THE HARDY SPACE H^1 IN THE RATIONAL DUNKL SETTING

JEAN-PHILIPPE ANKER, NÉJIB BEN SALEM, JACEK DZIUBAŃSKI, AND NABILA HAMDA

Abstract. This paper consists in a first study of the Hardy space H^1 in the rational Dunkl setting. Following Uchiyama's approach, we characterizee H^1 atomically and by means of the heat maximal operator. We also obtain a Fourier multiplier theorem for H^1. These results are proved here in the one-dimensional case and in the product case.

1. Introduction

Dunkl theory is a far reaching generalization of Euclidean Fourier analysis, which includes most special functions related to root systems, such as spherical functions on Riemannian symmetric spaces. It started in the late eighties with Dunkl's seminal article [7] and developed extensively afterwards. We refer to the lecture notes [18] for the rational Dunkl theory, to the lecture notes [15] for the trigonometric Dunkl theory, and to the books [4, 11] for the generalized quantum theories.

This paper deals with the real Hardy space H^1 in the rational Dunkl setting, where the underlying space is of homogeneous type in the sense of Coifman-Weiss. In such a setting, the theory of Hardy spaces goes back to the seventies [6, 12]. Here we follow Uchyama's approach [25] and we characterize the Hardy space H^1 in two ways, by means of the heat maximal operator and atomically. The first characterization, which requires precise heat kernel estimates, has lead us to a seemingly new observation, namely that the heat kernel has a rather slow decay in certain directions and is in particular not Gaussian in the present setting (see Remark 2.4). The second characterization is used to prove a Fourier multiplier theorem for H^1.

Throughout the paper we shall restrict to the one-dimensional case and to the product case. This restriction is due to our present lack of knowledge in general about the behavior of the Dunkl kernel on the one hand and about generalized translations on the other hand.

After this informal introduction, let us introduce some notation and state our main results. On $\mathbb{R}^n$ we consider the Dunkl operators

$$D_j f(\mathbf{x}) = \tfrac{\partial}{\partial x_j} f(\mathbf{x}) + \tfrac{k_j}{x_j}\big[f(\mathbf{x}) - f(\sigma_j \mathbf{x})\big] \qquad (j=1,2,\dots,n)$$

associated with the reflections

$$\sigma_j(x_1,x_2,\dots,x_j,\dots,x_n) = (x_1,x_2,\dots,-x_j,\dots,x_n) \tag{1.1}$$

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and the multiplicities $k_j \geq 0$. Their joint eigenfunctions constitute the Dunkl kernel

$$\mathbf{E}(\mathbf{x},\mathbf{y}) = \prod\nolimits_{j=1}^{n} E_{k_j}(x_j, y_j)\,, \tag{1.2}$$

where

$$\begin{aligned} E_k(x,y) &= \tfrac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\,\Gamma(\frac{1}{2})} \int_{-1}^{+1} du\,(1-u)^{k-1}(1+u)^k\, e^{\,xyu} \\ &= e^{\,xy} \underbrace{\tfrac{\Gamma(2k+1)}{\Gamma(k)\,\Gamma(k+1)} \int_0^1 dv\, v^{k-1}(1-v)^k\, e^{-2xyv}}_{{}_1F_1(k;\,2k+1;\,-2\,xy)} \end{aligned} \tag{1.3}$$

(see for instance [18, Example 2.34]). Here ${}_1F_1(a;b;z)$ is the confluent hypergeometric function, which is also known as the Kummer function and denoted by $M(a,b,z)$. Notice that $\mathbf{E}(\mathbf{x},\mathbf{y}) = e^{\langle\mathbf{x},\mathbf{y}\rangle}$ if all multiplicities k_j vanish.

Let us first define the Hardy space H^1 by means of the heat maximal operator. The Dunkl laplacian

$$\mathbf{L}f(\mathbf{x}) = \sum\nolimits_{j=1}^{n} D_j^2 f(\mathbf{x}) = \sum\nolimits_{j=1}^{n} \Big\{ \big(\tfrac{\partial}{\partial x_j}\big)^2 f(\mathbf{x}) + \tfrac{2k_j}{x_j}\tfrac{\partial}{\partial x_j} f(\mathbf{x}) - \tfrac{k_j}{x_j^2}\big[f(\mathbf{x}) - f(\sigma_j \mathbf{x})\big] \Big\}$$

is the infinitesimal generator of the heat semigroup

$$e^{\,t\mathbf{L}} \qquad (t>0),$$

which acts by linear self-adjoint operators on $L^2(\mathbb{R}^n, d\boldsymbol{\mu})$ and by linear contractions on $L^p(\mathbb{R}^n, d\boldsymbol{\mu})$, for every $1 \leq p \leq \infty$, where

$$d\boldsymbol{\mu}(\mathbf{x}) = d\mu_1(x_1) \ldots d\mu_n(x_n) = |x_1|^{2k_1} \ldots |x_n|^{2k_n}\, dx_1 \ldots dx_n \tag{1.4}$$

The heat semigroup consists of integral operators

$$e^{\,t\mathbf{L}} f(\mathbf{x}) = \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{y})\, \mathbf{h}_t(\mathbf{x},\mathbf{y})\, f(\mathbf{y})$$

associated with the heat kernel [17]

$$\mathbf{h}_t(\mathbf{x},\mathbf{y}) = \mathbf{c}_{\mathbf{k}}^{-1}\, t^{-\frac{\mathbf{N}}{2}}\, e^{-\frac{|\mathbf{x}|^2+|\mathbf{y}|^2}{4\,t}}\, \mathbf{E}\big(\tfrac{\mathbf{x}}{\sqrt{2t}}, \tfrac{\mathbf{y}}{\sqrt{2t}}\big)\,, \tag{1.5}$$

where

$$\mathbf{N} = n + \sum\nolimits_{j=1}^{n} 2\,k_j \tag{1.6}$$

is the homogeneous dimension and

$$\mathbf{c}_{\mathbf{k}} \,=\, 2^{\frac{\mathbf{N}}{2}} \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\, e^{-\frac{|\mathbf{x}|^2}{2}} \,=\, 2^{\mathbf{N}} \prod\nolimits_{j=1}^{n} \Gamma(k_j + \tfrac{1}{2})\,.$$

From this point of view, the Hardy space H^1 consists of all functions $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ whose maximal heat transform

$$\mathbf{h}_* f(\mathbf{x}) = \sup\nolimits_{t>0} \Big| \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{y})\, \mathbf{h}_t(\mathbf{x},\mathbf{y})\, f(\mathbf{y}) \Big| \tag{1.7}$$

belongs to $L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ and the norm is given by

$$\|f\|_{H^1} = \|\mathbf{h}_* f\|_{L^1}\,.$$

Let us turn next to the atomic definition of the Hardy space H^1. Notice that $\mathbb{R}^n$, equipped with the Euclidean distance $d(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|$ and with the measure $\boldsymbol{\mu}$, is a space of homogeneous type in the sense of Coifman-Weiss (see Appendix A). Recall that an atom is a measurable function $a : \mathbb{R}^n \to \mathbb{C}$ such that

- a is supported in a ball B,
- $\|a\|_{L^\infty} \lesssim \boldsymbol{\mu}(B)^{-1}$,
- $\displaystyle\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\, a(\mathbf{x}) = 0$.

By definition, the atomic Hardy space H^1_{atom} consists of all functions $f \in L^1(\mathbb{R}^n, d\boldsymbol{\mu})$ which can be written as $f = \sum_\ell \lambda_\ell\, a_\ell$, where the a_ℓ's are atoms and $\sum_\ell |\lambda_\ell| < +\infty$, and the norm is given by

$$\|f\|_{H^1_{\text{atom}}} = \inf \sum\nolimits_\ell |\lambda_\ell|\,,$$

where the infimum is taken over all atomic decompositions of f.

Our first main result is the following theorem.

Theorem 1.8. *The spaces H^1 and H^1_{atom} coincide and their norms are equivalent, i.e., there exists a constant $C > 0$ such that*

$$C^{-1}\, \|f\|_{H^1} \le \|f\|_{H^1_{\text{atom}}} \le C\, \|f\|_{H^1}\,.$$

The Fourier transform in the Dunkl setting is given by

$$\mathcal{F}f(\boldsymbol{\xi}) = \mathbf{c}_{\mathbf{k}}^{-1} \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\, f(\mathbf{x})\, \mathbf{E}(\mathbf{x}, -i\boldsymbol{\xi})\,. \tag{1.9}$$

It is an isometric isomorphism of $L^2(\mathbb{R}^n, d\boldsymbol{\mu})$ onto itself and the inversion formula reads

$$f(\mathbf{x}) = \mathcal{F}^2 f(-\mathbf{x})\,.$$

Notice that, if all multiplicities k_j vanish, then (1.9) boils down to the classical Fourier transform

$$\widehat{f}(\boldsymbol{\xi}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} d\mathbf{x}\, f(\mathbf{x})\, e^{-i\langle \mathbf{x}, \boldsymbol{\xi}\rangle}\,.$$

Our second main result is the following Hörmander type multiplier theorem (see [10] for the original multiplier theorem on L^p spaces).

Theorem 1.10. *Let $\chi = \chi(\boldsymbol{\xi})$ be a smooth radial function on $\mathbb{R}^n$ such that*

$$\chi(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } |\boldsymbol{\xi}| \in \left[\frac{1}{2}, 2\right], \\ 0 & \text{if } |\boldsymbol{\xi}| \notin \left(\frac{1}{4}, 4\right). \end{cases}$$

If a function $m = m(\boldsymbol{\xi})$ on $\mathbb{R}^n$ satisfies

$$M = \sup\nolimits_{t>0} \|\, \chi\, m(t\,.\,)\|_{W_2^{\mathbf{N}/2+\varepsilon}} < +\infty\,,$$

for some $\varepsilon > 0$, then the multiplier operator

$$\mathcal{T}_m f = \mathcal{F}^{-1}\{m\,(\mathcal{F}f)\}$$

is bounded on the Hardy space H^1 and

$$\|\, \mathcal{T}_m \,\|_{H^1 \to H^1} \lesssim M\,.$$

Here $W_2^\sigma(\mathbb{R}^n)$ denotes the classical L^2 Sobolev space on $\mathbb{R}^n$, whose norm is given by

$$\|g\|_{W_2^\sigma} = \Big\{ \int_{\mathbb{R}^n} d\mathbf{x}\, (1+|\mathbf{x}|^2)^\sigma\, |\widehat{g}(\mathbf{x})|^2 \Big\}^{1/2}\,.$$

Notice that the multiplier m is continuous and bounded, as $\frac{\mathbf{N}}{2} + \varepsilon > \frac{n}{2}$.

The theory of classical real Hardy spaces in $\mathbb{R}^n$ originates from the study of holomorphic functions of one variable in the upper half-plane. We refer the reader to the original works of Stein-Weiss [22], Burkholder-Gundy-Silverstein [3] and Fefferman-Stein [9]. An important contribution to this theory lies in the atomic decomposition introduced by Coifman [5] and extended to spaces of homogeneous type by Coifman-Weiss [6] (see also [12]). More information can be found in the book [21] and references therein.

Our paper is organized as follows. Section 2 is devoted to the heat kernel in dimension 1. There we analyze its behavior thoroughly and we remove a small part, in order to get Gaussian estimates similar to the Euclidean setting. These results are extended to the product case in Section 3. Section 4 is devoted to the proof of Theorem 1.8 and Section 5 to the proof of Theorem 1.10. Section 6 consists of 3 appendices. Appendix A contains information about the measure of balls, which is used throughout the paper. Appendices B and C are devoted to so-called folklore results in connection with Uchiyama's Theorem, which have been used for instance in [8].

This paper results from two independent research works, which were carried out by the first and third authors, respectively by the second and fourth authors, and which have been merged into a joint article.

2. Heat kernel estimates in dimension 1

Consider first the one-dimensional Dunkl kernel $E(x,y)=E_k(x,y)$. As the case $k=0$ is trivial, we may assume that $k>0$.

Lemma 2.1. (a) *$E(x,y)$ is a holomorphic function of $(x,y)\in\mathbb{C}^2$.*

(b) *$E(x,y)>0$ for every $x,y\in\mathbb{R}$.*

(c) *$E(x,y)$ has the following symmetry and rescaling properties:*

$$\begin{cases} E(x,y)=E(y,x) & \forall\, x,y\in\mathbb{C},\\ E(\lambda x,y)=E(x,\lambda y) & \forall\, \lambda,x,y\in\mathbb{C}. \end{cases}$$

(d) *For every $y\in\mathbb{C}$, $x\mapsto E(x,y)$ is an eigenfunction of the Dunkl operator*

$$Df(x)=f'(x)+\tfrac{k}{x}\,\{f(x)-f(-x)\}$$

and of the Dunkl laplacian

$$Lf(x)=D^2f(x)=f''(x)+\tfrac{2k}{x}\,f'(x)-\tfrac{k}{x^2}\,\{f(x)-f(-x)\}\,.$$

More precisely

$$D_xE(x,y)=y\,E(x,y)\quad\textit{and}\quad L_xE(x,y)=y^2\,E(x,y)\,.$$

(e) *As $xy\to 0$,*

$$E(x,y)=1+\mathrm{O}(|xy|)\,.$$

(f) *As $xy\to+\infty$,*

$$E(x,y)=\tfrac{2^k\,\Gamma(k+\frac12)}{\sqrt{\pi}}\,e^{xy}\,(xy)^{-k}\,\big\{1-\tfrac{k^2}{2}\tfrac{1}{xy}+\mathrm{O}\big(\tfrac{1}{x^2y^2}\big)\big\}\,.$$

(g) *As $xy\to-\infty$,*

$$E(x,y)=\tfrac{2^{k-1}\,k\,\Gamma(k+\frac12)}{\sqrt{\pi}}\,e^{-xy}\,(-xy)^{-k-1}\,\big\{1+\tfrac{k^2-1}{2}\tfrac{1}{xy}+\mathrm{O}\big(\tfrac{1}{x^2y^2}\big)\big\}\,.$$

Proof. The first four properties are known to hold in general. In dimension 1, they can be also deduced from the explicit expression (1.3), as does (e). As already observed in [20, Section 2] (see also [18, Example 5.1]), the asymptotics of $E(x,y)$ at infinity follow from the asymptotics of the confluent hypergeometric function, which read, let say for $0<a<b$,

$${}_1F_1(a;b;z)\,\sim\,\tfrac{\Gamma(b)}{\Gamma(a)}\,e^{\,z}\,z^{a-b}\,\sum\nolimits_{\ell=0}^{+\infty}\tfrac{(1-a)_\ell\,(b-a)_\ell}{\ell\,!}\,z^{-\ell}$$

as $z\to+\infty$ and

$$ {}_1F_1(a;b;z)\sim\frac{\Gamma(b)}{\Gamma(b-a)}\,|z|^{-a}\sum\nolimits_{\ell=0}^{+\infty}\frac{(a)_\ell\,(a-b+1)_\ell}{\ell\,!}\,|z|^{-\ell} $$

as $z\to-\infty$ (see for instance [1, (13.5.1)] or [14, (13.7.2)]). □

Consider next the one-dimensional heat kernel

$$ h_t(x,y)=c_k^{-1}\,t^{-k-\frac12}\,e^{-\frac{x^2+y^2}{4t}}\,E\big(\tfrac{x}{\sqrt{2t}},\tfrac{y}{\sqrt{2t}}\big)=c_k^{-1}\,t^{-k-\frac12}\,e^{-\frac{(x-y)^2}{4t}}\,{}_1F_1(k;2k+1;-\tfrac{xy}{t}), \tag{2.2} $$

where $c_k=2^{2k+1}\,\Gamma(k+\frac12)$.

