

FINITELY PRESENTED SUBGROUPS OF SYSTOLIC GROUPS ARE SYSTOLIC

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ABSTRACT. In this note we prove that every finitely presented subgroup of a systolic group is itself systolic.

1. INTRODUCTION

In early eighties, Gromov deduced several properties of Riemannian manifolds of non-positive sectional curvature without using Riemannian structure, but only the property of the induced distance function, which he called *CAT(0) inequality* [BGS85]. Gromov proved that for a cube complex equipped with the piecewise Euclidean metric, one can locally check CAT(0) condition in terms of combinatorial structure of the complex, see [BH99, Theorem II.5.20].

In [Hag03, JŚ06] there was introduced the following simplicial analogue of CAT(0) spaces. It is called simplicial non-positive curvature.

Definition 1. A simplicial complex is *flag* if every finite set of vertices that are pairwise connected by edges spans a simplex. A *loop of length m* in a simplicial complex X is a simplicial embedding of an m -cycle into X . An edge connecting two non-consecutive vertices of a loop is called a *diagonal*. The property that every loop of length at least four and less than m has a diagonal is called *m -largeness*. Let $m \geq 6$. A simply connected m -large flag simplicial complex is called *m -systolic*. We write only *systolic* instead of 6-systolic. A group acting properly and cocompactly by automorphisms on a systolic complex is called *systolic*.

Note that this definition of systolic complex differs from the original one, but is equivalent [JŚ06, Fact 1.2.(4) and Corollary 1.6].

The purpose of this note is to prove the following theorem.¹

Theorem 2. *Any finitely presented subgroup of a systolic group is systolic.*

Theorem 2 was proven by Wise for torsion-free systolic groups, see [Wis03, §5]. Wise considers the quotient of the systolic complex under the group action, and his proof does not generalize to groups with torsion. Note that Theorem 2 is not true if we replace “systolic” with “CAT(0)” in the statement, see [BRS07, §2.3.3].

1.1. Notation and outline. All the paths in any simplicial complex are taken in its one-skeleton. We use the symbol d_X to denote the shortest-path distance in one-skeleton of a simplicial complex X , where the length of each edge is 1. Given a subcomplex $Z \leq X$, the r -neighborhood of Z in X is defined as the simplicial span of all vertices r -close to Z , i.e.

$$N_X^r(Z) = \text{Span} \{x \in X^0 \mid d_X(x, Z^0) \leq r\}.$$

A neighborhood of a single vertex will also be called a *ball* around that vertex. Let G be a group acting properly and cocompactly by automorphisms on a systolic complex X . Let $H \leq G$ have a finite presentation $\langle \mathcal{S} | \mathcal{R} \rangle$ with \mathcal{S} symmetric. Let $\mathcal{C}_{\mathcal{S}}(H)$ be the oriented Cayley graph of H with respect to the generating set \mathcal{S} . This means that an edge connecting h and hs for $h \in H$ and a generator $s \in \mathcal{S}$ comes equipped with two orientations, one for s and one for s^{-1} , except when $s^2 = \mathbf{1}$, when there are two edges connecting h and hs . Denote by $\mathcal{C}_{\mathcal{S}}^X(H)$ such subdivision of $\mathcal{C}_{\mathcal{S}}(H)$ that there exists a simplicial H -equivariant map $\phi : \mathcal{C}_{\mathcal{S}}^X(H) \rightarrow X$. Let e_s denote the path between $\mathbf{1}$ and $s \in \mathcal{S}$ in $\mathcal{C}_{\mathcal{S}}^X(H)$ that comes from subdivision of the edge connecting $\mathbf{1}$ and s in $\mathcal{C}_{\mathcal{S}}(H)$. Let $x_0 = \phi(\mathbf{1})$ and $\gamma_s = \phi(e_s)$ for $s \in \mathcal{S}$. Denote by L the maximum of the lengths of γ_s . Denote also $\Gamma = \phi(\mathcal{C}_{\mathcal{S}}^X(H))$. We will frequently use s_1, \dots, s_m to denote generators from \mathcal{S} . We write $\gamma_{s_1 \dots s_m}$ for the path which is the concatenation $\gamma_{s_1} * (s_1 \gamma_{s_2}) * \dots * (s_1 \dots s_{m-1} \gamma_{s_m})$.

