroisiers and Forrester [\[2\]](#page-20-5), who considered general  $\beta$ -ensembles (with arbitrary  $\beta > 0$ , where  $\beta = 1$  corresponds to GOE and  $\beta = 2$  — to GUE; the interpolating ensembles [\(2\)](#page-2-0) are however not a special case of  $\beta$ -ensembles). Their work is based on the study of multivariate Hermite polynomials.

On the other hand, the potential range of applicability of the supersymmetric method seems to include many problems beyond the applicability of the orthogonal polynomial method (and even the method of multivariate orthogonal polynomials); see [\[14\]](#page-21-1).

Thus the supersymmetric method has several advantages over both perturbative methods and the method of orthogonal polynomials. The applications found during the last dozen years include an analysis of the density of states of a 3D band matrix model by Disertori, Pinson, and Spencer [\[4\]](#page-20-6), and the study of mixed moments of characteristic polynomials for a class of 1D band matrices by T. Shcherbina  $|13|$ ; see  $|14|$  for a review of other results.

Kalisch and Braak [\[9\]](#page-20-0) used the supersymmetric formalism to derive a formula for GOE, GUE and GSE density of states, and then applied saddlepoint analysis to derive the asymptotics of Theorems [1](#page-1-0) and [2.](#page-1-1) Their work is however on the physical level of rigor. A mathematically rigorous derivation of Theorem [1](#page-1-0) (as well as of its counterparts at the spectral edges) was given by Disertori [\[3\]](#page-20-7). The derivation of the integral formulæ by Kalisch–Braak and Disertori is based on the Hubbard-Stratonovich transformation.

Our contribution is three-fold. First, we derive a integral representation for the density of states using a different supersymmetric approach; our formalæ (Lemmata [1.1,](#page-4-0) [1.2,](#page-5-0) and [1.3\)](#page-5-1) seem simpler than the ones obtained via the Hubbard–Stratonovich transformation. We mention that a different approach avoiding the Hubbard–Stratonovich transformation was developed by Fyodorov in [\[7\]](#page-20-8).

Second, we perform a mathematically rigorous saddle-point analysis of both formalæ to derive the asymptotic expansions of Theorems [1,](#page-1-0) [2,](#page-1-1) and [3,](#page-2-1) thus extending the work of Disertori to GOE and to the interpolating ensembles. Although the results (at least, those pertaining to GUE and GOE) are not new, we believe that the methods can be applied to other problems intractable by other means; thus our third goal is a detailed and (relatively) self-contained exposition.

We remark that the method of the current paper can be probably applied to other Gaussian ensembles. As an example, we mention the anti-symmetric Hermitian ensemble of Mehta and Rosenzweig [\[11\]](#page-20-9), corresponding to  $r = 2$ in [\(2\)](#page-2-0).

To state the integral formulæ, we need some notation. Let  $E_{\epsilon} = E - i\epsilon$  and

$$
G(z) = (z - H)^{-1}
$$
;  $G(z; m, n) = (z - H)^{-1}(m, n)$ .

The density of states can be expressed in terms of G as follows:

<span id="page-4-1"></span>
$$
\rho(E) = \Im \lim_{\epsilon \to +0} \frac{1}{\pi N} \text{tr} \langle G(E_{\epsilon}) \rangle . \tag{3}
$$

The supersymmetric formalism is used to derive a (dual) integral representation for  $\langle \text{tr} G(E_{\epsilon}) \rangle$ .

<span id="page-4-0"></span>Lemma 1.1. For GUE,

$$
\frac{1}{N} \langle \text{tr } G(E_{\epsilon}) \rangle = \frac{(-1)^{N-1} N}{2\pi} \int_0^{\infty} ds \oint dz (iE - z + s)
$$
  
\n
$$
\exp \left[ -N(iE_{\epsilon} s + \frac{1}{2} s^2 - \ln s) \right] \exp \left[ -N(iE_{\epsilon} z - \frac{1}{2} z^2 + \ln z) \right] ,
$$

whereas

$$
i = \langle i \rangle = \frac{(-1)^{N-1}N}{2\pi} \int_0^\infty ds \oint dz \frac{iE - z + s}{s}
$$
  
exp  $\left[ -N(iE_\epsilon s + \frac{1}{2}s^2 - \ln s) \right]$  exp  $\left[ -N(iE_\epsilon z - \frac{1}{2}z^2 + \ln z) \right]$ .

<span id="page-5-0"></span>Lemma 1.2. For GOE,

$$
\frac{1}{N}\langle\operatorname{tr} G(E_{\epsilon})\rangle = \frac{(-1)^{N}2^{N}N^{2}}{8\pi^{2}}\int_{0}^{\infty} ds \int_{0}^{\infty} dt \int_{-1}^{1} d\alpha \oint dz
$$
  
\n
$$
z\frac{s+t}{st}(1-\alpha^{2})^{-3/2}\left[\frac{1}{N}+(z-iE_{\epsilon})^{2}-2(z-iE_{\epsilon})(s+t)+4st(1-\alpha^{2})\right]
$$
  
\n
$$
\exp\left[-N(iE_{\epsilon}(s+t)+s^{2}+t^{2}-\frac{1}{2}\ln s-\frac{1}{2}\ln t)\right]
$$
  
\n
$$
\exp\left[-N(iE_{\epsilon}z-\frac{1}{2}z^{2}+\ln z)\right]\exp\left[-N(2st\alpha^{2}-\frac{1}{2}\ln(1-\alpha^{2}))\right],
$$

whereas

$$
i = \langle i \rangle = \frac{(-1)^N 2^N N^2}{8\pi^2} \int_0^\infty ds \int_0^\infty dt \int_{-1}^1 d\alpha \oint dz
$$
  

$$
z \frac{1}{st} (1 - \alpha^2)^{-3/2} \left[ \frac{1}{N} + (z - iE_\epsilon)^2 - 2(z - iE_\epsilon)(s + t) + 4st(1 - \alpha^2) \right]
$$
  

$$
\exp \left[ -N(iE_\epsilon(s + t) + s^2 + t^2 - \frac{1}{2} \ln s - \frac{1}{2} \ln t) \right]
$$
  

$$
\exp \left[ -N(iE_\epsilon z - \frac{1}{2} z^2 + \ln z) \right] \exp \left[ -N(2st\alpha^2 - \frac{1}{2} \ln(1 - \alpha^2)) \right].
$$

