CHARACTERIZATIONS OF GRADED PRÜFER $\star$-MULTIPLICATION DOMAINS

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ABSTRACT. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain graded by an arbitrary grading torsionless monoid Γ, and $\star$ be a semistar operation on R. In this paper we define and study the graded integral domain analogue of $\star$-Nagata and Kronecker function rings of R with respect to $\star$. We say that R is a graded Prüfer $\star$-multiplication domain if each nonzero finitely generated homogeneous ideal of R is $\star_f$-invertible. Using $\star$-Nagata and Kronecker function rings, we give several different equivalent conditions for R to be a graded Prüfer $\star$-multiplication domain. In particular we give new characterizations for a graded integral domain, to be a PvMD.

1. Introduction

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded (commutative) integral domain graded by an arbitrary grading torsionless monoid Γ, that is Γ is a commutative cancellative monoid (written additively). Let $\langle\Gamma\rangle = \{a - b | a, b \in \Gamma\}$, be the quotient group of Γ, which is a torsionfree abelian group.

Let H be the saturated multiplicative set of nonzero homogeneous elements of R. Then $R_H = \bigoplus_{\alpha\in\langle\Gamma\rangle}(R_H)_\alpha$, called the *homogeneous quotient field of* R, is a graded integral domain whose nonzero homogeneous elements are units. For a fractional ideal I of R let I_h denote the fractional ideal generated by the set of homogeneous elements of R in I. It is known that if I is a prime ideal, then I_h is also a prime ideal (cf. [29, Page 124]). An integral ideal I of R is said to be homogeneous if $I = \bigoplus_{\alpha\in\Gamma}(I\cap R_\alpha)$; equivalently, if $I = I_h$. A fractional ideal I of R is *homogeneous* if sI is an integral homogeneous ideal of R for some $s \in H$ (thus $I \subseteq R_H$). For $f \in R_H$, let $C_R(f)$ (or simply $C(f)$) denote the fractional ideal of R generated by the homogeneous components of f. For a fractional ideal I of R with $I \subseteq R_H$, let $C(I) = \sum_{f\in I} C(f)$. For more on graded integral domains and their divisibility properties, see [3, 29].

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ and $N_v(H) = \{f \in R | C(f)^v = R\}$. (Definitions related to the v-operation will be reviewed in the sequel.) Then $N_v(H)$ is a saturated multiplicative subset of R by [4, Lemma 1.1(2)]. The graded integral domain analogue of the well known Nagata ring is the ring $R_{N_v(H)}$. In [4], Anderson and Chang, studied relationships between the ideal-theoretic properties of $R_{N_v(H)}$ and the homogeneous ideal-theoretic properties of R. For example it is shown that if R has a unit of nonzero degree, $Pic(R_{N_v(H)}) = 0$ and that R is a PvMD if and only if each ideal of $R_{N_v(H)}$ is extended from a homogeneous ideal of R, if and only if $R_{N_v(H)}$

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is a Prüfer (or Bézout) domain [4, Theorems 3.3 and 3.4]. Also, they generalized the notion of Kronecker function ring, (for `e.a.b.` star operations on R) and then showed that this ring is a Bézout domain [4, Theorem 3.5]. For the definition and properties of semistar-Nagata and Kronecker function rings of an integral domain see the interesting survey article [21]. Recall that the *Picard group (or the ideal class group)* of an integral domain D, is $Pic(D) = Inv(D)/Prin(D)$, where $Inv(D)$ is the multiplicative group of invertible fractional ideals of D, and $Prin(D)$ is the subgroup of principal fractional ideal of D.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be an integral domain, and $\star$ be a semistar operation on R. In Section 2 of this paper we study the homogeneous elements of $\mathrm{QSpec}^\star(R)$ denoted by h-$\mathrm{QSpec}^\star(R)$. We show that if $\star$ is a finite type semistar operation on R which sends homogeneous fractional ideals to homogeneous ones, and such that $R^\star \subsetneq R_H$, then each homogeneous quasi-$\star$-ideal of R, is contained in a homogeneous quasi-$\star$-prime ideal of R. One of key results in this paper is Proposition 2.3, which shows that if $R^\star \subsetneq R_H$, the $\widetilde{\star}$ sends homogeneous fractional ideals to homogeneous ones. We also define and study the Nagata ring of R with respect to $\star$. The $\star$-Nagata ring is defined by the quotient ring $R_{N_\star(H)}$, where $N_\star(H) = \{f \in R | C(f)^\star = R^\star\}$. Among other things, it is shown that $Pic(R_{N_\star(H)}) = 0$. In Section 3 we define and study the Kronecker function ring of R with respect to $\star$. The Kronecker function ring, inspired by [20, Theorem 5.1], is defined by $\mathrm{Kr}(R,\star) := \{0\} \cup \{f/g | 0 \neq f, g \in R,$ and there is $0 \neq h \in R$ such that $C(f)C(h) \subseteq (C(g)C(h))^\star\}$. It is shown that if $\star$ sends homogeneous fractional ideals to fractional ones, then $\mathrm{Kr}(R,\star)$ is a Bézout domain. In Section 3 we define the notion of graded Prüfer $\star$-multiplication domains and give several different equivalent conditions to be a graded P$\star$MD. A graded integral domain R, is called a *graded Prüfer $\star$-multiplication domain (graded P$\star$MD)* if every finitely generated homogeneous ideal of R is a $\star_f$-invertible, i.e., $(II^{-1})^{\star_f} = R^\star$ for each finitely generated homogeneous ideal I of R. Among other results we show that R is a graded P$\star$MD if and only if $R_{N_\star(H)}$ is a Prüfer domain if and only if $R_{N_\star(H)}$ is a Bézout domain if and only if $R_{N_\star(H)} = \mathrm{Kr}(R,\widetilde{\star})$ if and only if $\mathrm{Kr}(R,\widetilde{\star})$ is a flat R-module.

To facilitate the reading of the paper, we review some basic facts on semistar operations. Let D be an integral domain with quotient field K. Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D-submodules of K. Let $\mathcal{F}(D)$ be the set of all nonzero *fractional* ideals of D; i.e., $E \in \mathcal{F}(D)$ if $E \in \overline{\mathcal{F}}(D)$ and there exists a nonzero element $r \in D$ with $rE \subseteq D$. Let $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [30], a *semistar operation on* D is a map $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following three properties hold:

$\star_1$: $(xE)^\star = xE^\star$;
$\star_2$: $E \subseteq F$ implies that $E^\star \subseteq F^\star$;
$\star_3$: $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Let $\star$ be a semistar operation on the domain D. For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \cup F^\star$, where the union is taken over all finitely generated $F \in f(D)$ with $F \subseteq E$. It is easy to see that $\star_f$ is a semistar operation on D, and $\star_f$ is called *the semistar operation of finite type associated to* $\star$. Note that $(\star_f)_f = \star_f$. A semistar operation $\star$ is said to be of *finite type* if $\star = \star_f$; in particular $\star_f$ is of finite type. We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* of D, if $I^\star \cap D = I$; a *quasi-$\star$-prime* (ideal of D), if I is a prime quasi-$\star$-ideal of D; and a *quasi-$\star$-maximal* (ideal of D), if I is maximal in the set of all proper quasi-$\star$-ideals of D. Each quasi-$\star$-maximal ideal is a prime ideal. It was shown in [16, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi-$\star_f$-ideal of D is contained in a quasi-$\star_f$-maximal ideal of D. We denote by $\mathrm{QMax}^\star(D)$ (resp., $\mathrm{QSpec}^\star(D)$) the set of all quasi-$\star$-maximal ideals (resp., quasi-$\star$-prime ideals) of D.

If $\star_1$ and $\star_2$ are semistar operations on D, one says that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\mathcal{F}}(D)$ (cf. [30, page 6]). This is equivalent to saying that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$ for each $E \in \overline{\mathcal{F}}(D)$ (cf. [30, Lemma 16]). Obviously, for each semistar operation $\star$ defined on D, we have $\star_f \leq \star$. Let d_D (or, simply, d) denote the identity (semi)star operation on D. Clearly, $d_D \leq \star$ for all semistar operations $\star$ on D.

It has become standard to say that a semistar operation $\star$ is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$ for all E, $F \in \overline{\mathcal{F}}(D)$. ("Stable" has replaced the earlier usage, "quotient", in [30, Definition 21].) Given a semistar operation $\star$ on D, it is possible to construct a semistar operation $\widetilde{\star}$, which is stable and of finite type defined as follows: for each $E \in \overline{\mathcal{F}}(D)$,

$$E^{\widetilde{\star}} := \{x \in K | xJ \subseteq E, \text{ for some } J \subseteq R, J \in f(R), J^\star = D^\star\}.$$

It is well known that [16, Corollary 2.7]

$$E^{\widetilde{\star}} := \cap\{ED_P | P \in \mathrm{QMax}^{\star_f}(D)\}, \text{ for each } E \in \overline{\mathcal{F}}(D).$$

The most widely studied (semi)star operations on D have been the identity d, v, $t := v_f$, and $w := \widetilde{v}$ operations, where $A^v := (A^{-1})^{-1}$, with $A^{-1} := (R : A) := \{x \in K | xA \subseteq D\}$.

