

LOCAL BOUNDEDNESS PROPERTY FOR PARABOLIC BVP'S AND THE GAUSSIAN UPPER BOUND FOR THEIR GREEN FUNCTIONS

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ABSTRACT. In the present note, we give a concise proof for the equivalence between the local boundedness property for parabolic Dirichlet BVP's and the gaussian upper bound for their Green functions. The parabolic equations we consider are of general divergence form and our proof is essentially based on the gaussian upper bound by Daners [Da] and a Caccioppoli's type inequality. We also show how the same analysis enables us to get a weaker version of the local boundedness property for parabolic Neumann BVP's assuming that the corresponding Green functions satisfy a gaussian upper bound.

Key words : Dirichlet (Neumann) Green function ; gaussian upper bound ; local boundedness property ; Caccioppoli's type inequality.

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1. INTRODUCTION

We consider on Ω , a bounded Lipschitz domain of \mathbb{R}^n , the general divergence form parabolic operator

$$Lu = \partial_t u - \operatorname{div}(A(x, t)\nabla u + uB(x, t)) + C(x, t) \cdot \nabla u + d(x, t)u.$$

We assume that, where $Q = \Omega \times (0, +\infty)$, $A = (a_{ij}) \in L^\infty(Q)^{n \times n}$, $B, C \in L^\infty(Q)^n$, $d \in L^\infty(Q)$ and the following uniform ellipticity condition holds true

$$(1.1) \quad \lambda|\xi|^2 \leq a_{ij}(x, t)\xi_i\xi_j, \quad \text{a.e. } (x, t) \in Q \text{ and } \xi \in \mathbb{R}^n.$$

Henceforth $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$ (resp. $V = H^1(\Omega)$).

The bilinear form associated to the operator L takes the form

$$a(t, u, v) = \int_{\Omega} (A\nabla u \cdot \nabla v + uB \cdot \nabla v + vC \cdot \nabla u + duv) dx, \quad u, v \in V.$$

Let, where $0 \leq s < T$,

$$W = W(s, T, V, V') = \{u \in L^2((s, T), V); u' \in L^2((s, T), V')\}.$$

W is a Hilbert space continuously embedded in $C([s, T], H)$ (e.g. [DL], Section XVIII.1.2) and the following Green's formula holds true

$$(1.2) \quad \int_{s_1}^{s_2} \langle u'(s), v(s) \rangle ds + \int_{s_1}^{s_2} \langle u(s), v'(s) \rangle ds = u(s_2)v(s_2) - u(s_1)v(s_1), \quad u, v \in W,$$

where $s \leq s_1 < s_2 \leq T$ and $\langle \cdot, \cdot \rangle$ is the duality pairing between V and V' .

We say that $u \in W$ is a weak solution of the boundary value problem (abbreviated to BVP in the sequel)

$$(1.3) \quad Lu = 0 \text{ in } \Omega \times (s, T) \text{ and } u = 0 \text{ (resp. } A\nabla u + uB \cdot \nu = 0 \text{) on } \partial\Omega \times (s, T)$$

if

$$(1.4) \quad \langle u'(t), v \rangle + a(t, u(t), v) = 0 \quad \text{a.e., for any } v \in V.$$

Here $\nu = (\nu_1, \dots, \nu_n)$ is the outward unit normal vector to $\partial\Omega$.

Under the assumption (1.1), there exist operators $\mathcal{A}(t) \in \mathcal{B}(V, V')$ such that

$$\langle \mathcal{A}(t)u, v \rangle = a(t, u, v), \quad u, v \in V.$$

As it is noticed in [Da], the family $(\mathcal{A}(t))_{t \geq 0}$ generates an evolution system $(U(t, s))_{0 \leq s \leq t}$ of bounded operators on H . Specifically, for any $u_0 \in H$, $u(t) = U(t, s)u_0 \in W$ is the weak solution of the abstract Cauchy problem

$$(1.5) \quad \begin{cases} u'(t) + \mathcal{A}(t)u(t) = 0, & \text{in } (s, T), \\ u(s) = u_0, \end{cases}$$

in the sense that

$$-\int_s^T \langle u(t), v'(t) \rangle dt + \int_s^T a(t; u(t), v(t)) dt - \int_{\Omega} u_0 v(s) = 0,$$

for any $v \in W$ satisfying $v(T) = 0$.

We observe that the weak solution of the Cauchy problem (1.5) is also a weak solution of the BVP (1.3).

In the sequel, C denotes a generic constant that can depend on n , Ω and the coefficients of L .