<span id="page-5-1"></span>**Lemma 1.3.** For the interpolating ensemble  $(2)$ ,

$$
\frac{1}{N} \langle \text{tr } G(E_{\epsilon}) \rangle = \frac{(-1)^N 2^N N^2}{8\pi^2} \int_0^{\infty} ds \int_0^{\infty} dt \int_{-1}^1 d\alpha \oint dz
$$
\n
$$
z \frac{s+t}{st} (1-\alpha^2)^{-3/2} \left[ \frac{1}{N} + (z - iE_{\epsilon})^2 - (2 - r)(z - iE_{\epsilon})(s+t) + 4(1 - r)st(1 - \alpha^2) \right]
$$
\n
$$
\exp \left[ -N(iE_{\epsilon}(s+t) + (1 - \frac{r}{2})(s^2 + t^2) + rst - \frac{1}{2} \ln s - \frac{1}{2} \ln t) \right]
$$
\n
$$
\exp \left[ -N(iE_{\epsilon}z - \frac{1}{2}z^2 + \ln z) \right] \exp \left[ -N(2(1 - r)st\alpha^2 - \frac{1}{2} \ln(1 - \alpha^2)) \right],
$$

whereas

$$
i = \langle i \rangle = \frac{(-1)^N 2^N N^2}{8\pi^2} \int_0^\infty ds \int_0^\infty dt \int_{-1}^1 d\alpha \oint dz
$$
  

$$
z \frac{1}{st} (1 - \alpha^2)^{-3/2} \left[ \frac{1}{N} + (z - iE_\epsilon)^2 - (2 - r)(z - iE_\epsilon)(s + t) + 4(1 - r)st(1 - \alpha^2) \right]
$$
  

$$
\exp \left[ -N(iE_\epsilon(s + t) + (1 - \frac{r}{2})(s^2 + t^2) + rst - \frac{1}{2} \ln s - \frac{1}{2} \ln t) \right]
$$
  

$$
\exp \left[ -N(iE_\epsilon z - \frac{1}{2}z^2 + \ln z) \right] \exp \left[ -N(2(1 - r)st\alpha^2 - \frac{1}{2} \ln(1 - \alpha^2)) \right].
$$

In the three lemmata, the contour integral is along a counterclockwise contour about zero; the choice of the branch of the logarithm is not important, since it is multiplied by an integer number N in the exponent. We prove the lemmata in Section [2,](#page-6-0) and then apply saddle point analysis to derive the theorems in Section [3.](#page-13-0) We omit the proofs of Lemma [1.3](#page-5-1) and Theorem [3](#page-2-1) which are almost identical to the proofs of Lemma [1.2](#page-5-0) and Theorem [2,](#page-1-1) respectively.

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### <span id="page-6-0"></span>2 Integral representation

The proof of Lemmata [1.1](#page-4-0)[,1.2,](#page-5-0) and [1.3](#page-5-1) is based on the supersymmetric formalism. Let us introduce the notation.

For  $\mathbf{z}, \mathbf{z}' \in \mathbb{C}^N$ , set

$$
[\mathbf{z},\mathbf{z}']=\sum_{j=1}^N\bar{z}_jz'_j\;,\quad |\mathbf{z}|=\sqrt{[\mathbf{z},\mathbf{z}]}\;;
$$

in particular, for an  $N \times N$  matrix A,

$$
[\mathbf{z}, A\mathbf{z}] = \sum \bar{z}_j A_{jk} z_k .
$$

Also set

<span id="page-7-2"></span>
$$
D\mathbf{z} = \prod_{j=1}^{N} \frac{d\Re z_j \, d\Im z_j}{\pi} \,. \tag{4}
$$

Let  $\psi_1, \cdots, \psi_N, \bar{\psi}_1, \cdots, \bar{\psi}_N$  be anti-commuting variables, i.e.

$$
\psi_i \psi_j + \psi_j \psi_i = \psi_i \bar{\psi}_j + \bar{\psi}_j \psi_i = \bar{\psi}_i \bar{\psi}_j + \bar{\psi}_j \bar{\psi}_i = 0 ,
$$

and let

$$
[\boldsymbol{\psi},\boldsymbol{\psi}] = \sum \bar{\psi}_j \psi_j \ , \quad [\boldsymbol{\psi},A\boldsymbol{\psi}] = \sum \bar{\psi}_j A_{jk} \psi_k \ .
$$

Also let

$$
D\psi = \prod_{j=1}^N d\bar{\psi}_j d\psi_j .
$$

The supersymmetric (Berezin) integral is defined by the rules

$$
\int \psi_j d\psi_j = \int \bar{\psi}_j d\bar{\psi}_j = 1 , \quad \int d\psi_j = \int d\bar{\psi}_j = 0 .
$$

We start from the identities

<span id="page-7-0"></span>
$$
(H - E_{\epsilon})^{-1}(k, j) = i \int D\mathbf{z} \int D\psi \, e^{-i[\mathbf{z}, (E_{\epsilon} - H)\mathbf{z}] - i[\psi, (E_{\epsilon} - H)\psi]} z_k \bar{z}_j ,
$$
  
\n
$$
i = i \int D\mathbf{z} \int D\psi \, e^{-i[\mathbf{z}, (E_{\epsilon} - H)\mathbf{z}] - i[\psi, (E_{\epsilon} - H)\psi]} ,
$$
\n(5)

which are valid for any Hermitian matrix H and any  $\epsilon > 0$  (see Spencer [\[14,](#page-21-1) (4.20)]). We shall prove the first part of each of the lemmata using the first identity; the second part is similarly derived from the second identity.

*Proof of Lemma [1.1.](#page-4-0)* Taking the expectation of [\(5\)](#page-7-0) and summing over  $k = j$ , we obtain:

$$
\frac{1}{N} \langle \text{tr}\, G(E_{\epsilon}) \rangle = \frac{i}{N} \int D\mathbf{z} \int D\psi \left[ \mathbf{z}, \mathbf{z} \right] e^{-iE_{\epsilon}([\mathbf{z}, \mathbf{z}] + [\psi, \psi])} \langle e^{i([\mathbf{z}, H\mathbf{z}] + [\psi, H\psi])} \rangle . \tag{6}
$$

Using the identity

$$
\langle e^g \rangle = e^{\langle g^2 \rangle/2}
$$

(valid for any Gaussian random variable  $g$ ), we deduce

<span id="page-7-1"></span>
$$
\frac{1}{N} \langle \text{tr}\, G(E_{\epsilon}) \rangle = \frac{i}{N} \int D\mathbf{z} \int D\psi \left[ \mathbf{z}, \mathbf{z} \right] e^{-iE_{\epsilon}([\mathbf{z}, \mathbf{z}] + [\psi, \psi]} e^{-\frac{1}{2} \langle ([\mathbf{z}, H\mathbf{z}] + [\psi, H\psi])^2 \rangle} . \tag{7}
$$