Let $\star$ be a semistar operation on an integral domain D. We say that $\star$ is an `e.a.b.` *(endlich arithmetisch brauchbar) semistar operation* of D if, for all $E, F, G \in f(D)$, $(EF)^\star \subseteq (EG)^\star$ implies that $F^\star \subseteq G^\star$ ([20, Definition 2.3 and Lemma 2.7]). We can associate to any semistar operation $\star$ on D, an `e.a.b.` semistar operation of finite type $\star_a$ on D, called the `e.a.b.` *semistar operation associated to* $\star$, defined as follows for each $F \in f(D)$ and for each $E \in \overline{F}(D)$:

$$F^{\star_a} := \bigcup\{((FH)^\star : H^\star) | H \in f(R)\},$$

$$E^{\star_a} := \bigcup\{F^{\star_a} | F \subseteq E, F \in f(R)\}$$

[20, Definition 4.4 and Proposition 4.5] (note that $((FH)^\star : H^\star) = ((FH)^\star : H)$). It is known that $\star_f \leq \star_a$ [20, Proposition 4.5(3)]. Obviously $(\star_f)_a = \star_a$. Moreover, when $\star = \star_f$, then $\star$ is `e.a.b.` if and only if $\star = \star_a$ [20, Proposition 4.5(5)].

Let $\star$ be a semistar operation on a domain D. Recall from [17] that, D is called a *Prüfer $\star$-multiplication domain* (for short, a P$\star$MD) if each finitely generated ideal of D is *$\star_f$-invertible*; i.e., if $(II^{-1})^{\star_f} = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of PvMD; when $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain.

2. Nagata ring

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, $\star$ be a semistar operation on R, H be the set of nonzero homogeneous elements of R. An overring T of R, with $R \subseteq T \subseteq R_H$ will be called a *homogeneous overring* if $T = \bigoplus_{\alpha \in \langle\Gamma\rangle}(T \cap (R_H)_\alpha)$. Thus T is a graded integral domain with $T_\alpha = T \cap (R_H)_\alpha$.

In this section we study the homogeneous elements of $\text{QSpec}^\star(R)$, denoted by h-$\text{QSpec}^\star(R)$, and the graded integral domain analogue of $\star$-Nagata ring. Let h-$\text{QMax}^\star(R)$ denote the set of ideals of R which are maximal in the set of all proper homogeneous quasi-$\star$-ideals of R. The following lemma shows that, if $R^\star \subsetneq R_H$ and $\star = \star_f$ sends homogeneous fractional ideals to homogeneous ones, then h-$\text{QMax}^{\star_f}(R)$ is nonempty and each proper homogeneous quasi-$\star_f$-ideal is contained in a maximal homogeneous quasi-$\star_f$-ideal.

Lemma 2.1. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $\star$ a finite type semistar operation on R which sends homogeneous fractional ideals to homogeneous ones, and such that $R^\star \subsetneq R_H$. If I is a proper homogeneous quasi-$\star$-ideal of R, then I is contained in a proper homogeneous quasi-$\star$-prime ideal.*

Proof. Let $X := \{I | I \text{ is a homogeneous quasi-}\star\text{-ideal of } R\}$. Then it is easy to see that X is nonempty. Indeed, in this case $R^\star$ is a homogeneous overring of R, and if $u \in H$ is a nonunit in $R^\star$, then $uR^\star \cap R$ is a proper homogeneous quasi-$\star$-ideal of R. Also X is inductive (see proof of [16, Lemma 4.20]). From Zorn's Lemma, we see that every proper homogeneous quasi-$\star$-ideal of R is contained in some maximal element Q of X.

Now we show that Q is actually prime. Take $f, g \in H\backslash Q$ and suppose that $fg \in Q$. By the maximality of Q we have $(Q, f)^\star = R^\star$ (note that $(Q, f)^\star \cap R$ is a homogeneous quasi-$\star$-ideal of R and properly contains Q). Since $\star$ is of finite type, we can find a finitely generated ideal $J \subseteq Q$ such that $(J, f)^\star = R^\star$. Then $g \in gR^\star \cap R = g(J, f)^\star \cap R \subseteq Q^\star \cap R = Q$ a contradiction. Thus Q is a prime ideal. □

The following example shows that we can not drop the condition that, $\star$ sends homogeneous fractional ideals to homogeneous ones, in the above lemma.

Example 2.2. *Let k be a field and X, Y be indeterminates over k. Let $R = k[X, Y]$, which is a ($\mathbb{N}_0$-)graded Noetherian integral domain with $\deg X = \deg Y = 1$. Set $M := (X, Y + 1)$ which is a maximal non-homogeneous ideal of R. Let T be a DVR* [11]*, with maximal ideal N, dominating the local ring R_M. If $R_H \subseteq T$, then there exists a prime ideal P of R such that, $P \cap H = \emptyset$ and $N \cap R_H = PR_H$. Thus $M = N \cap R = N \cap R_H \cap R = PR_H \cap R = P$. Hence $M \cap H = \emptyset$, which is a contradiction, since $X \in M \cap H$. So that, $R_H \nsubseteq T$. Let $\star$ be a semistar operation on R defined by $E^\star = ET \cap ER_H$ for each $E \in \overline{\mathcal{F}}(R)$. Then clearly $\star = \star_f$ and $R^\star \subsetneq R_H$. If P is a nonzero prime ideal of R, such that $P \cap H = \emptyset$, then $P^{\star_f} \cap R = PT \cap PR_H \cap R = PT \cap P = P$. Thus P is a quasi-$\star_f$-prime ideal. On the other hand if P is any nonzero prime ideal of R such that $P \cap H \neq \emptyset$, then $PT = N^k$, for some integer $k \geq 1$. Therefore, if we assume that P is a quasi-$\star_f$-ideal of R, then we would have $P = PT \cap PR_H \cap R = PT \cap R = N^k \cap R \supseteq M^k$, which implies that $P = M$. Thus $\text{QSpec}^{\star_f}(R) = \{M\} \cup \{P \in \text{Spec}(R) | P \neq 0$ and $P \cap H = \emptyset\}$. Therefore by* [16, Lemma 4.1, Remark 4.5]*, we have $\text{QSpec}^{\widetilde{\star}}(R) = \{Q \in \text{Spec}(R) | 0 \neq Q \subseteq M\} \cup \{P \in \text{Spec}(R) | P \neq 0$ and $P \cap H = \emptyset\}$. Hence in the present example we have h-$\text{QSpec}^{\star_f}(R) = h$-$\text{QMax}^{\star_f}(R) = \emptyset$, and h-$\text{QSpec}^{\widetilde{\star}}(R) = h$-$\text{QMax}^{\widetilde{\star}}(R) = \{(X)\}$. Note that in this example h-$\text{QMax}^{\widetilde{\star}}(R) \nsubseteq \text{QMax}^{\widetilde{\star}}(R) = \text{QMax}^{\star_f}(R)$.*

From now on in this paper, we are interested and consider, the semistar operations $\star$ on R, such that $R^\star \subsetneq R_H$ and sends homogeneous fractional ideals to homogeneous ones. For any such semistar operation, if I is a homogeneous ideal of R, we have $I^{\star_f} = R^\star$ if and only if $I \nsubseteq Q$ for each $Q \in h\text{-QMax}^{\star_f}(R)$. Also if P is a quasi-$\star$-prime ideal of R, then either $P_h = 0$ or P_h is a quasi-$\star$-prime ideal of R. Indeed, if $P_h \neq 0$, then $P_h \subseteq (P_h)^\star \cap R \subseteq P^\star \cap R = P$, which implies that $P_h = (P_h)^\star \cap R$, since $(P_h)^\star \cap R$ is a homogeneous ideal.

The following proposition is the key result in this paper.

Proposition 2.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, $\widetilde{\star}$ sends homogeneous fractional ideals to homogeneous ones. In particular $h\text{-QMax}^{\widetilde{\star}}(R) \neq \emptyset$, and $R^{\widetilde{\star}}$ is a homogeneous overring of R.*

Proof. Let E be a homogenous fractional ideal of R. To show that $E^{\widetilde{\star}}$ is homogeneous let $f \in E^{\widetilde{\star}}$. Then $fJ \subseteq E$ for some finitely generated ideal J of R such that $J^\star = R^\star$. Suppose that $J = (g_1, \cdots, g_n)$. Using [4, Lemma 1.1(1)], there is an integer $m \geq 1$ such that $C(g_i)^{m+1}C(f) = C(g_i)^m C(fg_i)$ for all $i = 1, \cdots, n$. Since E is a homogeneous fractional ideal and $fg_i \in E$, we have $C(fg_i) \subseteq E$. Thus we have $C(g_i)^{m+1}C(f) \subseteq E$. Let $J_0 := C(g_1)^{m+1} + \cdots + C(g_n)^{m+1}$. Thus J_0 is a finitely generated homogeneous ideal of R such that $J_0^\star = R^\star$. Since $C(f)J_0 \subseteq E$, $C(f) \subseteq E^{\widetilde{\star}}$. Therefore $E^{\widetilde{\star}}$ is a homogeneous ideal. □

Lemma 2.4. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $\star$ a semistar operation on R which sends homogeneous fractional ideals to homogeneous ones. Then $\star_f$ sends homogeneous fractional ideals to homogeneous ones.*

Proof. Let E be a homogenous fractional ideal of R. Let $0 \neq x \in E^{\star_f}$. Then, there exists an $F \in f(R)$ such that $F \subseteq E$ and $x \in F^\star$. Suppose that F is generated by $y_1, \cdots, y_n \in R_H$. Let G be a homogeneous fractional ideal of R, generated by homogeneous components of $y_1, \cdots, y_n$. Note that $F \subseteq G \subseteq E$ and $x \in G^\star$. Thus homogeneous components of x belong to $G^\star \subseteq E^{\star_f}$. This shows that $E^{\star_f}$ is homogeneous. □

Note that the v-operation sends homogeneous fractional ideals to homogeneous ones by [3, Proposition 2.5]. Using the above two results, the t and w-operations also, send homogeneous fractional ideals to homogeneous ones.

