HURWITZ COMPLETE SETS OF FACTORIZATIONS IN THE MODULAR GROUP AND THE CLASSIFICATION OF LEFSCHETZ ELLIPTC FIBRATIONS OVER THE DISK

C. CADAVID[A], J. D. VÉLEZ[B], AND LUIS F. MORENO[C]

ABSTRACT. Given any matrix B in $SL(2,\mathbb{Z})$, we will describe an algorithm that provides at least one elliptic fibration over the disk, relatively minimal and Lefschetz, within each topological equivalence class, whose total monodromy is the conjugacy class of B.

1. Introduction

Locally holomorphic fibrations have received a great deal of attention due to the close relationship that exists between a 4-dimensional manifold M admitting a symplectic form and the existence of locally holomorphic fibrations over M (see [1], [5]). Such fibrations have been studied extensively by several authors: Over the sphere, by Moishezon [9], and over closed surfaces of arbitrary genus by Matsumoto [8]. Their classification over the disk, for the case when the total space is two dimensional, is carried out in [10], [6].

In [2], the authors studied distinguished factorizations in $SL(2,\mathbb{Z})$ in terms of conjugates of the matrix $U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, which naturally arise as the monodromy around a singular fiber in an elliptic fibration. In that article, it is proved that if M is one of the matrices in the *Kodaira's list*, and if $M = G_1 \cdots G_r$ where each G_i is a conjugate of U in $SL(2,\mathbb{Z})$, then after applying a finite sequence of Hurwitz moves, the product $G_1 \cdots G_r$ can be transformed into another product of the form $H_1 \cdots H_n G'_{n+1} \cdots G'_r$ where $H_1 \cdots H_n$ is some fixed shortest factorization of M in terms of conjugates of U, and $G'_{n+1} \cdots G'_r = Id_{2\times 2}$. We used this result to obtain necessary and sufficient conditions under which a relatively minimal elliptic fibration over the disk D without

Key words and phrases. Hurwitz equivalence, Elliptc fibrations, Modular group, Monodromy, Kodaira list.

MSC 58E05, 35K05.

multiple fibers, $\phi : S \to D$, admits a weak deformation into another such fibration having only one singular fiber.

In general, the problem of classification of elliptic fibrations over D which are *relatively minimal and Lefschetz strict* (see definition 3), up to topological equivalence, is equivalent to the problem of studying the set

$$\{(g_1, \ldots, g_n) : n \geq 0 \text{ and } g_i \in SL(2,\mathbb{Z}) \text{ is a conjugate of } U\} ,$$

where two n-tuples are identified if one can be obtained from the other by a finite sequence of Hurwitz moves followed by conjugation (see [1], and Definition 6). A satisfactory answer to the problem would comprise:

(1) A method by which given any $B \in SL(2,\mathbb{Z})$, one could obtain a subcollection of the set of all equivalent classes of factorization of B in terms of conjugates of U, modulo Hurwitz moves, has at least one representative in this subcollection.
(2) An algorithm to decide if two factorizations of B in conjugates of U are Hurwitz equivalent.

In this article we construct an algorithm that completely solves the first of these goals. Similar results were obtained in [8] and [9], for the case where the base is a closed surface.

The second goal seems to be a very difficult problem. It is known that some cases turned out to be undecidable (see [11]).

The article is organized as follows: in Section 2 we introduce the basic notions concerning elliptic fibrations over the unit disk and their classification. The central result is theorem 1 which relates the problem of classifying all of *special* elliptic fibrations over the disk to the problem of classifying their monodromy representations, up to conjugation and Hurwitz equivalence, in the modular group. Section 3 is devoted to the study of the relationship between *special* factorizations in $PSL(2,\mathbb{Z})$ and their liftings to $SL(2,\mathbb{Z})$. The next section deals with a combinatorial study of Hurwitz equivalence of special factorization in the modular group. The last section presents an algorithm for generating a relatively simple H-complete set of special factorizations of any given element in the modular group.

2. Elliptic fibrations over the disk and Hurwitz equivalence

Definition 1. *Let Σ be a compact, connected and oriented smooth two dimensional manifold (with or without boundary). A topological elliptic fibration over Σ is a smooth function $f : M \to \Sigma$ such that*

(1) *M is a compact, connected and oriented four dimensional smooth manifold (with or without boundary).*
(2) *f is surjective.*
(3) $f(\mathrm{int}(M)) = \mathrm{int}(\Sigma)$ *and* $f(\partial M) = \partial(\Sigma)$.
(4) *f has a finite number (possibly zero) of critical values $q_1, \ldots, q_n$ all contained in* $\mathrm{int}(\Sigma)$.
(5) *f is locally holomorphic, that is, for each $p \in \mathrm{int}(M)$ there exists orientation preserving charts from neighborhoods of p and $f(p)$, to open sets of $\mathbb{C}^2$ and $\mathbb{C}$ (endowed with their standard orientations), respectively, relative to which f is holomorphic.*
(6) *The preimage of each regular value is a smooth two dimensional manifold that is closed and connected, and of genus one.*

Two topological elliptic fibrations are regarded equivalent according to the following definition.