From Theorem 5.2 and Corollary 5.3 in [Da], we have the following estimates

$$(1.6) \quad \|U(t, s)\|_{2,2} \leq 1,$$

$$(1.7) \quad \|U(t, s)\|_{2,\infty} \leq C(t-s)^{-\frac{n}{4}},$$

$$(1.8) \quad \|U(t, s)\|_{1,\infty} \leq C(t-s)^{-\frac{n}{2}}.$$

Here $\|\cdot\|_{p,q}$ denotes the natural norm of $\mathcal{B}(L^p(\Omega), L^q(\Omega))$.

Therefore, according to the Dunford-Pettis theorem, $U(t, s)$ is an integral operator with kernel $G(\cdot, t; \cdot, s) \in L^\infty(\Omega \times \Omega)$, $0 \leq s \leq t$:

$$U(t, s)u_0(x) = \int_{\Omega} G(x, t; y, s)u_0(y)dy, \quad u_0 \in H.$$

In the sequel we call G the Dirichlet (resp. Neumann) Green function when $V = H_0^1(\Omega)$ (resp. $V = H^1(\Omega)$).

If

$$\mathcal{G}(x, t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, \quad x \in \mathbb{R}^n, \quad t > 0,$$

is the usual gaussian kernel, we set

$$\mathcal{G}_c(x, t) = c^{-1} \mathcal{G}(\sqrt{c}x, t), \quad c > 0.$$

From Theorem 6.1 in [Da], we know that G satisfies the gaussian upper bound

$$(1.9) \quad G(x, t; y, s) \leq \mathcal{G}_C(x - y, t - s).$$

We need to introduce some notations. If $x \in \bar{\Omega}$ and $t > s$, we set

$$\begin{aligned} I(r) &= I(r)(t) = (t - r^2, t], \\ Q(r) &= Q(r)(x, t) = [B(x, r) \cap \Omega] \times I_r. \end{aligned}$$

Here, r is a small parameter always chosen in such a way that $t - kr^2$ remains in the time interval under consideration, for different values of k appearing in the sequel.

Following [HK], we say that L has the local boundedness property if, for any weak solution u of (1.3) and any $x \in \bar{\Omega}$, we have

$$\|u\|_{L^\infty(Q(r))} \leq Cr^{-\frac{n+2}{2}} \|u\|_{L^2(Q(2r))}.$$

We aim to prove the following result.

Theorem 1.1. *If L possesses the local boundedness property, then G satisfies the gaussian upper bound (1.9). Conversely, under the additional assumption that $A \in L^\infty((0, \infty), W^{1,\infty}(\Omega)^{n \times n})$, if the Dirichlet Green function G satisfies the gaussian upper bound (1.9), then L possesses the local boundedness property.*

We mention that S. Hofmann and S. Kim [HK] proved the equivalence between the local boundedness property for a parabolic system and the gaussian upper bound for the corresponding fundamental solution.

2. PROOF OF THE MAIN THEOREM

Henceforth, the L^p -norm, $1 \leq p \leq \infty$, is denoted by $\|\cdot\|_p$.

Proof of Theorem 1.1. Let us first assume that L has the local boundedness property and we pick $u_0 \in H$. As $u = U(t, s)u_0$ is a weak solution of equation (1.4), we have, by using the local boundedness property,

$$\begin{aligned} |u(x, t)|^2 &\leq C(\sqrt{t-s})^{-(n+2)} \int_{t-(\sqrt{t-s})^2}^t \int_{B(x, \sqrt{t-s}) \cap \Omega} u^2(y, \tau) dy d\tau \quad \text{a.e.} \\ &\leq C(t-s)^{-\frac{n+2}{2}} \int_{t-(\sqrt{t-s})^2}^t \|U(\tau, s)u_0\|_2^2 d\tau \\ &\leq C(t-s)^{-\frac{n}{2}} \|u_0\|_2^2 \quad (\text{by (1.6)}). \end{aligned}$$

Then

$$\|U(t, s)u_0\|_\infty \leq C(t-s)^{-\frac{n}{4}} \|u_0\|_2.$$

In other words, we proved

$$\|U(t, s)\|_{2, \infty} \leq C(t-s)^{-\frac{n}{4}}.$$

This is exactly the estimate (1.7), which in turn implies, by duality, (1.8). The gaussian upper bound follows then from Theorem 6.1 in [Da] and its proof.

The proof of the converse is based on the following parabolic Caccioppoli's type inequality, that we prove later.

Lemma 2.1. *Let u be a weak solution of (1.3). Then*

$$\sup_{\tau \in I_r} \int_{B(x, r) \cap \Omega} u^2(\cdot, \tau) + \int_{Q(\theta_1 r)} |\nabla u|^2 \leq C\theta r^{-2} \int_{Q(\theta_2 r)} u^2,$$

for any $0 < \theta_1 < \theta_2$, where $\theta = (\theta_2^2 - \theta_1^2)^{-1} + (\theta_2 - \theta_1)^{-2}$.