Proposition 2.3. (a) *$h_t(x,y)$ is a C^∞ function of $(t,x,y)\in(0,+\infty)\times\mathbb{R}^2$.*

(b) *$h_t(x,y)>0$ for every $t>0$ and $x,y\in\mathbb{R}$.*

(c) *$h_t(x,y)$ has the following symmetry and rescaling properties:*

$$ \begin{cases} h_t(x,y)=h_t(y,x) & \forall\,x,y\in\mathbb{R},\\ h_{\lambda^2t}(\lambda x,\lambda y)=|\lambda|^{-2k-1}\,h_t(x,y) & \forall\,\lambda\in\mathbb{R}^*,\ \forall\,t>0,\ \forall\,x,y\in\mathbb{R}. \end{cases} $$

(d) *$h_t(x,y)$ satisfies the heat equation*

$$ \begin{cases} \partial_t h_t(x,y)=L_y h_t(x,y),\\ \lim_{t\searrow0} h_t(x,y)\,|y|^{2k}dy=\delta_x(y). \end{cases} $$

(e) *The heat kernel has the following global behavior:*

$$ h_t(x,y)\asymp\begin{cases} t^{-k-\frac12}\,e^{-\frac{x^2+y^2}{4t}} & \text{if } |xy|\le t,\\ t^{-\frac12}\,(xy)^{-k}\,e^{-\frac{(x-y)^2}{4t}} & \text{if } xy\ge t,\\ t^{\frac12}\,(-xy)^{-k-1}\,e^{-\frac{(x+y)^2}{4t}} & \text{if } -xy\ge t, \end{cases} $$

and the following asymptotics:

$$ h_t(x,y)=\begin{cases} c_k^{-1}\,t^{-k-\frac12}\,e^{-\frac{x^2+y^2}{4t}}\,\big\{1+\mathrm{O}\big(\frac{|xy|}{t}\big)\big\} & \text{if } \frac{xy}{t}\to0,\\ \frac{1}{2\sqrt\pi}\,e^{-\frac{(x-y)^2}{4t}}\,t^{-\frac12}\,(xy)^{-k}\,\big\{1-k^2\,\frac{t}{xy}+\mathrm{O}\big(\frac{t^2}{x^2y^2}\big)\big\} & \text{if } \frac{xy}{t}\to+\infty,\\ \frac{k}{2\sqrt\pi}\,e^{-\frac{(x+y)^2}{4t}}\,t^{\frac12}\,(-xy)^{-k-1}\,\big\{1+\mathrm{O}\big(-\frac{t}{xy}\big)\big\} & \text{if } \frac{xy}{t}\to-\infty. \end{cases} $$

(f) *The following gradient estimates hold for the heat kernel:*

$$ \big|\tfrac{\partial}{\partial y}h_t(x,y)\big|\lesssim\begin{cases} t^{-k-\frac32}\,(|x|+|y|)\,e^{-\frac{x^2+y^2}{4t}} & \text{if } |xy|\le t,\\ \big\{t^{-\frac32}\,|x-y|+t^{-\frac12}\,|y|^{-1}\big\}\,(xy)^{-k}\,e^{-\frac{(x-y)^2}{4t}} & \text{if } xy\ge t,\\ \big\{t^{-\frac12}\,|x+y|+t^{\frac12}\,(|x|^{-1}+|y|^{-1})\big\}\,(-xy)^{-k-1}\,e^{-\frac{(x+y)^2}{4t}} & \text{if } -xy\ge t. \end{cases} $$

Proof. The first five properties follow from the expression (2.2) and from Lemma 2.1. Let us turn to the proof of (f). By differentiating (2.2) with respect to y and by using the well-known formula

$$ \tfrac{d}{dz}\,{}_1F_1(a;b;z)=\tfrac{a}{b}\,{}_1F_1(a+1;b+1;z) $$

(see for instance [1, (13.4.8)] or [14, (13.3.15)]), we get

$$ \tfrac{\partial}{\partial y}h_t(x,y)=c_k^{-1}\,t^{-k-\frac12}\,e^{-\frac{(x-y)^2}{4t}}\,\big\{\tfrac{x-y}{2t}\,{}_1F_1(k;2k+1;-\tfrac{xy}{t})-\tfrac{k}{2k+1}\,\tfrac{x}{t}\,{}_1F_1(k+1;2k+2;-\tfrac{xy}{t})\big\}. $$

We conclude by using again the behavior of the confluent hypergeometric function. □

Remark 2.4. *It follows from Proposition 2.3.(e) and Appendix A that*

$$h_t(x,x) \asymp \mu(B(x,\sqrt{t}\,))^{-1} \quad \textit{and} \quad h_t(x,-x) \asymp \mu(B(x,\sqrt{t}\,))^{-1}\,\tfrac{t}{t+x^2}$$

for every $t>0$ *and* $x\in\mathbb{R}$. *Observe in particular that the heat kernel has no global Gaussian behavior and decays rather slowly in certain directions. This phenomenon is even more striking in the product case* (3.1), *where*

$$\mathbf{h}_t(\mathbf{x},\mathbf{y}) \asymp \boldsymbol{\mu}(B(\mathbf{x},\sqrt{t}\,))^{-1}\,\tfrac{t}{t+|\mathbf{x}-\mathbf{y}|^2}$$

if $t>0$, $\mathbf{x}\in\mathbb{R}^n$ *and* $\mathbf{y}=(-x_1,x_2,\dots,x_n)$.

Let us eventually introduce a variant of the heat kernel with a Gaussian behavior. Given two smooth bump functions χ_1 and χ_2 on $\mathbb{R}$ such that

$$\begin{cases} 0\le\chi_1\le 1, \\ \chi_1=1 \text{ on } \big[-1,+\tfrac12\big], \\ \operatorname{supp}\chi_1\subset\big[-2,+\tfrac23\big], \end{cases} \qquad\text{and}\qquad \begin{cases} 0\le\chi_2\le 1, \\ \chi_2=1 \text{ on } \big[0,+\tfrac12\big], \\ \operatorname{supp}\chi_2\subset\big[-1,+1\big], \end{cases}$$

consider the smooth cutoff function

$$\chi_t(x,y) = \begin{cases} \chi_1\big(\frac{x+y}{x}\big)\,\chi_2\big(\frac{t}{x^2}\big) & \text{if } x\neq 0, \\ 0 & \text{if } x=0, \end{cases}$$

and the truncated heat kernel

$$H_t(x,y) = \{1-\chi_t(x,y)\}\,h_t(x,y) \qquad \forall\,t>0,\ \forall\,x,y\in\mathbb{R}.$$

Remark 2.5. *The truncated heat kernel* $H_t(x,y)$ *inherits the following properties of the heat kernel* $h_t(x,y)$:

(a) *Smoothness*: $H_t(x,y)$ *is a* C^∞ *function of* $(t,x,y)\in(0,+\infty)\times\mathbb{R}^2$.
(b) *Non-negativity*: $H_t(x,y)\ge 0$ *for every* $t>0$ *and* $x,y\in\mathbb{R}$.
(c) *Rescaling*: $H_{\lambda^2 t}(\lambda x,\lambda y)=|\lambda|^{-2k-1}H_t(x,y)$ *for every* $\lambda\in\mathbb{R}^*$, $t>0$ *and* $x,y\in\mathbb{R}$.
(d) *Approximation of identity*: $\lim_{t\searrow 0} H_t(x,y)\,|y|^{2k}\,dy=\delta_x(y)$ *for every* $x,y\in\mathbb{R}$.

Theorem 2.6. *The following estimates hold for the truncated heat kernel* $H_t(x,y)$.

(a) *On-diagonal estimate*:

$$H_t(x,x)\asymp \mu(B(x,\sqrt{t}\,))^{-1} \qquad \forall\,t>0,\ \forall\,x\in\mathbb{R}.$$

(b) *Off-diagonal Gaussian estimate*:

$$0\le H_t(x,y)\lesssim \mu(B(x,\sqrt{t}\,))^{-1}\,e^{-\frac{(x-y)^2}{c\,t}} \qquad \forall\,t>0,\ \forall\,x,y\in\mathbb{R}.$$

(c) *Gradient estimate*:

$$\big|\tfrac{\partial}{\partial y}H_t(x,y)\big|\lesssim t^{-\frac12}\,\mu(B(x,\sqrt{t}\,))^{-1}\,e^{-\frac{(x-y)^2}{c\,t}} \qquad \forall\,t>0,\ \forall\,x,y\in\mathbb{R}.$$

(d) *Lipschitz estimates*:

$$|H_t(x,y)-H_t(x,y')|\lesssim \mu(B(x,\sqrt{t}\,))^{-1}\,\tfrac{|y-y'|}{\sqrt{t}} \qquad \forall\,t>0,\ \forall\,x,y,y'\in\mathbb{R},$$

with the following improvement, if $|y-y'|\le\frac12\,|x-y|$:

$$|H_t(x,y)-H_t(x,y')|\lesssim \mu(B(x,\sqrt{t}\,))^{-1}\,e^{-\frac{(x-y)^2}{c\,t}}\,\tfrac{|y-y'|}{\sqrt{t}}\,.$$

Here c denotes some positive constant and the ball measure has the following behavior, according to Appendix A:

$$\mu(B(x,\sqrt{t})) \asymp \begin{cases} t^{k+\frac{1}{2}} & \textit{if } |x| \le \sqrt{t}, \\ |x|^{2k}\sqrt{t} & \textit{if } |x| \ge \sqrt{t}. \end{cases}$$

Proof. As far as (a), (b), (c) are concerned, the case $x=0$ follows immediately from the previous heat kernel estimates. Thus we may assume that $x \ne 0$ and reduce furthermore to $x=1$ by rescaling.

(a) is immediate:

$$H_t(1,1) = h_t(1,1) \asymp \left\{ \begin{array}{ll} t^{-\frac{1}{2}} & \text{if } t \le 1 \\ t^{-k-\frac{1}{2}} & \text{if } t \ge 1 \end{array} \right\} \asymp \mu(B(1,\sqrt{t}))^{-1}.$$

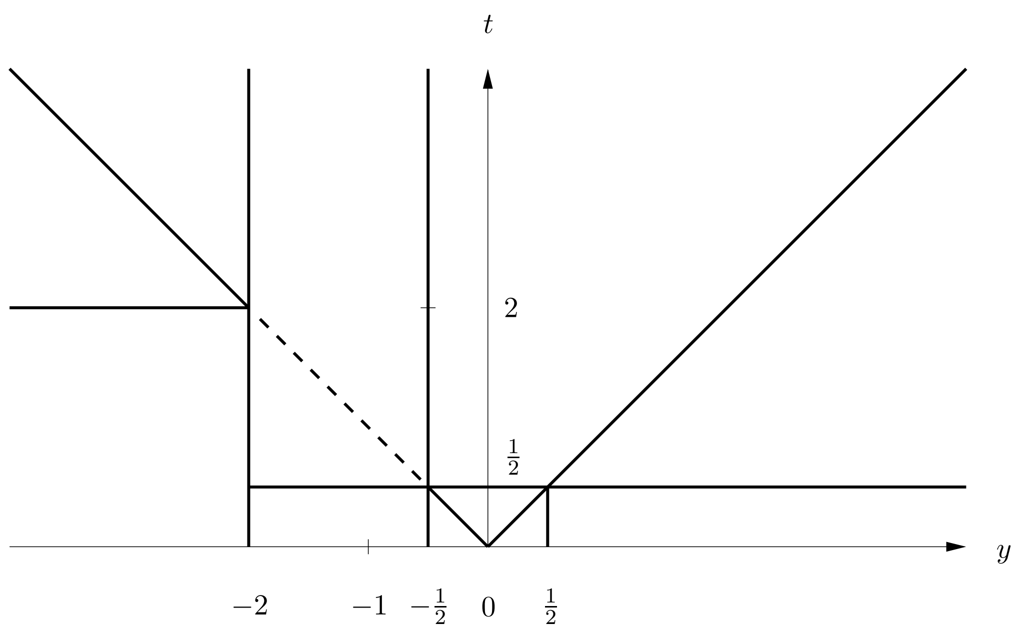

FIGURE 1. Cases and subcases considered in the proofs of (b) and (c)

Let us next prove (b).

