

The Classification of Real Singularities Using SINGULAR Part II: The Structure of the Equivalence Classes of the Unimodal Singularities

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ABSTRACT. In the classification of real singularities by Arnold et al. (1985), normal forms, as representatives of equivalence classes under right equivalence, are not always uniquely determined. We describe the complete structure of the equivalence classes of the unimodal real singularities of corank 2. In other words, we explicitly answer the question which normal forms of different type are equivalent, and how a normal form can be transformed within the same equivalence class by changing the value of the parameter. This provides new theoretical insights into these singularities and has important consequences for their algorithmic classification.

1. Introduction

This article is the second part of a series of articles on the algorithmic classification of real singularities up to modality 1 and corank 2. The first part (Marais and Steenpaß, 2015) covers the splitting lemma and the simple singularities. All the algorithms presented there have been implemented in the computer algebra system SINGULAR (Decker et al., 2015a) as a library called `realclassify.lib` (Marais and Steenpaß, 2015).

Both real and complex singularities have been extensively studied (see Siersma, 1974; Arnold, 1976; Bruce and Gaffney, 1982; Gibson and Hobbs, 1993; Frühbis-Krüger, 1999; Greuel et al., 2007; Frühbis-Krüger and Neumer, 2010). Our work is based on the classifications of complex and real singularities of small modality up to stable equivalence by Arnold et al. (1985). Two power series $f, g \in \mathbb{K}[[x_1, \dots, x_n]]$ with a critical point at the origin and critical value 0 are complex (if $\mathbb{K} = \mathbb{C}$) or real (if $\mathbb{K} = \mathbb{R}$) equivalent, denoted by $f \stackrel{\mathbb{K}}{\sim} g$, if there exists a \mathbb{K} -algebra automorphism ϕ of $\mathbb{K}[[x_1, \dots, x_n]]$ such that $\phi(f) = g$. They are stable (complex or real) equivalent if they become (complex or real) equivalent after the direct addition of non-degenerate quadratic terms.

In this article, we focus on the unimodal singularities of corank 2. Their complex and real normal forms can be found in Table 1. Just as for the simple singularities (cf. Marais and Steenpaß, 2015), it turns out that the complex singularity types split up into one or several real subtypes and that the normal forms of the real subtypes belonging to the same complex type differ from each other only in the signs of some terms. We therefore sometimes refer to the complex singularity

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TABLE 1. Normal forms of singularities of modality 1 and corank 2 as given in Arnold et al. (1985)

