REDUCIBILITY OF NILPOTENT COMMUTING VARIETIES

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ABSTRACT. Let $\mathcal{N}_n$ be the set of nilpotent n by n matrices over an algebraically closed field k. For each $r \geq 2$, let $C_r(\mathcal{N}_n)$ be the variety consisting of all pairwise commuting r-tuples of nilpotent matrices. It is well-kown that $C_2(\mathcal{N}_n)$ is irreducible for every n. We study in this note the reducibility of $C_r(\mathcal{N}_n)$ for various values of n and r. In particular it will be shown that the reducibility of $C_r(\mathfrak{gl}_n)$, the variety of commuting r-tuples of n by n matrices, implies that of $C_r(\mathcal{N}_n)$ under certain condition. Then we prove that $C_r(\mathcal{N}_n)$ is reducible for all $n, r \geq 4$. The ingredients of this result are also useful for getting a new lower bound of the dimensions of $C_r(\mathcal{N}_n)$ and $C_r(\mathfrak{gl}_n)$. Finally, we investigate values of n for which the variety $C_3(\mathcal{N}_n)$ of nilpotent commuting triples is reducible.

1. Introduction

1.1. Let $\mathfrak{gl}_n$ be the Lie algebra consisting all n by n matrices over the field k. It will be also considered as an affine space of dimension n^2 throughout this note. For each $r \geq 2$ and a closed subvariety V of $\mathfrak{gl}_n$. Define in general

$$C_r(V) = \{(v_1, \ldots, v_r) \in V^r \mid [v_i, v_j] = 0 \ , \ 1 \leq i \leq j \leq r\},$$

a *commuting variety*[1] of r-tuples over V. If $V = \mathfrak{gl}_n$ then $C_r(V)$ is well-known as an ordinary commuting variety of r-tuples in V. The study of ordinary commuting varieties was originated by the work of Motzkin and Taussky [MT] and then developed by Gerstenhaber [G]. Nowadays, one can find applications of commuting varieties in many branches of mathematics such as functional analysis, representation theory, and geometry [HO][SFB][Ba][BI]. Most of studies have focused on certain contexts, specially when V is either $\mathfrak{gl}_n$ or $\mathcal{N}_n$. We will always call $C_r(\mathcal{N}_n)$ a nilpotent commuting variety in this paper. Unlike $C_r(\mathfrak{gl}_n)$, very little is known for $C_r(\mathcal{N}_n)$. There is a conjecture that reducibility behaviors of both are similar [Yo], however, we could not find enough results on $C_r(\mathcal{N}_n)$ to confirm this analogy.

We are motivated by the connection of nilpotent commuting varieties to cohomology for Frobenius kernels of an algebraic group. In particular, the variety $C_r(\mathcal{N}_n)$ is homeomorphic to the spectrum of the cohomology ring for $(GL_n)_r$ [SFB]. Our goal in this note is to determine the reducibility of $C_r(\mathcal{N}_n)$ for $r \geq 3$. (Note that the case $r = 2$ is well-established by work of Basili, Baranovski, Premet [B][Ba][Pr]). As a consequence, our results provide further evidences to affirming the above conjecture.

To be convenient, we now review what have been done for $C_r(\mathfrak{gl}_n)$ with $r \geq 3$. It is shown to be irreducible for all r and $n \leq 3$ by Gerstenhaber [G], see also [Gu]. The first author verified the reducibility for $n, r \geq 4$ in [Gu]. So the most interesting case is when $r = 3$ which has been contributed by many studies of Guralnick, Sethuraman, Yakimova, Han, and Šivic, [Gu][GS][Ya][H][Si] and remained an open problem. More explicitly, $C_3(\mathfrak{gl}_n)$ is known to be irreducible for $n \leq 10$ and reducible for all $n \geq 30$, (some arguments require that $k = \mathbb{C}$ [HO][Si]). On the contrary, the irreducibility for $C_r(\mathcal{N}_n)$ when $n \leq 3$ is shown in a paper of the second author [N]. Almost nothing on nilpotent commuting varieties for higher ranks has appeared in literature.[2]

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[1] This is a generalization for the concept of the variety of commuting r-tuples of matrices in [G] and [Gu].