We pursue the proof by assuming that $A \in L^\infty((0, \infty), W^{1, \infty}(\Omega)^{n \times n})$ and G satisfies the gaussian upper bound (1.9). We consider $\varphi \in C^\infty(\mathbb{R})$ satisfying, $0 \leq \varphi \leq 1$, $\varphi = 1$ in a neighborhood of $I(\frac{5}{4}r)$, $\varphi = 0$ in a neighborhood of $(-\infty, t - (\frac{3}{2}r)^2)$ and $|\varphi'| \leq cr^{-2}$. Let $\psi \in C_c^\infty(B(x, \frac{3}{2}r))$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in a neighborhood of $B(x, \frac{5}{4}r)$ and $|\partial^\alpha \psi| \leq cr^{-|\alpha|}$, for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$, for some universal constant c .

Let $u \in W$ be a weak solution of (1.3) and $v = \varphi\psi$. In light of the identity $L(vu) = vLu + [L, v]u$ ¹, Duhamel's formula and the fact that $vu = 0$ on $\partial\Omega \times (0, +\infty)$, we get

$$(2.1) \quad (vu)(z, \tau) = \int_s^\tau \int_\Omega G(z, \tau; y, \rho) f(y, \rho) dy d\rho, \quad \text{a.e. } (z, \tau) \in Q(r).$$

Here we set $f = [L, v]u = f_1 + f_2$, with

$$\begin{aligned} f_1 &= u\varphi'\psi \\ f_2 &= \varphi\nabla\psi \cdot (A\nabla u + uB) - \varphi\text{div}(uA\nabla\psi) + \varphi uC \cdot \nabla\psi. \end{aligned}$$

We need in the sequel that $f_1, f_2 \in L^2(\Omega)$. This explains why we assumed that $A \in L^\infty((0, \infty), W^{1, \infty}(\Omega)^{n \times n})$.

¹Here $[\cdot, \cdot]$ is the usual commutator.

Since $\text{supp}(f_1) \subset Q_1(r) = [B(x, \frac{3}{2}r) \cap \Omega] \times (t - (\frac{3}{2}r)^2, t - (\frac{5}{4}r)^2)$, we show by elementary calculations

$$\begin{aligned} \left| \int_s^\tau \int_\Omega G(z, \tau; y, \rho) f_1(y, \rho) dy d\rho \right|^2 &= \left| \int_{Q_1(r)} G(z, \tau; y, \rho) f_1(y, \rho) dy d\rho \right|^2 \\ &\leq \int_{Q_1(r)} G^2(z, \tau; y, \rho) dy d\rho \int_{Q_1(r)} f_1^2(y, \rho) dy d\rho \\ &\leq \int_{Q_1(r)} \mathcal{G}_C^2(z - y, \tau - \rho) dy d\rho \|f_1\|_{L^2(Q(\frac{3}{2}r))}^2 \\ &\leq Cr^{-n} \|f_1\|_{L^2(Q(\frac{3}{2}r))}^2 \\ &\leq Cr^{-n-2} \|u\|_{L^2(Q(\frac{3}{2}r))}^2. \end{aligned}$$

Let $Q_2(r) = [\{B(x, \frac{3}{2}r) \setminus B(x, \frac{5}{4}r)\} \cap \Omega] \times I(\frac{3}{2}r)$. Similar calculations to those in pages 489 and 490 of [HK] give

$$\int_{Q_2(r)} \mathcal{G}_C^2(z - y, \tau - \rho) dy d\rho \leq Cr^{2-n}.$$

Hence

$$\begin{aligned} \left| \int_s^\tau \int_\Omega G(z, \tau; y, \rho) f_2(y, \rho) dy d\rho \right|^2 &= \left| \int_{Q_2(r)} G(z, \tau; y, \rho) f_2(y, \rho) dy d\rho \right|^2 \\ &\leq \int_{Q_2(r)} G^2(z, \tau; y, \rho) dy d\rho \int_{Q_2(r)} f_2^2(y, \rho) dy d\rho \\ &\leq \int_{Q_2(r)} \mathcal{G}_C^2(z - y, \tau - \rho) dy d\rho \|f_2\|_{L^2(Q(\frac{3}{2}r))}^2 \\ &\leq Cr^{-n+2} \|f_2\|_{L^2(Q(\frac{3}{2}r))}^2 \\ &\leq Cr^{-n} \left(\|\nabla u\|_{L^2(Q(\frac{3}{2}r))}^2 + r^{-2} \|u\|_{L^2(Q(\frac{3}{2}r))}^2 \right). \end{aligned}$$