Definition 2. *Two topological elliptic fibrations $f_1 : M_1 \to \Sigma_1$ and $f_2 : M_2 \to \Sigma_2$ are* topologically equivalent*, written as $f_1 \sim_{Top} f_2$, if there exist orientation preserving diffeomorphisms $H : M_1 \to M_2$ and $h : \Sigma_1 \to \Sigma_2$, such that $h \circ f_1 = f_2 \circ H$.*

Definition 3. *A topological elliptic fibration $f : M \to \Sigma$ will be called*
(1) Relatively minimal *if none of its fibers contains an embedded sphere with selfintersection -1.*
(2) Lefschetz strict *if for each critical point p (necessarily contained in $\mathrm{int}(M)$) of f there exist charts as in condition 5 above relative to which f takes the form $(z_1, z_2) \to z_1^2 + z_2^2$, and f is injective when restricted to the set of critical points.*

If $f : M \to \Sigma$ satisfies both conditions, we will say that f is a *special fibration.*

We notice that being *special* is preserved by topological equivalence.

In what follows we will only consider special elliptic fibrations over the closed unit disk, $D = \{z \in C : |z| \leq 1\}$, endowed with its standard orientation.

Definition 4. *Let G be a group. Any n-tuple of elements of G, $\alpha = (g_1, \ldots, g_n)$, $n \geq 0$, will be called a* factorization*. The only 0-tuple (the empty tuple) will be denoted by $(\ )$. The element $g_1 \cdots g_n$ will be called the* product *of the factorization, and will be denoted by prod(α). When α is empty, we define its product as the identity element of G.*

Given any g in G, we will say that α is a factorization *of g if its product is equal to g.*

If $A \subset G$, $F(A, G)$ will denote the set formed by all factorizations in G whose entries are all in A.

We will be interested in the case where G is $SL(2,\mathbb{Z})$ and $A = C(U)$ is the set of all conjugates of the element $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. ($U$ represents the monodromy around a critical point in any special fibration, as explained below.)

Definition 5. *We will say that a factorization in* $SL(2,\mathbb{Z})$ *is* special *if it belongs to* $F(C(U), SL(2,\mathbb{Z}))$. *This set will be denoted simply by* $F(U)$.

Definition 6. *Let* G *be a group, and* $n \geq 2$. *For any integer* $1 \leq i \leq n-1$, *a Hurwitz right move, at position* i, *is the function* $H_i : G^n \to G^n$ *defined as*

$$H_i(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n).$$

The inverse function is called a Hurwitz left move, at position i, *which is given by*

$$H_i^{-1}(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n).$$

When $H_i(g_1, \ldots, g_n) = (g_1', \ldots, g_n')$ *(resp.* $H_i^{-1}(g_1, \ldots, g_n) = (g_1', \ldots, g_n')$ *) we will say that* $(g_1', \ldots, g_n')$ *is obtained from* $(g_1, \ldots, g_n)$ *by a Hurwitz right move (respectively, by a Hurwitz left move) at position* i.

If $\alpha' = (g_1', \ldots, g_m')$ *is obtained from* $\alpha = (g_1, \ldots, g_n)$ *by a successive applications of finite Hurwitz moves, we will say that* α *and* α' *are* H-equivalent, *which we denote by* $\alpha \sim_H \alpha'$. *In this case, it follows immediately that* $n = m$ *and* $g_1' \cdots g_n' = g_1 \cdots g_n$, *and therefore their product is the same. If, moreover, there exists an element* h *such that* $\alpha \sim_H (h^{-1} g_1 h, \ldots, h^{-1} g_n h)$, *we will say that* α *and* α' *are* $C+H$-equivalent. *This will be denoted by* $\alpha \sim_{C+H} \alpha'$.

The set of classes $F(U)/\sim_H$, and $F(U)/\sim_{C+H}$ will be denoted by ε_H, and ε_{C+H}, respectively. It is clear that being $C+H$-equivalent is weaker than being H-equivalent.

2.1. **Hurwitz complete sets.** As in definition 4, $C(B)$ denotes the conjugacy class in $SL(2,\mathbb{Z})$ of the matrix B. Let us notice that if $\alpha = (G_1, \ldots, G_r)$ in $F(U)$ has product B , then any other element α' in the H-equivalence class of α also has product B. On the other hand, if α' is just $C+H$-equivalent to α, then its product belongs to the conjugacy class of B.

Definition 7. *For any matrix* B *in* $SL(2,\mathbb{Z})$, *a subset of* $F(U)$ *will be called* H*-complete (respectively,* $H+C$*-complete) if it contains at least one representative within each class of equivalence under the relation* $\sim_H$, *(respectively, under* $\sim_{C+H}$*).*

Let $f : M \to D$ be any special fibration over the disk. Let us denote by q_0 the point $(1, 0)$, and by C the boundary of the disk with its standard counterclockwise orientation. As usual,

$$\rho : \pi_1(D - \{q_1, \ldots, q_n\}, q_0) \to SL(2, \mathbb{Z})$$

will stand for the *monodromy representation* where we have identified the mapping class group of T^2, a fixed model of the regular fiber, with $SL(2, \mathbb{Z})$. The mapping ρ is an anti-homomorphism determined by its action on any basis of the rank n free group $\pi_1(D - \{q_1, \ldots, q_n\}, q_0)$. We may take $\{[\gamma_1], \ldots, [\gamma_n]\}$ the standard basis consisting of the classes of clockwise oriented, pairwise disjoint arcs where each γ_i surrounds exclusively the critical value q_i, $i = 1, \ldots, n$. We may choose the γ_i's in such a way that (for an appropriate numbering of the q_i's) the product $[\gamma_1] \cdots [\gamma_n]$ equals the class of C. The conjugacy class in $SL(2, \mathbb{Z})$ of $\rho([C])$ is called the *total monodromy* of the fibration. It can be readily seen that this is a well defined notion.