[2] We are aware of the results in the Ph.D. dissertation of Young, [Yo], however they have not been published.

In an alternative language, let n_r be the least integer such that $C_r(\mathfrak{gl}_n)$ is reducible. Obviously, we only consider this number for $r \geq 3$ as $C_2(\mathfrak{gl}_n)$ is always irreducible. It is implied from previous paragraph that $n_r = 4$ for all $r \geq 4$, and n_3 is in the interval $[10, 30]$ [HO]. Similarly, let n'_r be the least integer such that $C_r(\mathcal{N}_{n'_r})$ is reducible. One might be interested in whether $n'_r = n_r$ for all $r \geq 3$. Answering this question is one of the targets of this paper.

1.2. **Main results.** The note is organized as follows. We show in Section 3 that the irreducibility of $C_r(\mathcal{N}_n)$ implies that of $C_r(\mathfrak{gl}_n)$ under certain condition, cf. Theorem 3.2.2. As a consequence, we then have $n'_r = 4$ (hence $n'_r = n_r$) for each $r > 3$ and n'_3 is in the interval $[10, 30]$ as n_3 is, cf. Corollary 3.2.3. Next, we generalize this result by showing that $C_r(\mathcal{N}_n)$ is in fact reducible for all $n, r \geq 4$, cf. Theorem 3.4.1. The strategy is similar to that for $C_r(\mathfrak{gl}_n)$ in [Gu] and [GS]. Explicitly, we show in Section 3.3 that if $C_r(\mathcal{N}_n)$ is irreducible then the subalgebra of $\mathfrak{gl}_n$ generated by any r-tuple $(x_1, \ldots, x_r)$ in it has dimension no more than $n-1$. This verification is a strong evidence supporting the statement that both varieties $C_r(\mathfrak{gl}_n)$ and $C_r(\mathcal{N}_n)$ are simultaneously reducible (or irreducible) for every n and r.

In Section 4, we give new lower bounds for the dimensions of (nilpotent) commuting varieties. It is known that $C_r(\mathcal{N}_n)$ always has an irreducible component of dimension $n^2 - n + (r-1)(n-1)$, cf. Theorem 3.1.1. Thus this number is a lower bound of $\dim C_r(\mathcal{N}_n)$. Our method is analyzing the subvariety $V_P = G \cdot \mathfrak{u}_P^r$ where $\mathfrak{u}_P$ is the Lie algebra of U_P, the unipotent subgroup of a certain parabolic subgroup P of $GL_n(k)$. The dimension of V_P is computed in the Proposition 4.1.1. We then point out the values of n and r such that $\dim V_P$ is greater than the above lower bound; hence establish a new lower bound for the dimension of $C_r(\mathcal{N}_n)$, cf. Theorem 4.1.2, Corollary 4.1.3. In other words, $C_r(\mathcal{N}_n)$ is not equidimensional for most of values of n and r. Analogous result is also obtained for $C_r(\mathfrak{gl}_n)$ in Theorem 4.2.1.

We restrict our attetion in the last section to the case when $r = 3$. Basically, we narrow down the interval for possible values of n'_3. Recall from earlier that $n'_3 \leq 30$. In this section, we will lower this upper bound down to 16. In particular, using the method and ingredients in [Gu], applied for $C_3(\mathfrak{gl}_n)$, we show that $C_3(\mathcal{N}_n)$ is reducible if $n = 4s \geq 16$, Theorem 5.2.1.

2. Notation

2.1. Let k be an algebraically closed field. We always fix $G = GL_n(k)$ be the general linear algebraic group defined over k. Then denote by $\mathfrak{g} = \mathfrak{gl}_n$, the Lie algebra of G. Sometime we write $\mathfrak{gl}_n$ to distinguish with other general linear Lie algebras of different ranks. The Lie subalgebra $\mathfrak{t}_n$ of $\mathfrak{gl}_n$, consisting all the diagonal n by n matrices, is called Cartan subalgebra. The nilpotent cone denoted by $\mathcal{N}_n$ is a subvariety of $\mathfrak{gl}_n$ of dimension $n^2 - n$.

