

VALUES OF PAIRS INVOLVING ONE QUADRATIC FORM AND ONE LINEAR FORM AT S -INTEGRAL POINTS

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ABSTRACT. We prove the existence of S -integral solutions of simultaneous diophantine inequalities for pairs (Q, L) involving one quadratic form and one linear form satisfying some arithmetico-geometric conditions. This result generalises previous results of Gorodnik and Borel-Prasad. The proof uses Ratner's theorem for unipotent actions on homogeneous spaces combined with an argument of strong approximation.

1. INTRODUCTION

The theory of unipotent flows on homogeneous spaces is a powerful tool used to solve many difficult problems in number theory and more particularly in diophantine approximation. One of the great achievement of those so-called dynamical methods is the proof made by G.A. Margulis of the Oppenheim conjecture: *Let Q be a nondegenerate indefinite real quadratic form in $n \geq 3$ variables which is not proportional to a form with rational coefficients then $Q(\mathbb{Z}^n)$ is dense in \mathbb{R} .* A similar Oppenheim type problem concerns the existence of integral solutions of simultaneous diophantine inequalities involving one quadratic form and one linear form. More precisely given a pair (Q, L) and $(a, b) \in \mathbb{R}^2$ the problem is to find sufficient conditions which guarantees the existence of a nonzero integral vector in $x \in \mathbb{Z}^n$ such that

(A) For any $\varepsilon > 0$ one has simultaneously $|Q(x) - a| < \varepsilon$ and $|L(x) - b| < \varepsilon$

This condition is equivalent to ask the density of the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^n\}$ in \mathbb{R}^2 . The first result in that direction is due to S.G Dani and G.A. Margulis [DM90] and concerns the dimension 3 for a pair (Q, L) consisting of one nondegenerate indefinite quadratic form and a nonzero linear form in dimension 3 such that the cone $\{Q = 0\}$ intersects tangentially the plane $\{L = 0\}$ and no linear combination of Q and L^2 is rational. Under those conditions they proved using the original method used to prove the Oppenheim conjecture that the set $\{(Q(x), L(x)) : x \in \mathbb{Z}^3\}$ is dense in \mathbb{R}^2 . In higher dimension, the density for pairs holds if one replaces the previous transversality condition by the assumption that $Q|_{L=0}$ is indefinite, this result is due to A.Gorodnik [Gor04]:

Theorem 1.1 (Gorodnik). *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and L a nonzero linear form in dimension $n \geq 4$ satisfying the the following conditions*

- (1) Q is nondegenerate.
- (2) $Q|_{L=0}$ is indefinite.
- (3) No linear combination of Q and L^2 is rational.

Then the set $F(\mathcal{P}(\mathbb{Z}^n))$ is dense in \mathbb{R}^2 where $\mathcal{P}(\mathbb{Z}^n)$ is the set of primitive integer vectors.

The conclusion of the theorem implies immediately that the set $F(\mathbb{Z}^n)$ is dense in \mathbb{R}^2 . The proof of this theorem reduced to the case of the dimension 4. The condition (1) is a sufficient condition to ensure that we have $F(\mathbb{R}^n) = \mathbb{R}^2$ and this is a conjecture that this

condition can be weakened in order to make it necessary (see §7). The most important obstruction to prove density for pairs is that the identity component of the stabilizer of a pair (Q, L) is no longer maximal among the connected Lie subgroups of $\mathrm{SL}(4, \mathbb{R})$ in contrast with the case of the isotropy groups $\mathrm{SO}(3, 1)^\circ$ or $\mathrm{SO}(2, 2)^\circ$.

The stabilizer of the pair (Q, L) is defined by the following subgroup of G ,

$$\mathrm{Stab}(Q, L) = \{h \in \mathrm{SO}(Q) \mid L(hx) = L(x)\}.$$

It is not difficult to see that there exists $g \in G$ such that $(Q, L) = (Q_0^g, L_0^g)$ for some canonical pairs (Q_0, L_0) given explicitly (see [Gor04], Proposition 2). Clearly one has $\mathrm{Stab}(Q, L) = g\mathrm{Stab}(Q_0, L_0)g^{-1}$ and we are reduced to study the stabilizer of canonical pairs (Q_0, L_0) . The pairs such that $Q|_{L=0}$ is nondegenerate (resp. degenerate) are said to be of type (I) (resp. II). The proof of Theorem 1.1 is divided in two parts following each type and consists to apply Ratner's orbit closure theorem, and to study the action of $\mathrm{Stab}(Q_0, L_0)$ on the dual space of \mathbb{C}^4 . A remarkable fact is that the density is proved without showing the density of the orbit closure of the stabilizer in the homogeneous space G/Γ . Indeed the intermediate subgroups which possess non-trivial irreducible components have closed orbits in G/Γ , in particular they are not maximal. However, one is able to classify all the complex semisimple Lie algebras in $\mathfrak{sl}(4, \mathbb{C})$, and Gorodnik used this classification to check density case by case using the constrain on rationality given by the condition (3). The situation for pairs of type (II) is more complicated compared with the pairs of type (I) since the dual action of the stabilizer has three irreducible components for the pairs of type (II), instead of two for the pairs of type (I).

We are going to show an S -arithmetic generalisation of this result for pairs of type (I). Our proof is influenced by the work of Borel-Prasad on generalised the Oppenheim conjecture for quadratic forms ([BP92]) and of course also by Gorodnik's proof of theorem 1.1.