Remark 1. *If $f : M \to D$ is any special fibration over the disk, since each singular fiber has a single ordinary double point (of type I_1, in Kodaira´s classification* [7] *) the monodromy around any of these fibers is in the conjugacy class of $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $SL(2, \mathbb{Z})$.*

Special fibrations over D can be classified up to conjugation and Hurwitz moves. More precisely:

Theorem 1. *Let $f_1 : M_1 \to D$ and $f_2 : M_2 \to D$ be two special fibrations. Let us fix monodromy representations ρ and ρ', and basis $\{[\gamma_1], \ldots, [\gamma_n]\}$, and $\{[\gamma'_1], \ldots, [\gamma'_n]\}$, for f_1 and f_2, respectively. Let $g_i = \rho([\gamma_i])$, and $g'_i = \rho'([\gamma'_i])$. Then, f_1 and f_2 are topologically equivalent if and only if $\alpha = (g_1, \ldots, g_n)$ and $\alpha' = (g'_1, \ldots, g'_n)$ are equivalent under the equivalence relation $\sim_{C+H}$ (Definition 6).*

For a proof see [4].

Hence, the elements of ε_{C+H} are in bijective correspondence with topological equivalency classes of special fibrations over the disk. Therefore, in order to classify these fibrations, it suffices to describe the elements of ε_{C+H}. In this article *we present an algorithm that for any given matrix B in $SL(2, Z)$ produces an H-complete set of factorizations of B.* In general, this set could be redundant in the sense that it might contain more that one representative in some equivalence classes. Since $C + H$-equivalence is weaker than H-equivalence, it is clear that this set is also $H + C$-complete. *Therefore, for any given B, this algorithm will provide at least one special elliptic fibration over the disk* *within each topological equivalence class, whose total monodromy is the conjugacy class of* B.

3. Special factorizations in $PSL(2,\mathbb{Z})$

Even though it is well known that $SL(2,\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

it is important for our purposes that a decomposition of any matrix in $SL(2,\mathbb{Z})$ as product of powers of S and U (or equivalently, as a product of powers of S and $R = SU$) can be achieved algorithmically. This is the content of the next proposition.

Proposition 1. *Every matrix in* $SL(2,\mathbb{Z})$ *can be written as a product of powers of* S *and* U. *Moreover, there is an algorithm that given any matrix* B *in* $SL(2,\mathbb{Z})$ *yields one of such factorizations.*

Proof. For any matrix A, $U^n A$ is the matrix obtained from A by performing the row operation corresponding to adding n times the second row to the first, while SA is the matrix obtained from A by performing the row operation corresponding to interchanging the first and second row, and multiplying the first row by -1.

For any matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, since $\det(B) = 1$, the entries a and c must be relatively prime. If $|c| < |a|$, by the euclidean algorithm, if $a = cn + r$, then by premultiplying by U^{-n} we obtain a matrix of the form $U^{-n}B = \begin{pmatrix} r & b' \\ c & d \end{pmatrix}$

with $b' = b - nd$. In case where $|a| < |c|$, we may first multiply by S to interchange the rows. Thus, in any case, premultiplying by U^{-n}, or by $U^{-n}S$, has the effect of putting B in the form $\begin{pmatrix} r & b' \\ c & d \end{pmatrix}$, where $\operatorname{lcd}(a,c) = \operatorname{lcd}(c,r)$ (lcd denotes the lest common divisor). By successively premultiplying by S, and suitable powers of U, we may transform B into a matrix of the form $B' = \begin{pmatrix} \pm 1 & m \\ 0 & k \end{pmatrix}$. That is, $B' = PB$, where P is a product of S and powers of U. Since $B'$is in $SL(2,\mathbb{Z})$, k must be equal to ± 1. Therefore, $B' = \pm I_2 U^{\pm m}$. Since $S^2 = -I_2$, then $B = P^{-1}(\pm I_2)U^{\pm m}$. □

The modular group, $SL(2,\mathbb{Z})/\{\pm I_2\}$, will be denoted by $PSL(2,\mathbb{Z})$. For the sake of brevity, we will denote this group simply by $\mathcal{M}$.The classes of S, U and R will be denoted by ω, u and b, respectively. Note that $b = \omega u$. It is a well known fact

$$\mathcal{M}= \left\langle \omega, b \,\middle|\, \omega^2 = b^3 = 1 \right\rangle .$$

The following corollary is an immediate consequence of the previous proposition.

Corollary 1. *There is an algorithm that expresses any element in $\mathcal{M}$ as a product of positive powers of ω and b.*

Let $\pi : SL(2, \mathbb{Z}) \to \mathcal{M}$ denote the canonical homomorphism to the quotient.

