# Enlargement of filtration and predictable representation property for semi-martingales

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#### Abstract

We present two examples of loss of the predictable representation property for semi-martingales by enlargement of the reference filtration. First of all we show that the predictable representation property for a semi-martingale X does not transfer from the reference filtration  $\mathbb{F}$  to a larger filtration  $\mathbb{G}$  if the information starts growing up to a positive time. Then we study the case  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  when there exists a second special semi-martingale Y enjoying the predictable representation property with respect to  $\mathbb{H}$ . We establish conditions under which the triplet (X, Y, [X, Y]) enjoys the predictable representation property with respect to  $\mathbb{G}$ .

**Keywords:** Semi-martingales, predictable representations property, enlargement of filtration, completeness of a financial market

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# 1 Introduction

Given a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, P)$  and a fixed semi-martingale X on it, a classical problem in stochastic analysis is the investigation of conditions which allow to represent every  $L^{\infty}(\Omega, \mathcal{F}_T, P)$ -random variable as the sum of an  $\mathcal{F}_0$ -measurable random variable and a stochastic integral with respect to X. When there exists an equivalent local-martingale measure Q for X that representation follows as soon as X enjoys the *predictable representation property* (in short, p.r.p.) with respect to the filtration  $\mathbb{F}$  under  $Q^{-1}$ , that is any  $(Q, \mathbb{F})$ -local martingale can be written in the form  $m + \int_0^t \xi_s dX_s$  where m is  $\mathcal{F}_0$ -measurable and  $\xi$  is  $\mathbb{F}$ -predictable (see Definition 13.1 and Theorem 13.4 in [20]). The equivalence between the representation of all  $(Q, \mathbb{F})$ -local martingales and the uniqueness of Q, modulo  $\mathcal{F}_0$ , is the content of a classical martingale representation result (see Theorem 13.9 in [20]). In particular, under the stronger hypothesis that Q is the unique equivalent martingale measure for X,  $\mathcal{F}_0$  is trivial and X enjoys the  $(Q, \mathbb{F})$ -p.r.p. (see [19] and Theorem 13.4 in [20]).

If the reference filtration  $\mathbb{F}$  coincides with the natural filtration of X, then p.r.p. holds

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<sup>&</sup>lt;sup>1</sup> when not necessary we will avoid to mention Q

either for Brownian motion and Poisson process or, under suitable assumptions, for diffusions with jumps and in some particular non Markovian contexts (see e.g. [21] and [10] and references therein).

Assuming that the p.r.p. for X holds, an interesting question is about maintenance of that property with respect to an enlarged filtration  $\mathbb{G}$  in the following sense. If  $\tilde{Q}$  is an equivalent local-martingale measure for X with respect to  $\mathbb{G}$ , does X enjoy the  $(\tilde{Q}, \mathbb{G})$ p.r.p.? When the answer is positive, we say that *p.r.p. transfers from*  $\mathbb{F}$  to  $\mathbb{G}$ . In this case  $\tilde{Q}$  is the unique equivalent local-martingale measure, modulo  $\mathcal{G}_0$ , for X with respect to  $\mathbb{G}$ . Otherwise p.r.p. cannot hold with respect to any equivalent local-martingale measure for X with respect to  $\mathbb{G}$ , and we say that *p.r.p. disappears*.

This issue is related to the problem of enlargement of filtration, that is the investigation of conditions under which, on a given probability space,  $(P, \mathbb{F})$ -semi-martingales are also  $(P, \mathbb{G})$ -semi-martingales (see the classical [27] and more recently [28]).

Maintenance or loss of the p.r.p. with respect to an enlarged filtration appear both in the literature. Let us now recall two well-known examples of enlargement of filtration. In [1] it is shown that the p.r.p. is preserved in case of *initial enlargement* of  $\mathbb{F}$  defined by  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(G)$ , with G a random variable satisfying the condition  $P(G \in |\mathcal{F}_t)(\omega) \sim$  $P(G \in \cdot)$ , for almost all  $\omega$ . This assumption enables to show that X enjoys the  $(Q^{\mathbb{G}}, \mathbb{G})$ p.r.p. under a suitable equivalent martingale measure  $Q^{\mathbb{G}}$  on  $\mathcal{G}_T$ . At the same time, a Brownian motion B fails to exhibit the p.r.p. in case of progressive enlargement of its natural filtration  $\mathbb{F}^B$  obtained by the observation of the occurrence of a positive random time  $\tau$ , which is not an  $\mathbb{F}^B$ -stopping time, that is when  $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s^B \vee \sigma(\tau \wedge s)$ . In particular this occurs when  $\tau$  is a continuous random variable independent of B, and consequently integrals with respect to B are not enough to represent all the  $\mathbb{G}$ -martingales. Indeed one has to add integrals with respect to the compensated default process  $\mathbb{I}_{\tau\leq \cdot} - \int_0^{\tau\wedge \cdot} \lambda_s ds$ , with  $\lambda = f/G$ , where f and G are the density function and the survival function of  $\tau$ , respectively (this result can be viewed as a simple application of Theorem 7.5.5.1 in [26]). We stress that the two examples above differ; more precisely the initial sigma-algebra in the latter remains trivial, whereas in the former the enlargement already takes place up to the initial time: that is the starting sigma-algebra of the new filtration is different from  $\mathcal{F}_0$ .

In general, the fact that the progressive enlargement by a random time destroys the p.r.p. is well-known; we refer to [25] for conditions under which the p.r.p. can be achieved by considering a larger number of driving processes.

In this paper X is a special  $(P, \mathbb{F})$ -semi-martingale which admits a unique equivalent martingale measure. Therefore  $\mathcal{F}_0$  is trivial and X enjoys the p.r.p. with respect to  $\mathbb{F}$ . Without selecting particular models, we present two results where the p.r.p. disappears when the reference filtration is enlarged by keeping trivial the sigma-algebra at time zero. Our aim is to give a contribution to the investigation of the link between the p.r.p. and the reference filtration of the semi-martingale. It is well-known that p.r.p. may fail with respect to the natural filtration (see Example 23.11 in [35]). At the same time in some cases the property holds with respect to a filtration larger than the natural one (see Remark 3.7 in Section 3).

In our first result no assumption is introduced on the source of randomness giving rise to the enlargement. We prove that, if there exists an equivalent martingale measure for X with respect to the enlarged filtration  $\mathbb{G}$ , then the p.r.p. disappears whenever the set  $\{t \in [0, T] : \mathcal{F}_t \subsetneq \mathcal{G}_t\}$  has a positive minimum. In our second result the enlargement is obtained by adding the information given by a new filtration  $\mathbb{H}$  such that  $\mathcal{H}_0$  is trivial. More precisely  $\mathbb{G}$  coincides with  $\mathbb{F} \vee \mathbb{H}$ . We assume that there exists a  $(P, \mathbb{H})$ -semi-martingale Y enjoying the p.r.p. with respect to  $\mathbb{H}$ . Moreover the martingale part N of Y is  $(P, \mathbb{G})$ -strongly orthogonal to the martingale part M of X, and X and Y satisfy a technical assumption which provides them the *structure condition*. We also introduce suitable assumptions assuring the existence of the *minimal martingale measures* for X and Y. In this setting we prove that the  $(P, \mathbb{G})$ -semi-martingale (X, Y, [X, Y]) admits a unique equivalent martingale measure Q. More precisely we show that the triplet (X, Y, [X, Y]) is a *basic set of*  $(Q, \mathbb{G})$ -*orthogonal martingales* so that three is the *multiplicity of*  $\mathbb{G}$  in the sense of Davis and Varaiya (see [11]). Therefore the p.r.p. by the enlargement of filtration does not hold for X.

We stress that the issue addressed in this paper naturally emerges in the analysis of financial markets, where the discounted price of the risky asset is modeled as a semimartingale. In this setting generally the payoffs are random variables measurable with respect to the final sigma-algebra of the reference filtration and the investment strategy is a predictable process. Then, when the market is free of arbitrage, the p.r.p. of the discounted asset price process provides the *completeness of the market* that is the perfect replication of all the essentially bounded payoffs (see [17]). In their seminal papers Harrison and Pliska clearly state that completeness is a joint property of the filtration and of the asset price process and in particular they argued that the structure of the filtration should influence completeness. They also provided the original version of the *II Theorem of Asset* Pricing (see [18], [19]), even if their statement is not completely correct in the definition of self-financial strategies (to clarify this fact see [31] and the Appendix in [23] and [7] about the distinction between vector completeness and component completeness, which is at the origin of the imprecision of the Harrison and Pliska's result). Completeness has been widely studied when the reference filtration coincides with the natural one. However markets with default or markets with better informed agents are modeled by considering on the probability space a filtration larger than the natural one and most of them are not complete markets or even not arbitrage free (see e.g. [5] or [25] and [16]).