2. MAIN RESULT

2.1. S -arithmetic setting. Let us recall what we mean by S -arithmetic setting by fixing some notations. Let k be a number field, that is a finite extension of \mathbb{Q} and let \mathcal{O} be the ring of integers of k . For every normalised absolute value $|\cdot|_s$ on k , let k_s be the completion of k at s . We identify s with the specific absolute value $|\cdot|_s$ on k_s defined by the formula $\mu(a\Omega) = |a|_s \mu(\Omega)$, where μ is any Haar measure on the additive group k_s , $a \in k_s$ and Ω is a measurable subset of k_s of finite measure. We denote by Σ_k the set of places of k .

In the sequel S is a finite set of Σ_s which contains the set S_∞ of archimedean places, k_S the direct sum of the fields $k_s (s \in S)$ and \mathcal{O}_S the ring of S -integers of k (i.e. the ring of elements $x \in k$ such that $|x|_s \leq 1$ for $s \notin S$). For s non-archimedean, the valuation ring of the local field k_s is defined to be $\mathcal{O}_s = \{x \in k \mid |x|_s \leq 1\}$.

In all the statements of the article, without loss of generality one can replace k by \mathbb{Q} but for sake of completeness we work with number fields.

Let (Q, L) be a pair consisting of one quadratic form and one nonzero linear form on k_S^n . Equivalently, (Q, L) can be viewed as a family $(Q_s, L_s) (s \in S)$, where Q_s is a quadratic form on k_s^n and L_s a nonzero linear form on k_s^n . The form Q is nondegenerate if and only if each Q_s is nondegenerate. We say that Q is isotropic if each Q_s is so, i.e. if there exists for every $s \in S$ an element $x_s \in k_s^n - \{0\}$ such that $Q_s(x_s) = 0$, in particular if s is a real place an isotropic form is also said to be indefinite. For any quadratic form Q , we denote by $\mathrm{rad}(Q)$ (resp. $c(Q)$) the radical (resp. the isotropy cone) of Q , by definition Q is nondegenerate (resp. isotropic) if and only if $\mathrm{rad}(Q) \neq 0$ (resp. $c(Q) \neq 0$). The form Q

is said to be rational (over k) if there exists a quadratic form Q_o on k^n and a unit c of k_S such that $Q = c.Q_o$, and irrational otherwise. For any $s \in S$ let K_s denote an algebraic closure of k_s . If G is a locally compact group, G° denotes the connected component of the identity in G .

2.2. Main result. Let be given a pair $F = (Q_s, L_s)_{s \in S}$ on k_S^n and let $(a, b) \in k_S^2$. We are interested in finding sufficient conditions which guarantees the existence of nontrivial S -integral solutions $x \in \mathcal{O}_S^n$ of the following simultaneous diophantine problem

$$(A_S) \text{ For any } \varepsilon > 0, |Q_s(x) - a_s|_s < \varepsilon \text{ and } |L_s(x) - b_s|_s < \varepsilon \text{ for each } s \in S.$$

Obviously as in the real case, we need to find sufficient conditions on F so that the set $F(\mathcal{O}_S^n)$ would be dense in k_S^2 . One have to be careful since the condition (A_S) is not equivalent to density contrarily to real pairs (see [BP92], §6 and our §7).

Our main result gives the required conditions for assertion (A_S) to hold when $(a, b) = (0, 0)$. In other words, we give sufficient conditions which implies that $F(\mathcal{O}_S^n)$ is not discrete around the origin in k_S^2 . It may be seen as a weak S -arithmetic version of Theorem 1.1,

Theorem 2.1. *Let $Q = (Q_s)_{s \in S}$ be a quadratic form on k_S^n and $L = (L_s)_{s \in S}$ be a linear form on k_S^n with $n \geq 4$ and $L_s \neq 0$ for all $s \in S$. Suppose that the pair $F = (Q, L)$ satisfies the following conditions,*

- (1) Q is nondegenerate.
- (2) $Q|_{L=0}$ is nondegenerate and isotropic.
- (3) For each $s \in S$ the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational given any α_s, β_s in k_s with $(\alpha_s, \beta_s) \neq (0, 0)$.

Then for any $\varepsilon > 0$, there exists $x \in \mathcal{O}_S^n - \{0\}$ such that

$$|Q_s(x)|_s < \varepsilon \text{ and } |L_s(x)|_s < \varepsilon \text{ for each } s \in S.$$

2.3. Remarks. (1) The proof of Theorem 2.1 reduces to dimension 4, (see § 3) this reduction is necessary since the proof relies essentially on classification of intermediate subgroups and requires low dimension¹. The key is the use of the weak approximation in k_S follows in the same lines ([BP92], Proposition 1.3).

(2) The proof of Theorem 2.1 relies on Ratner's Theorem which gives a precise description of the closure orbits of lattices under the action of a connected Lie group generated by its unipotent one parameter subgroups. We need to apply an S -adic version of Ratner's theorem in order to find an integral solution simultaneously at all places. We treat first the case when $S = S_\infty$, by restriction of scalar we can use results of [Gor04] to elucidate the structure of the intermediate subgroups. This is exactly where we need to work in dimension 4, indeed the proof relies on the classifications of semisimple Lie algebras in \mathfrak{sl}_4 which contains the Lie algebra of the stabilizer. For a general finite set of places S containing both archimedean and nonarchimedean places, the use of strong approximation for number fields suffices to complete the picture.

¹The classification of intermediate Lie subgroups in dimension greater than five becomes rapidly unfeasible when the dimension increases.

(3) For Theorem 2.1 even if we assume that $\alpha Q + \beta L^2$ is irrational, it can be possible that the pencil form $\alpha_s Q_s + \beta_s L_s^2$ is rational for some place s , in this situation it is not possible to apply Ratner's theorem. It can be possible that the result is still true in this situation but there are serious obstacles to (see § 7).