2.2. It is well-known that G acts on $\mathfrak{g}$ by conjugation, which we denote by a dot "$\cdot$". One can consider the nilpotent cone $\mathcal{N}_n$ of $\mathfrak{g}$ as a G-variety under this action. The variety $\mathcal{N}_n$ is the union of finitely many orbits.

A nilpotent element (matrix) in $\mathfrak{g}$ is called *regular* if it is conjugated with the principal Jordan normal form denoted by x_{reg}. Then every regular matrix is nonderogatory in the sense of [Gu]. Furthermore, each element of the form $x + kI_n$ with x regular is nonderogatory. The regular orbit $\mathcal{O}_{\text{reg}} = G \cdot x_{\text{reg}}$ is dense in the nilpotent cone $\mathcal{N}_n$. We also denote by $z(x)$ the centralizer of x in $\mathfrak{g}$. For later convenience, we write z_{reg} for the centralizer of x_{reg}. It is well-known that $\dim z_{\text{reg}} = n$ and

$$z_{\text{reg}} = kI_n \oplus kx_{\text{reg}} \oplus kx_{\text{reg}}^2 \oplus \cdots \oplus kx_{\text{reg}}^{n-1}$$

as a vector space. Here I_n is the identity n by n matrix. In fact we can consider z_{reg} as the polynomial algebra $k[x_{\text{reg}}]$. It follows that the intersection

$$z_{\text{nilreg}} = z_{\text{reg}} \cap \mathcal{N}_n = kx_{\text{reg}} \oplus \cdots \oplus kx_{\text{reg}}^{n-1} = k\left\langle x_{\text{reg}} \right\rangle$$

as an algebra without a unity. As a remark, our arguments in this note, especially in Subsection 3.3 and 3.4, will heavily make use of non-unity commutative algebra generated by $x_1, \ldots, x_m$. So we fix the notation $k\langle x_1, \ldots, x_m\rangle$ for this algebra.

For each subset V of $\mathfrak{g}^m$ with some $m \geq 1$, we always write $\overline{V}$ for the closure of V in the Zariski topology.

3. Nilpotent commuting r-tuples

In this section we provide some general properties for both $C_r(\mathcal{N}_n)$ and $C_r(\mathfrak{g})$. Most of proofs for the latter variety will be omitted.

3.1. **Easy Theorem.** We first introduce a well-known criterion for varieties $C_r(\mathcal{N}_n)$ and $C_r(\mathfrak{g})$ to be irreducible. Denote by N_n^r (resp. G_n^r) be the closure of the subset of $C_r(\mathcal{N}_n)$ (resp. $C_r(\mathfrak{g})$) where the first matrix is regular (resp. nonderogatory).

Theorem 3.1.1. *Suppose $n, r \geq 2$. The varieties $C_r(\mathcal{N}_n)$ and $C_r(\mathfrak{g})$ is irreducible if and only if*

$$C_r(\mathcal{N}_n) = N_n^r \quad \text{and} \quad C_r(\mathfrak{g}) = G_n^r,$$

which are of dimensions $n^2 - n + (r-1)(n-1)$ and $n^2 + (r-1)n$ respectively.

Proof. We only give a proof for $C_r(\mathcal{N}_n)$ as that for $C_r(\mathfrak{g})$ is very similar and can be deduced from [GS, Proposition 6]. First, one can easily see that N_n^r is the closure of $G \cdot (x_{\mathrm{reg}}, z_{\mathrm{nilreg}}, ..., z_{\mathrm{nilreg}})$. This variety is irreducible as it is the image of the morphism

$$\begin{aligned} G \times z_{\mathrm{reg}}^{r-1} &\to \mathcal{N}_n^r \\ (g, x_1, \ldots, x_{r-1}) &\mapsto g \cdot (x_{\mathrm{reg}}, x_1, \ldots, x_{r-1}). \end{aligned}$$

Conversely, consider the projection to the first factor

$$p : C_r(\mathcal{N}_n) \to \mathcal{N}_n.$$

As the regular orbit $\mathcal{O}_{\mathrm{reg}}$ is an open dense subset of $\mathcal{N}_n$, so is $p^{-1}(\mathcal{O}_{\mathrm{reg}}) = G \cdot (x_{\mathrm{reg}}, z_{\mathrm{nilreg}}, ..., z_{\mathrm{nilreg}})$. As $C_r(\mathcal{N}_n)$ is irreducible, the first equality follows. Then we have

$$\dim C_r(\mathcal{N}_n) = \dim \mathcal{O}_{\mathrm{reg}} + (r-1)\dim(z_{\mathrm{nilreg}}) = n^2 - n + (r-1)(n-1).$$

This completes our proof. □

3.2. We can now establish the connection between the variety of nilpotent commuting r-tuples and that of general commuting r-tuples. We first consider a lemma related to G_n^r and N_n^r.