From  $(1)$ ,

$$
\langle ([\mathbf{z}, H\mathbf{z}] + [\boldsymbol{\psi}, H\boldsymbol{\psi}])^2 \rangle = \frac{1}{N} \left( [\mathbf{z}, \mathbf{z}]^2 - [\boldsymbol{\psi}, \boldsymbol{\psi}]^2 + 2 \sum_{j,k=1}^N z_j \bar{\psi}_j \bar{z}_k \psi_k \right) .
$$

Let U be a unitary matrix such that  $U_{1j} = \overline{z}_j/|\mathbf{z}|$ . Denote

$$
\bar{\phi}_i = \sum_j \bar{U}_{ij} \bar{\psi}_j \ , \quad \phi_i = \sum_j U_{ij} \psi_j \ .
$$

Then

$$
[\boldsymbol{\phi},\boldsymbol{\phi}]=[\boldsymbol{\psi},\boldsymbol{\psi}]\;,\quad \sum z_j\bar{\psi}_j=|\mathbf{z}|\bar{\phi}_1\;,\quad \sum z_j\psi_j=|\mathbf{z}|\phi_1\;,\quad D\boldsymbol{\phi}=D\boldsymbol{\psi}\;,
$$

in particular,

$$
\sum_{j,k=1}^N z_j \bar{\psi}_j \bar{z}_k \psi_k = [\mathbf{z}, \mathbf{z}] \bar{\phi}_1 \phi_1.
$$

Returning to [\(7\)](#page-7-1), we obtain:

$$
\frac{1}{N}\langle\operatorname{tr} G(E_{\epsilon})\rangle = \frac{i}{N}\int D\mathbf{z}\int D\widetilde{\boldsymbol{\phi}}d\bar{\phi}_{1}d\phi_{1}\left[\mathbf{z},\mathbf{z}\right]e^{-iE_{\epsilon}([\mathbf{z},\mathbf{z}]+[\widetilde{\boldsymbol{\phi}},\widetilde{\boldsymbol{\phi}}])}e^{-iE_{\epsilon}\bar{\phi}_{1}\phi_{1}}\ne^{-\frac{1}{2N}\left([\mathbf{z},\mathbf{z}]^{2}-[\widetilde{\boldsymbol{\phi}},\widetilde{\boldsymbol{\phi}}]^{2}\right)}e^{-\frac{1}{2N}\left(2[\mathbf{z},\mathbf{z}]\bar{\phi}_{1}\phi_{1}-2\bar{\phi}_{1}\phi_{1}[\widetilde{\boldsymbol{\phi}},\widetilde{\boldsymbol{\phi}}]\right)},
$$

where we have set  $\widetilde{\phi}$  to be the Grassmanian vector  $\phi$  without the first coordinate. Integrating over  $\phi_1$  (and  $\bar{\phi}_1$ ), we obtain:

$$
\int d\bar{\phi}_1 d\phi_1 e^{-iE_{\epsilon}\bar{\phi}_1\phi_1 - \frac{1}{N}[\mathbf{z},\mathbf{z}] \bar{\phi}_1\phi_1 - \frac{1}{N}\bar{\phi}_1\phi_1[\tilde{\phi},\tilde{\phi}]} = iE_{\epsilon} + \frac{1}{N}[\mathbf{z},\mathbf{z}] - \frac{1}{N}[\tilde{\phi},\tilde{\phi}].
$$

To integrate over  $\widetilde{\phi}$ , we use a Grassmanian version of polar coordinates:

<span id="page-8-0"></span>
$$
\int D\widetilde{\phi}F([\widetilde{\phi},\widetilde{\phi}]) = (-1)^{N-1} \frac{(N-1)!}{2\pi i} \oint \frac{F(z)}{z^N} dz \tag{8}
$$

(the contour encircles the origin counterclockwise). For the sake of completeness, let us prove this formula, after stating it as

**Lemma 2.1.** For any analytic function  $F$  and an  $N$ -component Grassmann vector  $\phi$ ,

$$
\int D\phi F([\phi,\phi]) = (-1)^N \frac{N!}{2\pi i} \oint \frac{F(z)}{z^{N+1}} dz ,
$$

where the contour encircles the origin counterclockwise.

Proof. First,

$$
F([\phi,\phi]) = \sum_{j=0}^{N} \frac{F^{(j)}(0)}{j!} [\phi,\phi]^{j}.
$$

Only the last term contributes to the integral, thus

$$
\int D\phi F([\phi,\phi]) = \frac{F^{(N)}(0)}{N!} \int D\phi[\phi,\phi]^N = (-1)^N F^{(N)}(0) ,
$$

where on the second step we opened all the brackets and applied the integration rules. Now the statement follows from Cauchy's formula.  $\Box$ 

Applying [\(8\)](#page-8-0), we obtain:

$$
\frac{1}{N} \langle \text{tr}\, G(E_{\epsilon}) \rangle = \frac{i}{N} (-1)^{N-1} \frac{(N-1)!}{2\pi i} \int D\mathbf{z} \oint \frac{dz}{z^N} [\mathbf{z}, \mathbf{z}]
$$
\n
$$
\left( iE_{\epsilon} + \frac{1}{N} [\mathbf{z}, \mathbf{z}] - \frac{z}{N} \right) \exp \left\{ -iE_{\epsilon} [\mathbf{z}, \mathbf{z}] - \frac{1}{2N} [\mathbf{z}, \mathbf{z}]^2 - iE_{\epsilon} z + \frac{1}{2N} z^2 \right\} .
$$
\n(9)

Now we pass to polar coordinates in z using the formula

$$
\int_{\mathbb{C}^N} D\mathbf{z} F([\mathbf{z}, \mathbf{z}]) = \frac{1}{(N-1)!} \int_0^\infty F(s) s^{N-1} ds , \qquad (10)
$$

and obtain:

$$
\frac{1}{N} \langle \text{tr}\, G(E_{\epsilon}) \rangle = \frac{(-1)^{N-1}}{2\pi N} \int_0^{\infty} ds \oint dz (iE_{\epsilon} + \frac{s-z}{N})
$$

$$
\exp\left\{-iE_{\epsilon} s - \frac{1}{2N} s^2\right\} \exp\left\{-iE_{\epsilon} z + \frac{1}{2N} z^2\right\} \frac{s^N}{z^N} . \quad (11)
$$

The change of variables  $s \to Ns$ ,  $z \to Nz$  concludes the proof of Lemma [1.1.](#page-4-0)

Proof of Lemma [1.2.](#page-5-0) Similarly to the proof of Lemma [1.1,](#page-4-0)