(5) Unfortunately we are not able to show the density of $F(\mathcal{O}_S^n)$ under the conditions of theorem 2.1 with our method. We are also even unable to show that $|Q_s(x)|_s$ and $|L_s(x)|_s$ are both nonzero for any $s \in S$ and $x \in \mathcal{O}_S^n$ as in the conclusion of Theorem 2.1. We discuss those issues in § 7.

(6) In the real case, one can hope to relax condition (2) by only asking $\alpha Q + \beta L^2$ to be isotropic as it is conjectured by Gorodnik (see § 7, Conjecture 7.1). The major issue is that reduction to lower dimension fails to hold.

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3. WEAK AND STRONG APPROXIMATION, REDUCTION TO DIMENSION 4

3.1. Weak approximation in number fields and Grassmannian varieties. Number fields satisfy a nice *local-global principle* called the weak approximation which can be seen as a refinement of the Chinese remainder theorem.

Theorem 3.1 (Weak approximation in number fields). *Let S be a finite set of Σ_k . Let given $\alpha_s \in k_s$ for each $s \in S$. Then there exists an $\alpha \in k$ which is arbitrarily close to α_s for all $s \in S$ with respect to the s -adic topology.*

Proof. (See e.g. [Lang2], Theorem1, p.35)

One can reformulate this theorem as follows: the diagonal embedding $k \hookrightarrow \prod_{s \in S} k_s$ is dense, the product being equipped with the product of the s -adic topologies.

Definition 3.2 (Weak approximation in algebraic varieties). Let X be an algebraic variety defined over k , then X is said to satisfies weak approximation property with respect to S if the diagonal embedding $X(k) \hookrightarrow \prod_{s \in S} X(k_s)$ is dense for the S -adic topology.

To prove reduction we need to introduce a useful class of algebraic varieties which satisfies weak approximation,

Definition 3.3. Let V be a k -vector space of dimension $n \geq 1$ and for each $1 \leq m \leq n$ let us define the set

$$\mathcal{G}_m(V) = \{k\text{-vector subspaces } W \subset V \text{ with } \dim W = m\}.$$

This is an algebraic variety defined over k called the Grassmannian variety, if $V = k^n$ we just denote it as $\mathcal{G}_{n,m}(k)$.

Proposition 3.4. *Let be given two integers $1 \leq m \leq n$, then the Grassmannian variety $\mathcal{G}_{n,m}(k)$ satisfies weak approximation with respect to S , that is,*

$$\mathcal{G}_{n,m}(k) \hookrightarrow \prod_{s \in S} \mathcal{G}_{n,m}(k_s) \text{ is dense.}$$

Proof. Let be given a family $(V_s)_{s \in S}$ of k_s -vector subspaces of dimension m in k_s^n for each $s \in S$. Each of these subspaces V_s are determined by m linearly independent vectors in k_s^n . For each of the V_s , the coefficients of these vectors in the standard basis of k_s^n give rise to a $m \times n$ -matrix A_s with coefficients in k_s . By weak approximation property in k_S^{nm} we obtain a matrix $B \in \mathcal{M}_{m,n}(k)$ such that for any $s \in S$, B_s is arbitrarily close to A_s . Let V' be the vector subspace generated by the n columns of B , obviously $V \in \mathcal{G}_{n,m}(k)$ and V'_s is arbitrary close to V_s for all $s \in S$.

3.2. Reduction of Theorem 2.1 to the dimension 4.

Proposition 3.5. *Let $F = (Q, L)$ be a pair consisting of a quadratic form Q and a nonzero linear form L in k_S^n ($n \geq 5$) such that*

- (1) Q is nondegenerate
- (2) $Q|_{L=0}$ is isotropic
- (3) Any quadratic form $\alpha_s Q_s + \beta_s L_s^2$ with α_s, β_s in k_s such that $(\alpha_s, \beta_s) \neq (0, 0)$ for all $s \in S$ is irrational.

Then there exists a k -rational subspace V of k^n of codimension 1 such that $F|_{V_S}$ satisfies the conditions (1)(2)(3), moreover V can be chosen such that $Q|_{\{L=0\} \cap V_S}$ is nondegenerate.

Proof. When s is an archimedean real place, it is proved in ([Gor04], Proposition 4) that there exists a subspace V_s of k^n of codimension 1 such that $F_s|_{V_s}$ verifies conditions (1)(2)(3). In the case of archimedean complex places and nonarchimedean places, one may replace the condition $Q_s|_{L_s=0}$ of type (I) which only valid for real places by equivalent condition that $Q_s|_{L_s=0}$ is nondegenerate which is valid for all $s \in S$. Therefore there exists a subspace V_s of k^n of codimension 1 such that $F_s|_{V_s}$ verifies conditions (1)(2)(3), the proof of the latter existence of V_s for non-archimedean places in S is identical to the real places (see [Gor04], Proposition 4). Hence for any $s \in S$ we may find V_s a subspace of k^n of codimension 1 so that the conditions (1)(2)(3) are satisfied by $F_s|_{V_s}$ and one can choose V_s to be such that $Q_s|_{\{L_s=0\} \cap V_s}$ is nondegenerate.