Lemma 3.2.1. *For each $r \geq 2$, we have $N_n^r \subset G_n^r \cap \mathcal{N}_n^r$. Moreover,*

$$N_n^r + (kI_n)^r \subset G_n^r.$$

Proof. As the open set $\mathcal{O}_{\mathrm{reg}} = \mathcal{N}_n \cap \{\text{nonderogatory elements of } \mathfrak{g}\}$, we then have

$$G \cdot (x_{\mathrm{reg}}, z_{\mathrm{nilreg}}, \ldots, z_{\mathrm{nilreg}}) \subset G_n^r \cap \mathcal{N}_n^r.$$

It follows that $N_n^r \subset G_n^r \cap \mathcal{N}_n^r$. Therefore, $N_n^r + (kI_n)^r \subset G_n^r + (kI_n)^r = G_n^r$ since $x_{\mathrm{reg}} + kI_n$ is nonderogatory. □

Theorem 3.2.2. *Suppose $n \geq 1$ is a number such that $C_r(\mathfrak{gl}_m)$ is irreducible for all $m < n$. If $C_r(\mathcal{N}_n)$ is irreducible then so is $C_r(\mathfrak{gl}_n)$.*

Proof. Consider a tuple $(x_1, \ldots, x_r)$ in $C_r(\mathfrak{gl}_n)$. If there exists an x_ℓ with $1 \le \ell \le r$ which has at least 2 distinct eigenvalues then all x_i can be decomposed into at least 2 blocks of sizes m and $n-m$ with $m < n$. Hence, the assumption implies that $(x_1, \ldots, x_r) \in G_m^r \times G_{n-m}^r$ so that it is in G_n^r.[3] Otherwise, every x_i would be of the following form

$$\lambda_i I_n + y_i$$

where $\lambda_i \in k$ and $y_i \in \mathcal{N}_n$. In other words, each tuple $(x_1, \ldots, x_r)$ then would belong to the variety $C_r(\mathcal{N}_n + kI_n)$. This would therefore imply that

$$C_r(\mathfrak{gl}_n) = G_n^r \cup C_r(\mathcal{N}_n + kI_n). \tag{1}$$

Note in addition that

$$C_r(\mathcal{N}_n + kI_n) = C_r(\mathcal{N}_n) + (kI_n)^r.$$

Hence if $C_r(\mathcal{N}_n)$ is irreducible, then $C_r(\mathcal{N}_n) = N_n^r$ by Theorem 3.1.1. So we must have by Lemma 3.2.1

$$C_r(\mathcal{N}_n + kI_n) = N_n^r + (kI_n)^r \subset G_n^r.$$

Hence (1) implies that $C_r(\mathfrak{gl}_n) = G_n^r$, the irreducibility follows. □

Recall that n_r (for $r > 2$) is the least integer such that $C_r(\mathfrak{gl}_n)$ is reducible. Then the theorem implies that $C_r(\mathcal{N}_{n_r})$ is also reducible. This doe not tell us much about the reducibility of $C_r(\mathcal{N}_n)$, however it does give some information on n'_r as following

Corollary 3.2.3. *Suppose $r > 2$ and let n'_r be the least integer such that $C_r(\mathcal{N}_{n'_r})$ is reducible. Then we have n'_3 is in the interval $[4, 29]$ and $n'_r = 4$ for all $r > 3$.*

Proof. The Theorem 3.2.2 shows that $n'_r \le n_r$. Note that since n_3 is in the interval $[11, 29]$, $n'_3 \le 29$. It is also known that $C_r(\mathcal{N}_n)$ is irreducible for $n \le 3$ and $r \ge 1$. It follows the result for n'_3. The other result also follows from the fact that $n_r = 4$ for all $r \ge 4$. □

3.3. We have shown that $C_r(\mathcal{N}_4)$ is reducible for all $r \ge 4$. Now we would like to extend this result for higher rank n. To do so, we first need to prove below the analogs for results of Motzkin and Taussky, Guralnick and Sethuraman for nilpotent commuting matrices. The first one is the "nilpotent version" of the first author in [Gu, Theorem 1].