Applying Lemma 2.1 with $\theta_1 = \frac{3}{2}$ and $\theta_2 = 2$, we find

$$\|\nabla u\|_{L^2(Q(\frac{3}{2}r))}^2 \leq Cr^{-2} \|u\|_{L^2(Q(2r))}^2$$

and then

$$\left| \int_s^\tau \int_\Omega G(z, \tau; y, \rho) f_2(y, \rho) dy d\rho \right|^2 \leq Cr^{-n-2} \|u\|_{L^2(Q(2r))}^2.$$

We end up getting

$$|u(z, \tau)| \leq Cr^{-\frac{n+2}{2}} \|u\|_{L^2(Q(2r))}^2, \quad \text{a.e. } (z, \tau) \in Q(r).$$

□

The analysis we carry out in the converse part of the theorem above is no longer valid for the Neumann Green function because in that case there is an additional term in (2.1). Precisely, we have in place of (2.1)

$$(2.2) \quad (vu)(z, \tau) = \int_s^\tau \int_\Omega G(z, \tau; y, \rho) f(y, \rho) dy d\rho + \int_s^\tau \int_{\partial\Omega} G(z, \tau; \sigma, \rho) g(\sigma, \rho) d\sigma d\rho, \quad \text{a.e. } (z, \tau) \in Q(r).$$

Here $g = -u\varphi A\nabla\psi \cdot \nu$.

Indeed, contrary to the Dirichlet case where vu satisfies a homogeneous boundary condition, in the Neumann case vu obeys to the following non homogeneous Neumann boundary condition

$$A\nabla(uv) + (uv)B \cdot \nu = -u\varphi A\nabla\psi \cdot \nu = g.$$

However, g is identically equal to zero if ψ is chosen compactly supported in Ω . So we can repeat the same argument as in the Dirichlet case to derive the following interior local boundedness property: there is a constant $C > 0$ so that, for any weak solution u of (1.3) in the Neumann case and any $x \in \Omega$, we have

$$\|u\|_{L^\infty(Q(r))} \leq Cr^{-\frac{n+2}{2}} \|u\|_{L^2(Q(2r))}, \quad 0 < r < \text{dist}(x, \partial\Omega).$$

We can also derive a weaker version of the local boundedness property. Let

$$\Sigma(r) = [B(x, r) \cap \partial\Omega] \times I(r).$$

The fact that G is dominated by a gaussian kernel implies in a straightforward manner that the following estimate holds true:

$$G(z, \tau, \sigma, \rho) \leq C|z - \sigma|^{-n+1/2}(\tau - \rho)^{-1/4}.$$

Using this estimate, we easily obtain

$$\left| \int_s^\tau \int_{\partial\Omega} G(z, \tau; \sigma, \rho) g(\sigma, \rho) d\sigma d\rho \right|^2 \leq Cr^{-2n-1} \|u\|_{L^2(\Sigma(3r/2))}^2.$$

Therefore, we can assert that if the Neumann Green function satisfies a gaussian upper bound then it satisfies the following local boundedness property:

$$(2.3) \quad \|u\|_{L^\infty(Q(r))} \leq Cr^{-\frac{n+2}{2}} (\|u\|_{L^2(Q(2r))} + r^{-1} \|u\|_{L^2(\Sigma(3r/2))}).$$

The boundary term in (2.3) can be removed. To do that, we pick $\chi \in C_c^\infty(B(x, 7r/4))$ satisfying $0 \leq \chi \leq 1$, $\chi = 1$ in a neighborhood of $B(x, 3r/2)$ and $|\nabla\chi| \leq cr^{-1}$. Since the trace operator $w \in H^1(\Omega) \rightarrow w|_{\partial\Omega}$ is continuous,

$$\|u\|_{L^2(\Sigma(3r/2))} \leq \|\chi u\|_{L^2(\partial\Omega \times I(3r/2))} \leq C\|\chi u\|_{H^1(\Omega \times I(3r/2))}.$$

On the other hand, by the help of Lemma 2.1, we show

$$\|\chi u\|_{H^1(\Omega \times I(3r/2))} \leq C\|\chi u\|_{H^1(Q(7r/2))} \leq Cr^{-1} \|u\|_{L^2(Q(2r))}.$$

Hence

$$\|u\|_{L^2(\Sigma(3r/2))} \leq Cr^{-1} \|u\|_{L^2(Q(2r))}.$$

This estimate in (2.3) yields the following local boundedness property.

$$\|u\|_{L^\infty(Q(r))} \leq Cr^{-\frac{n+6}{2}} \|u\|_{L^2(Q(2r))}.$$

We point out that the second term in the right hand side of (2.2) has been forgotten in the proof of Theorem 3.24 in [CK, p. 2855].