- **Case 1.** Assume that $|y| \le t$.

◦ **Subcase 1.1.** Assume that t is bounded above, say $t \le \frac{1}{2}$. Then

$$H_t(1,y) \le h_t(1,y) \asymp t^{-k-\frac{1}{2}}\, e^{-\frac{1+y^2}{4t}} = t^{-\frac{1}{2}}\, e^{-\frac{(1-y)^2}{8t}}\, t^{-k}\, e^{-\frac{1+y^2}{8t}}\, e^{-\frac{y}{4t}}$$

is bounded above by

$$\mu(B(1,\sqrt{t}))^{-1}\, e^{-\frac{(1-y)^2}{8\,t}}$$

as $t^{\frac{1}{2}} \asymp \mu(B(1,\sqrt{t}))$, $t^{-k} \lesssim e^{\frac{1}{8t}} \le e^{\frac{1+y^2}{8t}}$ and $e^{\frac{y}{4t}} \asymp 1$.

◦ **Subcase 1.2.** Assume that t is bounded below, say $t \ge \frac{1}{2}$. Then

$$H_t(1,y) \le h_t(1,y) \asymp t^{-k-\frac{1}{2}}\, e^{-\frac{1+y^2}{4t}} = t^{-k-\frac{1}{2}}\, e^{-\frac{(1-y)^2}{4t}}\, e^{-\frac{y}{2t}}$$

with $t^{k+\frac{1}{2}} \asymp \mu(B(1,\sqrt{t}))$ and $e^{\frac{y}{2t}} \asymp 1$.

• **Case 2.** Assume that y is close to $-x=-1$, say $-2\le y\le -\frac12$.

◦ **Subcase 2.1.** If $t\le\frac12\,(\le -y)$, then

$$H_t(1,y)=0\,.$$

◦ **Subcase 2.2.** If t is bounded below, say $t\ge\frac12$, we argue as in Subcase 1.2.

• **Case 3**. Assume that $y\ge t$.

◦ **Subcase 3.1.** Assume that t is bounded below, say $(y\ge)t\ge\frac12$. Then

$$H_t(1,y)\le h_t(1,y)\asymp t^{-\frac12}\,y^{-k}\,e^{-\frac{(1-y)^2}{4t}}\le t^{-k-\frac12}\,e^{-\frac{(1-y)^2}{4t}}$$

with $t^{k+\frac12}\asymp\mu(B(1,\sqrt{t}\,))$.

◦ **Subcase 3.2.** Assume that $y\ge\frac12\ge t$. Then

$$H_t(1,y)\le h_t(1,y)\asymp t^{-\frac12}\,y^{-k}\,e^{-\frac{(1-y)^2}{4t}}\lesssim t^{-\frac12}\,e^{-\frac{(1-y)^2}{4t}}$$

with $t^{\frac12}\asymp\mu(B(1,\sqrt{t}\,))$.

◦ **Subcase 3.3.** Assume that $t\le y\le\frac12$. Then

$$H_t(1,y)\le h_t(1,y)\asymp t^{-\frac12}\,y^{-k}\,e^{-\frac{(1-y)^2}{4t}}$$

is bounded above by

$$\mu(B(1,\sqrt{t}\,))^{-1}\,e^{-\frac{(1-y)^2}{8\,t}}$$

as $t^{\frac12}\asymp\mu(B(1,\sqrt{t}\,))$ and $y^{-k}\le t^{-k}\lesssim e^{\frac{1}{32\,t}}\le e^{\frac{(1-y)^2}{8t}}$.

• **Case 4**. Assume that $y\le -t\,(<0)$ and that y stays away from -1, say $y\notin\bigl(-2,-\frac12\bigr)$. Notice that $(1+y)^2\ge\frac{(1-y)^2}{9}$ if and only if $y\notin\bigl(-2,-\frac12\bigr)$.

◦ **Subcase 4.1.** Assume that $2\le t\le -y$. Then

$$H_t(1,y)\le h_t(1,y)\asymp t^{\frac12}\,(-y)^{-k-1}\,e^{-\frac{(1+y)^2}{4t}}\le t^{-k-\frac12}\,e^{-\frac{(1-y)^2}{36\,t}}$$

with $t^{k+\frac12}\asymp\mu(B(1,\sqrt{t}\,))$.

◦ **Subcase 4.2.** Assume that $t\le 2\le -y$. Then

$$H_t(1,y)\le h_t(1,y)\asymp t^{\frac12}\,(-y)^{-k-1}\,e^{-\frac{(1+y)^2}{4t}}\lesssim t^{-\frac12}\,e^{-\frac{(1-y)^2}{36\,t}}$$

with $t^{\frac12}\asymp\mu(B(1,\sqrt{t}\,))$.

◦ **Subcase 4.3.** Assume that $t\le -y\le\frac12$. Then

$$H_t(1,y)\le h_t(1,y)\asymp t^{\frac12}\,(-y)^{-k-1}\,e^{-\frac{(1+y)^2}{4t}}\le t^{-k-\frac12}\,e^{-\frac{(1+y)^2}{8t}}\,e^{-\frac{(1-y)^2}{72\,t}}$$

is bounded above by

$$\mu(B(1,\sqrt{t}\,))^{-1}\,e^{-\frac{(1-y)^2}{72\,t}}$$

as $t^{\frac12}\asymp\mu(B(1,\sqrt{t}\,))$ and $t^{-k}\lesssim e^{\frac{1}{32\,t}}\le e^{\frac{(1+y)^2}{8t}}$.

The proof of (c) follows the same pattern. To begin with, observe that the derivative

$$\tfrac{\partial}{\partial y}\bigl\{1-\underbrace{\chi_1(1+y)\,\chi_2(t)}_{\chi_t(1,y)}\bigr\}=-\,\chi_1'(1+y)\,\chi_2(t)$$

of the cut-off is bounded and vanishes unless $y\in\bigl(-3,-2\bigr)\cup\bigl(-\frac12,0\bigr)$ and $t\le1$. According to the subcases 1.1, 4.2 and 4.3 above, the contribution of $\frac{\partial}{\partial y}\{1-\chi_t(1,y)\}\,h_t(1,y)$ to $\frac{\partial}{\partial y}H_t(1,y)$ is bounded by

$$\mu(B(1,\sqrt{t}))^{-1}\,e^{-\frac{(1-y)^2}{c\,t}}\le t^{-\frac12}\,\mu(B(1,\sqrt{t}))^{-1}\,e^{-\frac{(1-y)^2}{c\,t}}$$

Thus it remains for us to estimate the contribution of $\{1-\chi_t(1,y)\}\,\frac{\partial}{\partial y}h_t(1,y)$.

• **Case 1.** Assume that $|y|\le t$.

◦ **Subcase 1.1.** Assume that $t\le\frac12$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\bigl|\tfrac{\partial}{\partial y}h_t(1,y)\bigr|&\lesssim t^{-k-\frac32}(1+|y|)\,e^{-\frac{1+y^2}{4t}}\lesssim \overbrace{t^{-k-\frac12}\,e^{-\frac{1+y^2}{8t}}\,e^{-\frac{y}{4t}}}^{\text{bounded}}\,t^{-1}\,e^{-\frac{(1-y)^2}{8t}}\\&\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}))^{-1}\,e^{-\frac{(1-y)^2}{8t}}\end{aligned}$$

◦ **Subcase 1.2.** Assume that $t\ge\frac12$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\bigl|\tfrac{\partial}{\partial y}h_t(1,y)\bigr|&\lesssim t^{-k-\frac32}(1+|y|)\,e^{-\frac{1+y^2}{4t}}\lesssim t^{-k-1}\,e^{-\frac{(1-y)^2}{8t}}\,\overbrace{\bigl(\tfrac{1+y^2}{t}\bigr)^{\frac12}\,e^{-\frac{1+y^2}{8t}}\,e^{-\frac{y}{4t}}}^{\text{bounded}}\\&\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}))^{-1}\,e^{-\frac{(1-y)^2}{8t}}\end{aligned}$$

• **Case 2.** Assume that $-2\le y\le-\frac12$.

◦ **Subcase 2.1.** If $t\le\frac12\,(\le-y)$, then

$$\{1-\chi_t(1,y)\}\,\tfrac{\partial}{\partial y}h_t(1,y)=0\,.$$

◦ **Subcase 2.2.** If t is bounded below, say $t\ge\frac12$, we argue as in Subcase 1.2.

• **Case 3**. Assume that $y\ge t$.

◦ **Subcase 3.1.** Assume that $(y\ge)t\ge\frac12$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\bigl|\tfrac{\partial}{\partial y}h_t(1,y)\bigr|&\lesssim\bigl\{t^{-\frac32}\,|1-y|+t^{-\frac12}\,y^{-1}\bigr\}\,y^{-k}\,e^{-\frac{(1-y)^2}{4t}}\\&\lesssim t^{-k-1}\,e^{-\frac{(1-y)^2}{8t}}\,\underbrace{\Bigl\{1+\tfrac{|1-y|}{\sqrt{t}}\,e^{-\frac{(1-y)^2}{8t}}\Bigr\}}_{\text{bounded}}\\&\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}))^{-1}\,e^{-\frac{(1-y)^2}{8t}}\end{aligned}$$

◦ **Subcase 3.2.** Assume that $y\ge\frac12\ge t$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\bigl|\tfrac{\partial}{\partial y}h_t(1,y)\bigr|&\lesssim\bigl\{t^{-\frac32}\,|1-y|+t^{-\frac12}\,y^{-1}\bigr\}\,y^{-k}\,e^{-\frac{(1-y)^2}{4t}}\\&\lesssim t^{-1}\,e^{-\frac{(1-y)^2}{8t}}\,\underbrace{\Bigl\{\sqrt{t}+\tfrac{|1-y|}{\sqrt{t}}\,e^{-\frac{(1-y)^2}{8t}}\Bigr\}}_{\text{bounded}}\\&\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}))^{-1}\,e^{-\frac{(1-y)^2}{8t}}\end{aligned}$$

∘ **Subcase 3.3.** Assume that $t\le y\le\frac12$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\big|\tfrac{\partial}{\partial y}h_t(1,y)\big| &\lesssim \big\{t^{-\frac32}|1-y|+t^{-\frac12}y^{-1}\big\}\,y^{-k}\,e^{-\frac{(1-y)^2}{4t}}\\ &\lesssim t^{-1}\,e^{-\frac{(1-y)^2}{8t}}\underbrace{t^{-k-\frac12}\,e^{-\frac{1}{32t}}}_{\text{bounded}}\\ &\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}\,))^{-1}\,e^{-\frac{(1-y)^2}{8t}}\end{aligned}$$

• **Case 4**. Assume that $y\le -t\,(<0)$ and that $y\notin\big(-2,-\frac12\big)$. Recall that $(1+y)^2\ge\frac{(1-y)^2}{9}$ if and only if $y\notin\big(-2,-\frac12\big)$.

∘ **Subcase 4.1.** Assume that $2\le t\le -y$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\big|\tfrac{\partial}{\partial y}h_t(1,y)\big| &\lesssim \big\{t^{-\frac12}|1+y|+t^{\frac12}\tfrac{1+|y|}{|y|}\big\}\,|y|^{-k-1}\,e^{-\frac{(1+y)^2}{4t}}\\ &\lesssim t^{-k-1}\,e^{-\frac{(1+y)^2}{8t}}\underbrace{\tfrac{|1+y|}{\sqrt{t}}\,e^{-\frac{(1+y)^2}{8t}}}_{\text{bounded}}\\ &\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}\,))^{-1}\,e^{-\frac{(1-y)^2}{72\,t}}\end{aligned}$$

∘ **Subcase 4.2.** Assume that $t\le 2\le -y$. Then

$$\begin{aligned}\{1-\chi_t(1,y)\}\,\big|\tfrac{\partial}{\partial y}h_t(1,y)\big| &\lesssim \big\{t^{-\frac12}|1+y|+t^{\frac12}\tfrac{1+|y|}{|y|}\big\}\,|y|^{-k-1}\,e^{-\frac{(1+y)^2}{4t}}\\ &\lesssim t^{-1}\,e^{-\frac{(1+y)^2}{8t}}\underbrace{\Big\{\tfrac{|1+y|}{\sqrt{t}}\,e^{-\frac{(1+y)^2}{8t}}+\sqrt{t}\Big\}}_{\text{bounded}}\\ &\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}\,))^{-1}\,e^{-\frac{(1-y)^2}{72\,t}}\end{aligned}$$

∘ **Subcase 4.3.** Assume that $t\le -y\le\frac12$. Then

$$\begin{aligned}\chi_t(1,y)\,\big|\tfrac{\partial}{\partial y}h_t(1,y)\big| &\lesssim \big\{t^{-\frac12}|1+y|+t^{\frac12}\tfrac{1+|y|}{|y|}\big\}\,|y|^{-k-1}\,e^{-\frac{(1+y)^2}{4t}}\\ &\lesssim t^{-1}\,e^{-\frac{(1+y)^2}{8t}}\underbrace{t^{-k-\frac12}\,e^{-\frac{1}{32t}}}_{\text{bounded}}\\ &\lesssim t^{-\frac12}\,\mu(B(1,\sqrt{t}\,))^{-1}\,e^{-\frac{(1-y)^2}{72\,t}}\end{aligned}$$

Eventually, (d) is an immediate consequence of (c). For every $y''\in[\,y,y'\,]$, we have indeed

$$e^{-\frac{(x-y'')^2}{c\,t}}\le 1\,.$$

Moreover, if $|y-y'|\le\frac12|x-y|$, then $|x-y''|\ge|x-y|-|y-y''|\ge|x-y|-|y-y'|\ge\frac12|x-y|$, hence

$$e^{-\frac{(x-y'')^2}{c\,t}}\le e^{-\frac{(x-y)^2}{4\,c\,t}}\,.$$

□

Remark 2.7. *Contrarily to $h_t(x,y)$, $H_t(x,y)$ is not symmetric in the space variables x,y. Nevertheless, according to the following result, we may replace $\mu(B(x,\sqrt{t}))$ by $\mu(B(y,\sqrt{t}))$ in the estimates (b), (c) and in the second estimate (d).*

Lemma 2.8. *For every $\varepsilon>0$, there exists $C>0$ such that*

$$\tfrac{\mu(B(x,\sqrt{t}))}{\mu(B(y,\sqrt{t}))} \le C\, e^{\,\varepsilon\,\frac{(x-y)^2}{t}} \qquad \forall\, x,y\in\mathbb{R},\ \forall\, t>0.$$

Proof. By rescaling (see Appendix A), we can reduce to the case $t=1$. The estimate

$$\tfrac{\mu(B(x,1))}{\mu(B(y,1))} \lesssim e^{\varepsilon\,(x-y)^2}$$

is obvious if x and y are bounded or if $|x|/|y|$ is bounded from above. In the remaining case, let say when $|x|\ge 1+2|y|$, we have $|x|\le|x-y|+|y|\le|x-y|+\frac{1}{2}|x|$, hence $|x|\le 2\,|x-y|$. Furthermore, as $|x-y|\ge|x|-|y|\ge 1$, we have $|x|\le 2\,(x-y)^2$. Thus

$$\tfrac{\mu(B(x,1))}{\mu(B(y,1))} \lesssim \mu(B(x,1)) \asymp (|x|+1)^{2k} \lesssim e^{\,\frac{\varepsilon}{2}|x|} \lesssim e^{\,\varepsilon\,(x-y)^2}.$$

□

Next proposition, which will be used in the proof of Theorem 1.8, shows that the truncated heat kernel $H_t(x,y)$ captures the main features of the heat kernel $h_t(x,y)$.