Proposition 3.3.1. *Suppose $x, y \in \mathcal{N}_n$ with $[x, y] = 0$. Let $\mathcal{A}_{x,y} = k\langle x, y\rangle$, the algebra generated by x, y. Then $\dim \mathcal{A} \le n-1$.*

Proof. Let take a pair (x, y) satisfying the hypothesis, i.e., $(x, y) \in C_2(\mathcal{N}_n)$. Then we consider the new pair $(x + I_n, y)$ which is in $C_2(\mathfrak{g})$ since adding the identity matrix does not change the commutativity of x with y. Recall from [Gu, Theorem 1] that $\dim \mathcal{A}_{x+I_n,y} \le n$. On the other hand, as I_n is linearly independent with x and y, we must have

$$\dim \mathcal{A}_{x+I_n,y} = \dim \mathcal{A}_{x,y} + 1 \le n.$$

It follows that $\dim \mathcal{A}_{x,y} \le n-1$, which completes our proof. □

In order to fit in our context, a generalization of the above result to r-tuples is necessary. This can be done by adapting the argument in [GS, Theorem 7]. Then we obtain a "nilpotent version" of it as follows.

Theorem 3.3.2. *For every r-tuple $(x_1, \ldots, x_r)$ in N_n^r let $\mathcal{A} = k\langle x_1, \ldots, x_r\rangle$. Then we have $\mathcal{A}$ is contained in an $(n-1)$-dimensional commutative subalgebra of $\mathfrak{g}$. In particular, $\dim \mathcal{A} \le n-1$.*

[3] This argument is slightly modified from the proof of Lemma 2.4 in [H]

Proof. First note that $N_n^r \subset G_n^r \cap \mathcal{N}_n^r$ from Lemma 3.2.1. Then suppose $(x_1, \ldots, x_r) \in N_n^r$, [GS, Proposition 4] implies that $\dim \mathcal{A} \leq n$. In fact the dimension of $\mathcal{A}$ must be less than or equal to $n-1$. For otherwise, assume that $\dim \mathcal{A} = n$, then the algebra associated with $(x_1 + I_n, x_2, \ldots, x_r) \in R$ would have dimension $n+1$ by the same argument in Proposition 3.3.1, which was a contradiction. This completes our theorem. □

Corollary 3.3.3. *If the variety $C_r(\mathcal{N}_n)$ is irreducible then any commutative subalgebra of $\mathfrak{g}$ generated by r elements in $C_r(\mathcal{N}_n)$ has dimension at most $n-1$.*

Proof. Follows immediately from Theorem 3.1.1 and Theorem 3.3.2 above. □

3.4. **The case $n, r \geq 4$.** We now apply our calculations in previous subsection to show the reducibility of $C_r(\mathcal{N}_n)$ for $n, r \geq 4$.

Fix a positive integer n. Let P be the parabolic subgroup of G corresponding to the partition $[m, m]$ if $n = 2m$, or $[m+1, m]$ if $n = 2m+1$. Set $\mathfrak{u}_P$ be the Lie algebra of the unipotent radical of P. More explicitly, every element of $\mathfrak{u}_P$ is of the form

$$\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$$

where 0 is the zero m by m matrix and A is an m by m matrix (or m by $m+1$ matrix if n is odd). Observe that $\mathfrak{u}_P$ is a commutative subalgebra of $\mathfrak{g}$. Moreover, we have

$$\dim \mathfrak{u}_P = \begin{cases} m^2 & \text{if } n = 2m, \\ m(m+1) & \text{if } n = 2m+1. \end{cases}$$

which is greater than $n-1$ if $n \geq 4$. Therefore, we obtain the main result of this section.