The outline of our proof is as follows. We find a neighborhood N of Γ in X such that every loop in Γ can be contracted in N . But new loops can appear in N . We thus have to consider Y , a disjoint union of H -translates of some large ball in X modulo appropriate equivalence relation. Then Y encodes an appropriate neighborhood of Γ and moreover does not give rise to any new homotopically nontrivial loop. Finally, we extend

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¹The author was told that the result was also proven independently in [GHMP13].

Y in appropriate category to a maximal H -cocompact simply connected flag simplicial complex and prove that it is 6-large.

2. PROOF OF THE THEOREM

We proceed in several steps as mentioned above. In the first step, we find a constant R such that loops in Γ are homotopically trivial in $N_X^R(\Gamma)$. Important properties of loops in Γ deduced in the proof of Step 1 are collected in Fact 1, since we will need them later on.

Step 1. *There exists a constant $R < \infty$ such that every loop in Γ is homotopically trivial in $N_X^{R-L}(\Gamma)$.*

Proof. After replacing a loop with its H -translate, it is enough to consider loops in Γ containing a point at distance at most L from x_0 . We distinguish two main cases.

Case (1). Loop of the form $\gamma_{s_1 \dots s_m}$ with $s_1 \dots s_m x_0 = x_0$. There are three subcases.

- (a) The word $s_1 \dots s_m$ belongs to \mathcal{R} . Because \mathcal{R} is finite, there is a number R_1 such that every such loop is homotopically trivial in $N_X^{R_1}(z)$ for every vertex $z \in \gamma_{s_1 \dots s_m}$.
- (b) The word $s_1 \dots s_m = \mathbf{1}$ but it does not belong to \mathcal{R} . Then $s_1 \dots s_m$ is a concatenation of conjugates of relators from \mathcal{R} , but each such conjugate is homotopically trivial in $N_X^{R_1}(\Gamma)$ hence the whole loop is homotopically trivial in $N_X^{R_1}(\Gamma)$.
- (c) The point x_0 is fixed by $s_1 \dots s_m$ but $s_1 \dots s_m \neq \mathbf{1}$. Without loss of generality, we can assume that $s_1 \dots s_m$ is the shortest representative of the corresponding group element since all the other representatives differ from the shortest by concatenation with words considered in Subcases (1.a, 1.b). By properness of G -action and hence of H -action, the number of elements $h \in H$ fixing x_0 is finite. Hence we can choose a constant R'_1 such that $\gamma_{s_1 \dots s_m}$ is homotopically trivial in $N_X^{R'_1}(z)$ for every vertex $z \in \gamma_{s_1 \dots s_m}$.

Case (2). Loop coming from a path $\gamma_{s_1 \dots s_m}$ with self-intersection $x \notin Hx_0$. Without loss of generality we can assume that $x = \gamma_{s_1} \cap (s_1 \dots s_{m-1} \gamma_{s_m})$. Figure 1 shows such configuration. Observe that in this case, $d_X(x_0, s_1 \dots s_m x_0) \leq 2L$. By properness of the H -action, there is an upper bound N such that $s_1 \dots s_m = p_1 \dots p_k$, where the number k of terms $p_i \in \mathcal{S}$ is at most N . Since

$$s_1 \dots s_m p_k^{-1} \dots p_1^{-1} = \mathbf{1},$$

the big loop $\gamma_{s_1 \dots s_m p_k^{-1} \dots p_1^{-1}}$ is a loop from Cases (1.a, 1.b), hence it can be contracted in $N_X^{R_1}(\Gamma)$. Thus to contract the original loop it suffices to contract

$$(\diamond) \quad \gamma_{p_1 \dots p_k} * \gamma,$$

where γ is a path in Γ from $s_1 \dots s_m x_0$ to the intersection x of γ_{s_1} and $s_1 \dots s_{m-1} \gamma_{s_m}$ concatenated with a path from x to x_0 . But the total length of the loop (\diamond) is at most $(N+2)L$. Let R_2 be a number such that any loop $\gamma_{p_1 \dots p_k} * \gamma$ of type (\diamond) can be contracted in $N_X^{R_2}(z)$ for any vertex $z \in \gamma_{p_1 \dots p_k} * \gamma$.