Proposition 2.9. *The maximal operator*

$$Q_*f(x) = \sup\nolimits_{t>0}\Big|\int_{\mathbb{R}} d\mu(y)\, Q_t(x,y)\, f(y)\Big|,$$

associated with the error

$$Q_t(x,y) = h_t(x,y) - H_t(x,y) = \chi_t(x,y)\, h_t(x,y) \ge 0\,,$$

is bounded from $L^1(\mathbb{R},d\mu)$ into itself.

Proof. It suffices to check that

$$\sup\nolimits_{y\in\mathbb{R}} \int_{\mathbb{R}} d\mu(x)\, \sup\nolimits_{t>0} Q_t(x,y) < +\infty\,.$$

The case $y=0$ is trivial, as $\chi_t(x,0)$ and hence $Q_t(x,0)$ vanish, for every $t>0$ and $x\in\mathbb{R}$. Consider next the case $y\in\mathbb{R}^*$, which reduces to $y=1$ by rescaling. Then $\chi_t(x,1)$ and $Q_t(x,1)$ vanish, unless $t<9$ and $-3<x<-\frac{1}{3}$, and in this range (see Proposition 2.3)

$$h_t(x,1) \asymp t^{\frac{1}{2}}\, e^{-\frac{(x+1)^2}{4t}}$$

is bounded. Hence

$$\int_{\mathbb{R}} d\mu(x)\, \sup\nolimits_{t>0} Q_t(x,1) \lesssim \int_{-3}^{-\frac{1}{3}} dx\, \sup\nolimits_{0<t<9} h_t(x,1) < +\infty\,.$$

□

3. Heat kernel estimates in the product case

According to (1.5) and (1.2), the heat kernel in the product case splits up into one-dimensional heat kernels:

$$\mathbf{h}_t(\mathbf{x},\mathbf{y}) = \prod\nolimits_{j=1}^{n} h_t^{(j)}(x_j,y_j)\,. \tag{3.1}$$

By expanding

$$h_t^{(j)}(x_j,y_j) = \underbrace{\{1-\chi_t(x_j,y_j)\}\,h_t^{(j)}(x_j,y_j)}_{H_t^{(j)}(x_j,y_j)} + \underbrace{\chi_t(x_j,y_j)\,h_t^{(j)}(x_j,y_j)}_{Q_t^{(j)}(x_j,y_j)},$$

we get

$$\mathbf{h}_t(\mathbf{x},\mathbf{y}) = \mathbf{H}_t(\mathbf{x},\mathbf{y}) + \mathbf{P}_t(\mathbf{x},\mathbf{y}).$$

Here

$$\mathbf{H}_t(\mathbf{x},\mathbf{y}) = \prod\nolimits_{j=1}^{n} H_t^{(j)}(x_j,y_j)$$

and $\mathbf{P}_t(\mathbf{x},\mathbf{y})$ is the sum of all possible products

$$\widetilde{\mathbf{P}}_t(\mathbf{x},\mathbf{y}) = \prod\nolimits_{j=1}^{n} p_t^{(j)}(x_j,y_j)\,,$$

where each factor $p_t^{(j)}(x_j,y_j)$ is equal to $H_t^{(j)}(x_j,y_j)$ or $Q_t^{(j)}(x_j,y_j)$, and at least one factor $p_t^{(j)}(x_j,y_j)$ is equal to $Q_t^{(j)}(x_j,y_j)$. Notice the rescaling property

$$\mathbf{h}_{\lambda^2 t}(\lambda\mathbf{x},\lambda\mathbf{y}) = |\lambda|^{-\mathbf{N}}\,\mathbf{h}_t(\mathbf{x},\mathbf{y}) \qquad \forall\,\lambda\in\mathbb{R}^*,\ \forall\,t>0,\ \forall\,\mathbf{x},\mathbf{y}\in\mathbb{R}^n,$$

and similarly for the other product kernels. The following estimates follow from the one-dimensional case (see Theorem 2.6 and Remark 2.7).

Theorem 3.2. (a) *On-diagonal estimate:*

$$\mathbf{H}_t(\mathbf{x},\mathbf{x}) \asymp \boldsymbol{\mu}(\mathbf{B}(\mathbf{x},\sqrt{t}))^{-1} \qquad \forall\,t>0,\ \forall\,\mathbf{x}\in\mathbb{R}^n.$$

(b) *Off-diagonal Gaussian estimate:*

$$0 \le \mathbf{H}_t(\mathbf{x},\mathbf{y}) \lesssim \max\big\{\mu(\mathbf{B}(\mathbf{x},\sqrt{t})),\mu(\mathbf{B}(\mathbf{y},\sqrt{t}))\big\}^{-1}\,e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}$$

for every $t>0$ and for every $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.

(c) *Gradient estimate:*

$$|\nabla_{\mathbf{y}}\mathbf{H}_t(\mathbf{x},\mathbf{y})| \lesssim t^{-\frac{1}{2}}\max\big\{\mu(\mathbf{B}(\mathbf{x},\sqrt{t})),\mu(\mathbf{B}(\mathbf{y},\sqrt{t}))\big\}^{-1}\,e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}$$

for every $t>0$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.

(d) *Lipschitz estimates:*

$$|\mathbf{H}_t(\mathbf{x},\mathbf{y})-\mathbf{H}_t(\mathbf{x},\mathbf{y}')| \lesssim \boldsymbol{\mu}(\mathbf{B}(\mathbf{x},\sqrt{t}))^{-1}\,\frac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\,,$$

for every $t>0$ and $\mathbf{x},\mathbf{y},\mathbf{y}'\in\mathbb{R}^n$, with the following improvement, if $|\mathbf{y}-\mathbf{y}'|\le\frac{1}{2}|\mathbf{x}-\mathbf{y}|$:

$$|\mathbf{H}_t(\mathbf{x},\mathbf{y})-\mathbf{H}_t(\mathbf{x},\mathbf{y}')| \lesssim \max\big\{\mu(\mathbf{B}(\mathbf{x},\sqrt{t})),\mu(\mathbf{B}(\mathbf{y},\sqrt{t}))\big\}^{-1}\,e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}\,\frac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\,.$$

Let us turn to the analog of Proposition 2.9 in the product case.

Proposition 3.3. *The maximal operator*

$$\mathbf{P}_*f(\mathbf{x}) = \sup\nolimits_{t>0}\Big|\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{y})\,\mathbf{P}_t(\mathbf{x},\mathbf{y})\,f(\mathbf{y})\Big|\,,$$

is bounded from $L^1(\mathbb{R}^n,\boldsymbol{\mu})$ into itself.

Proof. We will show again that

$$\sup_{\mathbf{y}\in\mathbb{R}^n}\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}\mathbf{P}_t(\mathbf{x},\mathbf{y})\,<\,+\infty\,,$$

but the proof will be more involved in the product case than in the one-dimensional case. Let us begin with some observations. First of all, by using the symmetries

$$H_t^{(j)}(x_j,y_j)=H_t^{(j)}(-x_j,-y_j)\quad\text{and}\quad Q_t^{(j)}(x_j,y_j)=Q_t^{(j)}(-x_j,-y_j)$$

and by interchanging variables, we may reduce to products of the form

$$\widetilde{\mathbf{P}}_t(\mathbf{x},\mathbf{y})=\underbrace{Q_t^{(1)}(x_1,y_1)\,\dots\,Q_t^{(n')}(x_{n'},y_{n'})}_{\mathbf{Q}'_t(\mathbf{x}',\mathbf{y}')}\underbrace{H_t^{(n'+1)}(x_{n'+1},y_{n'+1})\,\dots\,H_t^{(n)}(x_n,y_n)}_{\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')}$$

where $1\le n'\le n$ and $0\le y_1\le\ldots\le y_{n'}$. Next we may assume that, for every $1\le j\le n'$,

$$y_j>0\,,\quad -3y_j<x_j<-\tfrac{1}{3}y_j\quad\text{and}\quad x_j^2>t,$$

because otherwise $\chi_t(x_j,y_j)$ and hence $Q_t^{(j)}(x_j,y_j)$ vanish. Eventually, by rescaling, we may reduce to the case $y_1=1$. Consequently, t is bounded by $x_1^2<9\,y_1^2=9$ and each factor $Q_t^{(j)}(x_j,y_j)$ is bounded by

$$t^{\frac12}\,(-x_jy_j)^{-k_j-1}\,e^{-\frac{(x_j+y_j)^2}{4\,t}}\,\mathbb{1}_{\left(-3y_j,-\frac13y_j\right)}(x_j)\,\lesssim\,t^{\frac12}\,y_j^{-2k_j-2}\,\mathbb{1}_{\left(-3y_j,-\frac13y_j\right)}(x_j)\,.$$

Thus, on the one hand, the integral

$$\begin{aligned}\mathbf{I}'(\mathbf{y}')&=\int_{\mathbb{R}^{n'}}d\boldsymbol{\mu}'(\mathbf{x}')\,\sup_{t>0}t^{-\frac{n'}{2}}\,\mathbf{Q}'_t(\mathbf{x}',\mathbf{y}')\\&\lesssim\int_{-3}^{-\frac13}d\mu_1(x_1)\,y_2^{-2k_2-2}\int_{-3y_2}^{-\frac13y_2}d\mu_2(x_2)\,\dots\,y_{n'}^{-2k_{n'}-2}\int_{-3y_{n'}}^{-\frac13y_{n'}}d\mu_{n'}(x_{n'})\end{aligned}$$

is bounded, uniformly in $\mathbf{y}'$. On the other hand, let us prove the uniform boundedness of

$$\mathbf{I}''(\mathbf{y}'')=\int_{\mathbb{R}^{n''}}d\boldsymbol{\mu}''(\mathbf{x}'')\,\sup_{0<t<9}t^{\frac{n'}{2}}\,\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')\,,$$

when $n''=n-n'>0$. For this purpose, let us deduce from the Gaussian estimate

$$\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')\lesssim\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',\sqrt{t}))^{-1}\,e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{c\,t}}$$

that

$$\sup_{0<t<9}t^{\frac{n'}{2}}\,\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')\lesssim|\mathbf{x}''-\mathbf{y}''|\,\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}\,e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{18\,c}}\,.$$

Assume first that $|\mathbf{x}''-\mathbf{y}''|\ge\sqrt{t}$ with $0<t<9$. Then, by using (6.4),

$$\begin{aligned}t^{\frac{n'}{2}}\,\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')&\lesssim t^{\frac{n'}{2}}\,\frac{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))}{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',\sqrt{t}))}\,\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}\,e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{c\,t}}\\&\lesssim|\mathbf{x}''-\mathbf{y}''|\,\underbrace{\Bigl(\tfrac{|\mathbf{x}''-\mathbf{y}''|}{\sqrt{t}}\Bigr)^{\mathbf{N}''}e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{2\,c\,t}}}_{\lesssim1}\,\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}\,e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{18\,c}}\,.\end{aligned}$$

Assume next that $0<|\mathbf{x}''-\mathbf{y}''|\le\sqrt{t}\,(\le 3)$. Then, by using again (6.4),

$$t^{\frac{n'}{2}}\,\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')\lesssim t^{\frac{n'}{2}}\overbrace{\frac{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))}{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',\sqrt{t}))}}^{\lesssim\left(\frac{|\mathbf{x}''-\mathbf{y}''|}{\sqrt{t}}\right)^{n''}}\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}\,\overbrace{e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{c\,t}}}^{\asymp 1}$$
$$\lesssim \underbrace{t^{\frac{n'-1}{2}}}_{\lesssim 1}\underbrace{\left(\frac{|\mathbf{x}''-\mathbf{y}''|}{\sqrt{t}}\right)^{n''-1}}_{\lesssim 1}|\mathbf{x}''-\mathbf{y}''|\,\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}\underbrace{e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{18\,c}}}_{\asymp 1}.$$