Assume that $n \geq 5$. Let be given $s \in S$ and V_s a k -subspace of codimension 1 in k_s^n such that the restriction of Q_s on V_s is non-degenerate and isotropic. Let us define $H_s := SO(Q_s)$ the k_s -algebraic subgroup of the orthogonal group, $H_s(K_s)$ is a connected Lie group over the algebraic closure K_s of k_s . F

Following Borel and Prasad ([BP92] Proposition 1.3), let us consider the k_s -action of the group H_s on the Grassmanian variety $\mathcal{G}_{n-1,n}$ of the hyperplanes over k_s . Then we use the fact that the orbit $H_s(K_s)V$ is open in $\mathcal{G}_{n-1,n}(K_s)$ for the analytic topology. The fact that the fibration π of the orbit of V_s under $H_s(k_s)$ by the isotropy group of V_s has local k_s -cross-section σ implies that $H_s(k_s)V_s$ is also open in $\mathcal{G}_{n-1,n}(k_s)$ for the analytic topology.

$$\begin{array}{ccc} \text{Stab}_G(V_s) & \longrightarrow & H_s \xrightarrow{\pi} H_s.V_s \hookrightarrow \mathcal{G}_{n-1,n} \\ & & \swarrow \text{dotted arrow} \\ & & \sigma \end{array}$$

Moreover by weak approximation in k_S we can find a rational subspace in V' of codimension 1 in k^n such that $V' \otimes_k k_s$ is arbitrarily close to V_s for all $s \in S$, in particular they belong to the same open orbit under H_s . We have established that $F_s|_{V_s}$ satisfies conditions (1) and (2), it is equivalent to say that

$$\text{rad}(Q_s) \cap V_s = \{0\} \text{ and } c(Q_s|_{L_s=0}) \cap V_s \neq \{0\}. (*)$$

The condition (2) remains true if we replace V_s by any subspace sufficiently close to V_s . Since the subspace $\text{rad}(Q_s)$ is invariant under the action of the orthogonal group $\text{SO}(Q_s)$, the condition (1) above is verified by any element of $\mathcal{G}_{n-1,n}(k_s)$ which lies in the orbit of V_s under $\text{SO}(Q_s)$. In particular, $V' \otimes_k k_s$ satisfies $(*)$ for each $s \in S$. Hence we obtain a k -rational subspace V' of k^n such that $F|_{V'_s}$ satisfies the conditions (1)(2). It remains to find such V such that in addition $F|_{V_s}$ satisfies condition (3). Let us put

$$\mathcal{V} = \{V \in \mathcal{G}_{n-1,n}(k) \mid F|_{V_s} \text{ satisfies conditions (1)(2)}\}.$$

It is nonempty because it contains V' . Suppose there exists no V in \mathcal{V} for which $F|_{V_s}$ satisfies condition (3), that is to say that for any $V \in \mathcal{V}$, it should exist some $s \in S$ and some $(\alpha_s, \beta_s) \in k_s^2 - \{(0,0)\}$, such that the quadratic form $\alpha_s Q_s(x) + \beta_s L_s(x)|_{V_s}^2$ is rational. Let us consider the regular map $f : k_s^n \rightarrow k_s$ given by

$$f : x \mapsto \alpha_s Q_s(x) + \beta_s L_s(x)^2.$$

Clearly f is a polynomial function on K_s^n . For each $V \in \mathcal{V}$ we have $f(V(k)) \subset k$ and the Zariski density of $V(k)$ in \bar{k}_s^n implies that f is defined over k . In other words, $\alpha_s Q_s(x) + \beta_s L_s(x)^2$ is rational over k , contradiction. Hence there exists $V \in \mathcal{V}$ such that $F|_{V_s}$ satisfies condition (3).

Corollary 3.6. *It suffices to prove Theorem 2.1 for $n = 4$.*

Proof. It follows from the proposition by descending induction on n .

3.3. Adeles and strong approximation for number fields. The set of adeles \mathbb{A} of k is the subset of the direct product $\prod_{s \in \Sigma_k} k_s$ consisting of those $x = (x_s)$ such that $x \in \mathcal{O}_s$ for almost all $s \in \Sigma_k$. The set of adeles \mathbb{A} is a locally topological ring with respect to the adèle topology given by the base of open sets of the form $\prod_{s \in S} U_s \times \prod_{s \notin S} \mathcal{O}_s$ where $S \subset \Sigma_k$ is finite with $S \supset S_\infty$ and U_s are open subsets of k_s for each $s \in S$. For any subset $S \subset \Sigma_k$ finite with $S \supset S_\infty$, the ring of S -integral adeles is defined by:

$$\mathbb{A}(S) = \prod_{s \in S} k_s \times \prod_{s \notin S} \mathcal{O}_s, \text{ thus we can see that } \mathbb{A} = \bigcup_{S \supset S_\infty} \mathbb{A}(S).$$

We define also \mathbb{A}_S to be the image of \mathbb{A} onto $\prod_{s \notin S} k_s$, clearly $\mathbb{A} = k_S \times \mathbb{A}_S$.

Theorem 3.7 (Strong approximation). *If $S \neq \emptyset$ the image of k under the diagonal embedding is dense in \mathbb{A}_S .*

4. STABILIZERS OF PAIRS (Q, L)

For each $s \in S$ let us define $G_s = \text{SL}_4(k_s)$, $G_S = \prod_{s \in S} \text{SL}_4(k_s) = \text{SL}_4(k_S)$. Let $F = (Q, L)$ be a pair on k_S^4 satisfying the conditions (1)(2)(3) of Theorem 2.1.

For every $s \in S$ we realize Q_s on a four-dimensional quadratic vector space (W_s, Q_s) over k_s equipped with the standard basis $\mathcal{B} = \{e_1, \dots, e_4\}$. For each $s \in S$, let us define H_s the stabilizer of the pair F_s under the action of G_s , in other words

$$H_s = \{g \in G_s \mid Q_s \circ g = Q_s, L_s \circ g = L_s\}.$$

Equivalently one can write $H_s = \{g \in \text{SO}(Q_s) \mid L_s \circ g = L_s\}$, clearly it is a linear algebraic group defined over k_s . Also let us define $V_s = \{L_s = 0\}$, it is an hyperplane of W_s which induces a quadratic isotropic subspace $(V_s, Q_s|_{V_s})$ of dimension 3 in W_s . We have two cases following $(V_s, Q_s|_{V_s})$ is nondegenerate or not. If s is a real place the first case corresponds to pairs of type (I) in the terminology of [Gor04].