Definition 8. *A factorization $\alpha = (g_1, \ldots, g_n)$ in $\mathcal{M}$ will be called* special *if each g_i is a conjugate of u.*

A special factorization $\alpha = (A_1, \ldots, A_n)$ in $SL(2, \mathbb{Z})$ will be called a lift *of α, if $\pi(A_i) = g_i$ for each i.*

We observe that each special factorization $\alpha = (g_1, \ldots, g_n)$ in $\mathcal{M}$ has exactly one lift. Indeed, if $g_i = a_i u a_i^{-1}$, then its preimages are $\pm A_i U A_i^{-1}$, where A_i is any preimage of a_i. But only $A_i U A_i^{-1}$ is a conjugate of U, since the trace$(-A_i U A_i^{-1}) = -2$, and every conjugate of U has trace 2. The lift α will be denoted by lift(α).

Now, in $\mathcal{M}$, if α' is obtained from α by performing a Hurwitz move, then lift(α') can be obtained from lift(α) by the corresponding move. Reciprocally, Hurwitz moves in $SL(2, \mathbb{Z})$ can be transformed into Hurwitz moves in $\mathcal{M}$ via π. Therefore, $\alpha \sim_H \alpha'$ if and only if lift$(\alpha) \sim_H$ lift(α'). From this, it follows that H-complete sets for a matrix in $SL(2, \mathbb{Z})$ can be obtained from H-complete sets for $\pi(B)$ in $\mathcal{M}$. More precisely:

Proposition 2. *Let A be an element of $SL(2, \mathbb{Z})$. If $\mathcal{S}$ is an H-complete set for $\pi(A)$ then the collection*

$$\mathcal{R} = \{\textit{lift}(\alpha) : \alpha \in \mathcal{S} \textit{ and } \textit{prod}(\textit{lift}(\alpha)) = A\}$$

is an H-complete set for A.

Proof. The proposition follows from the obvious observation that if $\alpha \sim_H \alpha'$ in $SL(2, \mathbb{Z})$ then prod(α) =prod(α'). □

4. H-complete sets in $\mathcal{M}$

In this section, $\mathcal{M}$ will be identified with the free product

$$\mathbb{Z}_2 * \mathbb{Z}_3 = \left\langle \omega, b \,\middle|\, \omega^2 = b^3 = 1 \right\rangle .$$

There is a unique automorphism ϕ of $\mathcal{M}$ that sends ω into itself and b into b^2.Let us denote by $c_b : \mathcal{M} \to \mathcal{M}$ conjugation by b, i.e., $c_b(z) =$

bzb^{-1}, and by h the composition $h = c_b \circ \phi$. The problem of finding H-complete sets in $\mathcal{M}$ in terms of conjugates of $u = \omega b$ is equivalent, via h, to the problem of finding H-complete sets of elements in terms of conjugates of $h(u) = b\omega b$.

It is important to have a symbol for the empty word: We will denote it by 1.

It is a standard fact that each element a in $\mathcal{M}$ can be written uniquely as a product $a = t_k \cdots t_1$, where each t_i is either ω, b, or b^2 and no consecutive pair $t_i t_{i+1}$ is formed either by two powers of b or two copies of ω. .We call the product $t_k \cdots t_1$ the *reduced expression of* a, and we call k the *length* of a, denoted by $l(a)$. Let $z = t'_1 \cdots t'_l$ be the reduced expression of z. If exactly the first $m-1$ terms of z cancel with those of a, i.e. $t'_i = t_i^{-1}$, for $1 \leq i \leq m-1$, and if $m \leq \min(k, l)$, then $az = t_k \cdots t_m t'_m \cdots t'_l$ and $t_m t'_m$ has to be equal to a non trivial power of b. This is because if t_m were not a power of b then it would have to be ω and therefore t_{m-1} would be a first or second power of b, and so would be t'_{m-1}. Hence, t'_m would also have to be ω but in this case there would be m instead of $m-1$ cancellations at the juncture of a and z. Thus, t_m and t'_m are both powers of b and since there are exactly $m-1$ cancellations their product must be non trivial. Thus, the reduced expression for az is of the form

$$az = t_k \cdots t_{m+1} b^r t'_{m+1} \cdots t'_l,\ r = 1 \text{ or } 2, \text{ if } m \leq \min(k, l). \tag{4.1}$$

Let s_1 denote the element $b\omega b$. The shortest conjugates of s_1 in $\mathcal{M}$ are precisely $s_0 = b^2(b\omega b)b = \omega b^2$ and $s_2 = b(b\omega b)b^2 = b^2\omega$. The element s_1 is trivially a conjugate of itself of length 3. It can be easily seen that if g is a conjugate of greater length, its reduced expression is of the form $Q^{-1}s_1 Q$, where Q is a reduced word that begins with ω (see [3]), and $l(g) = 2l(Q) + 3$. We will call a conjugate of s_1 (*conjugate* will always mean conjugate of s_1 in $\mathcal{M}$) *short* if $g \in \{s_0, s_1, s_2\}$, otherwise it will be called *long*.

The following notion is the key ingredient for understanding the reduced expression of a product of conjugates of s_1.