		Complex normal form	Normal forms of real subtypes	Restrictions
Parabolic	X_9	$x^4 + ax^2y^2 + y^4$	$+x^4 + ax^2y^2 + y^4$ (X_9^{++})	$a^2 \neq +4^\dagger$
			$-x^4 + ax^2y^2 - y^4$ (X_9^{--})	
			$+x^4 + ax^2y^2 - y^4$ (X_9^{+-})	$a^2 \neq -4^\dagger$
			$-x^4 + ax^2y^2 + y^4$ (X_9^{-+})	
	J_{10}	$x^3 + ax^2y^2 + xy^4$	$x^3 + ax^2y^2 + xy^4$ (J_{10}^+)	$a^2 \neq +4^\dagger$
			$x^3 + ax^2y^2 - xy^4$ (J_{10}^-)	$a^2 \neq -4^\dagger$
Hyperbolic	J_{10+k}	$x^3 + x^2y^2 + ay^{6+k}$	$x^3 + x^2y^2 + ay^{6+k}$ (J_{10+k}^+)	$a \neq 0, k > 0$
			$x^3 - x^2y^2 + ay^{6+k}$ (J_{10+k}^-)	
	X_{9+k}	$x^4 + x^2y^2 + ay^{4+k}$	$+x^4 + x^2y^2 + ay^{4+k}$ (X_{9+k}^{++})	$a \neq 0, k > 0$
			$-x^4 - x^2y^2 + ay^{4+k}$ (X_{9+k}^{--})	
			$+x^4 - x^2y^2 + ay^{4+k}$ (X_{9+k}^{+-})	
			$-x^4 + x^2y^2 + ay^{4+k}$ (X_{9+k}^{-+})	
	$Y_{r,s}$	$x^2y^2 + x^r + ay^s$	$+x^2y^2 + x^r + ay^s$ ($Y_{r,s}^{++}$)	$a \neq 0, r, s > 4$
			$-x^2y^2 - x^r + ay^s$ ($Y_{r,s}^{--}$)	
			$+x^2y^2 - x^r + ay^s$ ($Y_{r,s}^{+-}$)	
			$-x^2y^2 + x^r + ay^s$ ($Y_{r,s}^{-+}$)	
	\tilde{Y}_r	$(x^2 + y^2)^2 + ax^r$	$+(x^2 + y^2)^2 + ax^r$ (\tilde{Y}_r^+)	$a \neq 0, r > 4$
			$-(x^2 + y^2)^2 + ax^r$ (\tilde{Y}_r^-)	
Exceptional	E_{12}	$x^3 + y^7 + axy^5$	$x^3 + y^7 + axy^5$	-
	E_{13}	$x^3 + xy^5 + ay^8$	$x^3 + xy^5 + ay^8$	-
	E_{14}	$x^3 + y^8 + axy^6$	$x^3 + y^8 + axy^6$ (E_{14}^+)	-
			$x^3 - y^8 + axy^6$ (E_{14}^-)	
	Z_{11}	$x^3y + y^5 + axy^4$	$x^3y + y^5 + axy^4$	-
	Z_{12}	$x^3y + xy^4 + ax^2y^3$	$x^3y + xy^4 + ax^2y^3$	-
	Z_{13}	$x^3y + y^6 + axy^5$	$x^3y + y^6 + axy^5$ (Z_{13}^+)	-
			$x^3y - y^6 + axy^5$ (Z_{13}^-)	
	W_{12}	$x^4 + y^5 + ax^2y^3$	$+x^4 + y^5 + ax^2y^3$ (W_{12}^+)	-
			$-x^4 + y^5 + ax^2y^3$ (W_{12}^-)	
W_{13}	$x^4 + xy^4 + ay^6$	$+x^4 + xy^4 + ay^6$ (W_{13}^+)	-	
		$-x^4 + xy^4 + ay^6$ (W_{13}^-)		

† Note that the restriction $a^2 \neq 4$ applies to the normal forms of the real subtypes X_9^{++} , X_9^{--} , and J_{10}^+ as well as to the normal forms of the complex types X_9 and J_{10} while the restriction $a^2 \neq -4$ applies to the normal forms of the real subtypes X_9^{+-} , X_9^{-+} , and J_{10}^- if we allow complex parameters.

types as main types. The hyperbolic type \tilde{Y}_r is an exception because it is complex equivalent to $Y_{r,r}$ and only occurs as a type on its own in the real classification.

Some of the normal forms in Table 1 are equivalent to others. Such equivalences occur both between different real subtypes and between normal forms with different values of the parameter a . However, there are no equivalences between different main types. To give an example, $x^4 - 4x^2y^2 + y^4$, the normal form of X_9^{++} with $a = -4$, is equivalent to $-x^4 + 10x^2y^2 - y^4$, the normal form of X_9^- with $a = 10$, via the coordinate transformation $x \mapsto c(x + y)$, $y \mapsto c(x - y)$ with $c = \frac{1}{\sqrt[3]{2}}$. Examples like this one have consequences for the algorithmic classification of real singularities. The question if the singularity in the example is of real type X_9^{++} or of real type X_9^- is not well-posed and the value of the parameter is not uniquely determined. Note that this problem does not occur for the simple singularities: By definition, their normal forms do not admit parameters, and there are no equivalences between different real subtypes except for the main types A_k where k is even, cf. Marais and Steenpaß (2015).

The goal of this article is to determine the complete structure of the equivalence classes for the unimodal real singularities of corank 2. Based on these results, we will present algorithms to determine the equivalence class of a given unimodal real singularity of corank 2 in a subsequent part of this series of articles. If T_1 and T_2 are subtypes of the same singularity main type T and if $g_1(a)$ and $g_2(a)$ are the normal forms of T_1 and T_2 , respectively, where a denotes the value of the parameter, then we are interested in the set of all pairs (u, v) such that $g_1(u)$ is equivalent to $g_2(v)$. This question can be asked in three different ways: If we consider complex values of u and v and complex coordinate transformations, we denote the corresponding set by $P_1(T_1, T_2)$, for real values of u and v , but still complex transformations by $P_2(T_1, T_2)$, and finally by $P_3(T_1, T_2)$ if we consider only real values of u and v and real transformations, cf. Definition 6. We study all three sets P_1 , P_2 , and P_3 for two reasons: First, we need P_1 and P_2 to compute P_3 which is the set we are mainly interested in for our purpose. Second, besides the real structure, we thus also obtain the complete complex structure of the equivalence classes for all the specific normal forms given in Table 1 which is an interesting result on its own.