This paper is organized as follows. In Section 2 we describe the mathematical setting and recall a classical result about the enlargement of filtration. Section 3 and Section 4 are devoted to our results. Section 4 is divided into two subsections: in the former we discuss the simpler case of martingales and in the latter we extend the result to semimartingales. In Section 5 we discuss possible connections of our results with some papers in the recent literature. Finally we devote Section 6 to avenues for future research in this area.

#### 2 Setting and notations

Let T > 0 be a finite time horizon and let  $X = (X_t)_{t \in [0,T]}$  be a real valued càdlàg squareintegrable semi-martingale defined on a given probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , with the filtration  $\mathbb{F}$  satisfying the usual conditions of right-continuity and completeness.

More precisely let X belong to the space  $S^2(P, \mathbb{F})$  of semi-martingales (see, e.g., [13]), i.e. let X be a special semi-martingale with canonical decomposition

$$X = X_0 + M + A \tag{1}$$

such that

$$E^{P}\left[X_{0}^{2} + [M]_{T} + |A|_{T}^{2}\right] < +\infty.$$
<sup>(2)</sup>

As usual M is a  $(P, \mathbb{F})$ -martingale, A is an  $\mathbb{F}$ -predictable process of finite variation,  $M_0 = A_0 = 0$ , |A| denotes the total variation process of A and [M] the quadratic variation process of M. Note that by integrability condition (2) it follows

$$E^{P}\left[\sup_{t\in[0,T]}X_{t}^{2}\right]<\infty.$$
(3)

In all the paper, given a filtered probability space  $(\Omega, \mathcal{A}, \mathbb{A}, R)$  and a square-integrable  $(R, \mathbb{A})$ -semi-martingale  $S = (S_t)_{t \in [0,T]}$  on it,  $L^2_0(\Omega, \mathcal{A}_T, R)$  will denote the subset of the centered elements of  $L^2(\Omega, \mathcal{A}_T, R)$  and  $\mathcal{L}^2(S, R, \mathbb{A})$  the space of the  $\mathbb{A}$ -predictable processes  $\xi$  such that

$$E^R \left[ \int_0^T \xi_t^2 d[S]_t \right] < +\infty.$$
(4)

Denote by  $\mathbb{P}(X, \mathbb{F})$  the set of probability measures on  $(\Omega, \mathcal{F}_T)$  under which X is a martingale and which are equivalent to  $P|_{\mathcal{F}_T}$ , the restriction of P to  $\mathcal{F}_T$ . Assume that the set  $\mathbb{P}(X, \mathbb{F})$  is a singleton, more precisely

**H1**) 
$$\mathbb{P}(X,\mathbb{F}) = \{P^X\}$$

Assumption H1) has two important consequences as already recalled in the introduction. The initial  $\sigma$ -algebra  $\mathcal{F}_0$  turns out to be *P*-trivial and the  $(P, \mathbb{F})$ -semi-martingale *X* enjoys the p.r.p. with respect to  $\mathbb{F}$  under  $P^X$ , that is any  $(P^X, \mathbb{F})$ -local martingale admits a representation of the form  $m + \int_0^t \xi_s dX_s$  where *m* is a constant and  $\xi$  is  $\mathbb{F}$ -predictable. In particular each *H* in  $L^2(\Omega, \mathcal{F}_T, P^X)$  admits  $P^X$ -a.s. the representation

$$H = H_0 + \int_0^T \xi_s^H dX_s,\tag{5}$$

with  $H_0$  a constant and  $\xi^H$  a process in  $\mathcal{L}^2(X, P^X, \mathbb{F})$ .

Let  $\mathcal{M}^2(P^X, \mathbb{F})$  be the space of the square-integrable  $(P^X, \mathbb{F})$ -martingales. Then for any Z in  $\mathcal{M}^2(P^X, \mathbb{F})$  an immediate application of (5) to  $H = Z_T$  proves that  $P^X$ -a.s.

$$Z_t = Z_0 + \int_0^t \xi_s^Z dX_s \tag{6}$$

with  $Z_0$  a constant and  $\xi^Z$  a process in the space  $\mathcal{L}^2(X, P^X, \mathbb{F})$ . Set

$$K^{2}(\Omega, \mathbb{F}, P^{X}, X) := \left\{ \int_{0}^{T} \xi_{s} dX_{s}, \ \xi \in \mathcal{L}^{2}(X, P^{X}, \mathbb{F}) \right\}.$$

$$(7)$$

Since  $\mathcal{F}_0$  is trivial, representation (5) is equivalent to the equality

$$L_0^2(\Omega, \mathcal{F}_T, P^X) = K^2(\Omega, \mathbb{F}, P^X, X).$$
(8)

**Definition 2.1.** A filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}$  on  $(\Omega, \mathcal{F}, P)$  under the standard hypotheses is an enlargement of the filtration  $\mathbb{F}$  if

$$\mathcal{F}_t \subset \mathcal{G}_t \text{ for all } t \in [0,T] \text{ and } \mathcal{F}_t \subsetneq \mathcal{G}_t \text{ for some } t \in [0,T].$$

Let Q be a probability measure on the space  $(\Omega, \mathcal{F})$  and  $\mathcal{M}(Q, \mathbb{F})$  the space of uniformly integrable  $(Q, \mathbb{F})$ -martingales. The following theorem holds.

**Theorem 2.2.** (Theorem 3 in [6]) Let  $\mathbb{G}$  be any enlargement of  $\mathbb{F}$ . Then the following conditions are equivalent

- (i)  $\mathcal{M}(Q, \mathbb{F}) \subset \mathcal{M}(Q, \mathbb{G})$  (immersion property);
- (ii) for any t in [0,T] and any bounded  $\mathcal{F}_T$ -measurable random variable Y,  $E^Q[Y \mid \mathcal{G}_t]$ is  $\mathcal{F}_t$ -measurable.

Under any of these conditions,  $\mathcal{F}_t = \mathcal{F}_T \cap \mathcal{G}_t$ .

# **3** Loss of the predictable representation property: a sufficient condition on the enlarged filtration

Let  $\mathbb{G}$  be any enlargement of  $\mathbb{F}$  and let  $\mathbb{P}(X, \mathbb{G})$  be the set of probability measures on  $(\Omega, \mathcal{G}_T)$  under which X is a  $\mathbb{G}$ -martingale and which are equivalent to  $P|_{\mathcal{G}_T}$ . Consider the assumption

**H2)**  $\mathbb{P}(X,\mathbb{G}) \neq \emptyset$ .

In this section for any Q in  $\mathbb{P}(X, \mathbb{G})$  the space  $K^2(\Omega, \mathbb{G}, Q, X)$  is defined analogously to (7).

A consequence of Theorem 2.2 is the following proposition.

Proposition 3.1. Assume H1) and H2) and set

$$u := \inf\{t \in [0, T] : \mathcal{F}_t \subsetneq \mathcal{G}_t\}.$$
(9)

Then when u = T it holds  $\mathcal{F}_T \subsetneq \mathcal{G}_T$  and when u < T it holds

$$\mathcal{F}_t \subsetneq \mathcal{G}_t \text{ for all } t \in (u, T].$$
(10)

Proof. Note that (9) is well-posed by Definition 2.1. Let Q be an element of  $\mathbb{P}(X, \mathbb{G})$  (existence follows by Hypothesis **H2**)). Since  $\mathcal{F}_T \subset \mathcal{G}_T$ , then  $Q|_{\mathcal{F}_T}$  belongs to  $\mathbb{P}(X, \mathbb{F})$ . Hypothesis **H1**) gives  $Q|_{\mathcal{F}_T} = P^X$  so that  $\mathcal{M}^2(Q, \mathbb{F}) = \mathcal{M}^2(P^X, \mathbb{F})$ . The last equality together with the representation property (6) and the definition of Q implies  $\mathcal{M}^2(Q, \mathbb{F}) \subset \mathcal{M}^2(Q, \mathbb{G})$  and by a density argument the immersion property follows<sup>2</sup>. Then, fixed  $T' \leq T$ , applying Theorem 2.2 equality  $\mathcal{F}_t = \mathcal{F}_{T'} \cap \mathcal{G}_t$  easily follows, for all  $t \in [0, T']$ . The latter identity proves relation (10) in the case u < T. In fact  $\mathcal{F}_{T'} = \mathcal{G}_{T'}$  for some  $T' \in (u, T]$  would imply  $\mathcal{F}_t = \mathcal{G}_t$  for all  $t \in [0, T']$ , in contradiction to the definition of u. Finally when u = T the inclusion in (10) is trivial.