Now that we have estimated $t^{\frac{n'}{2}}\mathbf{H}''_t(\mathbf{x}'',\mathbf{y}'')$, let us split up the integral

$$\mathbf{I}''(\mathbf{y}'')=\sum\nolimits_{j\in\mathbb{Z}}\mathbf{I}''_j(\mathbf{y}'')$$

according to the decomposition $\mathbb{R}^{n''}\smallsetminus\{0\}=\bigsqcup_{j\in\mathbb{Z}}\underbrace{\{\mathbf{x}''\in\mathbb{R}^{n''}\,|\,2^{j-\frac12}\le|\mathbf{x}''-\mathbf{y}''|<2^{j+\frac12}\}}_{\Omega_j}$. Let us show that

$$|\,\mathbf{I}''_j(\mathbf{y}'')|\lesssim 2^{-|j|}\,.$$

If $j\ge 0$, we have indeed

$$\mathbf{I}''_j(\mathbf{y}'')\lesssim\int_{\Omega_j}d\boldsymbol{\mu}''(\mathbf{x}'')\underbrace{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}}_{\asymp\,\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',2^j))^{-1}}\underbrace{|\mathbf{x}''-\mathbf{y}''|\,e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{18\,c}}}_{\lesssim 2^{-j}}\lesssim\underbrace{\frac{\boldsymbol{\mu}''(\Omega_j)}{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',2^j))}}_{\lesssim 1}2^{-j}$$

and, if $j\le 0$,

$$\mathbf{I}''_j(\mathbf{y}'')\lesssim\int_{\Omega_j}d\boldsymbol{\mu}''(\mathbf{x}'')\underbrace{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',|\mathbf{x}''-\mathbf{y}''|))^{-1}}_{\asymp\,\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',2^j))^{-1}}\underbrace{|\mathbf{x}''-\mathbf{y}''|}_{\lesssim 2^j}\underbrace{e^{-\frac{|\mathbf{x}''-\mathbf{y}''|^2}{18\,c}}}_{\lesssim 1}\lesssim\underbrace{\frac{\boldsymbol{\mu}''(\Omega_j)}{\boldsymbol{\mu}''(\mathbf{B}(\mathbf{y}'',2^j))}}_{\lesssim 1}2^{j}\,.$$

By summing up over $j\in\mathbb{Z}$, we obtain the uniform boundedness of $\mathbf{I}''(\mathbf{y}'')$. □

4. Proof of Theorem 1.8

Theorem 1.8 relies on the following result due to Uchiyama [25].

Theorem 4.1. *Assume that a set X is equipped with*

- *a* quasi-distance $\widetilde{d}$ *i.e. a distance except that the triangular inequality is replaced by the weaker condition*
$$\widetilde{d}(x,y)\le A\,\{\widetilde{d}(x,z)+\widetilde{d}(z,y)\}\qquad\forall\,x,y,z\in X,$$
- *a measure μ whose values on quasi-balls satisfy*
$$\tfrac{r}{A}\le\mu(\widetilde{B}(x,r))\le r\qquad\forall\,x\in X,\,\forall\,r>0\,,$$
- *a continuous kernel $K_r(x,y)\ge 0$ such that, for every $r>0$ and $x,y,y'\in X$,*
  - $K_r(x,x)\ge\frac{1}{A\,r}\,,$
  - $K_r(x,y)\le r^{-1}\big(1+\frac{\widetilde{d}(x,y)}{r}\big)^{-1-\delta}\,,$
  - $\big|K_r(x,y)-K_r(x,y')\big|\le r^{-1}\big(1+\frac{\widetilde{d}(x,y)}{r}\big)^{-1-2\delta}\big(\frac{\widetilde{d}(y,y')}{r}\big)^{\delta}$ *when* $\widetilde{d}(y,y')\le\frac{r+\widetilde{d}(x,y)}{4A}\,.$

Here $A\ge 1$ and $\delta>0$. Then the following definitions of the Hardy space $H^1(X)$ are equivalent:

- Maximal definition: *$H^1(X)$ consists of all functions $f\in L^1(X)$ such that*
$$K_*f(x)=\sup\nolimits_{r>0}\Big|\int_X d\mu(y)\,K_r(x,y)\,f(y)\Big|$$
belongs to $L^1(X)$ and the norm $\|f\|_{H^1}$ is comparable to $\|K_*f\|_{L^1}$.

- Atomic definition: $H^1(X)$ *consists of all functions* $f\in L^1(X)$ *which can be written as* $f=\sum_\ell \lambda_\ell a_\ell$*, where the* a_ℓ*'s are atoms* ([1]) *and* $\sum_\ell|\lambda_\ell|<+\infty$*, and the norm* $\|f\|_{H^1}$ *is comparable to the infimum of* $\sum_\ell|\lambda_\ell|$ *over all such representations.*

Going back to $X=\mathbb{R}^n$, equipped with the Euclidean distance $d(\mathbf{x},\mathbf{y})=|\mathbf{x}-\mathbf{y}|$ and the measure (1.4), set

$$\widetilde{d}(\mathbf{x},\mathbf{y})=\inf \boldsymbol{\mu}(B) \qquad \forall\, \mathbf{x},\mathbf{y}\in\mathbb{R}^n,$$

where the infimum is taken over all closed balls B containing $\mathbf{x}$ and $\mathbf{y}$, and

$$K_r(\mathbf{x},\mathbf{y})=\mathbf{H}_t(\mathbf{x},\mathbf{y}), \qquad \forall\, r>0,\ \forall\, \mathbf{x},\mathbf{y}\in\mathbb{R}^n, \tag{4.2}$$

where $t=t(\mathbf{x},r)$ is defined by $\boldsymbol{\mu}(B(\mathbf{x},\sqrt{t}\,))=r$. In Appendixes B and C, we check that these data satisfy the assumptions of Uchiyama's Theorem with $\delta=\frac{1}{\mathbf{N}}$. Actually the conditions in Theorem 4.1 are obtained up to constants and they can be achieved by considering suitable multiples of $\boldsymbol{\mu}$ and $K_r(\mathbf{x},\mathbf{y})$. Thus the conclusion of Uchiyama's Theorem hold for the quasi-distance $\widetilde{d}$ and for the maximal operator K_*.

On the one hand, d and $\widetilde{d}$ define the same Hardy space H^1, as balls and quasi-balls are comparable. Let us elaborate. For every $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and $t>0$, we have

$$|\mathbf{x}-\mathbf{y}|\le\sqrt{t} \quad\Longrightarrow\quad \widetilde{d}(\mathbf{x},\mathbf{y})\le r \quad\Longrightarrow\quad |\mathbf{x}-\mathbf{y}|\lesssim\sqrt{t}\,,$$

where $r=\boldsymbol{\mu}(B(\mathbf{x},\sqrt{t}\,))$. The first implication is an immediate consequence of the definition of $\widetilde{d}$ and the second one is obtained by combining Lemma 6.6.(b) in Appendix B with (6.4) in Appendix A. Hence there exists a constant $c>0$ such that

$$B(\mathbf{x},\sqrt{t}\,)\subset\widetilde{B}(\mathbf{x},r)\subset B(\mathbf{x},c\sqrt{t}\,)$$

and these sets have comparable measures, according to Appendix A.

On the other hand, the maximal operators K_* and $\mathbf{H}_*$ coincide and they define the same Hardy space H^1 as the heat maximal operator $\mathbf{h}_*$, according to Propositions 2.9 and 3.3. Indeed, for every $f\in L^1(\mathbb{R}^n,d\boldsymbol{\mu})$, the integrals

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\;\mathbf{h}_*f(\mathbf{x}) \quad\text{and}\quad \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\;\mathbf{H}_*f(\mathbf{x})$$

differ at most by a multiple of $\|f\|_{L^1}$, which is itself controlled by either integral above, as $\mathbf{h}_t(\mathbf{x},\mathbf{y})\,d\boldsymbol{\mu}(\mathbf{y})$ and $\mathbf{H}_t(\mathbf{x},\mathbf{y})\,d\boldsymbol{\mu}(\mathbf{y})$ are approximations of the identity.

In conclusion, the atomic Hardy space H^1 associated with Euclidean balls coincide with the Hardy space H^1 defined by the heat maximal operator $\mathbf{h}_*$.

□

5. Proof of Theorem 1.10

The proof of Theorem 1.10 requires some weighted estimates in Dunkl analysis, which are well-known in the Euclidean setting corresponding to $\mathbf{k}=0$. Let us first prove a weak analog of the Euclidean estimate

$$\|(1+|\boldsymbol{\xi}|)^\sigma\,\widehat{f}(\boldsymbol{\xi})\|_{L^1(d\boldsymbol{\xi})}\lesssim\|f\|_{W_2^{\sigma+n/2+\varepsilon}}\,.$$

Lemma 5.1. *For every* $\ell\in\mathbb{N}$ *and* $r>0$*, there is a constant* $C=C_{\ell,r}>0$ *such that*

$$\sup\nolimits_{\boldsymbol{\xi}\in\mathbb{R}^n}(1+|\boldsymbol{\xi}|)^\ell\,|\mathcal{F}f(\boldsymbol{\xi})|\le C\,\|f\|_{C^\ell}\,,$$

for every $f\in C^\ell(\mathbb{R}^n)$ *with* $\operatorname{supp}f\subset B(0,r)$.

[1] Recall that an atom is a measurable function $a:X\to\mathbb{C}$ such that a is supported in a quasi-ball $\widetilde{B}$, $\|a\|_{L^\infty}\lesssim\mu(\widetilde{B})^{-1}$ and $\int_X d\mu\,a=0$.

Proof. By using the Riemann-Lebesgue lemma for the Fourier transform (1.9), we get

$$\begin{aligned}\sup_{\boldsymbol{\xi}\in\mathbb{R}^n}(1+|\boldsymbol{\xi}|)^\ell\,|\mathcal{F}f(\boldsymbol{\xi})| &\lesssim \sup_{\boldsymbol{\xi}\in\mathbb{R}^n}\Big(1+\sum\nolimits_{j=1}^n|\xi_j|^\ell\Big)\,|\mathcal{F}f(\boldsymbol{\xi})|\\ &\le \|f\|_{L^1(d\boldsymbol{\mu})}+\sum\nolimits_{j=1}^n\|D_j^\ell f\|_{L^1(d\boldsymbol{\mu})}\,.\end{aligned}$$

The last expression is bounded by $\|f\|_{C^\ell}$ as, by induction on ℓ, $\operatorname{supp}(D_j^\ell f)\subset B(0,r)$ and $\|D_j^\ell f\|_{L^\infty}\lesssim\|f\|_{C^\ell}$. ☐

Corollary 5.2. *For every* $\ell\in\mathbb{N}$, $r>0$ *and* $\varepsilon>0$, *there is a constant* $C=C_{\ell,r,\varepsilon}>0$ *such that*

$$\|\,(1+|\boldsymbol{\xi}|)^{\ell-\mathbf{N}/2-\varepsilon}\,\mathcal{F}f(\boldsymbol{\xi})\,\|_{L^2(d\boldsymbol{\mu}(\boldsymbol{\xi}))}\le C\,\|f\|_{W_2^{\ell+n/2+\varepsilon}}\,,$$

for every $f\in W_2^{\ell+n/2+\varepsilon}(\mathbb{R}^n)$ *with* $\operatorname{supp}f\subset B(0,r)$.

Proof. This result is deduced from Lemma 5.1, by using on the left hand side the finiteness of the integral

$$\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\boldsymbol{\xi})\,(1+|\boldsymbol{\xi}|)^{-\mathbf{N}-2\varepsilon}$$

and on the right hand side the Euclidean Sobolev embedding theorem. ☐

Proposition 5.3. *For every* $\sigma>0$, $r>0$ *and* $\varepsilon>0$, *there is a constant* $C=C_{\sigma,r,\varepsilon}>0$ *such that*

$$\|\,(1+|\boldsymbol{\xi}|)^{\sigma}\,\mathcal{F}f(\boldsymbol{\xi})\,\|_{L^2(d\boldsymbol{\mu}(\boldsymbol{\xi}))}\le C\,\|f\|_{W_2^{\sigma+\varepsilon}}\,,$$

for every $f\in W_2^{\sigma+\varepsilon}(\mathbb{R}^n)$ *with* $\operatorname{supp}f\subset B(0,r)$.

Proof. Let $\chi\in C_c^\infty(\mathbb{R}^n)$. Following an argument due to Mauceri-Meda [13], this result is obtained by interpolation between the L^2 estimate

$$\|\mathcal{F}(\chi f)\|_{L^2(d\boldsymbol{\mu})}=\text{const.}\,\|\chi f\|_{L^2(d\boldsymbol{\mu})}\lesssim\|f\|_{L^2(d\mathbf{x})}\,,$$

which is deduced from Plancherel's formula, and the following estimate for $\ell\in\mathbb{N}$ large, which is deduced from Corollary 5.2 :

$$\|\,(1+|\boldsymbol{\xi}|)^{\ell-\mathbf{N}/2-\varepsilon'}\mathcal{F}(\chi f)(\boldsymbol{\xi})\,\|_{L^2(d\boldsymbol{\mu}(\boldsymbol{\xi}))}\lesssim\|\chi f\|_{W_2^{\ell+n/2+\varepsilon'}}\lesssim\|f\|_{W_2^{\ell+n/2+\varepsilon'}}\,.$$

☐

By using the Cauchy-Schwartz inequality, we deduce eventually the following result.

Corollary 5.4. *For every* $\sigma>0$, $r>0$ *and* $\varepsilon>0$, *there is a constant* $C=C_{\sigma,r,\varepsilon}>0$ *such that*

$$\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\boldsymbol{\xi})\,(1+|\boldsymbol{\xi}|)^{\sigma}\,|\mathcal{F}f(\boldsymbol{\xi})|\le C\,\|f\|_{W_2^{\sigma+\mathbf{N}/2+\varepsilon}}\,,$$

for every $f\in W_2^{\sigma+\mathbf{N}/2+\varepsilon}(\mathbb{R}^n)$ *with* $\operatorname{supp}f\subset B(0,r)$.