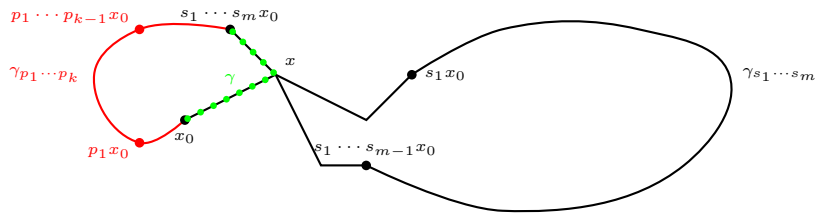


FIGURE 1. Loop from Case (2); possibly $x = x_0$ or $x = s_1 \dots s_m x_0$. If both equalities hold, this example is covered in Case (1).

Alltogether, we find a constant $R' = \max\{R_1, R'_1, R_2\}$ such that every loop in Γ is homotopic in $N_X^{R'}(\Gamma)$ to a trivial loop. Hence $R = R' + L$ works. \square

We will call loops from Cases (1.a, 1.c) and loops (\diamond) from Case (2) *short loops*. From the proof of Step 1 we deduce the following.

Fact 1. *Every loop in Γ is a concatenation of conjugates of short loops. Let γ be a short loop in Γ . Then for every vertex z on γ , the loop γ is fully contained in the ball $N_X^{R-L}(z)$.* \square

We are now ready to define Y . For every $h \in H$, denote $B_h^0 = N_X^R(hx_0)^0$ and denote the copy of the vertex $v \in X$ in B_h^0 by v^h . Let \sim be an equivalence relation on $\coprod_{h \in H} B_h^0$ generated by $v^h \sim u^g$ if and only if $v = u$ in X and $g^{-1}h \in \mathcal{S}$. Let

$$Y^0 = \left(\coprod_{h \in H} B_h^0 \right) / \sim.$$

Note that B_h^0 injects into Y^0 . For $y \in Y^0$ we write \bar{y} for the vertex of X such that $y = \bar{y}^h$ for some $h \in H$. Next, we define Y^1 . We connect two vertices $y, z \in Y^0$ by an edge if there exist representatives \bar{y}^h, \bar{z}^g for y and z with $h = g$ and \bar{y}, \bar{z} adjacent. Let Y be the flag completion of Y^1 and let B_h be the simplicial span of $B_h^0 \leq Y$. We define a natural action of H on Y , which is induced from H -action on X . Note that $hB_1 = B_h$, hence H -action on Y is proper and cocompact.

Observe that by the construction of Y , there exists a proper H -equivariant map $f : Y \rightarrow X$. It is defined by $f(y) = \bar{y}$ for $y \in Y^0$ and extends to higher-dimensional simplices simplicially. Let us define local sections

$$(\heartsuit) \quad i_h : N_X^R(hx_0) \rightarrow B_h$$

by $i_h(u) = u^h$ for every $u \in N_X^R(hx_0)^0$ and every $h \in H$. By definition of Y , each map i_h is bijective on zero-skeleta. Furthermore, vertices $u^h, v^h \in B_h^0$ are adjacent if and only if $u, v \in N_X^R(hx_0)^0$ are adjacent. Hence i_h is well defined isomorphism between one-skeleta of $N_X^R(hx_0)$ and B_h . Since a flag complex is determined by its one-skeleton, i_h is an isomorphism. Note that B_h might not be a ball in Y .

Observe that for any $h \in H$ and $s \in \mathcal{S}$ the two maps i_h and i_{hs} agree on $N_X^R(hx_0) \cap N_X^R(hsx_0)$ because they agree on the zero-skeleton of that intersection. By that fact, there is a natural map $\varphi : \mathcal{C}_S^X(H) \rightarrow Y$, sending edge path he_s to $i_h(h\gamma_s)$. We can as well describe the map φ in terms of the H -action on Y , but the previous definition is more useful for us. Next step ensures that φ has good properties.