**Remark 3.2.** When  $u = \min\{t \in [0, T] : \mathcal{F}_t \subsetneq \mathcal{G}_t\}$  obviously relation (10) gets stronger, that is it holds  $\mathcal{F}_t \subsetneq \mathcal{G}_t$  for all  $t \in [u, T]$ .

<sup>&</sup>lt;sup>2</sup> see Proposition 3.1 in [24] for a different proof

**Remark 3.3.** The above result implies that if **H1**) holds then condition  $\mathcal{F}_T = \mathcal{G}_T$  forces  $\mathbb{P}(X, \mathbb{G})$  to be void. In financial context this result has a very intuitive appeal. When the existence of a martingale measure is equivalent to the absence of arbitrage (see e.g. [12]), if the market is complete, adding information without changing the payoffs' set generates arbitrage opportunities.

**Theorem 3.4.** Assume that  $\mathcal{G}_0$  is *P*-trivial and that *u* defined by (9) is a minimum. If **H2**) holds, then for any *Q* in  $\mathbb{P}(X, \mathbb{G})$ 

$$K^{2}(\Omega, \mathbb{G}, Q, X) \subsetneq L^{2}_{0}(\Omega, \mathcal{G}_{T}, Q).$$
(11)

*Proof.* Since by assumption  $\mathcal{G}_0$  is *P*-trivial, the hypothesis that u is a minimum implies u > 0. Let Q be an element of  $\mathbb{P}(X, \mathbb{G})$  and let A be a non-trivial set such that  $A \notin \mathcal{F}_u$  and  $A \in \mathcal{G}_u$ . Consider the random variable

$$L := \mathbb{I}_A - E^Q[\mathbb{I}_A \mid \mathcal{F}_u].$$

 $\mathbb{I}_A$  is not  $\mathcal{F}_u$ -measurable, so that L is a non trivial element of  $L^2_0(\Omega, \mathcal{G}_T, Q)$ . If L were in  $K^2(\Omega, \mathbb{G}, Q, X)$  then it would exist a  $\mathbb{G}$ -predictable process  $\eta^L$  in the set  $\mathcal{L}^2(X, Q, \mathbb{G})$  such that

$$L = \int_0^T \eta_s^L dX_s, \quad Q\text{-a.s.}.$$
 (12)

But we claim that L satisfies

$$E^{Q}\left[L\int_{0}^{T}\eta_{s}dX_{s}\right] = 0$$
(13)

for every  $\mathbb{G}$ -predictable  $\eta$  in  $\mathcal{L}^2(X, Q, \mathbb{G})$ . Together with representation (12), this would imply  $E^Q[L^2] = 0$ , i.e. L = 0, Q-a.s..

In order to prove (13) it is useful to rewrite  $E^Q \left[ L \int_0^T \eta_s dX_s \right]$  as

$$E^{Q}\left[E^{Q}\left[L\left(\int_{0}^{T}\mathbb{I}_{s< u}\eta_{s}dX_{s}+\eta_{u}\Delta X_{u}\right)\mid\mathcal{F}_{u}\right]\right]+E^{Q}\left[E^{Q}\left[L\int_{0}^{T}\mathbb{I}_{s>u}\eta_{s}dX_{s}\mid\mathcal{G}_{u}\right]\right].$$

Then equality (13) immediately follows taking into account that  $\int_0^T \mathbb{I}_{s < u} \eta_s dX_s$  and  $\eta_u \Delta X_u$  are  $\mathcal{F}_u$ -measurable and L is  $\mathcal{G}_u$ -measurable so that  $E^Q \left[ L \int_0^T \eta_s dX_s \right]$  coincides with

$$E^{Q}\left[\left(\int_{0}^{u^{-}}\eta_{s}dX_{s}+\eta_{u}\Delta X_{u}\right) E^{Q}\left[L\mid\mathcal{F}_{u}\right]\right]+E^{Q}\left[LE^{Q}\left[\int_{u^{+}}^{T}\eta_{s}dX_{s}\mid\mathcal{G}_{u}\right]\right]$$

and finally considering that  $E^{Q}[L \mid \mathcal{F}_{u}] = 0 = E^{Q}\left[\int_{u^{+}}^{T} \eta_{s} dX_{s} \mid \mathcal{G}_{u}\right].$ 

**Corollary 3.5.** Assume **H1**). Then, under the hypotheses of the previous theorem, the *p.r.p.* for X is not preserved by the enlargement of filtration from  $\mathbb{F}$  to  $\mathbb{G}$ .

*Proof.* X enjoys p.r.p. with respect to  $\mathbb{F}$  under  $Q|_{\mathcal{F}_T} = P^X$  but the strict inclusion (11) implies that X doesn't enjoy the (Q,  $\mathbb{G}$ )-p.r.p..

**Remark 3.6.** Theorem 3.4 cannot apply when  $\mathbb{G}$  is a quasi-left continuous filtration. In fact, in this case, since  $\mathcal{G}_u = \mathcal{G}_{u^-} = \mathcal{F}_{u^-}$ , the assumption that u is a minimum would imply the existence of a non trivial set in  $\mathcal{F}_{u^-}$  but not in  $\mathcal{F}_u$ .

As an immediate application of Theorem 3.4 one can take the progressive enlargement defined by  $\mathcal{G}_t = \bigcap_{s>t} \mathcal{F}_s \lor \sigma(\tau \land s)$  with  $\tau$  any positive random variable independent of  $\mathcal{F}_T$  taking values in a finite set.

**Remark 3.7.** Assumption that u is a minimum in Theorem 3.4 cannot be dropped as proved by the following example.

Let B be a Brownian motion, then the process  $\int_0^{\cdot} sgn(B_s)dB_s$  is still a Brownian motion enjoying both the  $\mathbb{F}^{|B|}$ -p.r.p. and the  $\mathbb{F}^B$ -p.r.p. (see Chapter 6 in [30]). Moreover

$$\inf\{t \in [0,T] : \mathcal{F}_t^{|B|} \subsetneq \mathcal{F}_t^B\} = 0,$$

and the set  $\{t \in [0,T] : \mathcal{F}_t^{|B|} \subsetneq \mathcal{F}_t^B\}$  doesn't have a minimum. However the p.r.p. of  $\int_0^{\cdot} sgn(B_s) dB_s$  transfers from  $\mathbb{F}^{|B|}$  to  $\mathbb{F}^B$ .

# 4 Adding the reference filtration of a semi-martingale

In this section  $\mathbb{H}$  is a filtration on  $(\Omega, \mathcal{F}, P)$  satisfying the usual conditions of rightcontinuity and completeness with  $\mathcal{H}_0$  a trivial  $\sigma$ -algebra. Moreover Y is a  $(P, \mathbb{H})$ -semimartingale enjoying the p.r.p. with respect to  $\mathbb{H}$  under an equivalent martingale measure. The enlarged filtration is defined by

$$\mathbb{G} := \mathbb{F} \vee \mathbb{H}.$$

Next it is proved that there exists  $Q \in \mathbb{P}(X, Y, [X, Y], \mathbb{G})$  such that (X, Y, [X, Y]) enjoys the  $(Q, \mathbb{G})$ -p.r.p..

As a byproduct of this result a martingale representation property is derived: any  $(P, \mathbb{G})$ square-integrable martingale can be uniquely represented up to a constant as sum of integrals with respect to the martingales M, N and [M, N], where M and N are the martingale parts of X and Y, respectively. This result is closely related to that presented in [36]. In that paper the author works with the filtrations generated by two independent, quasi-left continuous semi-martingales. Here this regularity for the trajectories is not required and in place of the independence of the two semi-martingales the strong orthogonality of their martingale parts is assumed. Indeed, under the assumptions of this section, independence of  $\mathbb{F}$  and  $\mathbb{H}$  and strong orthogonality of M and N are equivalent conditions (see point i) in Theorem 4.13).

First, the case when X and Y are  $(P, \mathbb{G})$ -strongly orthogonal martingales is considered. Then the general case follows by using a key result: the martingale parts M and N of X and Y enjoy the  $(P, \mathbb{F})$ -p.r.p. and the  $(P, \mathbb{H})$ -p.r.p. respectively.