Theorem 3.4.1. *The variety $C_r(\mathcal{N}_n)$ is reducible for all $n, r \geq 4$.*

Proof. It immediately follows from Theorem 3.3.2 and the above observation. □

This result not only strengthens the Corollary 3.2.3 earlier but also confirms the statement that the reducibility behavior of the variety of nilpotent commuting matrices is similar to that of the variety of general commuting matrices in the case $n, r \geq 4$. Another analogy between these two varieties we can see is the following

Conjecture 3.4.2. *If $C_r(\mathfrak{gl}_n)$ is irreducible then so is $C_r(\mathcal{N}_n)$.*

There are many evidences for this statement to be true such as when $r = 2$, or $n = 2, 3$ (for which they are both irreducible), or $n, r \geq 4$ (for which they are both reducible). In other words, the only case where this is significant is when $r = 3$. It is obvious that the conjecture follows immediately from the equality $N_n^r = G_n^r \cap \mathcal{N}_n^r$ for each $n, r \geq 2$. One inclusion was shown in Lemma 3.2.1, we claim that the other one is also true.

4. Lower bound for the dimension of commuting varieties

In this section we explore further on the dimensions of $C_r(\mathcal{N}_n)$ and $C_r(\mathfrak{g})$. Recall from the Theorem 3.1.1 that the lower bound for $\dim C_r(\mathcal{N}_n)$ is $n^2 - n + (r-1)(n-1)$ and for $C_r(\mathfrak{g})$ is $n^2 + (r-1)n$. We shall increase these bounds for the case when n and r are sufficiently large.

4.1. We first do so for $C_r(\mathcal{N}_n)$. Let $V_P = G \cdot \mathfrak{u}_P^r$. By Lemma 8.7(c) in [Jan], we know that V_P is a closed variety of $\mathfrak{g}^r$. Moreover, the variety V_P is a subset of $C_r(\mathcal{N}_n)$ since $\mathfrak{u}_P^r$ is commutative.

We begin with the dimension of this variety.

Proposition 4.1.1. *For each $r \geq 1$, we have*

$$\dim V_P = \begin{cases} (r+1)m^2 & \text{if } n = 2m, \\ (r+1)m(m+1) & \text{if } n = 2m+1. \end{cases}$$

Proof. Note first that V_P is irreducible as the moment morphism $G \times \mathfrak{u}_P^r \to G \cdot \mathfrak{u}_P^r$ is surjective. Now we consider the projection

$$p : G \cdot \mathfrak{u}_P^r \to V_P$$

Let $\mathcal{O} = G \cdot v$ be the Richardson orbit corresponding to P. Then we have $\mathcal{O}$ is an open dense subset of $G \cdot \mathfrak{u}_P$ [Jan, Lemma 8.8(a)]. Combining with the formula (2) in [Jan, 8.8], we obtain that

$$\dim G \cdot \mathfrak{u}_P = \dim(\mathcal{O}) = \dim(G/P) + \dim \mathfrak{u}_P^r = 2 \dim \mathfrak{u}_P^r. \tag{2}$$

As each fiber $p^{-1}(g \cdot v) = g(v, \mathfrak{u}_P, \ldots, \mathfrak{u}_P) \cong \mathfrak{u}_P^{r-1}$ with $g \in G$ is of dimension $(r-1)\dim \mathfrak{u}_P$, we have

$$\dim G \cdot \mathfrak{u}_P^r = \dim G \cdot \mathfrak{u}_P + \dim p^{-1}(g \cdot v) = (r+1) \dim \mathfrak{u}_P^r$$

Finally the result follows from the dimension of $\mathfrak{u}_P$. □

The computation above implies that the dimension of V_P is greater than that of N_n^r for sufficiently large n and r. In particular, we have the following.