<span id="page-9-0"></span>
$$
\frac{1}{N} \langle \text{tr}\, G(E_{\epsilon}) \rangle = \frac{i}{N} \int D\mathbf{z} \int D\psi \left[ \mathbf{z}, \mathbf{z} \right] e^{-iE_{\epsilon}([\mathbf{z}, \mathbf{z}] + [\psi, \psi])} e^{-\frac{1}{2} \langle ([\mathbf{z}, H\mathbf{z}] + [\psi, H\psi])^2 \rangle} . \tag{12}
$$

Now we have:

$$
\langle ([\mathbf{z}, H\mathbf{z}] + [\boldsymbol{\psi}, H\boldsymbol{\psi}])^2 \rangle
$$
  
=  $\frac{1}{N} \left\{ |\sum_j z_j^2|^2 + [\mathbf{z}, \mathbf{z}]^2 - [\boldsymbol{\psi}, \boldsymbol{\psi}]^2 + 2 \sum_j \overline{z}_j \overline{\psi}_j \sum_k z_k \psi_k + 2 \sum_j z_j \overline{\psi}_j \sum_k \overline{z}_k \psi_k \right\}$   
=  $\frac{1}{N} \left\{ 2[\mathbf{x}, \mathbf{x}]^2 + 2[\mathbf{y}, \mathbf{y}]^2 + 4[\mathbf{x}, \mathbf{y}]^2 + 4 \left[ \sum_j x_j \overline{\psi}_j \sum_k x_k \psi_k + \sum_j y_j \overline{\psi}_j \sum_k y_k \psi_k \right] \right\},$ 

where we use the decomposition  $z = x + iy$  of z into its real and imaginary parts.

Denote  $u=\frac{1}{2}$  $\frac{1}{2}(1+\frac{[{\bf x},{\bf y}]}{|{\bf x}||{\bf y}|});$  then

$$
u(1-u) = \frac{1}{|\mathbf{x}|^2 |\mathbf{y}|^2} (|\mathbf{x}|^2 |\mathbf{y}|^2 - [\mathbf{x}, \mathbf{y}]^2) .
$$

Define

$$
\tilde{\mathbf{x}} = |x|(\sqrt{u}\mathbf{e}_1 + \sqrt{1 - u}\mathbf{e}_2), \quad \tilde{\mathbf{y}} = |\mathbf{y}|(\sqrt{u}\mathbf{e}_1 - \sqrt{1 - u}\mathbf{e}_2),
$$

where  $e_j$  is the j-th vector of the standard basis. Then one can find an orthogonal map which takes  $x, y$  to  $\tilde{x}, \tilde{y}$ . Hence, similarly to the proof of Lemma [1.1,](#page-4-0) one can pass from  $\psi$  to a new Grassmann variable  $\phi$  and rewrite [\(12\)](#page-9-0) as

$$
\frac{1}{N}\langle\operatorname{tr} G(E_{\epsilon})\rangle = \frac{i}{N}\iint D\mathbf{x}D\mathbf{y}\int D\phi(|\mathbf{x}|^{2} + |\mathbf{y}|^{2})
$$
  
\n
$$
\exp\left\{-iE_{\epsilon}(|\mathbf{x}|^{2} + |\mathbf{y}|^{2} + [\phi,\phi]) - \frac{1}{2N}\Big[2|\mathbf{x}|^{4} + 2|\mathbf{y}|^{4} + 4|\mathbf{x},\mathbf{y}|^{2} - [\phi,\phi]^{2}\right.\right.
$$
  
\n+4( $|\mathbf{x}|^{2}+|\mathbf{y}|^{2}$ )( $u\bar{\phi}_{1}\phi_{1}+(1-u)\bar{\phi}_{2}\phi_{2}$ )+4 $\sqrt{u(1-u)}(|\mathbf{x}|^{2}-|\mathbf{y}|^{2})(\bar{\phi}_{2}\phi_{1}+\bar{\phi}_{1}\phi_{2})\Big]\Big\},$ 

where  $D \mathbf{x} D \mathbf{y}$  still incorporates the factor  $\pi^{-N}$  from [\(4\)](#page-7-2). Let  $\phi$  be the Grassmannian vector  $\phi$  whithout the first two coordinates. Then the above expression can be rewritten as

$$
\frac{1}{N} \langle \text{tr}\, G(E_{\epsilon}) \rangle = \frac{i}{N} \iint D\mathbf{x}D\mathbf{y} \int D\tilde{\boldsymbol{\phi}} \int d\bar{\phi}_1 d\phi_1 d\bar{\phi}_2 d\phi_2 (|\mathbf{x}|^2 + |\mathbf{y}|^2)
$$
  
\n
$$
\exp \left\{ -iE_{\epsilon} (|\mathbf{x}|^2 + |\mathbf{y}|^2 + [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}]) - \frac{1}{N} \Big[ |\mathbf{x}|^4 + |\mathbf{y}|^4 + 2[\mathbf{x}, \mathbf{y}]^2 - \frac{1}{2} [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}]^2 \Big] -iE_{\epsilon} (\bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2) + \frac{1}{N} \Big[ \bar{\phi}_1 \phi_1 \bar{\phi}_2 \phi_2 + [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}] (\bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2) -2(|\mathbf{x}|^2 + |\mathbf{y}|^2) (u\bar{\phi}_1 \phi_1 + (1-u)\bar{\phi}_2 \phi_2) - 2\sqrt{u(1-u)} (|\mathbf{x}|^2 - |\mathbf{y}|^2) (\bar{\phi}_2 \phi_1 + \bar{\phi}_1 \phi_2) \Big] \right\}.
$$

Let us first integrate over  $\phi_1$ ,  $\bar{\phi}_1$ ,  $\phi_2$ , and  $\bar{\phi}_2$ . Using the Taylor expansion  $e^z = 1 + z + z^2/2 + \cdots$  (the higher-order terms vanish due to anticommutativity), we obtain:

$$
\int d\bar{\phi}_1 d\phi_1 d\bar{\phi}_2 d\phi_2 \exp \left\{ \frac{1}{N} \bar{\phi}_1 \phi_1 \bar{\phi}_2 \phi_2 + \left( \frac{1}{N} [\tilde{\phi}, \tilde{\phi}] - iE_{\epsilon} \right) (\bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2) \right.\left. - \frac{2}{N} (|\mathbf{x}|^2 + |\mathbf{y}|^2) (u \bar{\phi}_1 \phi_1 + (1 - u) \bar{\phi}_2 \phi_2) \right.\left. - \frac{2}{N} \sqrt{u(1 - u)} (|\mathbf{x}|^2 - |\mathbf{y}|^2) (\bar{\phi}_2 \phi_1 + \bar{\phi}_1 \phi_2) \right\}= \frac{1}{N} + (\frac{1}{N} [\tilde{\phi}, \tilde{\phi}] - iE_{\epsilon})^2 - \frac{2}{N} (\frac{1}{N} [\tilde{\phi}, \tilde{\phi}] - iE_{\epsilon}) (|\mathbf{x}|^2 + |\mathbf{y}|^2) + \frac{4}{N^2} (|\mathbf{x}|^2 |\mathbf{y}|^2 - [\mathbf{x}, \mathbf{y}]^2).
$$
\n(13)