#### 4.1 The martingale case

First of all consider the following particular case: the process A in decomposition (1) is identically zero and therefore X coincides with M. Let N be a square-integrable  $(P, \mathbb{H})$ martingale on  $(\Omega, \mathcal{F}, P)$ . Make the following assumption

**H1'**) 
$$\mathbb{P}(M, \mathbb{F}) = \{P_{|\mathcal{F}_T}\}, \mathbb{P}(N, \mathbb{H}) = \{P_{|\mathcal{H}_T}\}.$$

It is useful to recall the definition of strongly orthogonal square-integrable martingales.

**Definition 4.1.** ([33]) Two square-integrable  $(P, \mathbb{G})$ -martingales U and V are  $(P, \mathbb{G})$ strongly orthogonal if their product UV is a uniformly integrable  $(P, \mathbb{G})$ -martingale such that  $U_0V_0 = 0$ .

**Lemma 4.2.** Under hypothesis H1')  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are *P*-independent and  $M_0N_0 = 0$  if and only if *M* and *N* are  $(P, \mathbb{G})$ -martingales and  $(P, \mathbb{G})$ -strongly orthogonal.

*Proof.* The necessary part of the statement is straightforward, since independence together with condition  $M_0N_0 = 0$  implies strong orthogonality of M and N.

In order to prove the sufficient condition recall that, since M and N are strongly orthogonal  $(P, \mathbb{G})$ -martingales, the process  $[M, N] = ([M, N]_t)_{t \in [0,T]}$  is a  $(P, \mathbb{G})$ -martingale. Moreover by **H1'** it follows that if  $A \in \mathcal{F}_T$  and  $B \in \mathcal{H}_T$  then

$$\mathbb{I}_A = P(A) + \int_0^T \xi_s^A dM_s, \quad \mathbb{I}_B = P(B) + \int_0^T \xi_s^B dN_s, \quad P\text{-a.s.}$$
(14)

for  $\xi^A$  and  $\xi^B$  in  $\mathcal{L}^2(M, P, \mathbb{F})$  and  $\mathcal{L}^2(N, P, \mathbb{H})$  respectively. The equalities in (14) imply that  $P(A \cap B)$  differs from P(A)P(B) by the expression

$$P(B)E^{P}\left[\int_{0}^{T}\xi_{s}^{A}dM_{s}\right] + P(A)E^{P}\left[\int_{0}^{T}\xi_{s}^{B}dN_{s}\right] + E^{P}\left[\int_{0}^{T}\xi_{s}^{A}dM_{s}\int_{0}^{T}\xi_{s}^{B}dN_{s}\right].$$

The above expression is null. In fact the  $(P, \mathbb{G})$ -martingale property of M and N and the integrability of the integrands  $\xi^A$  and  $\xi^B$  imply that the processes  $\int_0^{\cdot} \xi_s^A dM_s$  and  $\int_0^{\cdot} \xi_s^B dN_s$  are centered  $(P, \mathbb{G})$ -martingales. Moreover the stable subspaces generated by M and N respectively, are  $(P, \mathbb{G})$ -strongly orthogonal so that also the product  $\int_0^{\cdot} \xi_s^A dM_s \cdot \int_0^{\cdot} \xi_s^B dN_s$  is a centered  $(P, \mathbb{G})$ -martingale (see Lemma 2 and Theorem 36 page 180 in [33]).

Before stating the main theorem of this section it is convenient to recall two general results.

**Lemma 4.3.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be two independent filtrations on  $(\Omega, \mathcal{F}, P)$  and let U and V be two real processes  $\mathbb{A}$ -adapted and  $\mathbb{B}$ -adapted respectively. Then for all 0 < s < t

$$E^{P}\left[U_{t}V_{t}|\mathcal{A}_{s} \vee \mathcal{B}_{s}\right] = E^{P}\left[U_{t}|\mathcal{A}_{s}\right]E^{P}\left[V_{t}|\mathcal{B}_{s}\right].$$

**Lemma 4.4.** Let  $\mathbb{A}$  and  $\mathbb{B}$  be two filtrations on  $(\Omega, \mathcal{F}, P)$  under standard hypotheses of completeness and right-continuity. Consider the filtration  $\mathbb{D}$  defined at any time t by  $\mathcal{D}_t = \mathcal{A}_t \vee \mathcal{B}_t$ . If there exists a probability measure Q equivalent to P such that  $\mathbb{A}$  and  $\mathbb{B}$ are Q-independent, then  $\mathbb{D}$  satisfies the standard hypotheses.

*Proof.* See Lemma 2.2 in [2].

**Theorem 4.5.** Assume **H1'**) and suppose that M and N are  $(P, \mathbb{G})$ -strongly orthogonal martingales. Then  $\mathbb{G}$  is a standard filtration,  $\mathbb{P}((M, N, [M, N]), \mathbb{G}) = \{P|_{\mathcal{G}_T}\}$  and the following decomposition holds

$$L_0^2(\Omega, \mathcal{G}_T, P) = K^2(\Omega, \mathbb{G}, P, M) \oplus K^2(\Omega, \mathbb{G}, P, N) \oplus K^2(\Omega, \mathbb{G}, P, [M, N]).$$
(15)

*Proof.* The proof will be done in three steps:

(i) the first goal is to prove the  $(P, \mathbb{G})$ -p.r.p. for (M, N, [M, N]);

(ii) as a second point the following key result is proved: [M, N] is  $(P, \mathbb{G})$ -strongly orthogonal to M and to N;

- (iii) finally points (i) and (ii) allow to derive decomposition (15).
  - (i)  $(P, \mathbb{G})$ -strong orthogonality of M and N and Lemma 4.2 provide the P-independence of  $\mathcal{F}_T$  and  $\mathcal{H}_T$ , so that the standard conditions for  $\mathbb{G}$  immediately follow by Lemma 4.4. The  $(P, \mathbb{G})$ -p.r.p. for (M, N, [M, N]) is achieved by proving that  $\mathbb{P}((M, N, [M, N]), \mathbb{G}) =$  $\{P\}$ , or, equivalently, that for any  $Q \in \mathbb{P}((M, N, [M, N]), \mathbb{G}), P$  and Q coincide on the  $\pi$ -system

$$\{A \cap B, A \in \mathcal{F}_T, B \in \mathcal{H}_T\},\$$

which generates  $\mathcal{G}_T$ . To this end, note that equalities in (14) hold under Q so that  $Q(A \cap B)$  differs from P(A)P(B) by the expression

$$P(B)E^{Q}\left[\int_{0}^{T}\xi_{s}^{A}dM_{s}\right] + P(A)E^{Q}\left[\int_{0}^{T}\xi_{s}^{B}dN_{s}\right] + E^{Q}\left[\int_{0}^{T}\xi_{s}^{A}dM_{s}\int_{0}^{T}\xi_{s}^{B}dN_{s}\right].$$
(16)

The above expression is null. In fact **H1'**) implies  $Q|_{\mathcal{F}_T} = P|_{\mathcal{F}_T}$  and  $Q|_{\mathcal{H}_T} = P|_{\mathcal{H}_T}$  and this in turn implies that  $\int_0^{\cdot} \xi_s^A dM_s$  and  $\int_0^{\cdot} \xi_s^B dN_s$  are centered  $(Q, \mathbb{G})$ -martingales. Moreover, by definition of Q, [M, N] is a  $(Q, \mathbb{G})$ -martingale so that MN is a  $(Q, \mathbb{G})$ -martingale and therefore M and N are  $(Q, \mathbb{G})$ -strongly orthogonal martingales, so that also the product  $\int_0^{\cdot} \xi_s^A dM_s \cdot \int_0^{\cdot} \xi_s^B dN_s$  is a centered  $(Q, \mathbb{G})$ -martingale.