It it well-known that $\text{QMax}^{\star_f}(R) = \text{QMax}^{\widetilde{\star}}(R)$, see [5, Theorem 2.16], for star operation case, and [18, Corollary 3.5(2)], in general semistar operations. Although Example 2.2, shows that it may happen that $h\text{-QMax}^{\star_f}(R) \neq h\text{-QMax}^{\widetilde{\star}}(R)$, we have the following proposition whose proof is almost the same as [4, Theorem 2.16].

Proposition 2.5. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $\star$ a semistar operation on R such that $R^\star \subsetneq R_H$, which sends homogeneous fractional ideals to homogeneous ones. Then $h\text{-QMax}^{\star_f}(R) = h\text{-QMax}^{\widetilde{\star}}(R)$.*

Proof. Assume that $Q \in h\text{-QMax}^{\star_f}(R)$. Then since $\widetilde{\star} \leq \star_f$ by [18, Lemma 2.7(1)], we have $Q \subseteq Q^{\widetilde{\star}} \cap R \subseteq Q^{\star_f} \cap R = Q$, that is Q is a quasi-$\widetilde{\star}$-ideal. Suppose that $Q \notin h\text{-QMax}^{\widetilde{\star}}(R)$. Then Q is properly contained in some $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. So since $Q \in h\text{-QMax}^{\star_f}(R)$, using Lemma 2.1, we must have $P^{\star_f} = R^\star$. Thus there is some finitely generated ideal $F \subseteq P$ such that $F^\star = R^\star$. So for any $r \in R$, $rF \subseteq F \subseteq P$. But then, $r \in P^{\widetilde{\star}}$, so $R \subseteq P^{\widetilde{\star}}$, which implies that $P^{\widetilde{\star}} = R^{\widetilde{\star}}$, a contradiction. Therefore, we must have $Q \in h\text{-QMax}^{\widetilde{\star}}(R)$.

If $Q \in h\text{-QMax}^{\widetilde{\star}}(R)$, then $Q = Q^{\widetilde{\star}} \cap R \subseteq Q^{\star_f} \cap R \subseteq R$. Suppose that $Q^{\star_f} \cap R = R$, which implies that $Q^{\star_f} = R^\star$. Then there is a finitely generated ideal $F \subseteq Q$ such that $F^\star = R^\star$. Now for any $r \in R$, $rF \subseteq F \subseteq Q$. Therefore $R \subseteq Q^{\widetilde{\star}}$, and so $R = Q^{\widetilde{\star}} \cap R = Q$, which is a contradiction. So $Q^{\star_f} \cap R \subsetneq R$. Now, since $Q^{\star_f} \cap R$ is a homogeneous quasi-$\star_f$-ideal, there is a $P \in h\text{-QMax}^{\star_f}(R)$ such that $Q \subseteq Q^{\star_f} \cap R \subseteq P$. From the first half of the proof, we know that $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. So we must have $P = Q$. Therefore $Q \in h\text{-QMax}^{\star_f}(R)$. □

Park in [31, Lemma 3.4], proved that $I^w = \bigcap_{P \in h\text{-}QMax^w(R)} IR_{H\setminus P}$ for each homogeneous ideal I of R.

Proposition 2.6. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $\star$ a semistar operation on R such that $R^\star \subsetneq R_H$. Then $I^{\widetilde{\star}} = \bigcap_{P \in h\text{-QMax}^{\widetilde{\star}}(R)} IR_{H\setminus P}$ for each homogeneous ideal I of R. Moreover $I^{\widetilde{\star}} R_{H\setminus P} = IR_{H\setminus P}$ for all homogeneous ideal I of R and all $P \in h\text{-QMax}^{\widetilde{\star}}(R)$.*

Proof. By Proposition 2.3, $I^{\widetilde{\star}}$ is a homogeneous ideal. Also note that $\bigcap_{P \in h\text{-QMax}^{\widetilde{\star}}(R)} IR_{H\setminus P}$ is a homogeneous ideal of R. Let $f \in I^{\widetilde{\star}}$ be homogeneous. Then $fJ \subseteq I$ for some homogeneous finitely generated ideal J of R such that $J^\star = R^\star$. It is easy to see that $J^{\widetilde{\star}} = R^{\widetilde{\star}}$. Hence we have $J \nsubseteq P$ for all $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. Thus $f \in IR_{H\setminus P}$ for all $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. Conversely, let $f \in \bigcap_{P \in h\text{-QMax}^{\widetilde{\star}}(R)} IR_{H\setminus P}$ be homogeneous. Then $(I : f)$ is a homogeneous ideal which is not contained in any $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. Therefore $(I : f)^{\widetilde{\star}} = R^{\widetilde{\star}}$. So that there exist a finitely generated ideal $J \subseteq (I : f)$ such that $J^\star = R^\star$. Thus $fJ \subseteq I$, i.e., $f \in I^{\widetilde{\star}}$. The second assertion follows from the first one. □

Let D be a domain with quotient field K, and let X be an indeterminate over K. For each $f \in K[X]$, we let $c_D(f)$ denote the content of the polynomial f, i.e., the (fractional) ideal of D generated by the coefficients of f. Let $\star$ be a semistar operation on D. If $N_\star := \{g \in D[X] | g \neq 0 \text{ and } c_D(g)^\star = D^\star\}$, then $N_\star = D[X] \setminus \bigcup \{P[X] | P \in \text{QMax}^{\star_f}(D)\}$ is a saturated multiplicative subset of $D[X]$. The ring of fractions

$$\text{Na}(D, \star) := D[X]_{N_\star}$$

is called the $\star$-*Nagata domain (of D with respect to the semistar operation $\star$)*. When $\star = d$, the identity (semi)star operation on D, then $\text{Na}(D, d)$ coincides with the classical Nagata domain $D(X)$ (as in, for instance [28, page 18], [23, Section 33] and [18]).

Let $N_\star(H) = \{f \in R | C(f)^\star = R^\star\}$. It is easy to see that $N_\star(H)$ is a saturated multiplicative subset of R. Indeed assume $f, g \in N_\star(H)$. Then $C(f)^{n+1}C(g) = C(f)^n C(fg)$ for some integer $n \geq 1$ by [4, Lemma 1.1(2)], and $C(fg) \subseteq C(f)C(g)$. Thus $fg \in N_\star(H) \Leftrightarrow C(fg)^\star = R^\star \Leftrightarrow C(f)^\star = C(g)^\star = R^\star \Leftrightarrow f, g \in N_\star(H)$. Also it is easy to show that $N_\star(H) = N_{\star_f}(H) = N_{\widetilde{\star}}(H)$. We define the graded integral domain analogue of $\star$-Nagata ring, by the quotient ring $R_{N_\star(H)}$. When $\star = v$, $R_{N_\star(H)}$ was studied in [4], denoted by $R_{N(H)}$.

Lemma 2.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$, which sends homogeneous fractional ideals to homogeneous ones.*

(1) $N_\star(H) = R \setminus \bigcup_{Q \in h\text{-QMax}^{\star_f}(R)} Q$.

(2) $\mathrm{Max}(R_{N_\star(H)}) = \{QR_{N_\star(H)} | Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)\}$ *if and only if R has the property that if I is a nonzero ideal of R with $C(I)^\star = R^\star$, then $I \cap N_\star(H) \neq \emptyset$.*

Proof. (1) Let $x \in R$. Then $x \in N_\star(H) \Leftrightarrow C(x)^\star = R^\star \Leftrightarrow C(x) \nsubseteq Q$ for all $Q \in h\text{-}\mathrm{QMax}^{\star_f}(R) \Leftrightarrow x \notin Q$ for all $Q \in h\text{-}\mathrm{QMax}^{\star_f}(R) \Leftrightarrow x \in R \setminus \bigcup_{Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)} Q$.

(2) ($\Rightarrow$) Let I is a nonzero ideal of R with $C(I)^\star = R^\star$. Then $I \nsubseteq Q$ for all $Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)$, and hence $IR_{N_\star(H)} = R_{N_\star(H)}$. Thus $I \cap N_\star(H) \neq \emptyset$.

($\Leftarrow$) Let I be a nonzero ideal of R such that $I \subseteq \bigcup_{Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)} Q$. If $C(I)^{\star_f} = R^\star$, then, by assumption, there exists an $f \in I$ with $C(f)^\star = R^\star$. But, since $I \subseteq \bigcup_{Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)} Q$, we have $f \in Q$ for some $Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)$, a contradiction. Thus $C(I)^\star \subsetneq R^\star$, and hence $I \subseteq Q$ for some $Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)$. Thus $\{QR_{N_\star(H)} | Q \in h\text{-}\mathrm{QMax}^{\star_f}(R)\}$ is the set of maximal ideals of $R_{N_\star(H)}$ by [23, Proposition 4.8]. □

We will say that R satisfies property $(\#_\star)$ if, for any nonzero ideal I of R, $C(I)^\star = R^\star$ implies that there exists an $f \in I$ such that $C(f)^\star = R^\star$.

Example 2.8. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and let $\star$ be a semistar operation on R. If R contains a unit of nonzero degree, then R satisfies property $(\#_\star)$ (see* [4, Example 1.6] *for the case $\star = t$).*

The next result is a generalization of the fact that $I^{\widetilde{\star}} = I\,\mathrm{Na}(R, \star) \cap K$, where K is the quotient field of R [18, Proposition 3.4(3)].

Lemma 2.9. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$, with property $(\#_\star)$. Then $I^{\widetilde{\star}} = IR_{N_\star(H)} \cap R_H$ and $I^{\widetilde{\star}} R_{N_\star(H)} = IR_{N_\star(H)}$ for each homogeneous ideal I of R. In particular $R^{\widetilde{\star}}$ is integrally closed if and only if $R_{N_\star(H)}$ is integrally closed.*

Proof. If $I^{\widetilde{\star}} = IR_{N_\star(H)} \cap R_H$, then it is easy to see that $I^{\widetilde{\star}} R_{N_\star(H)} = IR_{N_\star(H)}$. Hence it suffices to show that $I^{\widetilde{\star}} = IR_{N_\star(H)} \cap R_H$.