Thus

$$
\frac{1}{N} \langle \text{tr } G(E_{\epsilon}) \rangle = \frac{i}{N} \iint D\mathbf{x} D\mathbf{y} \int D\tilde{\boldsymbol{\phi}} \left( |\mathbf{x}|^{2} + |\mathbf{y}|^{2} \right) \n\left( \frac{1}{N} + (\frac{1}{N} [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}] - iE_{\epsilon})^{2} - \frac{2}{N} (\frac{1}{N} [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}] - iE_{\epsilon}) (|\mathbf{x}|^{2} + |\mathbf{y}|^{2}) \n+ \frac{4}{N^{2}} (|\mathbf{x}|^{2} |\mathbf{y}|^{2} - [\mathbf{x}, \mathbf{y}]^{2}) \right) \n\exp \left\{ -iE_{\epsilon} ([\mathbf{x}, \mathbf{x}] + [\mathbf{y}, \mathbf{y}] + [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}]) \right. \n- \frac{1}{N} \left[ |\mathbf{x}|^{4} + |\mathbf{y}|^{4} + 2[\mathbf{x}, \mathbf{y}]^{2} - \frac{1}{2} [\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}]^{2} \right] \right\}.
$$
\n(14)

Now we integrate over  $\widetilde{\phi}$  using the formula [\(8\)](#page-8-0). We obtain:

$$
\frac{1}{N}\langle\operatorname{tr} G(E_{\epsilon})\rangle = \frac{i}{N}\frac{(-1)^{N}(N-2)!}{2\pi i}\iint D\mathbf{x}D\mathbf{y}\oint \frac{dz}{z^{N-1}}\left(|\mathbf{x}|^{2}+|\mathbf{y}|^{2}\right)
$$

$$
\left(\frac{1}{N}+(\frac{z}{N}-iE_{\epsilon})^{2}-\frac{2}{N}(\frac{z}{N}-iE_{\epsilon})(|\mathbf{x}|^{2}+|\mathbf{y}|^{2})\right)
$$

$$
+\frac{4}{N^{2}}(|\mathbf{x}|^{2}|\mathbf{y}|^{2}-[\mathbf{x},\mathbf{y}|^{2})\right)
$$

$$
\exp\left\{-iE_{\epsilon}([\mathbf{x},\mathbf{x}]+[\mathbf{y},\mathbf{y}]+z)\right\}
$$

$$
-\frac{1}{N}\left[|\mathbf{x}|^{4}+|\mathbf{y}|^{4}+2[\mathbf{x},\mathbf{y}|^{2}-\frac{1}{2}z^{2}\right]\right\}.
$$
 (15)

Finally, we pass to polar coordinates in **x** and **y**. Setting  $x = s\alpha$ ,  $y = t\beta$ , where  $s = |\mathbf{x}|$ ,  $y = |\mathbf{y}|$ , and  $\alpha, \beta \in S^{N-1}$  and using the formulæ

$$
\int_{\mathbb{R}^N} F(\mathbf{x}) D\mathbf{x} = \frac{N \pi^{N/2}}{\Gamma(\frac{N}{2} + 1)} \int_0^\infty ds \int_{S^{N-1}} d\sigma(\alpha) F(s\alpha)
$$

and

$$
\iint_{S^{N-1}\times S^{N-1}} F([\alpha,\beta])d\sigma(\alpha)d\sigma(\beta) = \frac{1}{B(\frac{N-1}{2},\frac{1}{2})}\int_{-1}^{1} F(\alpha_1)(1-\alpha_1)^{\frac{N-3}{2}}d\alpha_1,
$$

where  $\sigma$  is the invariant probability measure on  $S^{N-1}$ , and

$$
B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}
$$

is the Euler beta function, we obtain:

$$
\frac{1}{N} \langle \text{tr } G(E_{\epsilon}) \rangle = \frac{i}{N \pi^{N}} \frac{(-1)^{N} (N-2)!}{2 \pi i} \frac{\pi^{N} N^{2}}{\Gamma(N/2+1)^{2} B(\frac{N-1}{2}, \frac{1}{2})}
$$
\n
$$
\int_{0}^{\infty} ds \, s^{N-1} \int_{0}^{\infty} dt \, t^{N-1} \int_{-1}^{1} d\alpha \, (1 - \alpha^{2})^{\frac{N-3}{2}} \oint \frac{dz}{z^{N-1}}
$$
\n
$$
(s^{2} + t^{2}) \Big( \frac{1}{N} + (\frac{z}{N} - iE_{\epsilon})^{2} - \frac{2}{N} (\frac{z}{N} - iE_{\epsilon}) (s^{2} + t^{2}) + \frac{4}{N^{2}} s^{2} t^{2} (1 - \alpha^{2}) \Big)
$$
\n
$$
\exp \Big\{ - iE_{\epsilon} (s^{2} + t^{2} + z) - \frac{1}{N} \Big[ s^{4} + t^{4} + 2s^{2} t^{2} \alpha^{2} - \frac{1}{2} z^{2} \Big] \Big\} .
$$
\n(16)

The final change of variables  $s \leftarrow s^2/N$ ,  $t \leftarrow t^2/N$ ,  $z \leftarrow z/N$  concludes the proof.

## <span id="page-13-0"></span>3 Saddle point analysis

*Proof of Theorem [1.](#page-1-0)* We shall take  $\epsilon \rightarrow +0$  in Lemma [1.1](#page-4-0) and compute the asymptotics using saddle-point analysis. First we calculate the saddles. Set

$$
f(s) = iEs + \frac{1}{2}s^2 - \ln s
$$
.