Let us next prove analogs in the Dunkl setting of the Euclidean estimates

$$\int_{\mathbb{R}^n}d\mathbf{x}\,(1+|\mathbf{x}|)^\delta\,|f*g(\mathbf{x})|\,\le\int_{\mathbb{R}^n}d\mathbf{z}\,(1+|\mathbf{z}|)^\delta\,|f(\mathbf{z})|\int_{\mathbb{R}^n}d\mathbf{y}\,(1+|\mathbf{y}|)^\delta\,|g(\mathbf{y})|\,,$$

and

$$\int_{\mathbb{R}^n\smallsetminus B(\mathbf{y},r)}d\mathbf{x}\,|f(\mathbf{x}-\mathbf{y})|\,\lesssim\,r^{-\delta}\,\|(1+|\mathbf{x}|)^\delta f(\mathbf{x})\|_{L^1(d\mathbf{x})}\,.$$

Recall that Dunkl translations are defined via the Fourier transform (1.9) by

$$(\tau_{\mathbf{y}}f)(\mathbf{x})=\mathbf{c}_{\mathbf{k}}^{-1}\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\boldsymbol{\xi})\,\mathcal{F}f(\boldsymbol{\xi})\,\mathbf{E}(\mathbf{x},i\boldsymbol{\xi})\,\mathbf{E}(\mathbf{y},i\boldsymbol{\xi})$$

(see [17, 24, 19, 23]) and have an explicit integral representation

$$(\tau_{\mathbf{y}} f)(\mathbf{x}) = \int_{\mathbb{R}^n} d\boldsymbol{\nu}_{\mathbf{x},\mathbf{y}}(\mathbf{z})\, f(\mathbf{z})\,,$$

in dimension 1 (see [16, 23, 2]) and hence in the product case. Specifically,

$$d\boldsymbol{\nu}_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = d\nu^{(1)}_{x_1,y_1}(z_1)\, \dots\, d\nu^{(n)}_{x_n,y_n}(z_n)\,,$$

where

$$d\nu^{(j)}_{x_j,y_j}(z_j) \;=\; \begin{cases} \nu_j(x_j,y_j,z_j)\,|z_j|^{2k_j}\,dz_j & \text{if } \ x_j,y_j\in\mathbb{R}^*,\\ d\delta_{y_j}(z_j) & \text{if } \ x_j=0,\\ d\delta_{x_j}(z_j) & \text{if } \ y_j=0, \end{cases}$$

and

$$\begin{aligned} \nu_j(x_j,y_j,z_j) \;=\; & \tfrac{\Gamma(k_j+\frac12)}{\sqrt{\pi}\,2^{2k_j}\,\Gamma(k_j)}\, \tfrac{(x_j+y_j+z_j)(-x_j+y_j+z_j)(x_j-y_j+z_j)}{x_j\,y_j\,z_j}\\ & \times\, \tfrac{\{(|x_j|+|y_j|+|z_j|)(-|x_j|+|y_j|+|z_j|)(|x_j|-|y_j|+|z_j|)(|x_j|+|y_j|-|z_j|)\}^{k_j-1}}{|x_j\,y_j\,z_j|^{2k_j-1}}\\ & \times\, \mathbb{1}_{\left[\,\left||x_j|-|y_j|\right|,|x_j|+|y_j|\right]}(|z_j|)\,. \end{aligned}$$

Thus $\boldsymbol{\nu}_{\mathbf{x},\mathbf{y}}$ is a signed measure, which is supported in the product

$$\boldsymbol{\mathcal{I}}_{\mathbf{x},\mathbf{y}} = \mathcal{I}_{x_1,y_1}\times\,\dots\,\times\mathcal{I}_{x_n,y_n}$$

of one-dimensional sets

$$\begin{aligned} \mathcal{I}_{x_j,y_j} &= \left\{\, z_j\in\mathbb{R} \,\middle|\, \left||x_j|-|y_j|\right| \le |z_j| \le |x_j|+|y_j| \,\right\}\\ &= \left[-|x_j|-|y_j|,-\left||x_j|-|y_j|\right|\right] \cup \left[\left||x_j|-|y_j|\right|,|x_j|+|y_j|\right] \end{aligned}$$

and which is generically given by

$$d\boldsymbol{\nu}_{\mathbf{x},\mathbf{y}}(\mathbf{z}) = \underbrace{\nu_1(x_1,y_1,z_1)\,\dots\,\nu_n(x_n,y_n,z_n)}_{\boldsymbol{\nu}(\mathbf{x},\mathbf{y},\mathbf{z})}\, d\boldsymbol{\mu}(\mathbf{z})\,.$$

Moreover, it is known (see [16, 23, 2]) that

$$\sup_{\mathbf{x},\mathbf{y}\in\mathbb{R}^n} |\boldsymbol{\nu}_{\mathbf{x},\mathbf{y}}|(\mathbb{R}^n) < +\infty\,.$$

Lemma 5.5. *For every $\delta\ge 0$, $L^1((1+|\mathbf{x}|)^\delta\, d\boldsymbol{\mu}(\mathbf{x}))$ is an algebra with respect to the Dunkl convolution product*

$$f*g(\mathbf{x}) = \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{y})\,(\tau_{-\mathbf{y}}f)(\mathbf{x})\,g(\mathbf{y})\ .$$

Proof. By using the symmetries

$$\boldsymbol{\nu}(\mathbf{x},-\mathbf{y},\mathbf{z}) = \boldsymbol{\nu}(-\mathbf{z},-\mathbf{y},-\mathbf{x}) = \boldsymbol{\nu}(\mathbf{z},\mathbf{y},\mathbf{x})\,,$$

we have

$$f*g(\mathbf{x}) = \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{z})\,f(\mathbf{z}) \int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{y})\,g(\mathbf{y})\,\boldsymbol{\nu}(\mathbf{z},\mathbf{y},\mathbf{x})\,.$$

We conclude by estimating

$$\int_{\boldsymbol{\mathcal{I}}_{\mathbf{z},\mathbf{y}}} d\boldsymbol{\mu}(\mathbf{x})\,(1+|\mathbf{x}|)^\delta\,|\boldsymbol{\nu}(\mathbf{z},\mathbf{y},\mathbf{x})| \lesssim (1+|\mathbf{z}|)^\delta\,(1+|\mathbf{y}|)^\delta\,.$$

□

Lemma 5.6. *For every $\delta>0$, there is a constant $C>0$ such that, for every $\mathbf{y}\in\mathbb{R}^n$ and $r>0$,*

$$\int_{\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{y},r)}|(\tau_{-\mathbf{y}}f)(\mathbf{x})|\,d\boldsymbol{\mu}(\mathbf{x})\le C\,r^{-\delta}\,\|f\|_{L^1((1+|\mathbf{x}|)^\delta d\boldsymbol{\mu}(\mathbf{x}))}\,,$$

where

$$\mathcal{O}(\mathbf{y},r)=\{\,\mathbf{x}\in\mathbb{R}^n\mid ||x_j|-|y_j||\le r\ \ \forall\,1\le j\le n\,\}$$

is the orbit of the ball $B(\mathbf{y},r)$ under the group generated by the reflections (1.1).

Proof. As $\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{y},r)$ is contained in the union of the sets

$$A_j=\{\,\mathbf{x}\in\mathbb{R}^n\mid ||x_j|-|y_j||>n^{-1/2}\,r\,\}\qquad(j=1,\dots,n),$$

we have

$$\int_{\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{y},r)}|(\tau_{-\mathbf{y}}f)(\mathbf{x})|\,d\boldsymbol{\mu}(\mathbf{x})\;\le\;\sum\nolimits_{j=1}^{n}\int_{A_j}d\boldsymbol{\mu}(\mathbf{x})\int_{\boldsymbol{\mathcal{I}}_{\mathbf{x},\mathbf{y}}}d\boldsymbol{\mu}(\mathbf{z})\,|\boldsymbol{\nu}(\mathbf{x},-\mathbf{y},\mathbf{z})|\,|f(\mathbf{z})|\,.$$

As

$$|\mathbf{z}|\ge|z_j|\ge\big|\,|x_j|-|y_j|\,\big|>n^{-1/2}\,r$$

when $\mathbf{x}\in A_j$ and $\mathbf{z}\in\mathcal{I}_{\mathbf{x},\mathbf{y}}$, the latter expression is bounded above by

$$n^{\delta/2}\,r^{-\delta}\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\mathbf{z})\,|\mathbf{z}|^\delta\,|f(\mathbf{z})|\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\mathbf{x})\,|\boldsymbol{\nu}(\mathbf{x},-\mathbf{y},\mathbf{z})|\,.$$

We conclude by using the uniform estimate

$$\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\mathbf{x})\,|\boldsymbol{\nu}(\mathbf{x},-\mathbf{y},\mathbf{z})|=\int_{\mathbb{R}^n}d\boldsymbol{\mu}(\mathbf{x})\,|\boldsymbol{\nu}(\mathbf{z},\mathbf{y},\mathbf{x})|\le C.$$

□

Let us turn to the proof of Theorem 1.10, which consists in estimating

$$\|\mathbf{h}_*(\mathcal{T}_m a)\|_{L^1(d\boldsymbol{\mu})}\lesssim M\,,\tag{5.7}$$

for every atom a in the Hardy space H^1. By rescaling it suffices to prove (5.7) for any atom a associated with a unit ball $B(\mathbf{z},1)$. As $\mathbf{h}_*$ and $\mathcal{T}_m$ are bounded on $L^2(\mathbb{R}^n,d\boldsymbol{\mu})$, we have

$$\|\mathbf{h}_*(\mathcal{T}_m a)\|_{L^1(\mathcal{O}(\mathbf{z},2),d\boldsymbol{\mu})}\lesssim M\,.$$

Thus it remains for us to show that

$$\|\mathbf{h}_*(\mathcal{T}_m a)\|_{L^1(\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{z},2),d\boldsymbol{\mu})}\lesssim M\,.\tag{5.8}$$

For this purpose, let us introduce a dyadic partition of unity on the Dunkl transform side. More precisely, given a smooth radial function ψ on $\mathbb{R}^n$ such that

$$\operatorname{supp}\psi\subset\{\,\xi\in\mathbb{R}^n\mid\tfrac12\le|\boldsymbol{\xi}|\le2\,\}\quad\text{and}\quad\sum\nolimits_{\ell\in\mathbb{Z}}\psi(2^{-\ell}\boldsymbol{\xi})^2=1\quad\forall\,\boldsymbol{\xi}\in\mathbb{R}^n\smallsetminus\{0\}\,,$$

let us split up

$$e^{-t|\boldsymbol{\xi}|^2}m(\boldsymbol{\xi})=\sum\nolimits_{\ell\in\mathbb{Z}}\psi(2^{-\ell}\boldsymbol{\xi})\,e^{-t|\boldsymbol{\xi}|^2}\psi(2^{-\ell}\boldsymbol{\xi})\,m(\boldsymbol{\xi})\,.$$

Set

$$\begin{aligned}m_{t,\ell}(\boldsymbol{\xi})&=\overbrace{\psi(\boldsymbol{\xi})\,e^{-t|2^\ell\boldsymbol{\xi}|^2}}^{\psi_{t,\ell}(\boldsymbol{\xi})}\overbrace{\psi(\boldsymbol{\xi})\,m(2^\ell\boldsymbol{\xi})}^{m_\ell(\boldsymbol{\xi})},\\ f_{t,\ell}=\mathcal{F}^{-1}(m_{t,\ell})&=\underbrace{\mathcal{F}^{-1}(\psi_{t,\ell})}_{g_{t,\ell}}*\underbrace{\mathcal{F}^{-1}(m_\ell)}_{w_\ell}.\end{aligned}$$

Then $e^{-t|\boldsymbol{\xi}|^2}m(\boldsymbol{\xi})=\sum\nolimits_{\ell\in\mathbb{Z}}m_{t,\ell}(2^{-\ell}\boldsymbol{\xi})$. Consider the convolution kernel

$$F_{t,\ell}(\mathbf{x},\mathbf{y})=\tau_{-\mathbf{y}}\,\mathcal{F}^{-1}\{m_{t,\ell}(2^{-\ell}.)\}(\mathbf{x})=2^{\mathbf{N}\ell}(\tau_{-2^\ell\mathbf{y}}f_{t,\ell})(2^\ell\mathbf{x}).$$

Lemma 5.9. (a) *On the one hand, for every* $0\le\delta<\varepsilon$, *we have*

$$\int_{\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{z},2)} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|F_{t,\ell}(\mathbf{x},\mathbf{y})|\lesssim M\,2^{-\delta\ell}\qquad\forall\,\ell\in\mathbb{Z},\ \forall\,\mathbf{z}\in\mathbb{R}^n,\ \forall\,\mathbf{y}\in\mathcal{O}(\mathbf{z},1).$$

(b) *On the other hand,*

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|F_{t,\ell}(\mathbf{x},\mathbf{y})-F_{t,\ell}(\mathbf{x},\mathbf{y}')|\lesssim M\,2^{\ell}|\mathbf{y}-\mathbf{y}'|\qquad\forall\,\ell\in\mathbb{Z},\ \forall\,\mathbf{y},\mathbf{y}'\in\mathbb{R}^n.$$

Proof. On the one hand, as

$$\big|\partial_{\boldsymbol{\xi}}^{\boldsymbol{\alpha}}\big(\psi(\boldsymbol{\xi})e^{-t|\boldsymbol{\xi}|^2}\big)\big|\le C_{\boldsymbol{\alpha}}\qquad\forall\,t>0,\ \forall\,\boldsymbol{\xi}\in\mathbb{R}^n,$$

Lemma 5.1 yields the estimate

$$|g_{t,\ell}(\mathbf{x})|\le C_d\,(1+|\mathbf{x}|)^{-d}\qquad\forall\,\mathbf{x}\in\mathbb{R}^n,$$

which holds for any $d\in\mathbb{N}$ and which is uniform in $t>0$ and $\ell\in\mathbb{Z}$. On the other hand, Corollary 5.4 yields the estimate