($\subseteq$) Let $f \in I^{\widetilde{\star}} (\subseteq R_H)$, and let J be a finitely generated ideal of R such that $J^\star = R^\star$ and $fJ \subseteq I$. Then $C(J)^\star = R^\star$, and since R satisfies property $(\#_\star)$, there exists an $h \in J$ with $C(h)^\star = R^\star$. Hence $h \in N_\star(H)$ and $fh \in I$. Thus $f \in IR_{N_\star(H)} \cap R_H$.

($\supseteq$) Let $f = \frac{g}{h} \in IR_{N_\star(H)} \cap R_H$, where $g \in I$ and $h \in N_\star(H)$. Then $fh = g \in I$, and since $C(h)^{m+1}C(f) = C(h)^m C(fh)$ for some integer $m \geq 1$ by [4, Lemma 1.1(1)], we have $fC(h)^{m+1} \subseteq C(f)C(h)^{m+1} = C(h)^m C(fh) = C(h)^m C(g) \subseteq I$. Also note that $(C(h)^{m+1})^\star = R^\star$, since $C(h)^\star = R^\star$. Thus $f \in I^{\widetilde{\star}}$.

For the in particular case, assume that $R_{N_\star(H)}$ is integrally closed. Using [3, Proposition 2.1], R_H is a GCD-domain, hence is integrally closed. Therefore $R^{\widetilde{\star}} = R_{N_\star(H)} \cap R_H$ is integrally closed. Conversely, assume that $R^{\widetilde{\star}}$ is integrally closed. Then R_Q is integrally closed by [14, Proposition 3.8] for all $Q \in \mathrm{QSpec}^{\widetilde{\star}}(R)$. Let $QR_{N_\star(H)}$ be a maximal ideal of $R_{N_\star(H)}$ for some $Q \in h\text{-}\mathrm{QMax}^{\widetilde{\star}}(R)$. Then $(R_{N_\star(H)})_{QR_{N_\star(H)}} = R_Q$ is integrally closed. Thus $R_{N_\star(H)}$ is integrally closed. □

Lemma 2.10. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$, with property $(\#_\star)$. Then for each nonzero finitely generated homogeneous ideal I of R, I is $\star_f$-invertible if and only if, $IR_{N_\star(H)}$ is invertible.*

Proof. Let I be nonzero finitely generated homogeneous ideal of R, such that I is $\star_f$-invertible. Let $QR_{N_\star(H)} \in \operatorname{Max}(R_{N_\star(H)})$, where $Q \in h\text{-QMax}^{\widetilde{\star}}(R)$ by Lemma 2.7(2). Thus by [22, Theorem 2.23], $(IR_{N_\star(H)})_{QR_{N_\star(H)}} = IR_Q$ is invertible (is principal) in R_Q. Hence $IR_{N_\star(H)}$ is invertible by [23, Theorem 7.3]. Conversely, assume that I is finitely generated, and $IR_{N_\star(H)}$ is invertible. By flatness we have $I^{-1}R_{N_\star(H)} = (R : I)R_{N_\star(H)} = (R_{N_\star(H)} : IR_{N_\star(H)}) = (IR_{N_\star(H)})^{-1}$. Therefore, $(II^{-1})R_{N_\star(H)} = (IR_{N_\star(H)})(I^{-1}R_{N_\star(H)}) = (IR_{N_\star(H)})(IR_{N_\star(H)})^{-1} = R_{N_\star(H)}$. Hence $II^{-1} \cap N_\star(H) \neq \emptyset$. Let $f \in II^{-1} \cap N_\star(H)$. So that $R^\star = C(f)^\star \subseteq (II^{-1})^{\star_f} \subseteq R^\star$. Thus I is $\star_f$-invertible. □

Corollary 2.11. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$, with property $(\#_\star)$ and $0 \neq f \in R$. Then the following conditions are equivalent:*

(1) *$C(f)$ is $\star_f$-invertible.*
(2) *$C(f)R_{N_\star(H)}$ is invertible.*
(3) *$C(f)R_{N_\star(H)} = fR_{N_\star(H)}$.*

Proof. Exactly is the same as [4, Corollary 1.9]. □

Let $\mathbb{Z}$ be the additive group of integers. Clearly, the direct sum $\Gamma \oplus \mathbb{Z}$ of Γ with $\mathbb{Z}$ is a torsionless grading monoid. So if y is an indeterminate over $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, then $R[y, y^{-1}]$ is a graded integral domain graded by $\Gamma \oplus \mathbb{Z}$. In the following proposition we use a technique for defining semistar operations on integral domains, due to Chang and Fontana [9, Theorem 2.3].

Proposition 2.12. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with quotient field K, let y, X be two indeterminates over R and let $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Set $T := R[y, y^{-1}]$, $K_1 := K(y)$ and take the following subset of* $\operatorname{Spec}(T)$*:*

$$\triangle^\star := \{Q \in \operatorname{Spec}(T) |\ Q \cap R = (0) \text{ or } Q = (Q \cap R)R[y, y^{-1}] \text{ and } (Q \cap R)^{\star_f} \subsetneq R^\star\}.$$

Set $S^\star := T[X] \backslash (\bigcup\{Q[X] | Q \in \triangle^\star\})$ and:

$$E^{\star\prime} := E[X]_{S^\star} \cap K_1, \text{ for all } E \in \overline{\mathcal{F}}(T).$$

(a) *The mapping $\star\prime : \overline{\mathcal{F}}(T) \to \overline{\mathcal{F}}(T)$, $E \mapsto E^{\star\prime}$ is a stable semistar operation of finite type on T, i.e., $\widetilde{\star\prime} = \star\prime$.*
(b) *$(\widetilde{\star})\prime = (\star_f)\prime = \star\prime$.*
(c) *$(ER[y, y^{-1}])^{\star\prime} \cap K = E^{\widetilde{\star}}$ for all $E \in \overline{\mathcal{F}}(R)$.*
(d) *$(ER[y, y^{-1}])^{\star\prime} = E^{\widetilde{\star}}R[y, y^{-1}]$ for all $E \in \overline{\mathcal{F}}(R)$.*
(e) *$T^{\star\prime} \subsetneq T_{H'}$, where H' is the set of nonzero homogeneous elements of T, and $\star\prime$ sends homogeneous fractional ideals to homogeneous ones.*
(f) *$\operatorname{QMax}^{\star\prime}(T) = \{Q | Q \in \operatorname{Spec}(T)$ such that $Q \cap R = (0)$ and $c_R(Q)^{\star_f} = R^\star\} \cup \{PR[y, y^{-1}] | P \in \operatorname{QMax}^{\star_f}(R)\}$.*
(g) *$h\text{-QMax}^{\star\prime}(T) = \{PR[y, y^{-1}] | P \in h\text{-}\operatorname{QMax}^{\widetilde{\star}}(R)\}$.*
(h) *$(w_R)\prime = (t_R)\prime = (v_R)\prime = w_T$.*

Proof. Set $\nabla^\star := \{Q \in \operatorname{Spec}(T) |\ Q \cap R = (0)$ and $c_D(Q)^{\star_f} = R^\star$ or $Q = PR[y, y^{-1}]$ and $P \in \operatorname{QMax}^{\star_f}(D)\}$. Then it is easy to see that the elements of $\nabla^\star$ are the maximal elements of $\triangle^\star$ (see proof of [9, Theorem 2.3]). Thus

$$S^\star := T[X] \backslash (\bigcup\{Q[X] | Q \in \triangle^\star\}) = T[X] \backslash (\bigcup\{Q[X] | Q \in \nabla^\star\}).$$

(a) It follows from [9, Theorem 2.1 (a) and (b)], that $\star\prime$ is a stable semistar operation of finite type on T.

(b) Since $\text{QMax}^{\star_f}(D) = \text{QMax}^{\widetilde{\star}}(D)$, the conclusion follows easily from the fact that $S^{\widetilde{\star}} = S^{\star_f} = S^{\star}$.

(c) and (d) Exactly are the same as proof of [9, Theorem 2.3(c) and (d)].

(e) From part (d) we have $T^{\star\prime} = R^{\widetilde{\star}}R[y, y^{-1}] \subsetneq R_H R[y, y^{-1}] = T_{H'}$. The second assertion follows from Proposition 2.3, since $\widetilde{\star\prime} = \star\prime$ by (a).

(f) Follows from [9, Theorem 2.1(e)] and the remark in the first paragraph in the proof.

(g) Let $M \in h\text{-QMax}^{\star\prime}(T)$. Since $y, y^{-1} \in T$, clearly we have $M \cap R \neq (0)$. Then by (f), there is $P \in \text{QMax}^{\star_f}(R)$ such that $M \subseteq PR[y, y^{-1}]$. If $P \in h\text{-QMax}^{\widetilde{\star}}(R)$, then $M = PR[y, y^{-1}]$ and we are done. So suppose that $P \notin h\text{-QMax}^{\widetilde{\star}}(R)$. Then note that $P_h \in h\text{-QSpec}^{\widetilde{\star}}(R)$ and $M \subseteq P_h R[y, y^{-1}] = (PR[y, y^{-1}])_h$; hence $M = P_h R[y, y^{-1}]$, because M is a homogeneous maximal quasi-$\star\prime$-ideal. Note that in this case $P_h \in h\text{-QMax}^{\widetilde{\star}}(R)$ by [16, Lemma 4.1, Remark 4.5]. So that $M \in \{PR[y, y^{-1}] | P \in h\text{-}\,\text{QMax}^{\widetilde{\star}}(R)\}$. The other inclusion is trivial.