(ii) [M, N] is (P, G)-strongly orthogonal to the (P, G)-martingales M and N, if and only if [M, [M, N]] and [N, [M, N]] are uniformly integrable (P, G)-martingales. Recall that

$$[M,N]_t = \langle M^c, N^c \rangle_t + \sum_{s \le t} \Delta M_s \Delta N_s, \qquad (17)$$

where  $M^c$  and  $N^c$  are the continuous martingale part of M and N respectively. By Lemma 4.2  $M^c$  and  $N^c$  are independent  $(P, \mathbb{G})$ -martingales so that  $\langle M^c, N^c \rangle \equiv 0$ , since by definition  $\langle M^c, N^c \rangle$  is the unique  $\mathbb{G}$ -predictable process with finite variation such that  $M^c N^c - \langle M^c, N^c \rangle$  is  $\mathbb{G}$ -local martingale equal to 0 at time 0 (see Subsection 9.3.2. in [26]). Therefore

$$[M,N]_t = \sum_{s \le t} \Delta M_s \Delta N_s.$$
(18)

As a consequence

$$[M, [M, N]]_t = \sum_{s \le t} (\Delta M_s)^2 \Delta N_s.$$

Then for  $u \leq t$  one has

$$E^{P} \left[ \left[ M, \left[ M, N \right] \right]_{t} | \mathcal{G}_{u} \right] \\= E^{P} \left[ \sum_{s \leq u} (\Delta M_{s})^{2} \Delta N_{s} | \mathcal{G}_{u} \right] + E^{P} \left[ \sum_{u < s \leq t} (\Delta M_{s})^{2} \Delta N_{s} | \mathcal{G}_{u} \right] \\= \left[ M, \left[ M, N \right] \right]_{u} + \sum_{u < s \leq t} E^{P} \left[ (\Delta M_{s})^{2} \Delta N_{s} | \mathcal{G}_{u} \right] \\= \left[ M, \left[ M, N \right] \right]_{u} + \sum_{u < s \leq t} E^{P} \left[ (\Delta M_{s})^{2} | \mathcal{F}_{u} \right] E^{P} \left[ \Delta N_{s} | \mathcal{H}_{u} \right],$$

where the last equality follows by Lemma 4.3 since  $\mathbb{F}$  and  $\mathbb{H}$  are *P*-independent. Then the martingale property for [M, [M, N]] follows by observing that  $E^P[\Delta N_s | \mathcal{H}_u] = 0$ , for any s > u. Finally [M, [M, N]] is uniformly integrable, since it is a  $(P, \mathbb{G})$ -regular martingale.

Analogously one gets that [M, N] is  $(P, \mathbb{G})$ -strongly orthogonal to N.

(iii) By point (i) it follows that (see (8))

$$L_0^2(\Omega, \mathcal{G}_T, P) = K^2(\Omega, \mathbb{G}, P, (M, N, [M, N]))$$

or equivalently that for each H in  $L^2_0(\Omega, \mathcal{G}_T, P)$  there exist  $\gamma^H$  in  $\mathcal{L}^2(M, P, \mathbb{G})$ ,  $\kappa^H$  in  $\mathcal{L}^2(N, P, \mathbb{G})$  and  $\phi^H$  in  $\mathcal{L}^2([M, N], P, \mathbb{G})$ , such that P-a.s.

$$H = \int_0^T \gamma_s^H dM_s + \int_0^T \kappa_s^H dN_s + \int_0^T \phi_s^H d[M, N]_s.$$
(19)

This equality and point (ii) entail

$$K^{2}(\Omega, \mathbb{G}, P, [M, N]) = \left(K^{2}(\Omega, \mathbb{G}, P, (M, N))\right)^{\perp}$$

In fact the  $(P, \mathbb{G})$ -strong orthogonality of [M, N] to M and to N is equivalent to the orthogonality of any random variable of the form  $\int_0^T \phi_s d[M, N]_s$ , with  $\phi$  in  $\mathcal{L}^2([M, N], P, \mathbb{G})$ , to random variables of the form  $\int_0^T \gamma_s dM_s + \int_0^T \kappa_s dN_s$ , with  $\gamma$  in  $\mathcal{L}^2(M, P, \mathbb{G})$  and  $\kappa$  in  $\mathcal{L}^2(N, P, \mathbb{G})$  (see Lemma 2 and Theorem 36 page 180 in [33]), so that

$$K^{2}(\Omega, \mathbb{G}, P, [M, N]) \subset \left(K^{2}(\Omega, \mathbb{G}, P, (M, N))\right)^{\perp}$$

At the same time, by representation (19), any element of  $\left(K^2(\Omega, \mathbb{G}, P, (M, N))\right)^{\perp}$ is of the form  $\int_0^T \phi_s d[M, N]_s$  so that

$$\left(K^2(\Omega, \mathbb{G}, P, (M, N))\right)^{\perp} \subset K^2(\Omega, \mathbb{G}, P, [M, N]).$$

Finally, by the  $(P, \mathbb{G})$ -strong orthogonality of M and N,

$$K^{2}(\Omega, \mathbb{G}, P, (M, N)) = K^{2}(\Omega, \mathbb{G}, P, M) \oplus K^{2}(\Omega, \mathbb{G}, P, N).$$

**Remark 4.6.** Lemma 4.2 and Theorem 4.5 can be extended to the case when M and N take values in  $\mathbb{R}^m$  and  $\mathbb{R}^n$  respectively and  $M^i$  and  $N^j$  are  $(P, \mathbb{G})$ -strongly orthogonal martingales for all i = 1, ..., m, j = 1, ..., n.

For the sake of completeness, recall that an  $\mathbb{R}^m$ -valued square integrable martingale M enjoys the  $(P, \mathbb{F})$ -p.r.p. if each H in  $L^2(\Omega, \mathcal{F}_T, P)$  can be represented as vector stochastic integral that is

$$H = H_0 + \int_0^T \xi^H dM,$$

with  $H_0 \in \mathcal{F}_0$  and  $\xi^H = (\xi_1^H, ..., \xi_m^H)$  an m-dimensional  $\mathbb{F}$ -predictable process such that such that

$$E^P\left[\sum_{i,j}\int_0^T \xi_i^H(t)\xi_j^H(t)\,d[M^i,M^j]_t\right] < +\infty,$$

(see [7]).

On a probability space  $(\Omega, \mathcal{F}, P)$  let M be the process defined at time t in [0, T] by

$$M_t := B_t + Z \mathbb{I}_{\{t \ge t_0\}},\tag{20}$$

where B is a standard Brownian motion and Z a binary centered  $\mathcal{F}$ -measurable random variable independent of B. Write for shortness

$$H_t := Z \mathbb{I}_{\{t \ge t_0\}}.$$

Following a similar procedure as in Section 9.5.2 in [26], it can be proved that M enjoys the p.r.p. with respect to its natural filtration  $\mathbb{F}^M = \mathbb{F}^B \vee \mathbb{F}^H$  (under P).

In fact consider the Doléans-Dade exponentials

$$\mathcal{E}^{1,\varphi}_{\cdot} := \mathcal{E}\Big(\int_0^{\cdot} \varphi_s dB_s\Big), \quad \mathcal{E}^{2,\varphi}_{\cdot} := \mathcal{E}\Big(\int_0^{\cdot} \varphi_s dH_s\Big),$$

with  $\varphi \in L^2([0,T])$ . Both are square-integrable random processes and in particular  $\mathcal{E}\left(\int_0^t \varphi_s dH_s\right) = 1 + \varphi_{t_0} \mathbb{Z} \mathbb{I}_{\{t \ge t_0\}}$ . Moreover, by the product formula it follows

$$\mathcal{E}_T^{1,\varphi} \mathcal{E}_T^{2,\varphi} = 1 + \int_0^T \mathcal{E}_t^{1,\varphi} \mathcal{E}_{t^-}^{2,\varphi} \varphi_t dM_t.$$
(21)

The set  $\{\mathcal{E}_T^{1,\varphi}\mathcal{E}_T^{2,\varphi}, \varphi \in L^2([0,T])\}$  is a total set for  $L^2(\Omega, \mathcal{F}_T^M, P)$  and therefore (21) is equivalent to the p.r.p. for M with respect to  $\mathbb{F}^M$  (under P).

Let now N be defined by

$$N_t := B_t + U \mathbb{I}_{\{t \ge t_0\}} \tag{22}$$

and assume  $\tilde{B}$  and U independent of B and Z.

Then  $\mathbb{F}^M$  and  $\mathbb{F}^N$  are independent filtrations and M and N are square-integrable  $\mathbb{F}^M \vee \mathbb{F}^N$ -strongly orthogonal martingales each of them enjoying the p.r.p. with respect

to its natural filtration (under P). Moreover the covariation process [M, N] at time t satisfies

$$[M,N]_t = ZU\mathbb{I}_{\{t \ge t_0\}}$$

Theorem 4.5 applies with  $\mathbb{F} = \mathbb{F}^M$  and  $\mathbb{H} = \mathbb{F}^N$  so that any K in  $L^2(\Omega, \mathcal{F}_T^M \vee \mathcal{F}_T^N, P)$  can be represented P-a.s. as

$$K = K_0 + \int_0^T \gamma_t^K dM_t + \int_0^T \kappa_t^K dN_t + \Phi^K ZU,$$
 (23)

where  $\gamma^{K}$  and  $\kappa^{K}$  belong to  $\mathcal{L}^{2}(M, P, \mathbb{F}^{M} \vee \mathbb{F}^{N})$  and  $\mathcal{L}^{2}(N, P, \mathbb{F}^{M} \vee \mathbb{F}^{N})$  respectively and  $\Phi^{H}$  is a square-integrable random variable  $\mathcal{F}_{t_{0}^{-}}^{M} \vee \mathcal{F}_{t_{0}^{-}}^{N}$ -measurable.