Step 2. *The map φ factors through $\phi : \mathcal{C}_S^X(H) \rightarrow \Gamma$.*

Proof. We have to check that if two points of $\mathcal{C}_S^X(H)$ are identified under ϕ , they are also identified under φ . To see this, observe that if two paths γ_s and $h\gamma_{s'}$ in Γ cross, where $s, s' \in \mathcal{S}$ and $h \in H$, then there is a sequence of generators $p_1, \dots, p_k \in \mathcal{S}$ such that $p_1 \cdots p_k$ is the shortest word representing h . In particular $p_1 \cdots p_k e_{s'} = h e_s$. It follows from Fact 1 that the ball $N_X^{R-L}(p_1 \cdots p_l x_0)$ contains the whole path $\gamma_{p_1 \cdots p_k}$ for all $l = 0, 1, \dots, k$, where the empty word represents $\mathbf{1}$. Thus if we write $\bar{\gamma}_s$ for γ_s with opposite orientation, we have that $N_X^R(p_1 \cdots p_l x_0)$ contains the path $\gamma = (h\gamma_{s'}) * \gamma_{p_1 \cdots p_k} * \bar{\gamma}_s$ for all $l = 0, 1, \dots, k$. Hence $B_{p_1 \cdots p_l}$ contains $i_{p_1 \cdots p_l}(\gamma)$ for all $l = 0, 1, \dots, k$. Since two consecutive maps $i_{p_1 \cdots p_{l-1}}$ and $i_{p_1 \cdots p_l}$ agree on the intersection of their domains, the path $i_{\mathbf{1}}(\gamma) = i_h(\gamma)$ is contained in the intersection $\bigcap_{l=0}^k B_{p_1 \cdots p_l}$. Thus the point on e_s is identified with the appropriate point on $h e_{s'}$ under φ . This finishes the proof. \square

By Step 2, there exists a lift $f_\Gamma : \Gamma \rightarrow Y$ of the map $f : Y \rightarrow X$. Obviously f_Γ agrees with i_h on $\Gamma \cap N_X^R(hx_0)$. From now on, we identify Γ with its f_Γ -image.

Step 3. *The complex Y is simply connected.*

As mentioned in the outline, we first prove that Y encodes an appropriate neighborhood of Γ such that loops in Γ are homotopically trivial in Y . Then we exhibit a homotopy from any loop in Y to a loop in Γ .

Proof. Take any loop γ in $\Gamma \subseteq Y$. By Fact 1, it is a concatenation of short loops. In the same way as in the proof of Step 2 one can show that each short loop γ' is fully contained in B_h for each $h \in H$ such that $d_Y(hx_0, \gamma') \leq L$. Pick such $h \in H$. We know that B_h is isomorphic to $N_X^R(hx_0)$ via the map i_h from (\heartsuit) . Invoking Fact 1 once again, we see that the loop γ' is homotopically trivial in B_h , hence γ is homotopically trivial in Y .

Finally we need to show that every loop in Y is homotopic to a loop in Γ . Let $\beta : S^1 \rightarrow Y^1$ be a loop in the one-skeleton of Y . We identify S^1 with $I/\partial I$, where $I = [0, 1]$ is an interval. Let $0 \leq t_0 < t_1 < \dots < t_n < 1$ be cyclically ordered points on S^1 and h_0, h_1, \dots, h_n elements of H such that $\beta(t_i) \in Y^0$ and $\beta([t_i, t_{i+1}]) \in B_{h_i}$ for all $i = 0, 1, \dots, n$, where indices are taken modulo $n + 1$. Since $B_{h_{i-1}}$ and B_{h_i} both contain $\beta(t_i)$, there is a sequence of generators $s_1^i, \dots, s_{n(i)}^i \in \mathcal{S}$ such that $h_i = h_{i-1} s_1^i \cdots s_{n(i)}^i$ and $\beta(t_i) \in B_{h_{i-1} s_1^i \cdots s_{n(i)}^i}$ for all $i = 0, 1, \dots, n(i)$. Recall that empty word stands for $\mathbf{1}$. This means that there exist geodesics

$$\beta_l^i : (I, \partial I) \rightarrow (B_{h_{i-1} s_1^i \cdots s_{n(i)}^i}, \{\beta(t_i), h_{i-1} s_1^i \cdots s_{n(i)}^i x_0\}) \text{ for all } i = 0, 1, \dots, n \text{ and } l = 0, 1, \dots, n(i).$$

Recall that $h_{i-1} s_1^i \cdots s_{n(i)}^i x_0$ belongs also to $B_{h_{i-1} s_1^i \cdots s_{n(i)}^i}$. Since balls in systolic complexes are geodesically convex [JS06, §7], the image of β_l^i is contained in $B_{h_{i-1} s_1^i \cdots s_{n(i)}^i}$. Next, we can find some $\varepsilon > 0$ such that $t_i + n(i)\varepsilon < t_{i+1}$ for all i . After precomposing β with a map $S^1 \rightarrow S^1$, homotopic to the identity, we can assume that β is constant on $[t_i, t_i + n(i)\varepsilon]$ for all $i = 0, 1, \dots, n$. Hence