Then

$$
f'(s) = iE + s - \frac{1}{s} ,
$$

therefore the saddles are

<span id="page-13-1"></span>
$$
s_{\pm} = -\frac{iE}{2} \pm \sqrt{1 - \frac{E^2}{4}} \,. \tag{17}
$$

Similarly, for

$$
g(z) = iEz - \frac{1}{2}z^2 + \ln z
$$

we have

$$
g'(z) = iE - z + \frac{1}{z},
$$

so the saddles are

<span id="page-13-2"></span>
$$
z_{\pm} = \frac{iE}{2} \pm \sqrt{1 - \frac{E^2}{4}} \,. \tag{18}
$$

We deform the contours in  $s$  and  $z$  as follows:

<span id="page-13-3"></span>
$$
\Gamma_1: \quad s = \begin{cases} s_+ \tilde{s} , & 0 \le \tilde{s} \le A \\ s_+ A + \tilde{s} - A , & A \le \tilde{s} , \end{cases} \tag{19}
$$

and

<span id="page-13-4"></span>
$$
\Gamma_2: \quad z = e^{i\theta} \;, \quad 0 \le \theta \le 2\pi \;, \tag{20}
$$

where

$$
A = \begin{cases} 2, & |E| \le \sqrt{3} \\ \frac{1}{E^2 - 1}, & \sqrt{3} < |E| < 2 \end{cases}
$$

The contour in s passes through the saddle point  $s_{+}$ , whereas the contour in z passes through both  $z_+$  and  $z_+$ . The change of coutour is justified according to Cauchy's theorem.

<span id="page-14-0"></span>Claim 3.1. The minimum of  $\Re f(s)$  on the s-contour is achieved at  $s = s_+$ , *i.e.*  $\tilde{s} = 1$ .

Proof of Claim [3.1.](#page-14-0) We have:

$$
\Re f(x+iy) = -Ey + \frac{1}{2}(x^2 - y^2) - \frac{1}{2}\ln(x^2 + y^2).
$$

Therefore

$$
\frac{\partial}{\partial x} \Re f(x+iy) = x \left\{ 1 - \frac{1}{x^2 + y^2} \right\},\,
$$
  

$$
\frac{\partial}{\partial y} \Re f(x+iy) = -E - y \left\{ 1 + \frac{1}{x^2 + y^2} \right\}
$$

.

For  $x^2 + y^2 \ge 1$ , the x-derivative is positive, therefore  $\Re f(s(\tilde{s}))$  is increasing for  $\tilde{s} \geq A$ . Next,

$$
\frac{d}{d\tilde{s}}\Re f(s_{+}\tilde{s}) = \tilde{s}(\Re s_{+})^{2}(1-\tilde{s}^{-2}) - \Im s_{+}(E + \tilde{s}\Im s_{+}(1+\tilde{s}^{-2}))
$$

$$
= \tilde{s}(1 - E^{2}/4)(1 - \tilde{s}^{-2}) + \frac{E}{2}(E - \tilde{s}\frac{E}{2}(1+\tilde{s}^{-2}))
$$

$$
= \tilde{s}^{-1}\left\{\tilde{s}^{2}(1-\frac{E^{2}}{2}) + \tilde{s}\frac{E^{2}}{2} - 1\right\}.
$$

The quadtatic expression in the brackets has two roots, 1 and  $(E^2/2 - 1)^{-1}$ ; for  $|E| < \sqrt{2}$  the second root is negative, whereas for  $|E| > \sqrt{2}$  it is greater than A. Therefore  $\Re f(s(\tilde{s}))$  is decreasing for  $0 \leq \tilde{s} \leq 1$  and increasing for  $1 \leq \tilde{s} \leq A$ .  $1 \leq \tilde{s} \leq A.$ 

<span id="page-14-1"></span>Claim 3.2. The minimum of  $\Re g(z)$  on the z-contour is achieved at  $z = z_{\pm}$ , *i.e.* for the two values of  $\theta$  for which  $\sin \theta = \frac{E}{2}$  $\frac{E}{2}$ .

Proof of Claim [3.2.](#page-14-1) We have:

$$
\Re f(e^{i\theta}) = -E\sin\theta - \frac{1}{2}(\cos^2\theta - \sin^2\theta) ,
$$

hence

$$
\frac{d}{d\theta}\Re f(e^{i\theta}) = -\cos\theta(E - 2\sin\theta) .
$$

The claim easily follows.

 $\Box$ 

According to Claims [3.1](#page-14-0) and [3.2,](#page-14-1) the saddle-point approximation is justified, i.e. the asympototics of the integral

$$
I = \frac{(-1)^{N-1}N}{2\pi} \int_{\Gamma_1} ds \oint_{\Gamma_2} dz (iE - z + s) \exp \{-N(f(s) + g(z))\}
$$

is given (to arbitrary order in  $1/N$ ) by the contribution of the saddle points  $(s = s_+, z = z_+)$  and  $(s = s_+, z = z_-)$ . This follows from the general results on saddle-point approximation, see e.g. Fedoryuk [\[7,](#page-20-8) §4.1, Theorem 1.2].

Let us compute the contribution of the saddle points (up to order  $1/N$ ). Denote  $C_N = \frac{(-1)^{N-1}N}{2\pi}$  $\frac{1}{2\pi}$ . Then the second part of Lemma [1.1](#page-4-0) yields:

$$
i = \langle i \rangle = C_N \int_{\Gamma_1} ds \oint_{\Gamma_2} dz \frac{iE - z + s}{s} \exp \{-N(f(s) + g(z))\},
$$

therefore

$$
I = is_{+} + \frac{(-1)^{N-1}N}{2\pi} \int_{\Gamma_1} ds \oint_{\Gamma_2} dz \frac{iE - z + s}{s} (s - s_{+}) \exp \{-N(f(s) + g(z))\}.
$$

We have:

$$
f''(s) = 1 + s^{-2}
$$
,  $f'''(s) = -2s^{-3}$ ,

hence

$$
f(s_+) = iEs_+ + \frac{1}{2}s_+^2 - \ln s_+, \quad f''(s_+) = \frac{s_+^2 + 1}{s_+^2}, \quad f'''(s_+) = -\frac{2}{s_+^3}.
$$

Also, for

$$
\phi(s) = \frac{iE - z + s}{s}(s - s_+),
$$

we have:

$$
\phi'(s) = 1 + (iE - z)\frac{s_+}{s^2}, \quad \phi''(s) = -(iE - z)\frac{2s_+}{s^3},
$$

hence

$$
\phi(s_+) = 0
$$
,  $\phi'(s_+) = \frac{iE + s_+ - z}{s_+}$ ,  $\phi_2(s_+) = -2\frac{iE - z}{s_+}$ .