#### 4.2 The semi-martingale case

In the general setup of this section let Y be in the space of semi-martingales  $\mathcal{S}^2(P, \mathbb{H})$ with canonical decomposition

$$Y = Y_0 + N + D. (24)$$

Assume that  $\mathbb{P}(Y,\mathbb{H})$  is a singleton and more precisely  $\mathbb{P}(Y,\mathbb{H}) = \{P^Y\}$  so that  $\mathcal{H}_0$  is P-trivial and the semi-martingale Y enjoys the p.r.p. with respect to  $\mathbb{H}$  under  $P^Y$ .

It is convenient to write the main assumptions on X and Y as a unique condition:

**H1**") 
$$\mathbb{P}(X,\mathbb{F}) = \{P^X\}, \mathbb{P}(Y,\mathbb{H}) = \{P^Y\}$$

Some other assumptions will be considered. Denote by  $\overline{K^X(\mathbb{F})}$  the closure in  $L^1(\Omega, \mathcal{F}_T, P)$ of the set  $\left\{\int_0^T \xi_s dX_s, \ \xi \ \mathbb{F}$ -predictable, simple and bounded  $\right\}$  and by  $L^1_+(\Omega, \mathcal{F}_T, P)$  the set of the non-negative integrable random variables. Define analogously  $\overline{K^Y(\mathbb{H})}$  and  $L^1_+(\Omega, \mathcal{H}_T, P)$ .

**H3**) 
$$\overline{K^X(\mathbb{F})} \cap L^1_+(\Omega, \mathcal{F}_T, P) = \{0\}, \quad \overline{K^Y(\mathbb{H})} \cap L^1_+(\Omega, \mathcal{H}_T, P) = \{0\},$$

Assumption **H3**) together with condition (3) and its analogous for Y provide the socalled *structure condition* for X and Y (see Theorem 8 in [3]), that is the existence of a predictable process  $\alpha$  P-a.s. in the space  $L^2([0,T], \mathcal{B}([0,T]), d\langle M \rangle_t)$  and of a predictable process  $\delta$  P-a.s. in the space  $L^2([0,T], \mathcal{B}([0,T]), d\langle M \rangle_t)$  such that

$$A_t = \int_0^t \alpha_s \, d\langle M \rangle_s, \qquad D_t = \int_0^t \delta_s \, d\langle N \rangle_s. \tag{25}$$

**Remark 4.7.** It is to note that in order to get the structure condition for X hypothesis **H3**) could be omitted by assuming  $dP^X/dP|_{\mathcal{F}_T}$  in  $L^2_{loc}(\Omega, \mathcal{F}, P)$  (see Proposition 4 in [34]).

Finally two hypotheses assure regularity to the Doléans-Dade exponentials  $\mathcal{E}(-\int_0^{\cdot} \alpha_s dM_s)$ and  $\mathcal{E}(-\int_0^{\cdot} \delta_s dN_s)$ .

H4)  $\alpha \Delta M < 1$ , *P*-a.s. ,  $\delta \Delta N < 1$ , *P*-a.s. , H5)

$$E^{P}\left[\exp\left\{\frac{1}{2}\int_{0}^{T}\alpha_{t}^{2}d\langle M^{c}\rangle_{t}+\int_{0}^{T}\alpha_{t}^{2}d\langle M^{d}\rangle_{t}\right\}\right]<+\infty,$$
$$E^{P}\left[\exp\left\{\frac{1}{2}\int_{0}^{T}\delta_{t}^{2}d\langle N^{c}\rangle_{t}+\int_{0}^{T}\delta_{t}^{2}d\langle N^{d}\rangle_{t}\right\}\right]<+\infty.$$

In particular **H4**) implies that the  $(P, \mathbb{F})$ -local martingale  $\mathcal{E}(-\int_0^{\cdot} \alpha_s dM_s)$  is strictly positive and **H5**) implies that it is a  $(P, \mathbb{F})$ -martingale (see Theorem 9 in [32] for further details).

**Definition 4.8.** ([4]) A measure Q in  $\mathbb{P}(X, \mathbb{F})$  is a minimal martingale measure for X if any  $(P, \mathbb{F})$ -local martingale Z such that ZM is a  $(P, \mathbb{F})$ -local martingale is a  $(Q, \mathbb{F})$ -local martingale.

**Lemma 4.9.** Under the previous assumptions  $P^X$  is the minimal martingale measure for X.

*Proof.* By Proposition 3.1 in [4]  $\mathcal{E}(-\int_0^{\cdot} \alpha_s dM_s)$  coincides with the derivative of the minimal martingale measure for X. The assumed uniqueness of the equivalent martingale measure gives the result.

Before stating next result, it is useful to recall the general result known as Yoeurp's lemma.

**Lemma 4.10.** ([13])Let M be a local martingale null at time zero and A a càdlàg process with finite variation on every compact set. If A is predictable then [M, A] is a local martingale.

**Proposition 4.11.** Let H1''), H3), H4) and H5) be verified. Then M enjoys the p.r.p. with respect to the filtration  $\mathbb{F}$  under P.

Proof. Assume that  $K^2(\Omega, \mathbb{F}, P, M)^{\perp} \neq \{0\}$ , that is there exists a non trivial centered random variable  $V \in K^2(\Omega, \mathbb{F}, P, M)^{\perp}$  such that the  $(P, \mathbb{F})$ -martingale  $(V_t)_{t \in [0,T]}$  defined by  $V_t := E^P[V|\mathcal{F}_t]$  is  $(P, \mathbb{F})$ -strongly orthogonal to M. By Lemma 4.9  $(V_t)_{t \in [0,T]}$  is a  $(P^X, \mathbb{F})$ -local martingale. Then by **H1**") there exists a predictable process  $\xi$  such that, for all  $t \in [0, T]$ ,  $P^X$ -a.s.

$$V_t = \int_0^t \xi_u dX_u. \tag{26}$$

As a consequence, since the covariation processes are invariant under an equivalent change of measure,  $P^X$ -a.s. and P-a.s.

$$[V,X]_t = \int_0^t \xi_u d[X]_u.$$

Under P the process on the left hand side in the previous equality is a  $(P, \mathbb{F})$ -local martingale. In fact under P one has [V, X] = [V, M] + [V, A] and by construction [V, M] is a  $(P, \mathbb{F})$ -martingale and by Lemma 4.10 the process [V, A] is a  $(P, \mathbb{F})$ -local martingale. More precisely [V, X] is a  $(P, \mathbb{F})$ -martingale since for all t in [0, T]

$$|[V, X]_t| \le [V + X]_T + [V - X]_T$$

and the right hand side of the above inequality is integrable by the assumption (2) on X and the construction of  $(V_t)_{t \in [0,T]}$ . Then also the process  $(\int_0^t \xi_u d[X]_u)_{t \in [0,T]}$  is a  $(P, \mathbb{F})$ -martingale so that for all 0 < s < t < T and  $B \in \mathcal{F}_s$  it holds

$$E^P\left[I_B\int_s^t \xi_u d[X]_u\right] = 0$$

and therefore

$$\xi = 0, P(d\omega)d[X]_t(\omega)$$
-a.s.

Finally by the equivalence between P and  $P^X$  and the invariance of [X] under equivalent change of measure

$$\xi = 0, P^X(d\omega)d[X]_t(\omega)$$
-a.s. .

As a consequence by Ito isometry

$$\int_0^t \xi_u dX_u = 0, \quad P^X \text{-a.s.}$$
(27)

that is, by equality (26),  $V_T = 0$ ,  $P^X$ -a.s. which contradicts the supposed non-triviality of V.

**Remark 4.12.** In the particular case when X is the solution of an Ito equation with unique weak solution and  $\mathbb{F} = \mathbb{F}^X$  the above result immediately follows from Theorem 9.5.4.2 in [26].