- for all $i = 0, 1, \dots, n$ and $l = 1, 2, \dots, n(i)$ there is $H_l^i : [t_i + (l-1)\varepsilon, t_i + l\varepsilon] \times I \rightarrow B_{h_{i-1} s_1^i \cdots s_{n(i)}^i} \cap B_{h_{i-1} s_1^i \cdots s_l^i}$ with $H_l^i(t_i + (l-1)\varepsilon, t) = \beta_{l-1}^i(t)$ and $H_l^i(t_i + l\varepsilon, t) = \beta_l^i(t)$, where $H_l^i(-, 0)$ is constant path $\beta(t_i)$ and $H_l^i(-, 1)$ is a path $h_{i-1} s_1^i \cdots s_{l-1}^i \gamma_{s_l^i} \subseteq \Gamma$;
- for all $i = 0, 1, \dots, n$ there is $H^i : [t_i + n(i)\varepsilon, t_{i+1}] \times I \rightarrow B_{h_i}$ with $H^i(t_i + n(i)\varepsilon, t) = \beta_{n(i)}^i(t)$ and $H^i(t_{i+1}, t) = \beta_0^{i+1}(t)$, where $H^i(-, 0) = \beta|_{[t_i + n(i)\varepsilon, t_{i+1}]}$ and $H^i(-, 1)$ is a constant path $h_i x_0$.

Since the homotopies from above agree on the intersections of their domains, they glue together to a homotopy $H : S^1 \times I \rightarrow Y$ with $H(-, 0) = \beta$ and $H(-, 1)$ a loop in Γ . \square

In the following step, using f we extend Y to a systolic complex \overline{Y} , on which H still acts properly and cocompactly and is thus a systolic group.

We say that a pair (W, f_W) is an f -extension of Y if the following holds. The complex W is simply connected flag simplicial complexes containing Y such that $Y^0 = W^0$ and that the H -action on Y extends to an H -action on W . Furthermore, the map $f_W : W \rightarrow X$ is a simplicial H -equivariant map which extends f . Note that f_W maps an edge of W either to an edge or to a vertex of X .

Let \mathcal{F} be the family of all f -extensions of Y . Observe that \mathcal{F} is equipped with a natural partial order \leq , where $(W_1, f_{W_1}) \leq (W_2, f_{W_2})$ if there exists an H -equivariant embedding $i : W_1 \rightarrow W_2$ fixing Y such that $f_{W_2} \circ i = f_{W_1}$. The family \mathcal{F} is nonempty since it contains (Y, f) by Step 3. Let $(W_\lambda, f_{W_\lambda})_{\lambda \in \Lambda}$ be an increasing chain in \mathcal{F} . Then the union $(\bigcup_\lambda W_\lambda, \bigcup_\lambda f_{W_\lambda})$ is also in \mathcal{F} , so it is an upper bound for the chain $(W_\lambda, f_{W_\lambda})_{\lambda \in \Lambda}$. By Kuratowski-Zorn Lemma, there exists a maximal element $(\overline{Y}, \overline{f}) \in \mathcal{F}$.

Step 4. For a maximal element $(\overline{Y}, \overline{f}) \in \mathcal{F}$, the simplicial complex \overline{Y} is a systolic complex, equipped with a proper and cocompact H -action.

Proof. We claim that the valence in \overline{Y}^1 of each $y \in \overline{Y}^0$ is bounded from above. Recall that \overline{f} and f agree on $\overline{Y}^0 = Y^0$. Let $N_y \subseteq Y^0$ denote the set of all vertices adjacent to y in \overline{Y} . For every $y' \in N_y$ either $f(y') = f(y)$ or $f(y')$ is adjacent to $f(y)$. This means that $f(N_y)$ is contained in $N_X^1(f(y))$. In other words, we have $N_y \subseteq f^{-1}(N_X^1(f(y)))$. Because X is proper, the ball $N_X^1(f(y))$ is compact. But f is a proper map, hence the set $f^{-1}(N_X^1(f(y)))$ is compact and the claim is proven. In particular, \overline{Y} is a proper simplicial complex. Because the vertex set of Y and \overline{Y} coincide, the action of H on \overline{Y} is proper and cocompact. By definition, \overline{Y} is flag and simply connected. It remains to prove 6-largeness.