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,(1+|\mathbf{x}|)^{\delta}\,|w_\ell(\mathbf{x})|\lesssim M,$$

which holds uniformly in $\ell\in\mathbb{Z}$. By resuming the proof of Lemma 5.5, we deduce that

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,(1+|\mathbf{x}|)^{\delta}\,\sup_{t>0}|f_{t,\ell}(\mathbf{x})|\lesssim M. \tag{5.10}$$

We reach our first conclusion by rescaling and by using Lemma 5.6 :

$$\begin{aligned}\int_{\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{z},2)} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|F_{t,\ell}(\mathbf{x},\mathbf{y})|&\le\int_{\mathbb{R}^n\smallsetminus\mathcal{O}(\mathbf{y},1)} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|F_{t,\ell}(\mathbf{x},\mathbf{y})|\\ =\int_{\mathbb{R}^n\smallsetminus\mathcal{O}(2^\ell\mathbf{y},2^\ell)} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|(\tau_{-2^\ell\mathbf{y}}f_{t,\ell})(\mathbf{x})|&\lesssim M\,2^{-\delta\ell}.\end{aligned}$$

Let us turn to the proof of (b). This time we factorize

$$m_{t,\ell}(\boldsymbol{\xi})=\underbrace{\overbrace{\psi(\boldsymbol{\xi})\,e^{|\boldsymbol{\xi}|^2}e^{-t|2^\ell\boldsymbol{\xi}|^2}}^{\widetilde{\psi}_{t,\ell}(\boldsymbol{\xi})}\overbrace{\psi(\boldsymbol{\xi})\,m(2^\ell\boldsymbol{\xi})}^{m_\ell(\boldsymbol{\xi})}}_{\widetilde{m}_{t,\ell}(\boldsymbol{\xi})}\,e^{-|\boldsymbol{\xi}|^2},$$

and accordingly

$$f_{t,\ell}=\mathcal{F}^{-1}(m_{t,\ell})=\underbrace{\mathcal{F}^{-1}(\widetilde{m}_{t,\ell})}_{\widetilde{f}_{t,\ell}}*\underbrace{\mathcal{F}^{-1}(e^{-|\boldsymbol{\xi}|^2})}_{h}.$$

On the one hand, by resuming the proof of (5.10), we get

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|\widetilde{f}_{t,\ell}(\mathbf{x})|\lesssim M.$$

On the other hand, $\mathbf{h}(\mathbf{x},\mathbf{y})=(\tau_{-\mathbf{y}}h)(\mathbf{x})$ is the heat kernel at time $t=1$, which satisfies

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,|\mathbf{h}(\mathbf{x},\mathbf{y})-\mathbf{h}(\mathbf{x},\mathbf{y}')|\lesssim|\mathbf{y}-\mathbf{y}'|\qquad\forall\,\mathbf{y},\mathbf{y}'\in\mathbb{R}^n,$$

according to next lemma. After rescaling, we reach our second conclusion :

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,\sup_{t>0}|F_{t,\ell}(\mathbf{x},\mathbf{y})-F_{t,\ell}(\mathbf{x},\mathbf{y}')|\lesssim M\,2^{\ell}\,|\mathbf{y}-\mathbf{y}'|.$$

□

Lemma 5.11. *The following gradient estimate holds for the heat kernel:*

$$\int_{\mathbb{R}^n} d\boldsymbol{\mu}(\mathbf{x})\,|\nabla_{\mathbf{y}}\mathbf{h}_t(\mathbf{x},\mathbf{y})| \lesssim t^{-\frac{1}{2}} \qquad \forall\, t>0,\ \forall\, \mathbf{y}\in\mathbb{R}^n.$$

Proof. We can reduce to the one-dimensional case and moreover to $t=1$ by rescaling. It follows from our gradient estimates for the heat kernel in dimension 1 (see Proposition 2.3) that

$$\big|\tfrac{\partial}{\partial y} h_1(x,y)\big| \lesssim \tfrac{1}{1+|xy|^k}\, e^{-\frac{1}{8}(|x|-|y|)^2}.$$

- *Case 1:* Assume that $|y|\le 2$. Then $|\partial_y h_1(x,y)| \lesssim e^{-x^2/16}$, hence

$$\int_{-\infty}^{+\infty} dx\,|x|^{2k}\,\big|\tfrac{\partial}{\partial y} h_1(x,y)\big| \lesssim 1\,.$$

- *Case 2:* Assume that $|y|\ge 2$. Then $|x|/|y|\le 1+\frac{1}{2}\big||x|-|y|\big|$, hence

$$|x|^{2k}\,\big|\tfrac{\partial}{\partial y} h_1(x,y)\big| \lesssim \big(\tfrac{|x|}{|y|}\big)^k e^{-\frac{1}{8}(|x|-|y|)^2} \lesssim \big(1+\big||x|-|y|\big|\big)^k e^{-\frac{1}{8}(|x|-|y|)^2} \lesssim e^{-\frac{1}{16}(|x|-|y|)^2}$$

and

$$\int_{-\infty}^{+\infty} dx\,|x|^{2k}\,\big|\tfrac{\partial}{\partial y} h_1(x,y)\big| \lesssim \int_0^{+\infty} dx\, e^{-\frac{1}{16}(x-|y|)^2} \lesssim \int_{-\infty}^{+\infty} dz\, e^{-\frac{1}{16}z^2} \lesssim 1\,.$$

□

End of proof of Theorem 1.10. Let us split up and estimate

$$\begin{aligned}
|\mathbf{h}_*(\mathcal{T}_m a)(\mathbf{x})| &\le \sum\nolimits_{\ell\ge 0} |\mathbf{h}_*(\mathcal{T}_{\psi(2^{-\ell}.)^2 m} a)(\mathbf{x})| + \sum\nolimits_{\ell<0} |\mathbf{h}_*(\mathcal{T}_{\psi(2^{-\ell}.)^2 m} a)(\mathbf{x})| \\
&= \sum\nolimits_{\ell\ge 0} \sup\nolimits_{t>0} \Big| \int_{B(\mathbf{z},1)} d\boldsymbol{\mu}(\mathbf{y})\, F_{t,\ell}(\mathbf{x},\mathbf{y})\, a(\mathbf{y}) \Big| \\
&\quad + \sum\nolimits_{\ell<0} \sup\nolimits_{t>0} \Big| \int_{B(\mathbf{z},1)} d\boldsymbol{\mu}(\mathbf{y})\, \big\{F_{t,\ell}(\mathbf{x},\mathbf{y}) - F_{t,\ell}(\mathbf{x},\mathbf{z})\big\}\, a(\mathbf{y}) \Big| \\
&\le \sum\nolimits_{\ell\ge 0} \int_{B(\mathbf{z},1)} d\boldsymbol{\mu}(\mathbf{y})\, |a(\mathbf{y})|\, \sup\nolimits_{t>0} \big|F_{t,\ell}(\mathbf{x},\mathbf{y})\big| \\
&\quad + \sum\nolimits_{\ell<0} \int_{B(\mathbf{z},1)} d\boldsymbol{\mu}(\mathbf{y})\, |a(\mathbf{y})|\, \sup\nolimits_{t>0} \big|F_{t,\ell}(\mathbf{x},\mathbf{y}) - F_{t,\ell}(\mathbf{x},\mathbf{z})\big|\,.
\end{aligned}$$

Then (5.8) follows from Lemma 5.9. □

Example 5.1. *The Riesz transforms $\mathcal{R}_j = D_j(-\mathbf{L})^{-1/2}$ in the Dunkl setting correspond to the multipliers $\boldsymbol{\xi}_j/|\boldsymbol{\xi}|$, up to a constant. Hence, by Theorem 1.10, they are bounded operators on the Hardy space H^1.*

6. Appendixes

6.1. Appendix A: Measure of balls.

Recall that $k_1,\dots,k_n\ge 0$ and $\mathbf{N}=n+\sum_{j=1}^n 2k_j$. On $\mathbb{R}^n$, equipped with the Euclidean distance, the product measure

$$d\boldsymbol{\mu}(\mathbf{x}) = d\mu_1(x_1)\dots d\mu_n(x_n) = |x_1|^{2k_1}\dots|x_n|^{2k_n}\,dx_1\dots dx_n \tag{1.4}$$

has the following rescaling properties:

$$d\boldsymbol{\mu}(\lambda\mathbf{x}) = |\lambda|^{\mathbf{N}}\,d\boldsymbol{\mu}(\mathbf{x}) \qquad \forall\,\lambda\in\mathbb{R}^* \tag{6.1}$$

and

$$\boldsymbol{\mu}(\mathbf{B}(\lambda\mathbf{x},|\lambda|r)) = |\lambda|^{\mathbf{N}}\,\boldsymbol{\mu}(\mathbf{B}(\mathbf{x},r)) \qquad \forall\,\mathbf{x}\in\mathbb{R}^n,\ \forall\,\lambda\in\mathbb{R}^*. \tag{6.2}$$

Moreover

$$\boldsymbol{\mu}(\mathbf{B}(\mathbf{x},r)) \asymp r^n \prod\nolimits_{j=1}^{n}(|x_j|+r)^{2k_j}. \tag{6.3}$$

Hence

$$\left(\tfrac{R}{r}\right)^n \lesssim \tfrac{\boldsymbol{\mu}(\mathbf{B}(\mathbf{x},R))}{\boldsymbol{\mu}(\mathbf{B}(\mathbf{x},r))} \lesssim \left(\tfrac{R}{r}\right)^{\mathbf{N}} \qquad \forall\,\mathbf{x}\in\mathbb{R}^n,\ \forall\,R\geq r>0. \tag{6.4}$$

In particular, $\boldsymbol{\mu}$ is doubling, i.e.,

$$\boldsymbol{\mu}(\mathbf{B}(\mathbf{x},2r)) \asymp \boldsymbol{\mu}(\mathbf{B}(\mathbf{x},r)) \qquad \forall\,\mathbf{x}\in\mathbb{R}^n,\ \forall\,r>0. \tag{6.5}$$

Let us prove (6.3) and (6.4). In dimension $n=1$, we have

$$\mu(B(x,r)) \,=\, \int_{|x|-r}^{|x|+r} dy\,|y|^{2k}\,.$$

On the one hand, if $r\leq\frac{|x|}{2}$, we deduce that

$$\mu(B(x,r)) \asymp |x|^{2k}\int_{|x|-r}^{|x|+r} dy \,\asymp\, |x|^{2k}\,r\,.$$

On the other hand, if $|x|\leq 2r$, we estimate from above

$$\mu(B(x,r)) \leq \int_{-r}^{3r} dy\,|y|^{2k} \asymp r^{2k+1}$$

and from below

$$\mu(B(x,r)) \geq \int_{0}^{r} dy\,y^{2k} \asymp r^{2k+1}.$$

Thus $\mu(B(x,r))\asymp(|x|+r)^{2k}r$ in all cases and

$$\mu(B(x,r)) \asymp \left(\tfrac{|x|+R}{|x|+r}\right)^{2k}\tfrac{R}{r} \asymp \begin{cases} \left(\frac{R}{r}\right)^{2k+1} & \text{if } |x|\leq r,\\ \left(\frac{R}{|x|}\right)^{2k}\frac{R}{r} & \text{if } r\leq|x|\leq R,\\ \frac{R}{r} & \text{if } |x|\geq R. \end{cases}$$

The product case follows from the one-dimensional case, since the ball $\mathbf{B}(\mathbf{x},r)$ and the cube

$$\mathbf{Q}(\mathbf{x},r) = \prod\nolimits_{j=1}^{n} B(x_j,r)$$

have comparable measures. More precisely, we have

$$\mathbf{Q}(\mathbf{x},\tfrac{r}{\sqrt{n}}) \subset \mathbf{B}(\mathbf{x},r) \subset \mathbf{Q}(\mathbf{x},r),$$

with

$$\boldsymbol{\mu}(\mathbf{Q}(\mathbf{x},\tfrac{r}{\sqrt{n}})) \asymp \boldsymbol{\mu}(\mathbf{Q}(\mathbf{x},r)) \asymp r^n \prod\nolimits_{j=1}^{n}(|x_j|+r)^{2k_j}.$$

6.2. **Appendix B : Distances.**

The following result, which is used in Section 4, is certainly known among specialists. We include nevertheless a proof, for lack of reference and for the reader's convenience.