The formal definitions of these sets and other basic notations are introduced in Section 2, along with different ways how $P_1(T_1, T_2)$, $P_2(T_1, T_2)$, and $P_3(T_1, T_2)$ can be conveniently written down in concrete cases. The following sections are devoted to the computation of these sets for any two real subtypes T_1 and T_2 listed in Table 1. We first recall the definitions of (piecewise) weighted jets and filtrations in Section 3. They play a major role in the proof of Theorem 19, the main result of Section 4. This theorem allows us to restrict ourselves to a small subset of coordinate transformations, which we call a sufficient set, if we want to determine $P_1(T_1, T_2)$. It is thus the theoretic basis for Section 5 where we explain how $P_1(T_1, T_2)$, $P_2(T_1, T_2)$, and $P_3(T_1, T_2)$ can be computed using SINGULAR. We also give an example with explicit SINGULAR commands. These methods do not apply for the singularity type \tilde{Y}_r which is treated separately in Section 5.4. Section 6 contains the results of these computations in a concise form. Finally, we point out some remarkable aspects of the results in this article as well as their consequences for the algorithmic classification of the unimodal real singularities of corank 2 in Section 7. The maybe most surprising outcome is that the real subtype J_{10}^- is actually redundant whereas J_{10}^+ is not.

2. The Sets of Parameter Transformations P_1 , P_2 , and P_3

Let us start with some basic definitions. Throughout the rest of this article, let \mathbb{K} be, in each case, either \mathbb{R} or \mathbb{C} .

DEFINITION 1. Two power series $f, g \in \mathbb{K}[[x_1, \dots, x_n]]$ are called \mathbb{K} -equivalent, denoted by $f \stackrel{\mathbb{K}}{\sim} g$, if there exists a \mathbb{K} -algebra automorphism ϕ of $\mathbb{K}[[x_1, \dots, x_n]]$ such that $\phi(f) = g$.

Note that $\stackrel{\mathbb{K}}{\sim}$ is an equivalence relation on $\mathbb{K}[[x_1, \dots, x_n]]$. Arnold et al. (1985) give the following formal definition for normal forms w.r.t. this relation:

DEFINITION 2. Let $K \subset \mathbb{K}[[x_1, \dots, x_n]]$ be a union of equivalence classes w.r.t. the relation $\stackrel{\mathbb{K}}{\sim}$. A normal form for K is given by a smooth map

$$\Phi : B \longrightarrow \mathbb{K}[x_1, \dots, x_n] \subset \mathbb{K}[[x_1, \dots, x_n]]$$

of a finite-dimensional \mathbb{K} -linear space of parameters B into the space of polynomials for which the following three conditions hold:

- (1) $\Phi(B)$ intersects all the equivalence classes of K ;
- (2) the inverse image in B of each equivalence class is finite;
- (3) the inverse image of the whole complement to K is contained in some proper hypersurface in B .

REMARK 3. Note that the term *normal form* is subtly ambiguous. According to the above definition, a normal form is a smooth map where the inverse image of each equivalence class may contain more than one element, whereas the common meaning of this term rather refers to the polynomials which are the images under this map. We could be more precise and avoid this ambiguity by introducing a new term for either of the two meanings. However, we stay with the common usage of the term *normal form* in order to prevent confusion.

DEFINITION 4. Let $S \subset \text{Aut}_{\mathbb{K}}(\mathbb{K}[[x_1, \dots, x_n]])$ be a set of \mathbb{K} -algebra automorphisms of $\mathbb{K}[[x_1, \dots, x_n]]$ and let $f, g \in \mathbb{K}[[x_1, \dots, x_n]]$ be two power series.

- (1) We denote the set of all automorphisms in S which take f to g by $T_{\mathbb{K}}^S(f, g)$, i.e.

$$T_{\mathbb{K}}^S(f, g) := \{\phi \in S \mid \phi(f) = g\}.$$

- (2) If $S = \text{Aut}_{\mathbb{K}}(\mathbb{K}[[x_1, \dots, x_n]])$, we simply write $T_{\mathbb{K}}(f, g)$ for $T_{\mathbb{K}}^S(f, g)$, i.e.

$$T_{\mathbb{K}}(f, g) := \{\phi \in \text{Aut}_{\mathbb{K}}(\mathbb{K}[[x_1, \dots, x_n]]) \mid \phi(f) = g\}.$$

The above definition is the key ingredient for the definition of P_1 , P_2 , and P_3 . We also need the following notation.