Definition 9. *We will say that two conjugates g and h of s_1* join well *if $l(gh) \geq \max(l(g), l(h))$. Otherwise, we say they* join badly*.*

The notion of being a *special* factorization will be used in the following sense:

Definition 10. *A factorization $\alpha = (g_1, \ldots, g_n)$ in $\mathcal{M}$ is called* special *if each g_i is a conjugate of s_1. We say α is* well jointed *if each pair of elements g_i, g_{i+1} join well. Otherwise, we say that α is* badly jointed.

The empty factorization will be regarded as being special, and well jointed. Special factorizations with just one element will also be regarded as well jointed.

Remark 2. *We notice that the following identities hold:*
$s_2s_2 = b^2\omega b^2\omega$, $s_1s_1 = b\omega b^2\omega b$,
$s_0s_0 = \omega b^2\omega b^2$, $s_2s_1 = b^2\omega b\omega b$,
and, $s_1s_0 = b\omega b\omega b^2$ *and* $s_0s_2 = \omega b\omega$.
Hence, the corresponding factorizations in each case are well jointed. On the other hand, since $s_0s_1 = s_1s_2 = s_2s_0 = b$*, the corresponding factorizations are badly joined.*

The following propositions will be useful for the proof of one of the main results used for the construction of H-complete sets.

Proposition 3. *Let* g_1, g_2 *be conjugates of* s_1 *such that* g_1, g_2 *do not joint well. Then:*

(1) g_1, g_2 *are short conjugates or*
(2) (g_1, g_2) *may be transformed by a Hurwitz move into a new pair* (h_1, h_2) *such that* $\max\{0, l(h_1)-3\}+\max\{0, l(h_2)-3\} < \max\{0, l(g_1)-3\} + \max\{0, l(g_2) - 3\}$.

Proof. It follows from the proof of Proposition 4.15, [3]. □

Proposition 4. *Every spacial factorization* $\alpha = (g_1, \ldots, g_n)$ *can be trasformed by Hurwitz moves into a factorization* $\beta = (h_1, \ldots, h_n)$*(necessarily special, and with the same number of factors), satisfying:*

i) *Each* h_i *is short, or*
ii) β *is well jointed and at least one of the* h_i'*s is long.*

Proof. (See [3]) □

Proposition 5. *Every factorization* (g_1, g_2, g_3) *in which each* g_i *is short, and where* g_1, g_2 *join badly, is* H*-equivalent to a factorization* (g_1', g_2', g_3')*, where each* g_i' *is short, and* g_2', g_3' *join badly.*

Proof. The only pairs of short conjugates that do not join well are $(s_0, s_1), (s_1, s_2)$,
(s_2, s_0). It follows that for each s_i there exists an s_j such that (s_j, s_i) does not join well. We also notice that any two of these pairs are H-equivalent. Hence, for g_3, there is s_j such that (s_j, g_3) does not join well. Therefore, after a Hurwitz move performed on the pair (g_1, g_2), transforming it into (g_1', g_2'), with $g_2' = s_j$, then, the factorization (g_1', g_2', g_3') with $g_3' = g_3$, is Hurwitz equivalent to (g_1, g_2, g_3), and (g_2', g_3') does not join well. □

Proposition 6.

(1) Every factorization $(g_1,\dots,g_n)$ where each g_i is a short conjugate, and where not all pairs of elements g_i, g_{i+1} join well, is H-equivalent to a factorization $(g'_1,\dots,g'_n)$, where (g'_{n-1}, g_n) join badly.
(2) Every factorization $(g_1,\dots,g_n)$ in short conjugates where not all pairs of elements g_i, g_{i+1} join well is H-equivalent to a factorization $(g'_1,\dots,g'_n)$ in which $(g'_{n-1}, g_n) = (s_0, s_1)$.

Proof. For each factorization $\alpha = (g_1,\dots,g_n)$ $(n \geq 2)$ in short conjugates where not all pairs of elements g_i, g_{i+1} join well we associate the integer $k(\alpha) = n - \max\{r : (g_r, g_{r+1}) \text{ does not join well}\}$. The proof proceeds by induction on k. If $k = 1$, then (g_{n-1}, g_n) join badly, and the result follows. For $k_0 \geq 1$, let us suppose that the result is true for all α such that $k(\alpha) \leq k_0$. Let $\beta = (g_1,\dots,g_n)$ be a factorization with $k(\beta) = k_0 + 1$. This implies that (g_{n-k_0-1}, g_{n-k_0}) does not join well. Applying Proposition 5 we infer that $(g_{n-k_0-1}, g_{n-k_0}, g_{n-k_0+1})$ is H-equivalent to a factorization $(g'_{n-k_0-1}, g'_{n-k_0}, g'_{n-k_0+1})$ in short conjugates, such that $(g'_{n-k_0}, g'_{n-k_0+1})$ join badly. Summarizing, the original factorization β is H-equivalent to a factorization in short conjugates $\beta' = (g'_1,\dots,g'_n)$ in which $(g'_{n-k_0}, g'_{n-k_0+1})$ join badly. Clearly $k(\beta') < k(\beta)$, thus the proposition holds for β', i.e., β' is H-equivalent to another factorization in short conjugates $\beta'' = (g''_1,\dots,g''_n)$ in which (g''_{n-1}, g''_n) join badly. We conclude that the result also holds β, since β is Hurwitz equivalent to β''. This proves the first statement. The second assertion easily follows from the fact that all pairs of short conjugates that join badly are H-equivalent to (s_0, s_1). □