**Theorem 4.13.** Assume H1''), H3),H4), H5) and suppose that M and N are  $(P, \mathbb{G})$ -strongly orthogonal martingales. Then

i)  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are *P*-independent,  $\mathbb{G}$  fulfills the standard hypotheses and every *W* in  $\mathcal{M}^2(P,\mathbb{G})$  can be uniquely represented as

$$W_{t} = W_{0} + \int_{0}^{t} \gamma_{s}^{W} dM_{s} + \int_{0}^{t} \kappa_{s}^{W} dN_{s} + \int_{0}^{t} \phi_{s}^{W} d[M, N]_{s}, \quad P\text{-}a.s.$$

with  $\gamma^W$  in  $\mathcal{L}^2(M, P, \mathbb{G})$ ,  $\kappa^W$  in  $\mathcal{L}^2(N, P, \mathbb{G})$  and  $\phi^W$  in  $\mathcal{L}^2([M, N], P, \mathbb{G})$ ;

ii) there exists a probability measure Q on  $(\Omega, \mathcal{G}_T)$  such that (X, Y, [X, Y]) enjoys the p.r.p. with respect to  $\mathbb{G}$  under Q. More precisely every Z in  $\mathcal{M}^2(Q, \mathbb{G})$  can be uniquely represented as

$$Z_{t} = Z_{0} + \int_{0}^{t} \eta_{s}^{Z} dX_{s} + \int_{0}^{t} \theta_{s}^{Z} dY_{s} + \int_{0}^{t} \zeta_{s}^{Z} d[X, Y]_{s} \quad Q\text{-}a.s.,$$

with  $\eta^Z$  in  $\mathcal{L}^2(X, Q, \mathbb{G})$ ,  $\theta^Z$  in  $\mathcal{L}^2(Y, Q, \mathbb{G})$  and  $\zeta^Z$  in  $\mathcal{L}^2([X, Y], Q, \mathbb{G})$ .

*Proof.* By the previous proposition and its analogous for N, the martingales M and N satisfy condition **H1'**) so that Theorem 4.5 applies and the first statement is proved. Define Q on  $(\Omega, \mathcal{G}_T)$  by

$$\frac{dQ}{dP} := L^X \cdot L^Y$$

where

$$L^X := \frac{dP^X}{dP|_{\mathcal{F}_T}}, \qquad L^Y := \frac{dP^Y}{dP|_{\mathcal{H}_T}}.$$

The definition is well-posed since by point i)  $L^X \cdot L^Y$  is in  $L^1(\Omega, P, \mathcal{G}_T)$ .  $L^X$  and  $L^Y$  are strictly positive and therefore Q and  $P|_{\mathcal{G}_T}$  are equivalent measures. Moreover for all A in  $\mathcal{F}_T$  and B in  $\mathcal{H}_T$  it holds

$$Q(A \cap B) = E^P[\mathbb{I}_A L^X] E^P[\mathbb{I}_B L^Y],$$

since  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are independent under *P*. Using the equalities  $E^P[L^X] = 1 = E^P[L^Y]$  one immediately gets the *Q*-independence of  $\mathcal{F}_T$  and  $\mathcal{H}_T$ .

Finally X is a  $(Q, \mathbb{F})$ -martingale since  $Q|_{\mathbb{F}} = P^X$  and it is also a  $(Q, \mathbb{G})$ -martingale by the Q-independence of  $\mathbb{F}$  and  $\mathbb{H}$ . Analogously it can be shown that Y is a  $(Q, \mathbb{G})$ martingale. Since  $X_0$  and  $Y_0$  are constants, the second statement follows by applying Theorem 4.5 to the  $(Q, \mathbb{G})$ -strongly orthogonal martingales  $X - X_0$  and  $Y - Y_0$ .  $\Box$ 

**Remark 4.14.** Under the hypotheses of the above theorem **H2**) is verified. In fact Q belongs to  $\mathbb{P}(X, \mathbb{G})$ . This immediately follows recalling that Q is equivalent to  $P|_{\mathcal{G}_T}$ , coincides with  $P^X$  on  $\mathcal{F}_T$  and decouples  $\mathbb{F}$  and  $\mathbb{H}$  in such a way that the immersion property is verified.

**Remark 4.15.** The  $(P, \mathbb{G})$ -strong orthogonality of M and N in Theorem 4.13 can be weakened assuming the existence of a measure  $P^*$  equivalent to P such that M and Nare  $(P^*, \mathbb{G})$ -strongly orthogonal martingales. In this case, P in i) has to be replaced by  $P^*$ and Q in ii) has to be changed into the measure defined by  $\frac{dP^X}{dP^*|_{\mathcal{F}_T}} \frac{dP^Y}{dP^*|_{\mathcal{H}_T}} dP^*$ . Note that by Lemma 4.2  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are  $P^*$ -independent. On the other hand if there exists a measure R equivalent to P such that  $\mathcal{F}_T$  and  $\mathcal{H}_T$  are R-independent, then  $\frac{dP|_{\mathcal{F}_T}}{dR|_{\mathcal{F}_T}} \frac{dP|_{\mathcal{H}_T}}{dR|_{\mathcal{H}_T}} dR$  defines a measure with the same properties as the previous  $P^*$ .

Finally when  $\mathbb{H}$  coincides with  $\mathbb{F}^{Y}$  and Y is P-independent of  $\mathbb{F}$ , then the above conditions are verified with  $P^* = R = P$ .

**Corollary 4.16.** Under the hypotheses of the previous theorem, X and Y verify  $[X, Y]_t \equiv 0$  P-a.s. if and only if the  $(P, \mathbb{G})$ -semi-martingale (X, Y) enjoys the p.r.p. with respect to  $\mathbb{G}$ .

**Remark 4.17.** Note that the vanishing of the quadratic covariation as sufficient condition for the p.r.p. of a pair of orthogonal martingales each enjoying the p.r.p. with respect to its own filtration is already known (see [29]).

Corollary 4.16 implies that if X and Y in Theorem 4.13 are quasi-left continuous then the pair (X, Y) enjoys the p.r.p. with respect to G under Q. In fact a semi-martingale is quasi-left continuous if and only its jumps times are totally inaccessible so that, using

$$[X,Y]_t = \langle M^c, N^c \rangle_t + \sum_{s \le t} \Delta X_s \Delta Y_s$$
(28)

and the *P*-independence of  $\mathbb{F}$  and  $\mathbb{H}$ , it follows that  $[X, Y] \equiv 0$ . In particular the second addend at the right hand side is zero, since two independent processes cannot have common inaccessible jump times with positive probability.

In conclusion, under the hypotheses of Theorem 4.13,  $[X, Y] \neq 0$  if and only if X and Y share accessible jump times with positive probability.

Along the lines of Example 4.1 it is easy to construct a pair (X, Y) of semi-martingales satisfying H1"), H4) and H5) with strongly orthogonal martingale parts and covariation process not identically zero. In this case the structure condition is given by the model so that assumption H3) is superfluous.

Let M be defined as in (20). Note that

$$\langle M \rangle_t = t + E^P[Z^2 \mid \mathcal{F}_{t-}] \mathbb{I}_{\{t \ge t_0\}}$$

Consider a continuous function  $(\alpha_t)_{t \in [0,T]}$  in  $L^2([0,T])$  such that  $\alpha_{t_0} = 0$ . Then  $\int_0^T \alpha_s d\langle M \rangle_s = \int_0^T \alpha_s ds$ . Define the  $(P, \mathbb{F}^M)$ -semi-martingale X by

$$X_t := B_t + \int_0^t \alpha_s \, ds + Z \, \mathbb{I}_{\{t \ge t_0\}}.$$

X satisfies the structure condition with martingale part equal to M as well as conditions **H4**) and **H5**). Moreover X enjoys the p.r.p. with respect to  $\mathbb{F}^M$  under the measure  $P^X$  defined by

$$dP^X := L dP|_{\mathcal{F}^M_{\mathcal{T}}}$$

with  $L := \mathcal{E}(-\int_0^T \alpha_s dB_s)$ . The last statement follows by Example 4.1 considering that  $\mathbb{F}^M = \mathbb{F}^B \vee \mathbb{F}^H$ , the process  $(B_t + \int_0^t \alpha_s ds)_{t \in [0,T]}$  under  $P^X$  is a standard Brownian motion independent of Z and its natural filtration coincides with  $\mathbb{F}^B$  since  $\alpha$  is deterministic. Analogously, define a second semi-martingale

$$Y_t := \tilde{B}_t + \int_0^t \delta_s \, ds + U \, \mathbb{I}_{\{t \ge t_0\}}$$

with N as in (22) and the function  $(\delta_t)_{t \in [0,T]}$  continuous, in  $L^2([0,T])$  and such that  $\delta_{t_0} = 0$ . Then  $[X,Y]_t = [M,N]_t = ZU\mathbb{I}_{\{t \ge t_0\}}$ . In conclusion Theorem 4.13 applies to show that the triplet  $(X,Y,ZU\mathbb{I}_{\{\cdot \ge t_0\}})$  enjoys the p.r.p. with respect to  $\mathbb{F}^M \vee \mathbb{F}^N$ . The pair (X,Y) instead admits infinite equivalent martingale measures<sup>3</sup>.