(h) Suppose that $\star_f = t$. Note that if $M \in \text{QMax}^{\star\prime}(T)$, and $M \cap R \neq (0)$, then, $M = (M \cap R)[y, y^{-1}]$ and $M \cap R \in \text{QMax}^t(R)$ (cf. [24, Proposition 1.1]). Moreover, if $Q \in \text{Spec}(T)$ is such that $Q \cap R = (0)$, then Q is a quasi-t-maximal ideal of T if and only if $c_R(Q)^t = R$. Indeed, if Q is a quasi-t-maximal ideal of T, and $c_R(Q)^t \subsetneq R$, then there exists a quasi-t-maximal ideal P of R such that $c_R(Q)^t \subseteq P$. Hence $Q \subseteq P[y, y^{-1}]$, and therefore $Q = P[y, y^{-1}]$. Consequently $(0) = Q \cap R = P[y, y^{-1}] \cap R = P$ which is a contradiction. Conversely assume that $c_R(Q)^t = R$. Suppose Q is not a quasi-t-maximal ideal of T, and let M be a quasi-t-maximal ideal of T which contains Q. Since the containment is proper, we have $M \cap R \neq (0)$. Thus $M = (M \cap R)[y, y^{-1}]$ and $M \cap R \in \text{QMax}^t(R)$ (cf. [24, Proposition 1.1]). Since $Q \subseteq M$, $c_R(Q)$ is contained in the quasi-t-ideal $M \cap R$, so that $c_R(Q)^t \neq R$ which is a contradiction. Thus we showed that $\text{QMax}^t(T) = \{Q | Q \in \text{Spec}(T) \text{ such that } Q \cap R = (0) \text{ and } c_R(Q)^{\star_f} = R^{\star}\} \cup \{PR[y, y^{-1}] | P \in \text{QMax}^{\star_f}(R)\} = \text{QMax}^{\star\prime}(T)$, where the second equality is by (f). Thus using (a) and (b), we obtain $(w_R)\prime = (t_R)\prime = (v_R)\prime = w_T$. □

It is known that $Pic(D(X)) = 0$ [1, Theorem 2]. More generally, if $*$ is a star operation on D, then $Pic(\text{Na}(D, *)) = 0$, [26, Theorem 2.14]. Also in the graded case it is shown in [4, Theorem 3.3], that $Pic(R_{N_v(H)}) = 0$, where $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded integral domain containing a unit of nonzero degree. We next show in general that $Pic(R_{N_\star(H)}) = 0$.

Theorem 2.13. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then $Pic(R_{N_\star(H)}) = 0$.*

Proof. Let y be an indeterminate over R, and $T = R[y, y^{-1}]$. Using Proposition 2.12(e) and (g) and Lemma 2.7, we deduce that $\text{Max}(T_{N_{\star\prime}(H)}) = \{QT_{N_{\star\prime}(H)} | Q \in h\text{-QMax}^{\star_f}(R)\}$. Next since $\text{Max}((R_{N_\star(H)})(y)) = \{P(y) | P$ is a maximal ideal of $R_{N_\star(H)}\}$, [23, Proposition 33.1], we have $\text{Max}((R_{N_\star(H)})(y)) = \{(QR_{N_\star(H)})(y) | Q \in h\text{-QMax}^{\star_f}(R)\}$. Thus by a computation similar to the proof of [4, Lemma 3.2], we obtain the equality $T_{N_{\star\prime}(H)} = (R_{N_\star(H)})(y)$. The rest of the proof is exactly the same as proof of [4, Theorem 3.3], using Proposition 2.12. □

Let D be a domain and T an overring of D. Let $\star$ and $\star'$ be semistar operations on D and T, respectively. One says that T is $(\star,\star')$-*linked to* D (or that T is a $(\star,\star')$-*linked overring of* D) if

$$F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero finitely generated ideal F of D. (The preceding definition generalizes the notion of "t-linked overring" which was introduced in [13].) It is shown in [15, Theorem 3.8], that T is a $(\star,\star')$-linked overring of D if and only if $\mathrm{Na}(D,\star) \subseteq \mathrm{Na}(T,\star')$. We need a graded analogue of linkedness.

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and T be a homogeneous overring of R. Let $\star$ and $\star'$ be semistar operations on R and T, respectively. We say that T *is homogeneously* $(\star,\star')$-*linked overring of* R if

$$F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero homogeneous finitely generated ideal F of R. We say that T *is homogeneously t-linked overring of* R if T is homogeneously (t,t)-linked overring of R. Also it can be seen that T is homogeneously $(\star,\star')$-linked overring of R if and only if T is homogeneously $(\widetilde{\star},\widetilde{\star'})$-linked overring of R (cf. [15, Theorem 3.8]).

Example 2.14. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and let $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Let $P \in h\text{-QSpec}^{\widetilde{\star}}(R)$. Then, $R_{H\setminus P}$ is a homogeneously $(\star,\star')$-linked overring of R, for all semistar operation $\star'$ on $R_{H\setminus P}$. Indeed assume that F is a nonzero finitely generated homogeneous ideal of R such that $F^\star = R^\star$. Then we have $F^{\widetilde{\star}} = R^{\widetilde{\star}}$. Thus using Proposition 2.6, we have $FR_{H\setminus P} = F^{\widetilde{\star}}R_{H\setminus P} = R^{\widetilde{\star}}R_{H\setminus P} = R_{H\setminus P}$.*

Lemma 2.15. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and let T be a homogeneous overring of R. Let $\star$ (resp. $\star'$) be a semistar operation on R (resp. on T). Then, T is a homogeneously $(\star,\star')$-linked overring of R if and only if $R_{N_\star(H)} \subseteq T_{N_{\star'}(H)}$.*

Proof. Let $f \in R$ such that $C_R(f)^\star = R^\star$. Then by assumption $C_T(f)^{\star'} = (C_R(f)T)^{\star'} = R^{\star'}$. Hence $R_{N_\star(H)} \subseteq T_{N_{\star'}(H)}$. Conversely let F be a nonzero homogeneous finitely generated ideal of R such that $F^\star = R^\star$. Since R has a unit of nonzero degree we can choose an element $f \in R$ such that $C_R(f) = F$. From the fact that $C_R(f)^\star = R^\star$, we have that f is a unit in $R_{N_\star(H)}$ and so by assumption, f is a unit in $T_{N_{\star'}(H)}$. This implies that $C_T(f)^{\star'} = (C_R(f)T)^{\star'} = T^{\star'}$, i.e., $(FT)^{\star'} = T^{\star'}$. ◻

3. Kronecker function ring

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $*$ an `e.a.b.` star operation on R. The graded analogue of the well known Kronecker function ring (see [23, Theorem 32.7]) of R with respect to $*$ is defined by

$$\mathrm{Kr}(R,*) := \left\{ \frac{f}{g} \,\middle|\, f,g\in R,\ g\neq 0,\ \text{and } C(f)\subseteq C(g)^* \right\}$$

in [4]. The following lemma is proved in [4, Theorems 2.9 and 3.5], for an `e.a.b.` star operation $*$. We need to state it for `e.a.b.` semistar operations. Since the proof is exactly the same as star operation case, we omit the proof.

Lemma 3.1. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $\star$ an* `e.a.b.` *semistar operation on R, and*

$$\mathrm{Kr}(R,\star) := \left\{ \frac{f}{g} \,\middle|\, f,g \in R,\ g \neq 0,\ \text{and } C(f) \subseteq C(g)^\star \right\}.$$

Then

(1) $\mathrm{Kr}(R,\star)$ *is an integral domain.*

In addition, if R has a unit of nonzero degree, then,

(2) $\mathrm{Kr}(R,\star)$ *is a Bézout domain.*
(3) $I\,\mathrm{Kr}(R,\star) \cap R_H = I^\star$ *for every nonzero finitely generated homogeneous ideal I of R.*

Inspired by the work of Fontana and Loper in [20], we can generalize this definition of $\mathrm{Kr}(R,\star)$ to all semistar operations on R which send homogeneous fractional ideals, to homogeneous ones, provided that R has a unit of nonzero degree. Before doing that we need a lemma.

Lemma 3.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, $\star$ a semistar operation on R which sends homogeneous fractional ideals to homogeneous ones. Suppose that $a \in R$ is homogeneous and $B, F \in f(R)$, with B homogeneous and $F \subseteq R_H$, such that $aF \subseteq (BF)^\star$. Then there exists a homogeneous $T \in f(R)$ such that $aT \subseteq (BT)^\star$.*

Proof. Suppose that F is generated by $y_1, \cdots, y_n \in R_H$. Let $y_i = \sum t_{ij}$ be the decomposition of y_i to homogeneous elements for $i = 1, \cdots, n$. Then $ay_i \in (BF)^\star = (\sum y_i B)^\star \subseteq (\sum t_{ij} B)^\star$. Since $(\sum t_{ij} B)^\star$ is homogeneous we have $at_{ij} \in (\sum t_{ij} B)^\star$. Let T be the fractional ideal of R, generated by all homogeneous elements t_{ij}. So that $aT \subseteq (BT)^\star$ and $T \in f(R)$ is homogeneous. □

Theorem 3.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, $\star$ a semistar operation on R which sends homogeneous fractional ideals to homogeneous ones, and*

$$\mathrm{Kr}(R,\star) := \left\{ \frac{f}{g} \,\middle|\, \begin{array}{l} f,g \in R, g \neq 0,\ \text{and there is } 0 \neq h \in R \\ \text{such that } C(f)C(h) \subseteq (C(g)C(h))^\star \end{array} \right\}.$$

Then

(1) $\mathrm{Kr}(R,\star) = \mathrm{Kr}(R,\star_a)$.
(2) $\mathrm{Kr}(R,\star)$ *is a Bézout domain.*
(3) $I\,\mathrm{Kr}(R,\star) \cap R_H = I^{\star_a}$ *for every nonzero finitely generated homogeneous ideal I of R.*
(4) *If $f, g \in R$ are nonzero such that $C(f+g)^\star = (C(f) + C(g))^\star$, then $(f,g)\,\mathrm{Kr}(R,\star) = (f+g)\,\mathrm{Kr}(R,\star)$. In particular, $f\,\mathrm{Kr}(R,\star) = C(f)\,\mathrm{Kr}(R,\star)$ for all $f \in R$.*

Proof. It it clear from the definition that $\mathrm{Kr}(R,\star) = \mathrm{Kr}(R,\star_f)$. Thus using Lemma 2.4, we can assume, without loss of generality, that $\star$ is a semistar operation of finite type.