Therefore

$$
\sqrt{\frac{2\pi}{N}} \frac{e^{-Nf(s_+)} }{N(f''(s_+))^{3/2}} \left[ \frac{\phi''(s_+)}{2} - \frac{f'''(s_+) \phi'(s_+)}{2f''(s_+)} \right]
$$
  
=  $\sqrt{\frac{2\pi}{N}} \frac{1}{N} e^{-N(iEs_+ + \frac{1}{2}s_+^2 - \ln s_+)} \frac{s_+}{(s_+^2 + 1)^{3/2}} \left[ z - iE + \frac{iE + s_+ - z}{s_+^2 - 1} \right]$ 

This expression gives the contribution of  $s_{+}$  to the s-integral, up to corrections of order  $N^{-3/2}$ . The contribution of  $z_{\pm}$  to the z-integral is given by

$$
\sqrt{\frac{2\pi}{N}} \left[ \frac{e^{-Ng(z_+)}}{\sqrt{g''(z_+)}} \psi(z_+) + \frac{e^{-Ng(z_-)}}{\sqrt{g''(z_-)}} \psi(z_-) \right] ,
$$

where  $\psi(z)$  is the prefactor in the *z*-integral. This expression is equal to

$$
e^{-N(iEz_{+}-\frac{1}{2}z_{+}^2+\ln z_{+})}\frac{i z_{+} s_{+}}{(z_{+}^2+1)^{1/2}}\ .
$$

Note that the second term vanishes since  $\psi(z_+) = 0$ . Combining all the expressions, we obtain:

$$
I = \frac{E}{2} + i\sqrt{1 - \frac{E^2}{4}} - \frac{(-1)^N i}{4N(1 - E^2/4)} e^{-N[iE\sqrt{1 - E^2/4} + 2\ln(iE/2 + \sqrt{1 - E^2/4})]} + O(N^{-3/2}).
$$

We conclude the proof by taking the imaginary part and using [\(3\)](#page-4-1).

 $\Box$ 

.

*Proof of Theorem [2.](#page-1-1)* As in the proof of Theorem [2,](#page-1-1) we take the limit  $\epsilon \to +0$ in the formula from Lemma [1.2,](#page-5-0) and use saddle-point analysis to compute the asymptotics of the resulting integral

$$
I = \frac{(-1)^N 2^N N^2}{8\pi^2} \int_0^\infty ds \int_0^\infty dt \int_{-1}^1 d\alpha \oint dz \exp \{-N F(s, t, z, \alpha)\} \Phi(s, t, z, \alpha) ,
$$

where

$$
F(s, t, z, \alpha) = iE(s + t) + s^2 + t^2 - \frac{1}{2}\ln s - \frac{1}{2}\ln t
$$
  
+  $iEz - \frac{1}{2}z^2 + \ln z + 2st\alpha^2 - \frac{1}{2}\ln(1 - \alpha^2)$ 

and

$$
\Phi(s, t, z, \alpha) = z \frac{s+t}{st} (1 - \alpha^2)^{-3/2}
$$
  
 
$$
\times \left[ \frac{1}{N} + (z - iE_{\epsilon})^2 - 2(z - iE_{\epsilon})(s + t) + 4st(1 - \alpha^2) \right].
$$

The relevant saddles of F are given by  $\alpha = 0$ ,  $s = t = \frac{s_+}{2}$  $\frac{z_+}{2}$ ,  $z=z_{\pm}$ , where  $s_{+}$  is the same as in [\(17\)](#page-13-1) and  $z_{\pm}$  is as in [\(18\)](#page-13-2). We shall deform the contours so that they will pass through these saddles and the minimum of  $\Re F$  will be achieved only at these two points. We do it as follows: the  $\alpha$ -contour will remain the interval  $[-1, 1]$ . In the s and t-variables, we integrate along the contour  $\Gamma_1$  from [\(19\)](#page-13-3), with the modification

$$
A = \begin{cases} 2, & |E| \le \sqrt{\frac{5}{2}}\\ \frac{1}{\sqrt{E^2/2 - 1}}, & \sqrt{\frac{5}{2}} < |E| < 2 \end{cases}
$$

while in the z-variable, we integrate along  $\Gamma_2$  from [\(20\)](#page-13-4).

<span id="page-17-0"></span>**Claim 3.3.** For every  $s, t \in \Gamma_1$  and  $z \in \Gamma_2$ , the minimum of  $\Re F(s, t, z, \alpha)$ on [-1, 1] is achieved at the point  $\alpha = 0$ .

Proof of Claim [3.3.](#page-17-0) We have:

$$
\frac{\partial \Re F}{\partial \alpha} = 4\Re st\alpha + \frac{\alpha}{1 - \alpha^2} = \frac{\alpha}{1 - \alpha^2}((4\Re st + 1) - 4\Re st\alpha^2).
$$

The derivative vanishes at  $\alpha = 0$ , and at the two points  $\alpha_{\pm}$  given by

$$
\alpha^2 = 1 + \frac{1}{4\Re st} \; .
$$

For s, t on the contour  $\Gamma_1$ ,

$$
\Re st \ge A^2 \Re s_+^2 \ge 0 ,
$$

hence the two points are not in the domain [−1, 1]. Since  $\Re f$  tends to  $+\infty$  as  $\alpha \to \pm 1$ , the minimum is indeed at  $\alpha = 0$ . as  $\alpha \to \pm 1$ , the minimum is indeed at  $\alpha = 0$ .

According to Claim [3.3](#page-17-0) and Claims [3.1](#page-14-0) (modified for the new choice of A) and [3.2,](#page-14-1) the minimum of  $F$  is indeed achieved at the two saddles. Thus the asympototic contribution to I comes, to any order in  $N^{-1}$ , ony from the neighborhoods of these saddles.

Since

$$
\frac{(-1)^N 2^N N^2}{8\pi^2} \int_0^\infty ds \int_0^\infty dt \int_{-1}^1 d\alpha \oint dz \exp \{-N F(s, t, z, \alpha)\} \frac{\Phi(s, t, z, \alpha)}{s + t} = i
$$

according to the second half of Lemma [1.2,](#page-5-0) we rewrite  $I$  as

$$
I = i s_{+} + C_{N} \int_{0}^{\infty} ds \int_{0}^{\infty} dt \int_{-1}^{1} d\alpha \oint dz e^{-N(f(s) + f(t) + g(s, t, \alpha) + h(z))} \Phi_{1}(s, t, z, \alpha) ,
$$

where

$$
C_N = \frac{(-1)^N 2^N N^2}{8\pi^2},
$$
  
\n
$$
f(s) = iEs + s^2 - \frac{1}{2} \ln s,
$$
  
\n
$$
g(s, t, \alpha) = 2st\alpha^2 - \frac{1}{2} \ln(1 - \alpha^2),
$$
  
\n
$$
h(z) = iEz - \frac{1}{2}z^2 + \ln z,
$$
  
\n
$$
\Phi_1(s, t, z, \alpha) = z \frac{s - \frac{s_+}{2} + t - \frac{s_+}{2}}{st} (1 - \alpha^2)^{-3/2}
$$
  