**Corollary 4.18.** Under the hypotheses of Theorem 4.13 the p.r.p. for X is not preserved by the enlargement of filtration from  $\mathbb{F}$  to  $\mathbb{G}$ .

Proof.  $Q|_{\mathcal{F}_T} = P^X$  so that X enjoys the p.r.p. with respect to  $\mathbb{F}$  under  $Q|_{\mathcal{F}_T}$ . Moreover Q belongs to  $\mathbb{P}(X, \mathbb{G})$  but this set is not a singleton. In fact let H be any non trivial set in  $\mathcal{H}_T$  and set  $Z := \frac{I_H}{2Q(H)} + \frac{I_{H^c}}{2Q(H^c)}$ . Consider on  $\mathcal{G}_T$  the measure  $\widetilde{Q}$  defined as the unique extension of  $\widetilde{Q}(A \cap B) := Q(A)E^Q[Z\mathbb{I}_B]$ , with  $A \in \mathcal{F}_T$  and  $B \in \mathcal{H}_T$ . Then it is easy to see that  $\widetilde{Q}$  belongs to  $\mathbb{P}(X, \mathbb{G})$ .

### 5 Conclusions

In all the paper a fundamental role is played by the existence of an equivalent martingale measure for X with respect to  $\mathbb{G}$ . In the framework of mathematical finance this condition implies the financial market with the enlarged information  $\mathbb{G}$  to be free of arbitrage (see [12]). This property is given as hypothesis in Section 3 while in Section 4 it derives from the assumptions. As far as the results in Section 4 are concerned, note that it is common practice to complete the market by adding new components to the discounted asset price vector (as recent contributions in this sense see [9], [22] and [8]). Theorem 4.13 suggests in

<sup>&</sup>lt;sup>3</sup> e.g. if Z takes values in (-z, z) and U takes values in (-u, u), then to any choice of P(Z = z, U = u)in (0, 1/2) it corresponds a different joint law for (Z, U, ZU) which preserves the law of (Z, U)

particular how to complete the market with discounted asset price X when the available information  $\mathbb{F}$  is enlarged by the observation of an independent semi-martingale Y: it is enough to assume the processes Y and [X, Y] as the discounted prices of two new assets.

Amendinger, Becherer and Schweizer in [2], in the framework of maximization of utilities, generalizing a result in [1], show that the p.r.p. of X transfers from  $\mathbb{F}$  to  $\mathbb{G} = \mathbb{F} \vee \sigma(G)$ with G a  $\mathcal{F}$ -measurable random variable. The basic assumption in that paper is condition **D**) there exists a probability measure R on  $\mathcal{F}$  equivalent to P such that  $\mathcal{F}_T$  and  $\sigma(G)$  are R-independent.

Grorud and Pontier in [17] study existence and characterization of the risk-neutral probabilities for an insider trader with initial information. In particular under hypothesis H1) they show the equivalence between conditions H2) and D).

In the present paper, in the setting of Section 4, the role of condition  $\mathbf{D}$ ) is played by one of the equivalent conditions discussed in Remark 4.15. It is to note that condition  $\mathbf{D}$ ) in the setting of initial enlargement of filtrations allows to prove that p.r.p. is preserved, whereas its generalization to the progressive enlargement in Section 4 produces the loss of p.r.p.. Indeed a general result can be announced.

**Proposition 5.1.** Assume **H1**) and  $\mathbb{G} := \mathbb{F} \vee \mathbb{K}$  where  $\mathbb{K}$  is a filtration on  $(\Omega, \mathcal{F}, P)$ satisfying the standard conditions and such that  $\mathcal{K}_t \neq \mathcal{K}_0$  for some t in (0, T]. Let R be a probability measure on  $\mathcal{F}$  equivalent to P such that  $\mathcal{F}_T$  and  $\mathcal{K}_T$  are R-independent. Then the p.r.p. for X is not preserved by the enlargement of filtration from  $\mathbb{F}$  to  $\mathbb{G}$ .

*Proof.* Let  $\nu$  be a probability measure on  $\mathcal{K}_T$  equivalent to  $R|_{\mathcal{K}_T}$ , such that the derivative  $\frac{d\nu}{dR|_{\mathcal{K}_T}}$  is not  $\mathcal{K}_0$ -measurable and  $\nu|_{\mathcal{K}_0} = R|_{\mathcal{K}_0}$ . Define  $Q^{\nu}$  on  $\mathcal{G}_T$  by

$$dQ^{\nu} := \frac{dP^X}{dR|_{\mathcal{F}_T}} \frac{d\nu}{dR|_{\mathcal{K}_T}} dR.$$

 $Q^{\nu}|_{\mathcal{K}_T}$  is equivalent to  $P|_{\mathcal{K}_T}$ ,  $Q^{\nu}|_{\mathcal{F}_T}$  coincides with  $P^X$  and  $Q^{\nu}$  decouples  $\mathbb{F}$  and  $\mathbb{K}$ . The conclusion follows by Theorem 13.9 in [20].

The different role of condition D) under initial and progressive enlargement generates only an apparent paradox. In fact, let assume H1), that is  $\mathcal{F}_0$  is trivial and the  $(P^X, \mathbb{F})$ p.r.p. for X holds. Then all source of randomness by representing  $\mathbb{F}$ -local martingales lies in X. When one adds new information, either initially as in [1] or progressively as in Section 4 of this paper, this fact modifies two fundamental aspects of the interpretation of X in the representation of the G-local martingales. First, the randomness of X is no more sufficient. More precisely, by initial enlargement the starting value of the martingales in the enlarged filtration becomes random, whereas by progressive enlargement new stochastic integrators appear. Second, one has to extend the probability measure  $P^X$  to a measure R on  $\mathcal{G}_T$ , in order to get the  $(R, \mathbb{G})$ -p.r.p. for the driving process (eventually multidimensional), which in case of progressive enlargement contains X as a component and in case of initial enlargement is X itself.

Given a Lévy process X, Corcuera, Nualart and Schoutens in [8] construct a basic set of orthogonal martingales for  $\mathbb{F}^X$  (*Teugels martingales*) by an orthogonalization procedure, using X and its power jump processes  $\sum_{s \leq t} \Delta X_s^i$ ,  $i \geq 2$ . That result suggests that in the case of a pair of Lévy processes (X, Y) the representation of  $\mathbb{F}^X \vee \mathbb{F}^Y$ -local martingales

should involve X, Y the processes of the form  $\sum_{s \leq t} \Delta X_s^i \Delta Y_s^j$ ,  $i, j \geq 1$ . Here, in the more general framework of semi-martingales, Theorem 4.13 gives conditions for representing all  $\mathbb{F}^X \vee \mathbb{F}^Y$ -local martingales: it is sufficient to add to X and Y the process of the common jumps  $\sum_{s \leq t} \Delta X_s \Delta Y_s$ . One can observe that, under the hypotheses of Theorem 4.13, the family of processes given by X, Y and  $\sum_{s \leq t} \Delta X_s^i \Delta Y_s^j$ ,  $i, j \geq 1$  is invariant under covariation. In particular when X and Y are martingales this family is a *compensated-stable covariation family of martingales* in the space of square-integrable martingales (see [15], [14]). Moreover Theorem 4.13 provides the minimal number of martingales needed for the predictable representation of this family.

# 6 Perspectives

A natural development of the current results is to investigate whether Theorem 4.13 continues to hold under weaker conditions, allowing for instance to drop the hypotheses **H4**) and **H5**); this issue is the object of ongoing research. As in Remark 4.7, some regularity of the Girsanov derivatives should be sufficient to state the result and to extend it to the multidimensional case.

In a mathematical finance environment, an interesting and natural question is to investigate the possibility to obtain representation results in markets driven by processes sharing accessible jumps times with positive probability. We conjecture that this goal could be achieved exploiting the fact that, under a decoupling measure for the assets, the covariation process is an element of the base of orthogonal martingales in the sense of Davis and Varaiya ([11]). The validation of this conjecture is another topic of ongoing and future research.

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