Lemma 6.6. *Let (X,d,μ) be a metric measure space such that balls have finite positive measure and satisfy the* doubling property*, i.e.,*

$$\exists\,C>0,\ \forall\,x\in X,\ \forall\,r>0,\ \mu(B(x,2\,r)) \leq C\,\mu(B(x,r))\,.$$

Set

$$\widetilde{d}(x,y) = \inf \mu(B),$$

where the infimum is taken over all closed balls B containing x and y. Then

(a) $\widetilde{d}$ *is a quasi-distance,*

(b) $\widetilde{d}(x,y) \asymp \mu(B(x,d(x,y)))\ \ \forall\, x,y\in X,$

Moreover, if the measure μ *has no atoms and* $\mu(X)=+\infty$*, then*

(c) $\mu(\widetilde{B}(x,r)) \asymp r$*, for every* $x\in X$ *and* $r>0$*, where* $\widetilde{B}(x,r)$ *denotes the closed quasi-ball with center* x *and radius* r*.*

Proof. Let us first prove (b). Set $R=d(x,y)$. On the one hand, we have $\widetilde{d}(x,y)\le\mu(B(x,R))$, as x and y belong to $B(x,R)$. On the other hand, if x and y belong to a ball $B=B(z,r)$, then $R\le 2\,r$, hence $B(x,R)\subset B(z,3r)$ and $\mu(B(x,R))\le\mu(B(z,3r))\asymp\mu(B(z,r))$. By taking the infimum over all balls B containing both x and y, we conclude that $\mu(B(x,R))\lesssim\widetilde{d}(x,y)$. Let us next prove (a). For every $x,y,z\in X$, we have $d(x,y)\le d(x,z)+d(z,y)$. Assume that $r=d(x,z)\ge d(z,y)$. Then $x,y\in B(z,r)$. By using (b), we conclude that

$$\widetilde{d}(x,y)\le\mu(B(z,r)\asymp\widetilde{d}(z,x)\le\max\{\widetilde{d}(x,z),\widetilde{d}(z,y)\}\le\widetilde{d}(x,z)+\widetilde{d}(z,y)\,.$$

Let us eventually prove (c). Given $x\in X$, notice that $\mu(B(x,r))$ is an increasing càdlàg function of $r\in(0,+\infty)$ such that

$$\begin{cases}\mu(B(x,r))\searrow 0 & \text{as } r\searrow 0\,,\\ \mu(B(x,r))\nearrow +\infty & \text{as } r\nearrow +\infty\,.\end{cases}$$

Here we have used our additional assumptions. Let $x\in X$ and $r>0$. On the one hand, for every $y\in\widetilde{B}(x,r)$, we have $\mu(B(x,d(x,y))\asymp\widetilde{d}(x,y)\le r$. Hence

$$R=\sup\{\,d(x,y)\,|\,y\in\widetilde{B}(x,r)\}<+\infty\,.$$

Let $y\in\widetilde{B}(x,r)$ such that $d(x,y)\ge\frac{R}{2}$. Then $\widetilde{B}(x,r)\subset B(x,R)\subset B(x,2\,d(x,y))$. Hence

$$\mu(\widetilde{B}(x,r))\le\mu(B(x,2\,d(x,y))\asymp\mu(B(x,d(x,y))\asymp\widetilde{d}(x,y)\le r\,.$$

On the other hand,

$$T=\inf\{\,t>0\,|\,\mu(B(x,t))\ge r\}>0\,.$$

As $\mu(B(x,\frac{T}{2}))<r$, we have $\widetilde{d}(x,y)<r$, for every $y\in B(x,\frac{T}{2})$, hence $B(x,\frac{T}{2})\subset\widetilde{B}(x,r)$. Consequently,

$$r\le\mu(B(x,T))\asymp\mu(B(x,\tfrac{T}{2}))\le\mu(\widetilde{B}(x,r)).$$

□

6.3. Appendix C : Kernel bounds.

Recall from Section 4 that the kernels $K_r(\mathbf{x},\mathbf{y})$ and $\mathbf{H}_t(\mathbf{x},\mathbf{y})$ are related by

$$K_r(\mathbf{x},\mathbf{y})=\mathbf{H}_t(\mathbf{x},\mathbf{y})\,, \tag{4.2}$$

where $r=\boldsymbol{\mu}(B(\mathbf{x},\sqrt{t}))$. In this appendix, we check that the Gaussian estimates of $\mathbf{H}_t(\mathbf{x},\mathbf{y})$ in Theorem 3.2 imply the estimates of $K_r(\mathbf{x},\mathbf{y})$ required in Uchiyama's Theorem (Theorem 4.1). This result is certainly well-known among specialists. We include nevertheless a proof, for lack of reference and for the reader's convenience.

According to Appendices A and B, we may consider the quasi-distance $\widetilde{d}$ on $\mathbb{R}^n$ associated with the Euclidean distance $d(\mathbf{x},\mathbf{y})=|\mathbf{x}-\mathbf{y}|$ and the product measure (1.4). The on-diagonal lower estimate

$$K_r(\mathbf{x},\mathbf{x})\ge\tfrac{C_1}{r} \tag{6.7}$$

is an immediate consequence of Theorem 3.2.(a). For every $\delta>0$, the upper estimate

$$K_r(\mathbf{x},\mathbf{y})\le\tfrac{C_2}{r}\left(1+\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y})}{r}\right)^{-1-\delta} \tag{6.8}$$

follows from Theorem 3.2.(b), more precisely by combining

$$K_r(\mathbf{x},\mathbf{y}) \lesssim r^{-1}\, e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}$$

with

$$(6.9)\qquad \bigl(1+\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y})}{r}\bigr)^{1+\delta} \le \bigl(1+\tfrac{\mu(B(\mathbf{x},|\mathbf{x}-\mathbf{y}|))}{\mu(B(\mathbf{x},\sqrt{t}))}\bigr)^{1+\delta} \lesssim \bigl(1+\tfrac{|\mathbf{x}-\mathbf{y}|}{\sqrt{t}}\bigr)^{\mathbf{N}(1+\delta)} \lesssim e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}\,.$$

The main problem consists in checking the following Lipschitz estimate.

Lemma 6.10. *There exists $C_3>0$ and, for every $\delta>0$, there exists $C_4>0$ such that*

$$(6.11)\qquad \bigl|K_r(\mathbf{x},\mathbf{y})-K_r(\mathbf{x},\mathbf{y}')\bigr| \le \tfrac{C_4}{r}\bigl(1+\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y})}{r}\bigr)^{-1-\delta}\bigl(\tfrac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r}\bigr)^{\frac{1}{\mathbf{N}}}$$

if $\widetilde{d}(\mathbf{y},\mathbf{y}')\le C_3\max\{r,\widetilde{d}(\mathbf{x},\mathbf{y})\}$.

Proof. Let us begin with some observations. First of all, (6.11) follows from (6.8), as long as $\widetilde{d}(\mathbf{y},\mathbf{y}')\asymp r$. In this case, we have indeed

$$1+\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y})}{r}\asymp 1+\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y}')}{r}\,.$$

Next, notice that

$$\begin{cases} |\mathbf{x}-\mathbf{y}|\lesssim\sqrt{t} \iff \widetilde{d}(\mathbf{x},\mathbf{y})\lesssim r\,,\\ |\mathbf{x}-\mathbf{y}|\gtrsim\sqrt{t} \iff \widetilde{d}(\mathbf{x},\mathbf{y})\gtrsim r\,.\end{cases}$$

This follows indeed from the estimates

$$\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y})}{r}\asymp\tfrac{\mu(B(\mathbf{x},|\mathbf{x}-\mathbf{y}|))}{\mu(B(\mathbf{x},\sqrt{t}))}$$

and

$$\bigl(\tfrac{R}{r}\bigr)^n\lesssim\tfrac{\mu(B(\mathbf{x},R))}{\mu(B(\mathbf{x},r))}\lesssim\bigl(\tfrac{R}{r}\bigr)^{\mathbf{N}}\quad\text{if } r\lesssim R.$$

Similarly, we have

$$|\mathbf{y}-\mathbf{y}'|\lesssim|\mathbf{y}-\mathbf{x}| \iff \widetilde{d}(\mathbf{y},\mathbf{y}')\lesssim\widetilde{d}(\mathbf{y},\mathbf{x}).$$

In particular, there exists $C_3>0$ such that

$$|\mathbf{y}-\mathbf{y}'|\le\tfrac12|\mathbf{x}-\mathbf{y}|\quad\text{if}\quad\widetilde{d}(\mathbf{y},\mathbf{y}')\le C_3\,\widetilde{d}(\mathbf{x},\mathbf{y}).$$

Let us turn to the proof of (6.11) and assume first that $\widetilde{d}(\mathbf{x},\mathbf{y})\ge r$. In this case, $|\mathbf{x}-\mathbf{y}|\gtrsim\sqrt{t}$ and $\widetilde{d}(\mathbf{y},\mathbf{y}')\le C_3\,\widetilde{d}(\mathbf{x},\mathbf{y})$, hence $|\mathbf{y}-\mathbf{y}'|\le\frac12|\mathbf{x}-\mathbf{y}|$. Thus, according to Theorem 3.2.(d),

$$|K_r(\mathbf{x},\mathbf{y})-K_r(\mathbf{x},\mathbf{y}')|=|\mathbf{H}_t(\mathbf{x},\mathbf{y})-\mathbf{H}_t(\mathbf{x},\mathbf{y}')|$$

is bounded above by

$$\mu(B(\mathbf{x},\sqrt{t}))^{-1}\,e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}\,\tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\,.$$

After substituting $r=\mu(B(\mathbf{x},\sqrt{t}))$ and estimating

$$\bigl(1+\tfrac{\widetilde{d}(\mathbf{x},\mathbf{y})}{r}\bigr)^{1+\delta}\lesssim e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{2\,c\,t}}$$

as in (6.9), it remains for us to show that

$$(6.12)\qquad \tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\lesssim\bigl(\tfrac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r}\bigr)^{\frac{1}{\mathbf{N}}}\,e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{2\,c\,t}}\,.$$

If $|\mathbf{y}-\mathbf{y}'|\le\sqrt{t}$, then

$$\tfrac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r}\asymp\tfrac{\mu(B(\mathbf{y},|\mathbf{y}-\mathbf{y}'|))}{\mu(B(\mathbf{x},\sqrt{t}))}=\tfrac{\mu(B(\mathbf{y},|\mathbf{y}-\mathbf{y}'|))}{\mu(B(\mathbf{y},\sqrt{t}))}\,\tfrac{\mu(B(\mathbf{y},\sqrt{t}))}{\mu(B(\mathbf{x},\sqrt{t}))}$$

with

$$\tfrac{\mu(B(\mathbf{y},|\mathbf{y}-\mathbf{y}'|))}{\mu(B(\mathbf{y},\sqrt{t}))}\gtrsim\bigl(\tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\bigr)^{\mathbf{N}}$$

and

$$\tfrac{\mu(B(\mathbf{y},\sqrt{t}))}{\mu(B(\mathbf{x},\sqrt{t}))}\ge\tfrac{\mu(B(\mathbf{y},\sqrt{t}))}{\mu(B(\mathbf{y},|\mathbf{x}-\mathbf{y}|+\sqrt{t}))}\gtrsim\bigl(\tfrac{\sqrt{t}}{|\mathbf{x}-\mathbf{y}|+\sqrt{t}}\bigr)^{\mathbf{N}}=\bigl(1+\tfrac{|\mathbf{x}-\mathbf{y}|}{\sqrt{t}}\bigr)^{-\mathbf{N}}\gtrsim e^{-\frac{\mathbf{N}}{2}\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}\,.$$

If $|\mathbf{y}-\mathbf{y}'|\ge\sqrt{t}$, we argue similarly, estimating this time

$$\tfrac{\mu(B(\mathbf{y},|\mathbf{y}-\mathbf{y}'|))}{\mu(B(\mathbf{y},\sqrt{t}))} \gtrsim \big(\tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\big)^n \gtrsim \big(\tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\big)^{\mathbf{N}} \big(\tfrac{|\mathbf{x}-\mathbf{y}|}{\sqrt{t}}\big)^{-(\mathbf{N}-n)} \gtrsim \big(\tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\big)^{\mathbf{N}} e^{-\frac{\mathbf{N}}{4}\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}$$

and

$$\tfrac{\mu(B(\mathbf{y},\sqrt{t}))}{\mu(B(\mathbf{x},\sqrt{t}))} \gtrsim e^{-\frac{\mathbf{N}}{4}\frac{|\mathbf{x}-\mathbf{y}|^2}{c\,t}}\,.$$

Assume next that $\widetilde{d}(\mathbf{x},\mathbf{y}) \le r$. Then $|\mathbf{x}-\mathbf{y}| \lesssim \sqrt{t}$, $\widetilde{d}(\mathbf{y},\mathbf{y}') \le C_3\, r$ and (6.11) amounts to

$$\big|K_r(\mathbf{x},\mathbf{y}) - K_r(\mathbf{x},\mathbf{y}')\big| \lesssim r^{-1} \big(\tfrac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r}\big)^{\frac{1}{\mathbf{N}}}.$$

According to Theorem 3.2.(d),

$$|K_r(\mathbf{x},\mathbf{y}) - K_r(\mathbf{x},\mathbf{y}')| = |\mathbf{H}_t(x,\mathbf{y}) - \mathbf{H}_t(x,\mathbf{y}')| \lesssim \mu(B(\mathbf{x},\sqrt{t}))^{-1}\, \tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\,.$$

As

$$\mu(B(\mathbf{y},\sqrt{t})) \asymp \mu(B(\mathbf{x},\sqrt{t})) = r\,,$$

we have

$$\tfrac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r} \asymp \tfrac{\mu(B(\mathbf{y},|\mathbf{y}-\mathbf{y}'|))}{\mu(B(\mathbf{y},\sqrt{t}))}\,.$$

As $\frac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r} \le C_3$ and $\frac{\mu(B(\mathbf{y},|\mathbf{y}-\mathbf{y}'|))}{\mu(B(\mathbf{y},\sqrt{t}))}$ is bounded from below by a power of $\frac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}$, we deduce first that $|\mathbf{y}-\mathbf{y}'| \lesssim \sqrt{t}$ and next that

$$\tfrac{\widetilde{d}(\mathbf{y},\mathbf{y}')}{r} \gtrsim \big(\tfrac{|\mathbf{y}-\mathbf{y}'|}{\sqrt{t}}\big)^{\mathbf{N}}.$$

This concludes the proof of Lemma 6.10. □

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Université d'Orléans & CNRS, Fédération Denis Poisson (FR 2964) & Laboratoire MAPMO (UMR 6628), Bâtiment de mathématiques, B.P. 6759, 45067 Orléans cedex 2, France
E-mail address: anker@univ-orleans.fr

Université de Tunis El Manar, Faculté des Sciences de Tunis, LR11ES11 Analyse Mathématiques et Applications, 2092, Tunis, Tunisie
E-mail address: Nejib.Bensalem@fst.rnu.tn

Uniwersytet Wrocławski, Instytut Matematyczny, Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: Jacek.Dziubanski@math.uni.wroc.pl

Université de Tunis El Manar, Faculté des Sciences de Tunis, LR11ES11 Analyse Mathématiques et Applications, 2092, Tunis, Tunisie
E-mail address: hamdanabila@gmail.com