Proposition 7. *Every factorization $(g_1,\dots,g_n)$ in short conjugates is H-equivalent to another factorization in short conjugates, of the form $(g'_1,\dots,g'_m, s_0, s_1,\dots,s_0,s_1)$, $0 \leq m \leq n$, where there are $(n-m)/2$ pairs of s_0, s_1, and $(g'_1,\dots,g'_m)$ is well jointed.*

Proof. Let $\alpha = (g_1,\dots,g_n)$ be a factorization in short conjugates. Each factorization $\beta = (h_1,\dots,h_n)$ in short conjugates that is H-equivalent to α can be written uniquely as $(h_1,\dots,h_m, s_0, s_1,\dots,s_0,s_1)$ where there are $r \geq 0$ pairs s_0, s_1, and where $m \geq 0$ and $(h_{m-1}, h_m) \neq (s_0, s_1)$, if $m \geq 2$. The integer r will be denoted by $r(\beta)$ to indicate its dependence on β. Let $\gamma = (g'_1,\dots,g'_m, s_0, s_1,\dots,s_0,s_1)$ be a factorization in short conjugates, H-equivalent to α, such that $r(\gamma) \geq r(\beta)$ for any other factorization in short conjugates β, H-equivalent to α. Let us verify that $(g'_1,\dots,g'_m)$ is well jointed. If $(g'_1,\dots,g'_m)$ is badly jointed, and $m \geq 2$, by the second part of Proposition 6 there would be another factorization in short conjugates $(g''_1,\dots,g''_m)$ H-equivalent to $(g'_1,\dots,g'_m)$, and such that $(g''_{m-1},g''_m)=(s_0,s_1)$. Hence, γ would also be (and, therefore α), H-equivalent to a factorization in short conjugates $(g''_1,\dots,g''_{m-2},s_0,s_1,\dots,s_0,s_1)$ with $r(\gamma)+1$ pairs s_0,s_1, in contradiction with the maximality of γ. Thus, $(g'_1,\dots,g'_m)$ is well jointed. □

Proposition 8. *Each special factorization $(g_1,\dots,g_n)$ is H-equivalent to a factorization of the form $(g'_1,\dots,g'_m,s_0,s_1,\dots,s_0,s_1)$, where there are $r\geq 0$ pairs s_0,s_1, and where $(g'_1,\dots,g'_m)$ is well jointed. Moreover, $(g'_1,\dots,g'_m)$ is a factorization in short conjugates, whenever $r>0$.*

Proof. By Proposition 4, $(g_1,\dots,g_n)$ is H-equivalent to a factorization $\beta=(g'_1,\dots,g'_n)$ that either, is well jointed and at least one of the g'_is is a long conjugate, or it is badly jointed and all g'_is are short conjugates. In the first case, β already has the desired form, since the fact that the factors join well implies that $(g'_{n-1},g'_n)\neq(s_0,s_1)$, and consequently $r=0$. Now, in case β consists of short conjugates that join well, then it also has already the desired form for the same reason. Hence, let us suppose that β is a factorization in short conjugates that is badly joined. By Proposition 7, this factorization is H-equivalent to another one in short conjugates, of the form $(g''_1,\dots,g''_m,s_0,s_1,\dots,s_0,s_1)$, with $(n-m)/2$ pairs s_0,s_1, and where $(g''_1,\dots,g''_m)$ is well jointed. □

An immediate consequence is the following theorem.

Theorem 2. *For each $g\in\mathcal{M}$, the set of all special factorizations of g having either of the following two forms is H-complete:*

(1) $(g_1,\dots,g_m,s_0,s_1,\dots,s_0,s_1)$, *where there are $r>0$ pairs s_0,s_1, $(g_1,\dots,g_m)$ is well jointed, and each g_i is short.*
(2) $(g_1,\dots,g_p)$, *where this factorization is well jointed.*

5. An algorithm to produce H-complete sets

For h in the modular group, let us denote by $WJ(h)$ the set formed by all *special factorizations* of h that are well jointed, and by $WJS(h)$ the subset of factorizations in short conjugates. Remember that we regard the empty factorization ( ) as a *well jointed special factorization of the identity* 1, *in short conjugates.*

Since $s_0s_1=b$ and $b^3=1$, we have that $(s_0s_1)^{3k+l}$ equals 1 if $l=0$, b if $l=1$ and b^2 if $l=2$. According to Theorem 2, for any fixed element g, the union of the following four sets of factorizations of g is H-complete:

(1) $A=\{\alpha:\alpha$ is a *well jointed special* factorization of $g\}$.