Suppose for contradiction that there is some loop α of length four or five in \overline{Y} without diagonals. If \overline{f} maps α bijectively to $\alpha' = \overline{f}(\alpha) \subseteq X$, then there exist two non-consecutive vertices u' and v' of α' connected by a diagonal because X is systolic. Let u and v be the vertices of α mapped to u' and v' by \overline{f} . For every $h \in H$, we add an edge in Y between hu and hv and extend \overline{f} to the new edges naturally. Let us remind the reader that if for n different $h_1, \dots, h_n \in H$ all the sets $\{h_i u, h_i v\}$ coincide for $i = 1, 2, \dots, n$, we only add one edge between $h_1 u$ and $h_1 v$ instead of n . This remark will be applied two more times without mentioning it.

If \overline{f} is not bijective on α , there must be two vertices u, v of α , which are mapped by \overline{f} to the same vertex. If they are non-consecutive in α , we add edges between hu and hv for every $h \in H$ and extend f such that it maps any new edge to the common image of its endpoints. If u and v are consecutive, let $w \neq u$ be the other neighbor of v in α . Then we add edges between hu and hw for all $h \in H$. Note that since $\overline{f}(u) = \overline{f}(v)$, the point $\overline{f}(w)$ is either adjacent to or coincide with $\overline{f}(u)$ and hence we can extend \overline{f} to the newly added edges.

In all cases, we added an H -orbit of an edge to \overline{Y} . After a flag completion, we obtain a flag simplicial complex \hat{Y} on the set of vertices Y^0 , properly containing \overline{Y} , together with a map \hat{f} , extending \overline{f} , and equipped with an H -action, extending H -action on \overline{Y} . The complex \hat{Y} is also simply connected. Indeed, every edge e in $\hat{Y}^1 - \overline{Y}^1$ is a diagonal of a loop α in \overline{Y}^1 of length less than six. This means that e together with two consecutive edges of α form a triangle, which is filled after flag completion. Hence the path e is homotopic relative to its endpoints to a path of length two in \overline{Y} . In other words, any loop in \hat{Y} is homotopic to a loop in \overline{Y} and the latter is simply connected since it belongs to \mathcal{F} . Hence $(\overline{Y}, \overline{f}) \preceq (\hat{Y}, \hat{f}) \in \mathcal{F}$, which contradicts the maximality of $(\overline{Y}, \overline{f})$. \square

Remark 3. A first counterexample to Theorem 2 assuming only finitely generated instead of finitely presented subgroup is due to Stallings. Denote a free group of rank two generated by x and y by $\langle x, y \rangle$. In [Sta63] (see also [BRS07, §2.4.2]) Stallings proved that the kernel K of the homomorphism

$$\tau : \langle a, b \rangle \times \langle x, y \rangle \rightarrow \mathbf{Z}, \quad \tau(a) = \tau(b) = \tau(x) = \tau(y) = 1,$$

is finitely generated, but not finitely presentable. Hence K cannot be systolic. On the other hand, the direct product of two free groups of rank two is systolic, see [EP11].

Even in the case where G is hyperbolic, one cannot hope for a generalization of the theorem above. By the Rips Construction [Rip82], for any finitely presented group Q and arbitrary $\lambda > 0$, there exists a finitely presented $C'(\lambda)$ small cancellation group G and a short exact sequence $\{1\} \rightarrow N \rightarrow G \rightarrow Q \rightarrow \{1\}$, where N is finitely generated normal subgroup of G . Due to [Bie81], the group N is finitely presentable if and only if Q is finite. Hence, if we choose $Q = \mathbf{Z}$ and $\lambda = \frac{1}{6}$, then the Rips Construction gives a finitely presented $C'(1/6)$ small cancellation group G which is hyperbolic [Gro87] and $C(7)$ [LS77, Chapter V, §2]. By [Wis03], $C(7)$ group G is 7-systolic. But it has a finitely generated not finitely presentable subgroup N , hence a finitely generated non-systolic subgroup. In particular, systolic and even 7-systolic groups are not coherent in general.

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