Lemma 4.1. *Let be given a pair (Q, L) satisfying the conditions of Theorem 2.1 in dimension 4. Then the stabilizer of (Q, L) under the action of G is of the form (up to conjugation)*

$$H = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \text{SO}(Q|_{L=0}) \right\} \subseteq \text{SL}_4(\overline{k_s}).$$

In particular, H is semisimple. Moreover, any quadratic form \tilde{Q} which is H -invariant is of the form $\alpha Q + \beta L^2$ for some $\alpha, \beta \in k_S$ not both zero.

Proof. Since $(V_s, Q_s|_{V_s})$ is nondegenerate, one can write the the following decomposition $W_s = V_s \oplus V_s^\perp$ where V_s^\perp the orthogonal complement w.r.t. Q . Since $\dim V_s^\perp = 1$ there exists some nonzero u vector of W_s such that $V_s^\perp = \langle u \rangle$ with $L_s(u) \neq 0$. Moreover by definition L_s is H_s -invariant so V_s is H_s -invariant. Moreover any element of $h \in H_s$ is in particular an element of $SO(Q_s)$, that is, $h^T = h$ hence $V_s^\perp = \langle u \rangle$ is also H_s -invariant. Then for any $h \in H_s$, the restriction $h|_{V_s}$ induces an automorphism of V_s and $hu = u$. Let us put $w_4 = u$, and complete with a basis of V_s $\{w_1, w_2, w_3\}$ to obtain the following matrix representation of H_s up to an k_s -isomorphism of W_s ,

$$H_s \simeq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \text{SO}(Q_s|_{V_s}) \right\} \subseteq \text{SL}_4(\overline{k_s}).$$

It is well-known that the orthogonal group of a nondegenerate quadratic form is a semisimple group. The last statement is the Lemma 9 in [Gor04].

S -adic products. Now let $F = (Q_s, L_s)_{s \in S}$ be a pair satisfying the conditions of the main theorem. Let \mathcal{H}_s be the algebraic group defined over k_s such that $\mathcal{H}_s(k_s) = H_s$. Define H_s^+ to be the subgroup of H_s generated by its one-dimensional unipotent subgroups.

Let us put

$$H_S = \prod_{s \in S} H_s \text{ and } H_S^+ = \prod_{s \in S} H_s^+.$$

Therefore H_S is an algebraic subgroup of $\text{SL}_4(k_S)$ which leaves invariant the pair $F = (Q, L)$ with respect to the S -basis $\mathcal{B}' = \{w_1, w_2, w_3, w_4\}$ as in the previous lemma.

In other words, we have

$$H_S \simeq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \text{SO}(Q|_{V_S}) \right\} \subseteq \text{SL}_4(\overline{k_S}).$$

and

$$H_S^+ \simeq \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right) \mid A \in \text{SO}(Q|_{V_S})^+ \right\} \subseteq \text{SL}_4(\overline{k_S}).$$

5. TOPOLOGICAL RIGIDITY IN S -ADIC HOMOGENEOUS SPACES

5.1. Ratner's topological rigidity theorem for unipotent groups actions. Let $G_S = \text{SL}_4(k_S)$ and let Γ_S be the S -arithmetic subgroup of G_S given by $\Gamma_S = \text{SL}_4(\mathcal{O}_S)$. The ring \mathcal{O}_S is a lattice in k_S . Let us define Ω_S to be the quotient space given by G_S/Γ_S . It is the space of free \mathcal{O}_S -submodules of k_S^4 of maximal rank and determinant one. Then Ω_S is the homogeneous space of unimodular lattices of \mathcal{O}_S^4 , by lattice we mean a discrete subgroup of G_S of finite covolume. For every $s \in S$, let \mathcal{U}_s be a unipotent k_s -algebraic subgroup of SL_4/k_s and denote by $\mathcal{U} = \prod_{s \in S} \mathcal{U}_s(k_s)$ the associated unipotent subgroup of G_S .

We are interested in the left action of \mathcal{U} on the homogeneous space Ω_S and more particularly with the closure of such orbits. If $x \in \Omega_S$ it turns out that the closure of the orbit $\mathcal{U}x$ is also an orbit of x . The following result is the generalisation of Ratner's orbit closure theorem for S -products proven independently by Margulis-Tomanov and Ratner (see [MT94], [R93]).

Theorem 5.1 (Ratner's Theorem for S -adic groups). *Assume that \mathcal{U} is generated by its one-dimensional unipotent subgroups. Then for any $x \in \Omega_S$, there exists a closed subgroup $M = M(x) \subset G_S$ containing \mathcal{U} such that the closure of the orbit $\mathcal{U}x$ coincides with Mx and Mx admits M -invariant probability measure.*

In the real case, this result was conjectured by Raghunathan who stated it with $G = SL(3, \mathbb{R})$, $U = SO(2, 1)$ and $\Gamma = SL(3, \mathbb{Z})$. He noticed that the proof of this Conjecture gives a the Oppenheim conjecture. Despite appearances, the proof of this theorem is measure theoretic and consists to classify ergodic measures under the action of the unipotent flow on the homogeneous space Ω . In both proofs made by Margulis-Tomanov ([MT94]) and Ratner ([R93]) the notion of entropy plays a central role for the measure classification.