(2) $B = \{(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1) : (g_1, \ldots, g_m)$ is a *well jointed special* factorization of g in *short* conjugates and the number of pairs s_0, s_1 is of the form $3k$, with $k \geq 1\}$.
(3) $C = \{(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1) : (g_1, \ldots, g_m)$ is a *well jointed special* factorization of gb^2 in *short* conjugates and the number of pairs s_0, s_1 is of the form $3k+1$, with $k \geq 0\}$.
(4) $D = \{(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1) : (g_1, \ldots, g_m)$ is a *well jointed special* factorization of gb in *short* conjugates and the number of pairs s_0, s_1 is of the form $3k+2$, with $k \geq 0\}$.

In consequence:

Remark 3. *In order to find an H-complete set of special factorizations of an element g we need i) an algorithm that takes an element h in the modular group, and produces the set $WJ(h)$, and ii) an algorithm that extracts the subset $WJS(h)$.*

This second task is trivial, but the first one is less so. The key ingredient to formulate the algorithm in i) is discussed next.

Definition 11. *We define the* left part *of short conjugates of $b\omega b$ as $left(s_0) = left(\omega b^2) = \omega$, $left(s_1) = left(b\omega b) = b\omega$, $left(s_2) = left(b^2\omega) = b^2\omega$. For long conjugates, $left(P^{-1}b\omega bP) = P^{-1}b\omega$, where P is an element of the modular group that begins with ω.*

The following result is Lemma 2.4 in [8].

Proposition 9. *If $(h_1, \ldots, h_n)$ with $n \geq 1$ is a special factorization that is well jointed, its product $h_1 \cdots h_n$ begins with $left(h_1)$.*

Proof. See [8]. □

According to this result, if $(h_1, \ldots, h_n)$ is a well jointed special factorization of an element h in the modular group, then h_1 is either one of the following:

(1) ωb^2 if h begins with ω,
(2) $b\omega b$ if h begins with $b\omega$,
(3) $b^2\omega$ if h begins with $b^2\omega$,
(4) $P^{-1}b\omega bP$ if h begins with $P^{-1}b\omega$ for any P that begins with ω.

In particular, the element 1 has only one well jointed special factorization, namely the empty factorization. Also, b and b^2 admit no well jointed special factorization.

Now we give an algorithm, that we will call *FirstFactor*, which takes as input any element h in the modular group, with h not in the set $\{1, b, b^2\}$, and produces all possible candidates to be first factors in any well jointed special factorization of h. This algorithm outputs a set that contains:

a: ωb^2, if h begins with ω,
b: $b\omega b$, if h begins with $b\omega$,
c: $b^2\omega$, if h begins with $b^2\omega$
d: For each occurrence of $b\omega$ that is not at the beginning of h, the element $P^{-1}b\omega bP$, where P^{-1} is the initial section of h ending right before the occurrence of $b\omega$ starts.

Then we define another algorithm that we will call *Sibling*. This algorithm receives as input an ordered pair $((g_1, \dots, g_n), z)$, where $(g_1, \dots, g_n)$ is a special factorization and z is any element in the modular group. Then, *Sibling* takes the following actions:

(1) If $z = 1$, then *Sibling* outputs the set $\{((g_1, \dots, g_n), z)\}$.
(2) If z is b or b^2, then *Sibling* outputs the empty set $\{\ \}$.
(3) If z is different from 1, b and b^2, and $(g_1, \dots, g_n)$ is the empty factorization, then *Sibling* computes the (necessarily nonempty) set $F = \text{FirstFactor}(z)$, and then outputs the set $\{((g), g^{-1}z) : g \in F\}$.
(4) If z is different from 1, b and b^2, and $(g_1, \dots, g_n)$ is not the empty factorization, *Sibling* computes the (necessarily nonempty) set $F = \text{FirstFactor}(z)$ and then outputs the set $\{((g_1, \dots, g_n, g), g^{-1}z) : g \in F \text{ and } (g_n, g) \text{ join well}\}$.

We make the following elementary but important observations:

(1) For each pair $((h_1, \dots, h_n), z)$ formed by a factorization and any element z in the modular group, we call $h_1 \cdots h_n z$ *the product of the pair*. Then *Sibling* preserves products, i.e., each pair in *Sibling*$(((g_1, \dots, g_n), z))$ has the same product as the pair $((g_1, \dots, g_n), z)$. Notice that this statement is true even if z is b or b^2.
(2) If $((g_1, \dots, g_n), z)$, where $(g_1, \dots, g_n)$ is a well jointed special factorization with $n \geq 0$, then the first component of each element of *Sibling*$(((g_1, \dots, g_n), z))$ is a special factorization that is also well jointed. Notice that this is true even if z is b or b^2.
(3) (a) *Sibling*$(((g_1, \dots, g_n), \omega)) = \{((g_1, \dots, g_n, \omega b^2), b)\}$ for any special factorization $(g_1, \dots, g_n)$ with $n \geq 0$. Therefore

$$\mathit{Sibling}(\mathit{Sibling}(((g_1, \dots, g_n), \omega))) = \{\ \}$$

for any special factorization $(g_1, \dots, g_n)$ with $n \geq 0$.
(b) *Sibling*$(((g_1, \dots, g_n), z)) = \{\ \}$, if $z = b, b^2$ and $(g_1, \dots, g_n)$ is a special factorization with $n \geq 0$.
(c) Sibling$(((g_1, \dots, g_n), z)) = \{((g_1, \dots, g_n), z)\}$, if z is 1 and $(g_1, \dots, g_n)$ is a special factorization with $n \geq 0$.