Parts (2) and (3) are direct consequences of (1) using Lemma 3.1. For the proof of (1) we have two cases:

Case 1: Assume that $\star$ is an `e.a.b.` semistar operation of finite type. In this case, for $f, g, h \in R\backslash\{0\}$ we have

$$C(f)C(h) \subseteq (C(g)C(h))^{\star} \Leftrightarrow C(f) \subseteq C(g)^{\star}.$$

Therefore $\mathrm{Kr}(R, \star)$ -as defined in this theorem- coincides with $\mathrm{Kr}(R, \star)$ of an `e.a.b.` semistar operation $\star$, as defined in Lemma 3.1. Also in this case $\star = \star_a$ by [20, Proposition 4.5(5)]. Hence in this case (1) is true.

Case 2: General case. Let $\star$ be a semistar operation of finite type on R. By definition it is easy to see that, given two semistar operations on R with $\star_1 \leq \star_2$, then $\mathrm{Kr}(R, \star_1) \subseteq \mathrm{Kr}(R, \star_2)$. Using [20, Proposition 4.5(3)] we have $\star \leq \star_a$. Therefore $\mathrm{Kr}(R, \star) \subseteq \mathrm{Kr}(R, \star_a)$. Conversely let $f/g \in \mathrm{Kr}(R, \star_a)$. Then, by Case 1, $C(f) \subseteq C(g)^{\star_a}$. Set $A := C(f)$ and $B := C(g)$. Then $A \subseteq B^{\star_a} = \bigcup\{((BH)^{\star} : H)|H \in f(R)\}$. Suppose that A is generated by homogeneous elements $x_1, \cdots, x_n \in R$. Then there is $H_i \in f(R)$, such that $x_iH_i \subseteq (BH_i)^{\star}$ for $i = 1, \cdots, n$. Choose $0 \neq r_i \in R$ such that $F_i = r_iH_i \subseteq R$. Thus $x_iF_i \subseteq (BF_i)^{\star}$. Therefore Lemma 3.2 gives a homogeneous $T_i \in f(R)$ such that $x_iT_i \subseteq (BT_i)^{\star}$. Now set $T := T_1T_2 \cdots T_n$ which is a finitely generated homogeneous fractional ideal of R such that $AT \subseteq (BT)^{\star}$. Now since R has a unit of nonzero degree, we can find an element $h \in R$ such that $C(h) = T$. Then $C(f)C(h) \subseteq (C(g)C(h))^{\star}$. This means that $f/g \in \mathrm{Kr}(R, \star)$ to complete the proof of (1).

The proof of (4) is exactly the same as [4, Theorem 2.9(3)]. □

4. Graded P$\star$MDs

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain, $\star$ be a semistar operation on R, H be the set of nonzero homogeneous elements of R, and $N_{\star}(H) = \{f \in R | C(f)^{\star} = R^{\star}\}$. In this section we define the notion of graded Prüfer $\star$-multiplication domain (graded P$\star$MD for short) and give several characterization of it.

We say that a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ with a semistar operation $\star$, is a *graded Prüfer $\star$-multiplication domain (graded P$\star$MD)* if every nonzero finitely generated homogeneous ideal of R is a $\star_f$-invertible, i.e., $(II^{-1})^{\star_f} = R^{\star}$ for every nonzero finitely generated homogeneous ideal I of R. It is easy to see that a graded P$\star$MD is the same as a graded P$\star_f$MD by definition, and is the same as a graded P$\widetilde{\star}$MD by [22, Proposition 2.18]. When $\star = v$ we recover the classical notion of a *graded Prüfer v-multiplication domain (graded PvMD)* [2]. It is known that R is a graded PvMD if and only if R is a PvMD [2, Theorem 6.4].

Also when $\star = d$, a graded PdMD is called a *graded Prüfer domain* [4]. It is clear that every graded Prüfer domain is a graded PvMD and hence a PvMD. In particular every graded Prüfer domain is an integrally closed domain. Although R is a graded PvMD if and only if R is a PvMD, Anderson and Chang in [4, Example 3.6] provided an example of a graded Prüfer domain which is not Prüfer. It is known that if A, B, C are ideals of an integral domain D, then $(A+B)(A+C)(B+C) = (A+B+C)(AB+AC+BC)$. Thus $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a graded Prüfer domain if and only if every nonzero ideal of R generated by two homogeneous elements is invertible. We use this result in this section without comments.

The following proposition is inspired by [23, Theorem 24.3].

Proposition 4.1. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following conditions are equivalent:*

(1) *R is a graded Prüfer domain.*

(2) *Each finitely generated nonzero homogeneous ideal of R is a cancelation ideal.*
(3) *If A, B, C are finitely generated homogeneous ideals of R such that $AB = AC$ and A is nonzero, then $B = C$.*
(4) *R is integrally closed and there is a positive integer $n > 1$ such that $(a, b)^n = (a^n, b^n)$ for each $a, b \in H$.*
(5) *R is integrally closed and there exists an integer $n > 1$ such that $a^{n-1}b \in (a^n, b^n)$ for each $a, b \in H$.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are clear.

(3) $\Rightarrow$ (4) By the same argument as in the proof of part (2) $\Rightarrow$ (3), in [23, Proposition 24.1], we have that R is integrally closed in R_H. Therefore by [3, Proposition 5.4], R is integrally closed. Now if $a, b \in H$ we have $(a, b)^3 = (a, b)(a^2, b^2)$. Thus by (3) we obtain that $(a, b)^2 = (a^2, b^2)$.

(5) $\Rightarrow$ (1) If (5) holds then [23, Proposition 24.2], implies that each nonzero homogeneous ideal generated by two homogeneous elements is invertible. Therefore R is a graded Prüfer domain. □

The ungraded version of the following theorem is due to Gilmer (see [23, Corollary 28.5]).

Theorem 4.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then R is a graded Prüfer domain if and only if $C(f)C(g) = C(fg)$ for all $f, g \in R_H$.*

Proof. ($\Rightarrow$) Let $f, g \in R_H$. Then by [4, Lemma 1.1(1)], there exists some positive integer n such that $C(f)^{n+1}C(g) = C(f)^nC(fg)$. Now since R is a graded Prüfer domain, the homogeneous fractional ideal $C(f)^n$ is invertible. Thus $C(f)C(g) = C(fg)$ for all $f, g \in R_H$.

($\Leftarrow$) Let $\alpha \in H$ be a unit of nonzero degree. Assume that $C(f)C(g) = C(fg)$ for all $f, g \in R_H$. Hence R is integrally closed by [2, Theorem 3.7]. Now let $a, b \in H$ be arbitrary. We can choose a positive integer n such that $\deg(a) \neq \deg(\alpha^n b)$. So that $C(a + \alpha^n b) = (a, b)$. Hence, since $(a + \alpha^n b)(a - \alpha^n b) = a^2 - (\alpha^n b)^2$, we have $(a, b)(a, -b) = (a^2, -b^2)$. Consequently $(a, b)^2 = (a^2, b^2)$. Thus by Proposition 4.1, we see that R is a graded Prüfer domain. □

Lemma 4.3. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and P be a homogeneous prime ideal. Then, the following statements are equivalent:*

(1) *$R_{H\setminus P}$ is a graded Prüfer domain*
(2) *R_P is a valuation domain.*
(3) *For each nonzero homogeneous $u \in R_H$, u or u^{-1} is in $R_{H\setminus P}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $R_{H\setminus P}$ is a graded Prüfer domain. In particular $R_{H\setminus P}$ is a (graded) PvMD and each nonzero homogeneous ideal of $R_{H\setminus P}$ is a t-ideal. So that $h\text{-QMax}^t(R_{H\setminus P}) = \{PR_{H\setminus P}\}$. Thus by [10, Lemma 2.7], we see that $(R_{H\setminus P})_{PR_{H\setminus P}} = R_P$ is a valuation domain.

(2) $\Rightarrow$ (3) Let $0 \neq u \in R_H$. Thus by the hypothesis u or u^{-1} is in R_P. Thus u or u^{-1} is in $R_{H\setminus P}$.

(3) $\Rightarrow$ (1) Let I, J be two nonzero homogeneous ideals of $R_{H\setminus P}$ and assume that $I \nsubseteq J$. So there is a homogeneous element $a \in I\setminus J$. For each $b \in J$, we have $\frac{a}{b} \notin R_{H\setminus P}$, since otherwise we have $a = (\frac{a}{b})b \in J$. Thus by the hypothesis

$\frac{b}{a} \in R_{H\setminus P}$. Hence $b = (\frac{b}{a})a \in I$. Thus we showed that $J \subseteq I$, and so every two homogeneous ideal are comparable.