5.2. The structure of intermediate subgroups in the archimedean case. The application of Ratner's Theorem 5.1 above gives a nice description of the orbit closure under the action of a subgroup H^+ generated by unipotent one parameter subgroups in G_S . Considering a unimodular lattice $x \in \Omega_S$, we get that

$$\overline{H^+x} = Lx$$

for some closed connected subgroup L (depending on x) such that $H^+ \subset L \subset G$. Indeed, the solution of the Oppenheim conjecture relies essentially in proving that the orbit closure of any element G/Γ under the action of $SO(2, 1)^\circ$ is either closed or dense in G/Γ . The following theorem due to N. Shah gives extra information about the structure of the intermediate subgroups arising from Ratner's orbit closure theorem above,

Theorem 5.2 ([Shah], Prop. 3.2). *Let $G = \mathcal{G}(\mathbb{R})^\circ$ with \mathcal{G} an algebraic subgroup of $SL(n, \mathbb{C})$ defined over \mathbb{Q} and $\Gamma = G \cap SL(n, \mathbb{Z})$. Let H be a subgroup of $G(\mathbb{R})$ generated by its unipotent one parameter subgroups and assume that*

$$\overline{H\Gamma} = P\Gamma \text{ where } P \text{ is a closed connected subgroup of } G(\mathbb{R})$$

such that $P \cap \Gamma$ has finite covolume in P . Then $P = \tilde{P}(\mathbb{R})^\circ$ where \tilde{P} is the smallest \mathbb{Q} -subgroup of G whose group of real points contains H .

By combining Ratner's theorem and previous proposition, we obtain immediately the following corollary stated in the set-up of the previous proposition.

Corollary 5.3 ([BP92], Proposition 7.2). *Let $g \in G$ such that $x = g.0$ where 0 is the coset of Γ in G/Γ . Then $g^{-1}Lg = \tilde{P}(\mathbb{R})$ where \tilde{P} is the smallest subgroup of real points which contains $g^{-1}Hg$.*

To use the results of this section for our purposes, we need to reduce to the case when $k = \mathbb{Q}$. This can be done by using the functor of restriction of scalars for algebraic groups.

Proposition 5.4. *Let k be a number field. Given any algebraic group $G \subset GL_n(\bar{k})$ defined over k , there exists an algebraic group G' defined over \mathbb{Q} such that $G'(\mathbb{Q}) \simeq G(k)$.*

Proof. See e.g. [PR], §2.1.2 for a general construction for finite separable extensions.

Definition 5.5 (Restriction of scalars). We denote G' by $\mathcal{R}_{k/\mathbb{Q}}(G)$ and it is called the algebraic group associated to G obtained by restriction of scalars from k to \mathbb{Q} .

The operation $\mathcal{R}_{k/\mathbb{Q}}$ defines a functor from the category of k -groups to the category of \mathbb{Q} -groups. This functor has a nice arithmetic property regarding the set of integral points, given any algebraic group defined over k , we have²

$$\mathcal{R}_{k/\mathbb{Q}}(G)(\mathbb{Z}) \simeq G(\mathcal{O}_k)$$

Assume $G = SL_{n|k}$ view as a algebraic group defined over k and if all the places are archimedean i.e. $S = S_\infty$. The fact that $k_S = k \otimes_{\mathbb{Q}} \mathbb{R}$ implies $G_S = \mathcal{R}_{k/\mathbb{Q}}(G)(\mathbb{R})$ and the intermediate subgroups have a still a interesting structure by means of restriction of scalars

Proposition 5.6 ([BP92], Prop. 7.3). *Let H_s be a closed subgroup of $G_s = SL_n(k_s)$ for each $s \in S_\infty$ and H the product of the H_s . Then the smallest \mathbb{Q} -algebraic subgroup \mathcal{L} of \mathcal{G} whose group of real points contains H is of the form $\mathcal{L} = \mathcal{R}_{k/\mathbb{Q}}\mathcal{L}'$, where \mathcal{L}' is a connected k -subgroup of \mathcal{G} .*

6. PROOF OF THE THEOREM 2.1

Let $F = (Q, L)$ be a pair in k_S^n which satisfies the conditions of Theorem 2.1. After § 3, we know that it suffices to show it for $n = 4$. By condition (3) all the forms $\alpha_s Q_s + \beta_s L_s^2$ are irrational for each α_s, β_s in k_s such that $(\alpha_s, \beta_s) \neq (0, 0)$ for any $s \in S$. Let $g \in G_S$ be the matrix of the basis \mathcal{B}'_S in the standard basis of k_S^4 . By definition $g^{-1}H_S g$ leaves invariant the pair $F = (Q_s, L_s)_{s \in S}$, and H_S^+ is generated by one-dimensional unipotent subgroups. We consider Γ_S as an element of the homogeneous space Ω_S . By applying Ratner's Theorem 5.1, one obtains

$$\overline{g^{-1}H_S^+ g \Gamma_S} = P \Gamma_S \quad (1)$$

where P is a closed subgroup of G_S which contains $g^{-1}H_S^+ g$.