(d) Let $z \notin \{1, \omega, b, b^2\}$ and let $g \in \text{FirstFactor}(z)$. Let us see that $l(g^{-1}z) < l(z)$. Let z begin with ω and $g = \omega b^2$. Then z will be of the form $\omega b^\delta Q$, where $\delta = 1, 2$ and Q is a reduced word that is 1 or begins with ω. We have

$$g^{-1}z = (\omega b^2)^{-1}(\omega b^\delta\ Q) = (b\omega)(\omega b^\delta)Q = b^\gamma Q,$$

where γ is 0 or 2, and clearly $l(b^\gamma Q) < l(\omega b^\delta Q)$. Let z begin with $b\omega$ and $g = b\omega b$. In this case z is of the form $b\omega Q$, where Q is a reduced word that is either 1 or begins with b or b^2. We have $g^{-1}z = (b^2\omega b^2)(b\omega Q) = b^2 Q$, and clearly $l(b^2Q) < l(b\omega Q)$. Let z begin with $b^2\omega$ and let $g = b^2\omega$. It is clear that $l(g^{-1}z) < l(z)$ in this case. Let z begin with $P^{-1}b\omega$, where P is a reduced word that begins with ω, and let $g = P^{-1}b\omega bP$. Then z is of the form $P^{-1}b\omega Q$, where Q is a reduced word that is either 1 or begins with b^δ with $\delta = 1, 2$. Then $g^{-1}z = (P^{-1}b^2\omega b^2 P)(P^{-1}b\omega Q) = P^{-1}b^2 Q$. Clearly, $l(P^{-1}b^2Q) < l(P^{-1}b\omega Q)$.

Now we define another routine, that we will call *SiblingSets* that takes as input a set S whose elements are ordered pairs of the form $((g_1, \ldots, g_n), z)$, and outputs the set $\cup_{s\in S}\text{Sibling}(s)$. Notice that *SiblingSets* applied to the empty set gives the empty set. Finally, we define a routine, that we call *WellJointed* that takes an element h in the modular group as input, then calculates the result of applying $l(h) + 1$ times *SiblingSets* to the set $\{((\ ), h)\}$, i.e. calculates $T = \mathit{SiblingSets}^{l(h)+1}(\{((\ ), h)\})$, and then outputs the set formed by the first components of the ordered pairs in T.

By all the observations above, the algorithm WellJointed finds all possible well jointed special factorizations of any element h. By Remark 3 this is all we needed in order to find an H-complete set of special factorizations of an element g.

6. Acknowledgements

We thank the Universidad Nacional of Colombia and Universidad Eafit for their invaluable support.

References

[1] D. Auroux, *Mapping class group factorizations and symplectic 4-manifolds: some open problems.* Problems on Mapping Class Groups and Related Topics, B. Farb Ed., Amer. Math. Soc., Proc. Symp. Pure Math., 74, 123–132 (2006).

[2] Carlos A. Cadavid and Juan D. Vélez. *On a minimal factorization conjecture.* ISSN 0166–8641, **154** (15), 2786—2794 (2007).

[3] Robert Friedman and John W. Morgan. *Smooth Four-Manifolds and Complex Surfaces*, ISBN 978-3-540-57058-5, Springer, 1994.
[4] Robert E. Gompf, András Stipsicz. *4-manifolds and Kirby Calculus*. Graduate Studies in Mathematics. Volume 20. American Mathematical Society, Providence, Rhode Island.
[5] Robert E. Gompf. *Locally holomorphic maps yield symplectic structures*. Communications in analysis and geometry, ISSN 1019–8385, **13**(3), 511–525 (2005).
[6] A. G. Khovanskii and Smilka Zdravkovska. *Branched covers of S^2 and braid groups*, 1996.
[7] Kodaira, K. (1963). *On compact analytic surfaces: II*. The Annals of Mathematics, 77(3), 563-626.
[8] Yukio Matsumoto. *Diffeomorphism types of elliptic surfaces*. Topology, **Vol. 25**(No. 4), 549-563 (1986).
[9] Boris Moishezon. *Complex surfaces and connected sums of complex projective planes*, ISBN 0387083553, Springer–Verlag, 1977.
[10] Natanzon, S. M. (1993). *Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves*. Selecta Mathematica formerly Sovietica, 12(3), 251-291.
[11] Liberman, E., Teicher, M. (2005). The Hurwitz equivalence problem is undecidable. arXiv preprint math/0511153.

a Corresponding Author: Carlos Cadavid, Universidad EAFIT, Departamento de Ciencias Básicas, Bloque 38, Office 417 Carrera 49 No. 7 Sur -50, Medellín, Colombia, Phone: (57)(4)-2619500 Ext 9790, Fax:(57)(4) 2664284
E-mail address: ccadavid@eafit.edu.co.

b Juan D. Vélez, Escuela Matematicas, Universidad Nacional, Medellín Colombia
E-mail address: jdvelez14@gmail.com

c Lus Moreno l, EAFIT, Departamento de Ciencias Básicas, Bloque 38, Office 417 Medellín Colombia
E-mail address: lmorenos@eafit.edu.co.