Now Let (a, b) be an ideal generated by two homogeneous elements of $R_{H\setminus P}$. Now by the first paragraph $(a, b) = (a)$ or $(a, b) = (b)$. Thus (a, b) is invertible. Hence $R_{H\setminus P}$ is a graded Prüfer domain. □

Theorem 4.4. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent:*

1. *R is a graded P$\star$MD.*
2. *$R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QSpec}^{\widetilde{\star}}(R)$.*
3. *$R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QMax}^{\widetilde{\star}}(R)$.*
4. *R_P is a valuation domain for each $P \in h\text{-QSpec}^{\widetilde{\star}}(R)$.*
5. *R_P is a valuation domain for each $P \in h\text{-QMax}^{\widetilde{\star}}(R)$.*

Proof. (2) ⇒ (3) is trivial, and, (2) ⇔ (4) and (3) ⇔ (5), follow from Lemma 4.3.

(1) ⇒ (2) Let I be a nonzero finitely generated homogeneous ideal of R. Then I is $\widetilde{\star}$-invertible. Therefore, for each $P \in h\text{-QSpec}^{\widetilde{\star}}(R)$, since $II^{-1} \nsubseteq P$, we have $R_{H\setminus P} = (II^{-1})R_{H\setminus P} = IR_{H\setminus P}I^{-1}R_{H\setminus P} = (IR_{H\setminus P})(IR_{H\setminus P})^{-1}$. So that $IR_{H\setminus P}$ is invertible. Thus $R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QSpec}^{\widetilde{\star}}(R)$.

(3) ⇒ (1) Let I be a nonzero finitely generated homogeneous ideal of R. Suppose that I is not $\widetilde{\star}$-invertible. Hence there exists $P \in h\text{-QMax}^{\widetilde{\star}}(R)$ such that $II^{-1} \subseteq P$. Thus $R_{H\setminus P} = (IR_{H\setminus P})(IR_{H\setminus P})^{-1} = II^{-1}R_{H\setminus P} \subseteq PR_{H\setminus P}$, which is a contradiction. So that $II^{-1} \nsubseteq P$ for each $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. Therefore $(II^{-1})^{\widetilde{\star}} = R^{\widetilde{\star}}$, that is I is $\widetilde{\star}$-invertible, and hence R is a graded P$\star$MD. □

The ungraded version of the following theorem is due to Chang in the star operation case [8, Theorem 3.7], and is due to Anderson, Fontana, and Zafrullah in the case of semistar operations [6, Theorem 1.1].

Theorem 4.5. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then R is a graded P$\star$MD if and only if $(C(f)C(g))^{\widetilde{\star}} = C(fg)^{\widetilde{\star}}$ for all $f, g \in R_H$.*

Proof. (⇒) Let $f, g \in R_H$. Choose a positive integer n such that $C(f)^{n+1}C(g) = C(f)^nC(fg)$ by [4, Lemma 1.1(1)]. Thus $(C(f)^{n+1}C(g))^{\widetilde{\star}} = (C(f)^nC(fg))^{\widetilde{\star}}$. Since R is a graded P$\star$MD, the homogeneous fractional ideal $C(f)^n$ is $\widetilde{\star}$-invertible. Thus $(C(f)C(g))^{\widetilde{\star}} = C(fg)^{\widetilde{\star}}$ for all $f, g \in R_H$.

(⇐) Assume that $(C(f)C(g))^{\widetilde{\star}} = C(fg)^{\widetilde{\star}}$ for all $f, g \in R_H$. Let $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. Then using Proposition 2.6, we have $C(f)R_{H\setminus P}C(g)R_{H\setminus P} = C(f)C(g)R_{H\setminus P} = (C(f)C(g))^{\widetilde{\star}}R_{H\setminus P} = C(fg)^{\widetilde{\star}}R_{H\setminus P} = C(fg)R_{H\setminus P}$. Since $R_{H\setminus P}$ has a unit of nonzero degree, Theorem 4.2 shows that $R_{H\setminus P}$ is a graded Prüfer domain. Now Theorem 4.4, implies that R is a graded P$\star$MD. □

We now recall the notion of $\star$-valuation overring (a notion due essentially to P. Jaffard [25, page 46]). For a domain D and a semistar operation $\star$ on D, we say that a valuation overring V of D is a *$\star$-valuation overring of D* provided $F^\star \subseteq FV$, for each $F \in f(D)$.

Remark 4.6. *(1) Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Recall that for each $F \in f(R)$ we have*

$$F^{\star_a} = \bigcap\{FV | V \text{ is a } \star\text{-valuation overring of } R\},$$

by [19, Propositions 3.3 and 3.4 and Theorem 3.5].

(2) We have $N_\star(H) = N_{\widetilde{\star}_a}(H)$. Indeed, since $\widetilde{\star} \leq \widetilde{\star}_a$ by [20, Proposition 4.5], *we have $N_\star(H) = N_{\widetilde{\star}}(H) \subseteq N_{\widetilde{\star}_a}(H)$. Now if $f \in R\backslash N_\star(H)$ then, $C(f)^{\widetilde{\star}} \subsetneq R^{\widetilde{\star}}$. Thus there is a homogeneous quasi-$\widetilde{\star}$-prime ideal P of R such that $C(f) \subseteq P$. Let V be a valuation domain dominating R_P with maximal ideal M* [23, Corollary 19.7]. *Therefore V is a $\widetilde{\star}$-valuation overring of R by* [18, Theorem 3.9], *and $C(f)V \subseteq M$; so $C(f)^{(\widetilde{\star})_a} \subsetneq R^{(\widetilde{\star})_a}$ and $f \notin N_{\widetilde{\star}_a}(H)$. Thus we obtain that $N_\star(H) = N_{\widetilde{\star}_a}(H)$.*

In the following theorem we generalize a characterization of PvMDs proved by Arnold and Brewer [7, Theorem 3]. It also generalizes [8, Theorem 3.7], [4, Theorems 3.4 and 3.5], and [17, Theorem 3.1].

Theorem 4.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent:*

1. *R is a graded P$\star$MD.*
2. *Every ideal of $R_{N_\star(H)}$ is extended from a homogeneous ideal of R.*
3. *Every principal ideal of $R_{N_\star(H)}$ is extended from a homogeneous ideal of R.*
4. *$R_{N_\star(H)}$ is a Prüfer domain.*
5. *$R_{N_\star(H)}$ is a Bézout domain.*
6. *$R_{N_\star(H)} = \mathrm{Kr}(R,\widetilde{\star})$.*
7. *$\mathrm{Kr}(R,\widetilde{\star})$ is a quotient ring of R.*
8. *$\mathrm{Kr}(R,\widetilde{\star})$ is a flat R-module.*
9. *$I^{\widetilde{\star}} = I^{\widetilde{\star}_a}$ for each nonzero homogeneous finitely generated ideal of R.*

In particular if R is a graded P$\star$MD, then $R^{\widetilde{\star}}$ is integrally closed.

Proof. By Proposition 2.3 and Theorem 3.3, we have $\mathrm{Kr}(R,\widetilde{\star})$ is well-defined and is a Bézout domain.

(1) $\Rightarrow$ (2) Let $0 \neq f \in R$. Then $C(f)$ is $\widetilde{\star}$-invertible, because R is a graded P$\star$MD, and thus $fR_{N_\star(H)} = C(f)R_{N_\star(H)}$ by Corollary 2.11. Hence if A is an ideal of $R_{N_\star(H)}$, then $A = IR_{N_\star(H)}$ for some ideal I of R, and thus $A = (\sum_{f\in I} C(f))R_{N_\star(H)}$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Is the same as part (3) $\Rightarrow$ (1) in [4, Theorem 3.4].

(1) $\Rightarrow$ (4) Let A be a nonzero finitely generated ideal of $R_{N_\star(H)}$. Then by Corollary 2.11, $A = IR_{N_\star(H)}$ for some nonzero finitely generated homogeneous ideal I of R. Since R is a graded P$\star$MD, I is $\widetilde{\star}$-invertible, and thus $A = IR_{N_\star(H)}$ is invertible by Lemma 2.10.

(4) $\Rightarrow$ (5) Follows from Theorem 2.13.

(5) $\Rightarrow$ (6) Clearly $R_{N_\star(H)} \subseteq \mathrm{Kr}(R,\widetilde{\star})$. Since $R_{N_\star(H)}$ is a Bézout domain, then $\mathrm{Kr}(R,\widetilde{\star})$ is a quotient ring of $R_{N_\star(H)}$, by [23, Proposition 27.3]. If $Q \in h\text{-QMax}^{\widetilde{\star}}(R)$, then $Q\,\mathrm{Kr}(R,\widetilde{\star}) \subsetneq \mathrm{Kr}(R,\widetilde{\star})$. Otherwise $Q\,\mathrm{Kr}(R,\widetilde{\star}) = \mathrm{Kr}(R,\widetilde{\star})$, and hence there is an element $f \in Q$, such that $f\,\mathrm{Kr}(R,\widetilde{\star}) = \mathrm{Kr}(R,\widetilde{\star})$. Thus $\frac{1}{f} \in \mathrm{Kr}(R,\widetilde{\star})$. Therefore $R = C(1) \subseteq C(f)^{(\widetilde{\star})_a} \subseteq R^{(\widetilde{\star})_a}$, so that $C(f)^{(\widetilde{\star})_a} = R^{(\widetilde{\star})_a}$. Hence $f \in N_{(\widetilde{\star})_a}(H) = N_\star(H)$ by Remark 4.6(2). This means that $Q^{\widetilde{\star}} = R^{\widetilde{\star}}$, a contradiction. Thus $Q\,\mathrm{Kr}(R,\widetilde{\star}) \subsetneq \mathrm{Kr}(R,\widetilde{\star})$, and so there is a maximal ideal M of $\mathrm{Kr}(R,\widetilde{\star})$ such that

$Q\operatorname{Kr}(R,\star) \subseteq M$. Hence $M \cap R_{N_\star(H)} = QR_{N_\star(H)}$, by Lemma 2.7. Consequently $R_Q \subseteq \operatorname{Kr}(R,\widetilde{\star})_M$, and since R_Q is a valuation domain, we have $R_Q = \operatorname{Kr}(R,\widetilde{\star})_M$. Therefore $R_{N_\star(H)} = \bigcap_{Q\in h\text{-}\operatorname{QMax}^{\widetilde{\star}}(R)} R_Q \supseteq \bigcap_{M\in\operatorname{Max}(\operatorname{Kr}(R,\widetilde{\star}))} \operatorname{Kr}(R,\widetilde{\star})_M$. Hence $R_{N_\star(H)} = \operatorname{Kr}(R,\widetilde{\star})$.