Assume first that $S = S_\infty$, thus $\mathcal{O}_S^4 = \mathcal{O}^4$ and we simply write k_∞, H_∞ and G_∞ respectively for $k_{S_\infty}, H_{S_\infty}$ and G_{S_∞} . One notes also that H_∞^+ is nothing else than the component of the identity H_∞° . Using equality (1) one deduces that the set $F(\mathcal{O}^4)$ is dense in k_∞^2 . Indeed, we are going to adapt the proof of ([Gor04], Proposition 10) to the S_∞ -products, as follows³. We first reduce the ground field from k to the field of rational numbers. To achieve this we realise G_∞ as the group of real points of an algebraic group \mathcal{G} defined over \mathbb{Q} . In view of Proposition 5.6 this is given explicitly by taking $\mathcal{G} = R_{k/\mathbb{Q}}\mathrm{SL}_4$ where $R_{k/\mathbb{Q}}$ is the functor restriction of scalars of the field extension k/\mathbb{Q} and where SL_4 is regarded as the usual algebraic group over k . In other words, $G_\infty = \mathcal{G}(\mathbb{R})$ with \mathcal{G} an algebraic group defined over \mathbb{Q} . Now let us precise the structure of P . From Corollary 5.3 above, we infer that there exists an algebraic group \tilde{P} defined over \mathbb{Q} which is the smallest \mathbb{Q} -algebraic group whose group of real points \tilde{P} contains $g^{-1}H_\infty^\circ g$. In the other hand, Proposition 5.2 implies that $P = \tilde{P}(\mathbb{R})^\circ$ and the unipotent radical U of \tilde{P} is also defined over \mathbb{Q} . Thus equality (1) may be read as

$$\overline{g^{-1}H_\infty^\circ g \Gamma} = \tilde{P}(\mathbb{R})^\circ \Gamma. \quad (2)$$

Lemma 6.1. *For each $s \in S_\infty$, let P_s be the intersection of P with G_s . If P_s acts irreducibly on \mathbb{C}^4 , then $P_s = G_s$. Otherwise, $P_s = M_s U$ where*

$$M_s = u g_s^{-1} \left(\begin{array}{c|c} \mathrm{SL}_3 & 0 \\ \hline 0 & 1 \end{array} \right) g_s u^{-1} \text{ for some } u \in U_s.$$

²This result will not be used in the sequel.

³For more precisions, the reader is invited to read the original proof which is similar.

Proof the Lemma. This result is the core of the proof of Proposition 10 in [Gor04] for which we recall the outlines. If P_s acts irreducibly on \mathbb{C}^4 , then P_s is semisimple and the classification of irreducible semisimple Lie groups in SL_4 implies that P_s is equal either to G_s or $SO(B_s)$ for some nondegenerate form B_s (Proposition 7 and Lemma 8, [Gor04]). Such form B_s being H_s -invariant is necessarily of the form $\alpha_s Q_s + \beta_s L_s^2$ for some $(\alpha_s, \beta_s) \neq (0, 0)$ (Lemma 4.1). As seen before P_s is defined over \mathbb{Q} , so that B_s is forced to be rational which is a contradiction. Hence $P_s = G_s$. For the second assertion, we consider the induced action of P_s on the space \mathcal{L} of linear forms in \mathbb{C}^4 , it is reducible by hypothesis. There are only two P_s -invariant subspaces in \mathcal{L} , namely $\mathcal{L}_1 = \langle L_1, L_2, L_3 \rangle$ and $\mathcal{L}_2 = \langle L_4 \rangle$ where $L_i(x) = (gx)_i$ for $i = 1, \dots, 4$, note that $L_4 = L$. Since P_s is defined over \mathbb{Q} , one infers that M is semisimple thus admitting a Levi decomposition

$$P_s = M_s U_s$$

where M_s and U_s are respectively a Levi subgroup and the unipotent radical of P_s . The Levi subgroup M_s is defined over \mathbb{Q} since P_s is. Also as seen above, U is defined over \mathbb{Q} and Malcev's theorem ensures that the Levi subgroups are unique up to conjugacy (e.g. see §4.3 [OV]), in particular

$$g_s^{-1} H_s^\circ g_s \subseteq u^{-1} M_s u \text{ for some } u \in U_s. \quad (3)$$

Moreover this inclusion is strict because H_s is not defined over k . The latter fact and the maximality of $\mathrm{SO}(Q|_{L=0})$ in SL_3 ($Q|_{L=0}$ is isotropic) gives the equality

$$M_s = u g_s^{-1} \left(\begin{array}{c|c} \mathrm{SL}_3 & 0 \\ \hline 0 & 1 \end{array} \right) g_s u^{-1}.$$

This achieves the proof of the Lemma.

Let us define the subgroup

$$M'_s := u^{-1} M_s u = g_s^{-1} \left(\begin{array}{c|c} \mathrm{SL}_3 & 0 \\ \hline 0 & 1 \end{array} \right) g_s.$$

By the previous Lemma 6.1, one can rephrase equality (2) in the following way

$$\overline{g^{-1} H_\infty^\circ g \Gamma} = M'(\mathbb{R})^\circ U(\mathbb{R}) \Gamma. \quad (4)$$

Now let be given $(a, b) \in k_\infty^2$ and let us choose $x \in \mathcal{O}^4 - \langle g^{-1} e_4 \rangle$. It is not difficult to see that there exists $m \in M'(\mathbb{R})^\circ$ and $u \in U(\mathbb{R})$ such that

$$F(mux) = (Q(mux), L(mux)) = (a, b).$$

Using density in (4), we infer that there exists $h_n \in g^{-1} H_\infty^\circ g$ and $\gamma_n \in \Gamma$ such that

$$h_n \gamma_n \rightarrow mux \text{ as } n \rightarrow \infty.$$

We conclude that

$$F(\gamma_n x) = F(h_n \gamma_n x) \rightarrow F(umx) = (a, b) \text{ as } n \rightarrow \infty.$$

In other words, $F(\mathcal{O}^4)$ is dense in k_∞^2 and in particular this proves Case 1 when $S = S_\infty$.