Theorem 4.1.2. *If $n > 3$ and $r \geq 6$, or $n > 7$ and $r > 3$, then we have $\dim V_P > \dim N_n^r$.*

Proof. We prove by contradiction. From Propositions 3.1.1 and 4.1.1, it is equivalent to setting up

$$\frac{(r+1)n^2}{4} \leq n^2 - n + (r-1)(n-1) \quad \text{if } n \text{ is even}$$

or

$$\frac{(r+1)(n^2-1)}{4} \leq n^2 - n + (r-1)(n-1) \quad \text{if } n \text{ is odd.}$$

The last two inequalities can be rewritten as follows

$$r \leq \frac{3n^2 - 8n + 4}{(n-2)^2} = 3 + \frac{4}{n-2} \quad \text{if } n \text{ is even,}$$
$$r \leq \frac{3n^2 - 8n + 5}{n^2 - 4n + 3} = 3 + \frac{4}{n-3} \quad \text{if } n \text{ is odd.}$$

It is not true if $n > 3$ and $r > 5$, or $n > 7$ and $r > 3$. □

Corollary 4.1.3. *For every $n, r \geq 2$, we have $dimC_r(\mathcal{N}_n) \geq (r+1)\dim \mathfrak{u}_P$.*

For most values of n and r, we claim that V_P is an irreducible component of $C_r(\mathcal{N}_n)$.

Conjecture 4.1.4. *The variety V_P is an irreducible component of $C_r(\mathcal{N}_n)$ for all $n, r \geq 4$.*

4.2. Now the new lower bound for $\dim C_r(\mathfrak{g})$ can be obtained by slightly modifying the arguments in previous subsection. Indeed, let $\mathfrak{b}_P = \mathfrak{u}_P + kI_n$ and then we have $G \cdot \mathfrak{b}_P = G \cdot \mathfrak{u}_P + kI_n$ so $\dim G \cdot \mathfrak{b}_P = 2 \dim \mathfrak{u}_P + 1$ by (2). Consider $V = G \cdot \mathfrak{b}_P^r$. The theorem on dimension of fibers then gives us

$$\dim V = \dim G \cdot \mathfrak{b}_P + \dim \mathfrak{b}_P^{r-1} = (r+1)\dim \mathfrak{u}_P + r.$$

This establishes a new lower bound for the dimension of $C_r(\mathfrak{g})$ as follows.

Theorem 4.2.1. *The dimension of $C_r(\mathfrak{gl}_n)$ is $\geq (r+1)\dim \mathfrak{u}_P + r$.*

As analyzing earlier in Theorem 4.1.2, this number becomes significant, i.e., it is greater than the old lower bound $n^2 + (r-1)n$, when $n \geq 4$ and $r \geq 9$, or $n \geq 12$ and $r \geq 4$. One should note that there is no "good" upper bounds for the dimensions of both $C_r(\mathcal{N}_n)$ and $C_r(\mathfrak{gl}_n)$ when $r \geq 3$. So finding such numbers would be an interesting problem.

5. Nilpotent commuting triples

Recall that n_3' is the least integer such that $C_3(\mathcal{N}_{n_3'})$ is reducible. Similar to the story for the variety of commuting triple of matrices [GS][HO][H][Si], determining n_3' is a non-trivial problem. We prove in this section that the upper bound of n_3' could be less than 16. Our method and ingredients are mainly from [Gu]. So we first review some notation in that paper.

5.1. For each positive integer s. Let v be a $4s \times 4s$ matrix defined by

$$\begin{pmatrix} 0 & I_s & 0 & 0 \\ 0 & 0 & 0 & I_s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where 0 is the s by s zero matrix. Then we have $z(v)$ is of dimension $6s^2$ and contains all $(4s \times 4s)$-matrices of the form

$$\begin{pmatrix} 0 & A_1 & A_2 & A_3 \\ 0 & 0 & 0 & A_1 \\ 0 & 0 & 0 & A_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $A_i \in \mathfrak{gl}_s$. Set Γ is the set of all matrices of this form. As Γ is a linear affine space of dimension $4s^2$ and the commuting condition of a pair in Γ^2 is defined by s^2 equations, the dimension $\dim C_2(\Gamma) \geq 7s^2$.