\n
$$
\times \left[ \frac{1}{N} + (z - iE_\epsilon)^2 - 2(z - iE_\epsilon)(s + t) + 4st(1 - \alpha^2) \right].
$$

The addend  $1/N$  does not contribute to the asymptotics up to order  $1/N$ . Therefore we compute the leading term of the contribution of  $\alpha = 0$  to the integral

$$
\int e^{-N(2st\alpha^2 - \frac{1}{2}\ln(1-\alpha^2))} (1-\alpha^2)^{-3/2}
$$
\n
$$
[(z - iE)^2 - 2(z - iE)(s + t) + 4st(1 - \alpha^2)] d\alpha.
$$
\n(21)

It is equal to

$$
\sqrt{\frac{2\pi}{N}} \frac{1}{\sqrt{4st+1}} \left[ (z - iE)^2 - 2(z - iE)(s+t) + 4st \right] ,
$$

up to terms of higher order. Now we integrate over  $s$  and  $t$ , keeping the terms up to order  $1/N$ . The integral is given by

$$
I_2(z) = \int_0^\infty ds \int_0^\infty dt \, e^{-N(f(s) + f(t))} \Phi_2(s, t, z) \tag{22}
$$

where

$$
\Phi_2(s,t,z) = \frac{s - \frac{s_+}{2} + t - \frac{s_+}{2}}{st\sqrt{4st + 1}} \left[ (z - iE_\epsilon)^2 - 2(z - iE_\epsilon)(s + t) + 4st \right] ;
$$

we compute the contribution of  $s = t = s_+$  to order  $1/N$ , which is equal to

$$
\frac{2\pi}{N^2}e^{-2Nf(s_+/2)}\frac{1}{f''(s_+/2)^2}\left[\Phi_{2ss}(s_+/2,s_+/2,z)-\frac{f'''(s_+/2)}{f''(s_+/2)}\Phi_{2s}(s_+/2,s_+/2,z)\right]
$$
  
since  $\Phi_{2ss}(s_+/2,z_2)=0$ . This expression is equal to

since  $\Phi_2(s_+/2, s_+/2, z) = 0$ . This expression is equal to

$$
\frac{2\pi}{N^2} e^{-N(iEs_+ + s_+^2/2 - \ln\frac{s_+}{2})} \frac{2s_+^2}{(s_+^2 + 1)^{5/2}} \times \left[2s_+ - 2(z - iE) - \frac{3s_+}{s_+^2 + 1} \left((z - iE)^2 - 2s_+(z - iE) + s_+^2\right)\right].
$$

Then we compute the contribution of  $z = z_{\pm}$  to the integral over z,

$$
\oint dz e^{-Nh(z)} \Phi_3(z) ,
$$

where

$$
\Phi_3(z) = z \left[ 2s_+ - 2(z - iE) - \frac{3s_+}{s_+^2 + 1} \left( (z - iE)^2 - 2s_+(z - iE) + s_+^2 \right) \right] .
$$

The contribution comes from  $z_$ , since  $\Phi_3(z_+) = 0$ , and we only need the leading term,

$$
\sqrt{\frac{2\pi}{N}} \frac{1}{\sqrt{h''(z_-)}} e^{-Nh(z_-)} \Phi_3(z_-) .
$$

This final computation yields the answer:

$$
I = is_{+} + i \frac{\frac{iE}{2} - \sqrt{1 - \frac{E^2}{4}}}{4N(1 - \frac{E^2}{4})} + O(N^{-3/2}).
$$

In particular,

$$
\rho(E) = \Im \pi^{-1} I = \rho_{\rm sc}(E) - \frac{1}{4\pi^2 N \rho_{\rm sc}(E)} + O(N^{-3/2}).
$$



#### <span id="page-20-1"></span>References

- <span id="page-20-5"></span>[1] F. A. Berezin, Some remarks on the Wigner distribution, Teoret. Mat. Fiz. 17 (1973), 305–318.
- [2] P. Desrosiers, P. J. Forrester, Hermite and Laguerre  $\beta$ -ensembles: asymptotic corrections to the eigenvalue density, Nuclear Phys. B 743 (2006), no. 3, 307–332.
- <span id="page-20-7"></span><span id="page-20-6"></span>[3] M. Disertori, Density of states for GUE through supersymmetric approach, Rev. Math. Phys. 16 (2004), no. 9, 1191–1225.
- [4] M. Disertori, H. Pinson, T. Spencer, Density of states for random band matrices, Comm. Math. Phys. 232 (2002), no. 1, 83–124.
- <span id="page-20-4"></span>[5] M. V. Fedoryuk, The saddle-point method, "Nauka", Moscow, 1977. 368 pp.
- [6] P. J. Forrester, N. E. Frankel, T. M. Garoni, Asymptotic form of the density profile for Gaussian and Laguerre random matrix ensembles with orthogonal and symplectic symmetry, J. Math. Phys. 47 (2006), no. 2, 023301.
- <span id="page-20-8"></span>[7] Y. V. Fyodorov, Complexity of Random Energy Landscapes, Glass Transition, and Absolute Value of the Spectral Determinant of Random Matrices, Phys. Rev. Lett. 92, no. 24, 240601 (2004) and 93, 149901  $(2004).$
- <span id="page-20-3"></span>[8] T. M. Garoni, P. J. Forrester, N. E. Frankel, Asymptotic corrections to the eigenvalue density of the GUE and LUE, J. Math. Phys. 46 (2005), no. 10, 103301.
- <span id="page-20-0"></span>[9] F. Kalisch, D. Braak, Exact density of states for finite Gaussian random matrix ensembles via supersymmetry, J. Phys. A 35 (2002), no. 47, 9957–9969.
- <span id="page-20-2"></span>[10] M. L. Mehta, Random matrices, Pure and Applied Mathematics (Amsterdam), 142, Elsevier/Academic Press, Amsterdam, 2004, xviii+688 pp, ISBN: 0-12-088409-7.
- <span id="page-20-9"></span>[11] M. L. Mehta, N. Rosenzweig, Distribution laws for the roots of a random antisymmetric hermitian matrix, Nucl. Phys. A 109.2 (1968): 449–456.
- <span id="page-21-0"></span>[12] A. Pandey, M. L. Mehta, Gaussian ensembles of random Hermitian matrices intermediate between orthogonal and unitary ones, Comm. Math. Phys. 87.4 (1983): 449–468.
- <span id="page-21-2"></span><span id="page-21-1"></span>[13] T. Shcherbina, On the second mixed moment of characteristic polynomials of 1D band matrices, arXiv:1209.3385
- [14] T. Spencer, SUSY statistical mechanics and random band matrices, Quantum many body systems, 125–177, Lecture Notes in Math., 2051, Springer, Heidelberg, 2012.