Now let us assume $S \neq S_\infty$ and let be given $s \in S_f$. The set \mathcal{O}^4 is bounded in k_s^4 , thus for any neighbourhood U of the origin in k_s^4 one can find an integer $a_s \in \mathcal{O}_s$ such that $a_s \cdot \mathcal{O}^4 \subset U$. In other words, given any $\varepsilon > 0$ one can find $a_s \in \mathcal{O}_s$ such that :

$$|Q_s(a_s x)|_s \leq \varepsilon \text{ and } |L_s(a_s x)|_s \leq \varepsilon \text{ for all } x \in \mathcal{O}^4.$$

Thus for each $s \in S_f$, we can associate an integer $a_s \in \mathcal{O}_s$ satisfying the previous inequalities. By strong approximation one can find $a \in \mathcal{O}$ such that $|a|_s = |a_s|_s$ for all $s \in S_f$. Put $\|a\|_\infty = \max_{s \in S_\infty} |a|_s$, by the previous case we can find $x \in \mathcal{O}^4$ such that:

$$|Q_s(x)|_s \leq \varepsilon / \|a\|_\infty^2 \text{ and } |L_s(x)|_s \leq \varepsilon / \|a\|_\infty \text{ for all } s \in S_\infty.$$

We immediately obtain for all $s \in S_\infty$

$$|Q_s(a_s x)|_s = |a_s|_s^2 |Q_s(x)|_s \leq \varepsilon \text{ and } |L_s(a_s x)|_s = |a_s|_s |L_s(x)|_s \leq \varepsilon.$$

Hence given any $\varepsilon > 0$, we get a nonzero element $y = a.x \in \mathcal{O}_S^4$ satisfying the conclusion of Theorem 2.1, i.e.

$$|Q_s(y)|_s \leq \varepsilon \text{ and } |L_s(y)|_s \leq \varepsilon \text{ for all } s \in S.$$

7. COMMENTS AND OPEN PROBLEMS

Irrationality of the pencils forms. The rationality condition in Theorem 2.1, namely asking irrationality of all the pencils of Q and L^2 at all places of S is more restrictive than assuming irrationality of the pencils over k_S . Indeed the latter condition leaves the possibility that some pencil could be rational at some place(s) of S . In this case, using Ratner's theorem and therefore the classification of intermediate subgroups cannot be achieved by our methods. By analogy with the work of Borel and Prasad in the case of a family of quadratic forms $(Q_s)_{s \in S}$, it may be possible to apply strong approximation and avoiding reduction of dimension. The problem is that this method does not give integral solutions $x \in \mathcal{O}_S^n$ of inequalities $|Q(x)| \leq \varepsilon$ and $|L(x)| \leq \varepsilon$ but only nonzero integral solutions of the pencil forms $|\alpha Q(x) + \beta L^2(x)| \leq \varepsilon$ which may depend on the coefficients α and β . The most serious issue is to eliminate the dependance on the coefficients, that is, to replace the valid assertion

$$\forall \varepsilon > 0, \forall P \in \mathbb{P}^1(k_S), \exists x \in \mathcal{O}_S^n - \{0\}, |\tilde{Q}_P(x)|_S \leq \varepsilon$$

by the one we would like

$$\forall \varepsilon > 0, \exists x \in \mathcal{O}_S^n - \{0\}, \forall P \in \mathbb{P}^1(k_S), |\tilde{Q}_P(x)|_S \leq \varepsilon.$$

Indeed if one is able to do so, such x will satisfies those inequalities for both $P_1 = [0 : 1]$ and $P_2 = [1 : 0]$ and by homogeneity it would give the solution of our problem.

The problem of null values. The Theorem 2.1 is not conclusive regarding the existence of solutions leading to null values of either Q or L . Indeed we are not able to discard this possibility when s is a finite place, the reason is that the use of strong approximation does not provide information enough which guarantees the existence of a solution $x \in \mathcal{O}_S^n$ with $0 < |Q(x)|_s$ and $< 0 |L(x)|_s$ for all $s \in S$.

Towards density. It should be possible to obtain the density of $F(\mathcal{O}_S^n)$ for a pair $F = (Q, L)$ over k_S under the same assumptions generalising those of Theorem 1.1 i.e. without the condition $Q|_{L=0}$ is nondegenerate added here for our purpose. For this we need a analog of Lemma 6 of [Gor04] for nonarchimedean completions which has no clear reason to fail in characteristic zero. A significant difference with the classical Oppenheim conjecture is that the stabilizer of such pairs is no more maximal, and the classification of intermediate subgroups is much more involved. Unfortunately we are not able to prove Lemma 6.1 for non archimedean completions and to avoid the use of strong approximation.

An Open problem. We conclude by mentioning a conjecture of Gorodnik (see [Gor04], conjecture 15) which concerns the assumption (2) of Theorem 1.1 in the real case. It is conjectured that the condition $Q|_{L=0}$ is isotropic can be replaced by the weaker assumption that the pencil $\alpha Q + \beta L^2$ is isotropic for any real numbers α, β such that $(\alpha, \beta) \neq (0, 0)$.

Conjecture 7.1 (Gorodnik). Let $F = (Q, L)$ be a pair consisting of one nondegenerate quadratic and one nonzero linear form in dimension $n \geq 4$. Suppose that

- (1) For every $\beta \in \mathbb{R}$, $Q + \beta L^2$ is indefinite.
- (2) For every $(\alpha, \beta) \neq (0, 0)$, with $\alpha, \beta \in \mathbb{R}$, $\alpha Q + \beta L^2$ is irrational.

Then $F(\mathcal{P}(\mathbb{Z}^n))$ is dense in \mathbb{R}^2 .

The first condition is necessary for density to hold. The main issue is that this condition (contrarily to the condition that $Q|_{L=0}$ is indefinite) does not allow us to reduce to the four dimensional case. Hence all the strategy of the proof of Theorem 1.1 becomes needless regarding the impossibility to classify all the intermediate subgroups in higher dimension.

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