Proof of Lemma 2.1. We fix $0 < \theta_1 < \theta_2$ and we pick $\varphi \in C^\infty(\mathbb{R})$ satisfying $0 \leq \varphi \leq 1$, $\varphi = 1$ in a neighborhood of $I(\theta_1 r)$, $\varphi = 0$ in a neighborhood of $(-\infty, t - (\theta_2 r)^2)$ and $|\varphi'| \leq c(\theta_2^2 - \theta_1^2)^{-1} r^{-2}$. We take also $\psi \in C_c^\infty(B(x, 2r))$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in a neighborhood of $B(x, r)$ and $|\partial^\alpha \psi| \leq c((\theta_2 - \theta_1)r)^{-|\alpha|}$, for any $\alpha \in \mathbb{N}^n$, $|\alpha| \leq 2$. Here c is some universal constant.

We pick $\tau \in I(\theta_1 r)$. If u is a weak solution of (1.5), we obtain, after taking $v = u(t)\psi^2\varphi(t)$ as a test function in (1.4),

$$\langle u'(t), u(t)\psi^2\varphi(t) \rangle + a(t, u(t), u(t)\psi^2)\varphi(t) = 0 \quad \text{a.e.}$$

Hence

$$(2.4) \quad \int_{t-(\theta_2 r)^2}^\tau \langle u'(t), u(t)\psi^2\varphi(t) \rangle dt + \int_{t-(\theta_2 r)^2}^\tau a(t, u(t), u(t)\psi^2)\varphi(t) dt = 0.$$

By using Green's formula (1.2) between $s_1 = t - (\theta_2 r)^2$ and $s_2 = \tau$, we get in a straightforward manner that

$$(2.5) \quad \int_{t-(\theta_2 r)^2}^\tau \langle u'(t), u(t)\psi^2\varphi(t) \rangle dt = \frac{1}{2} \int_\Omega u^2(\tau)\psi^2\varphi(\tau) dx - \frac{1}{2} \int_{t-(\theta_2 r)^2}^\tau \int_\Omega u^2(t)\psi^2\varphi'(t) dx dt.$$

On the other hand, an elementary computation gives

$$(2.6) \quad a(t, u(t), u(t)\psi^2) = \int_{\Omega} A \nabla u \cdot \nabla u \psi^2 + \int_{\Omega} u E \cdot \psi \nabla u + \int_{\Omega} f_0 u^2,$$

where

$$\begin{aligned} E &= E(x, t) = 2A \nabla \psi + \psi B + \psi C, \\ f_0 &= f_0(x, t) = 2\psi B \cdot \nabla \psi + d\psi^2. \end{aligned}$$

Let $f_1 = f_0 \varphi - (1/2)\psi^2 \varphi'$. Then a combination of (2.4), (2.5) and (2.6) yields

$$(2.7) \quad \begin{aligned} &\frac{1}{2} \int_{\Omega} u^2(\tau) \psi^2 \varphi(\tau) dx + \int_{t-(\theta_2 r)^2}^{\tau} \int_{\Omega} A \nabla u \cdot \nabla u \psi^2 \varphi dx dt \\ &= - \int_{t-(\theta_2 r)^2}^{\tau} \int_{\Omega} u E \cdot \psi \nabla u \varphi dx dt - \int_{t-(\theta_2 r)^2}^{\tau} \int_{\Omega} f_1 u^2 dx dt. \end{aligned}$$

In light of the convexity inequality

$$- \int_{\Omega} u E \cdot \psi \nabla u dx \leq \frac{1}{2\lambda} \int_{\Omega} |E|^2 u^2 dx + \int_{\Omega} \frac{\lambda}{2} |\nabla u|^2 \psi^2 dx$$

and the ellipticity condition (1.1), we deduce from (2.7) that

$$\frac{1}{2} \int_{\Omega} u^2(\tau) \psi^2 \varphi(\tau) dx + \frac{\lambda}{2} \int_{t-(\theta_2 r)^2}^{\tau} \int_{\Omega} |\nabla u|^2 \psi^2 \varphi dx dt \leq \int_{t-(\theta_2 r)^2}^{\tau} \int_{\Omega} \left(\frac{1}{2\lambda} |E|^2 \varphi + |f_1| \right) u^2 dx dt$$

We complete the proof using the estimates on the derivatives of ψ and φ . □

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