5.2. Here comes the main result of this section.

Theorem 5.2.1. *If $n = 4s \geq 16$ then $C_3(\mathcal{N}_n)$ is reducible.*

Proof. It is obvious that v is a nilpotent matrix and $z(v) \cap \mathcal{N}_n$ contains Γ, so that $C_2(\Gamma) \subset C_2(z(v) \cap \mathcal{N}_n)$. It follows that

$$\dim C_3(\mathcal{N}_n) \geq \dim \overline{G \cdot (v, C_2(z(v) \cap \mathcal{N}_n))} \geq \dim \overline{G \cdot (v, C_2(\Gamma))}.$$

Moreover, use the Theorem for dimension of fiber we obtain that

$$\dim \overline{G \cdot (v, C_2(\Gamma))} \geq \dim G \cdot v + \dim C_2(\Gamma) \geq n^2 - 6s^2 + 7s^2 = n^2 + s^2.$$

Now suppose that $C_3(\mathcal{N}_n)$ is irreducible then we must have by Theorem 3.1.1

$$C_3(\mathcal{N}_n) = N_n^3$$

which implies that $\dim C_3(\mathcal{N}_n) = n^2 + n - 2$. Finally, we have

$$n^2 + n - 2 \geq n^2 + s^2 \Rightarrow s^2 - 4s + 2 \leq 0.$$

so that $s \leq 3$. Therefore, the variety $C_3(\mathcal{N}_n)$ must be reducible when $s \geq 4$, i.e., $n \geq 16$. $\square$

Remark 5.2.2. The theorem implies that $n_3' \leq 16$, which significantly improves the result in Corollary 3.2.3.

5.3. **Open problems.** It seems to be doable for verifying the following statements.

Conjecture 5.3.1. *If $C_r(\mathcal{N}_n)$ is reducible then so is $C_r(\mathcal{N}_{n+1})$.*

An argument for this claim will affirm the following.

Conjecture 5.3.2. *The variety $C_3(\mathcal{N}_n)$ is reducible for all $n \geq 16$.*

References

[B] R. Basili, *On the irreducibility of commuting varieties of nilpotent matrices*, J. Pure Appl. Algebra, **149** (2000), 107–120.

[Ba] V. Baranovsky, *The varieties of pairs of commuting nilpotent matrices is irreducible*, Transform. Groups, **6** (2001), 3–8.

[BI] R. Basili and A. Iarrobino, *Pairs of commuting nilpotent matrices, and Hilbert function*, J. Algebra, **320** (2008), 1235–1254.

[G] M. Gerstenhaber, *On dominance and varieties of commuting Matrices*, Ann. Math., **73** (1961), 324–348.

[Gu] R. M. Guralnick, *A note on commuting pairs of matrices*, Linear and Multilinear Alg., **31** (1992), 71–75.

[GS] R. M. Guralnick and B. A. Sethuraman,*Commuting pairs and triples of matrices and related varieties* , Linear Algebra Appl., **310** (2000), 139–148.

[H] Y. Han, *Commuting triples of matrices*, ELA. Electronis J. of Lin. Alg., **13** (2005), 274–343.

[HO] J. Holbrook and M. Omladic, *Approximating commuting operators*, Linear Alg. and its App., **327** (2001), 131–149.

[Jan] J. C. Jantzen and K-H. Neeb, *Lie Theory: Lie Algebras and Representations*, Progress in Mathematics, Birkhauser, **228** 2004.

[L] P. Levy, *Commuting varieties of Lie algebras over fields of prime characteristic*, J. Algebra, **250** (2002), 473–484.

[MT] T. Motzkin and O. Taussky, *Pairs of matrices with property L. II*, Trans. AMS., **80** (1955), 387–401.

[N] N. V. Ngo, *Commuting varieties of r-tuples over Lie algebras*, submitted.

[Pr] A. Premet, *Nilpotent commuting varieties of reductive Lie algebras*, Invent. Math., **154** (2003), 653–683.

[R] R. W. Richardson, *Commuting varieties of semisimple Lie algebras and algebraic groups*, Comp. Math., **38** (1979), 311–327.

[SFB] A. Suslin, E. M. Friedlander, and C. P. Bendel, *Infinitesimal 1-parameter subgroups and cohomology*, J. Amer. Math. Soc, **10** (1997), 693–728.

[Si] K. Šivic, *On varieties of commuting triples III*, Linear Alg. and its App., **437** (2012), 393–460.

[Ya] O. Yakimova, *Surprising properties of centralisers in classical Lie algebras*, Annales de l'institut Fourier, **59** (2009), 903–935.

[Yo] H. Young, *Components of Algebraic Sets of Commuting and Nearly Commuting Matrices*, Ph.D. Thesis at the University of Michigan (2010).

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