(6) ⇒ (7) and (7) ⇒ (8) are clear.

(8) ⇒ (6) Recall that an overring T of an integral domain S is a flat S-module if and only if $T_M = S_{M\cap S}$ for all $M \in \operatorname{Max}(T)$ by [32, Theorem 2].

Let A be an ideal of R such that $A\operatorname{Kr}(R,\widetilde{\star}) = \operatorname{Kr}(R,\widetilde{\star})$. Then there exists an element $f \in A$ such that $f\operatorname{Kr}(R,\widetilde{\star}) = \operatorname{Kr}(R,\widetilde{\star})$ using Theorem 3.3; so $\frac{1}{f} \in \operatorname{Kr}(R,\widetilde{\star}) = \operatorname{Kr}(R,\widetilde{\star}_a)$. Thus $R = C(1) \subseteq C(f)^{\widetilde{\star}_a} \subseteq R^{\widetilde{\star}_a}$, and so $C(f)^{\widetilde{\star}_a} = R^{\widetilde{\star}_a}$. Hence $C(f)^{\widetilde{\star}} = R^{\widetilde{\star}}$. Therefore $f \in A \cap N_\star(H) \neq \emptyset$. Hence, if P_0 is a homogeneous maximal quasi-$\widetilde{\star}$-ideal of R, then $P_0\operatorname{Kr}(R,\widetilde{\star}) \subsetneq \operatorname{Kr}(R,\widetilde{\star})$, and since $P_0R_{N_\star(H)}$ is a maximal ideal of $R_{N_\star(H)}$, there is a maximal ideal M_0 of $\operatorname{Kr}(R,\widetilde{\star})$ such that $M_0 \cap R = (M_0 \cap R_{N_\star(H)}) \cap R = P_0R_{N_\star(H)} \cap R = P_0$. Thus by (8), $\operatorname{Kr}(R,w)_{M_0} = R_{P_0} = (R_{N(H)})_{P_0R_{N(H)}}$.

Let M_1 be a maximal ideal of $\operatorname{Kr}(R,\widetilde{\star})$, and let P_1 be a homogeneous maximal quasi-$\widetilde{\star}$-ideal of R such that $M_1 \cap R_{N_\star(H)} \subseteq P_1R_{N_\star(H)}$. By the above paragraph, there is a maximal ideal M_2 of $\operatorname{Kr}(R,\widetilde{\star})$ such that $\operatorname{Kr}(R,\widetilde{\star})_{M_2} = (R_{N_\star(H)})_{P_1R_{N_\star(H)}}$. Note that $\operatorname{Kr}(R,\widetilde{\star})_{M_2} \subseteq \operatorname{Kr}(R,\widetilde{\star})_{M_1}$, M_1 and M_2 are maximal ideals, and $\operatorname{Kr}(R,\widetilde{\star})$ is a Prüfer domain; hence $M_1 = M_2$ (cf. [23, Theorem 17.6(c)]) and $\operatorname{Kr}(R,\widetilde{\star})_{M_1} = (R_{N_\star(H)})_{P_1R_{N(H)}}$. Thus

$$\operatorname{Kr}(R,\widetilde{\star}) = \bigcap_{M\in\operatorname{Max}(\operatorname{Kr}(R,\widetilde{\star}))} \operatorname{Kr}(R,\widetilde{\star})_M = \bigcap_{P\in h\text{-}\operatorname{QMax}^{\widetilde{\star}}(R)} (R_{N_\star(H)})_{PR_{N_\star(H)}} = R_{N_\star(H)}.$$

(6) ⇒ (9) Assume that $R_{N_\star(H)} = \operatorname{Kr}(R,\widetilde{\star})$. Let I be a nonzero homogeneous finitely generated ideal of R. Then by Lemma 2.9 and Theorem 3.3(3), we have $I^{\widetilde{\star}} = IR_{N_\star(H)} \cap R_H = I\operatorname{Kr}(R,\widetilde{\star}) \cap R_H = I^{\widetilde{\star}_a}$.

(9) ⇒ (1) Let a and b be two nonzero homogeneous elements of R. Then $((a,b)^3)^{\widetilde{\star}_a} = ((a,b)(a^2,b^2))^{\widetilde{\star}_a}$ which implies that $((a,b)^2)^{\widetilde{\star}_a} = (a^2,b^2)^{\widetilde{\star}_a}$. Hence $((a,b)^2)^{\widetilde{\star}} = (a^2,b^2)^{\widetilde{\star}}$ and so $(a,b)^2R_{H\setminus P} = (a^2,b^2)R_{H\setminus P}$ for each homogeneous maximal quasi-$\widetilde{\star}$-ideal P of R. On the other hand $R^{\widetilde{\star}} = R^{\widetilde{\star}_a}$ by (9). Hence $R^{\widetilde{\star}}$ is integrally closed. Thus $R^{\widetilde{\star}}R_{H\setminus P} = R_{H\setminus P}$ is integrally closed. Therefore by Proposition 4.1, $R_{H\setminus P}$ is a graded Prüfer domain for each homogeneous maximal quasi-$\star_f$-ideal of R. Thus R is a graded P$\star$MD by Theorem 4.4. □

The following theorem is a graded version of a characterization of Prüfer domains proved by Davis [12, Theorem 1]. It also generalizes [13, Theorem 2.10], in the t-operation, and [15, Theorem 5.3], in the case of semistar operations.

Theorem 4.8. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent:*

(1) *R is a graded P$\star$MD.*
(2) *Each homogeneously $(\star,t)$-linked overring of R is a PvMD.*
(3) *Each homogeneously $(\star,d)$-linked overring of R is a graded Prüfer domain.*
(4) *Each homogeneously $(\star,t)$-linked overring of R, is integrally closed.*
(5) *Each homogeneously $(\star,d)$-linked overring of R, is integrally closed.*

Proof. (1) $\Rightarrow$ (2) Let T be a homogeneously $(\star, t)$-linked overring of R. Thus by Lemma 2.15, we have $R_{N_\star(H)} \subseteq T_{N_v(H)}$. Since R is a graded P$\star$MD, by Theorem 4.7, we have $R_{N_\star(H)}$ is a Prüfer domain. Thus by [23, Theorem 26.1], we have $T_{N_v(H)}$ is a Prüfer domain. Hence, again by Theorem 4.7, we have T is a graded PvMD. Therefore using [2, Theorem 6.4], T is a PvMD.

(2) $\Rightarrow$ (4) $\Rightarrow$ (5) and (3) $\Rightarrow$ (5) are clear.

(5) $\Rightarrow$ (1) Let $P \in h\text{-QMax}^{\widetilde{\star}}(R)$. For a nonzero homogeneous $u \in R_H$, let $T = R[u^2, u^3]_{H \setminus P}$. Then $R_{H\setminus P}$ and T are homogeneous $(\star, d)$-linked overring of R by Example 2.14. So that $R_{H\setminus P}$ and T are integrally closed. Hence $u \in T$, and since $T = R_{H\setminus P}[u^2, u^3]$, there exists a polynomial $\gamma \in R_{H\setminus P}[X]$ such that $\gamma(u) = 0$ and one of the coefficients of γ is a unit in $R_{H\setminus P}$. So u or u^{-1} is in $R_{H\setminus P}$ by [27, Theorem 67]. Therefore by Lemma 4.3, $R_{H\setminus P}$ is a graded Prüfer domain. Thus R is a graded P$\star$MD by Theorem 4.4.

(1) $\Rightarrow$ (3) Is the same argument as in part (1) $\Rightarrow$ (2). □

The next result gives new characterizations of PvMDs for graded integral domains, which is the special cases of Theorems 4.4, 4.5, 4.7, and 4.8, for $\star = v$.

Corollary 4.9. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain with a unit of nonzero degree. Then, the following statements are equivalent:*

(1) *R is a (graded) PvMD.*
(2) *$R_{H\setminus P}$ is a graded Prüfer domain for each $P \in h\text{-QMax}^t(R)$.*
(3) *R_P is a valuation domain for each $P \in h\text{-QMax}^t(R)$.*
(4) *Every ideal of $R_{N_v(H)}$ is extended from a homogeneous ideal of R.*
(5) *$R_{N_v(H)}$ is a Prüfer domain.*
(6) *$R_{N_v(H)}$ is a Bézout domain.*
(7) *$R_{N_v(H)} = \mathrm{Kr}(R, w)$.*
(8) *$\mathrm{Kr}(R, w)$ is a quotient ring of R.*
(9) *$\mathrm{Kr}(R, w)$ is a flat R-module.*
(10) *Each homogeneously t-linked overring of R is a PvMD.*
(11) *Each homogeneously t-linked overring of R, is integrally closed.*
(12) *$(C(f)C(g))^w = C(fg)^w$ for all $f, g \in R_H$.*
(13) *$I^w = I^{w_a}$ for each nonzero homogeneous finitely generated ideal of R.*

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