REMARK 5. As usual, we denote the field of quotients $\text{Quot}(\mathbb{K}[a])$ by $\mathbb{K}(a)$. Let $f \in \mathbb{K}(a)[[x_1, \dots, x_n]]$ be a power series over this quotient field. Then f can be written as $f = \sum_{\nu \in \mathbb{N}^n} c_{\nu} \mathbf{x}^{\nu}$ with coefficients $c_{\nu} = \frac{p_{\nu}}{q_{\nu}} \in \mathbb{K}(a)$ where $p_{\nu}, q_{\nu} \in \mathbb{K}[a]$ are polynomials of minimal degree with this property and $q_{\nu} \neq 0$ for all $\nu \in \mathbb{N}^n$.

If we consider the polynomials p_{ν}, q_{ν} as polynomial functions $p_{\nu}, q_{\nu} : \mathbb{K} \rightarrow \mathbb{K}$, then we may also consider the coefficients c_{ν} as functions $c_{\nu} : \mathbb{K} \setminus V(q_{\nu}) \rightarrow \mathbb{K}$ where $V(q_{\nu})$ is the set of points where q_{ν} vanishes. Via this correspondence, we finally get power series $f(u) := \sum_{\nu \in \mathbb{N}^n} c_{\nu}(u) \mathbf{x}^{\nu} \in \mathbb{K}[[x_1, \dots, x_n]]$ for each value $u \in \mathbb{K} \setminus \bigcup_{\nu \in \mathbb{N}^n} V(q_{\nu})$.

We use the notation $f(u)$ throughout this paper. Likewise, we add the value of the parameter which occurs in the normal form as given in Table 1 in parentheses to the name of the singularity (sub-)type if we want to refer specifically to the corresponding equivalence class. For instance, we denote by $E_{14}(3)$ the (complex or real) right-equivalence class of $x^3 + y^8 + 3xy^6$.

For any specific singularity type T , we denote by $\text{NF}(T)$ its normal form as shown in Table 1, i.e. we write $\text{NF}(E_{14}(a)) = \text{NF}(E_{14}^+(a))$ for the polynomial $x^3 + y^8 + axy^6$ and $\text{NF}(E_{14}^-(5))$ for $x^3 - y^8 + 5xy^6$.

We can now state the main definition of this section.

DEFINITION 6.

- (1) Given power series $f, g \in \mathbb{C}(a)[[x_1, \dots, x_n]]$, we define the first set of parameter transformations of f and g as

$$P_1(f, g) := \{(u, v) \in \mathbb{C}^2 \mid f(u) \text{ and } g(v) \text{ are well-defined and } \mathbb{T}_{\mathbb{C}}(f(u), g(v)) \neq \emptyset\}.$$

- (2) Given power series $f, g \in \mathbb{R}(a)[[x_1, \dots, x_n]]$, we define the second set of parameter transformations of f and g as

$$P_2(f, g) := \{(u, v) \in \mathbb{R}^2 \mid f(u) \text{ and } g(v) \text{ are well-defined and } \mathbb{T}_{\mathbb{C}}(f(u), g(v)) \neq \emptyset\}.$$

- (3) Given power series $f, g \in \mathbb{R}(a)[[x_1, \dots, x_n]]$, we define the third set of parameter transformations of f and g as

$$P_3(f, g) := \{(u, v) \in \mathbb{R}^2 \mid f(u) \text{ and } g(v) \text{ are well-defined and } \mathbb{T}_{\mathbb{R}}(f(u), g(v)) \neq \emptyset\}.$$

REMARK 7.

- (1) Note that we have $P_3(f, g) \subseteq P_2(f, g) \subseteq P_1(f, g)$ for any two power series $f, g \in \mathbb{R}(a)[[x_1, \dots, x_n]]$.
- (2) For any two unimodal singularity (sub-)types T_1, T_2 and $i \in \{1, 2, 3\}$, we simply write $P_i(T_1, T_2)$ instead of $P_i(\text{NF}(T_1(a)), \text{NF}(T_2(a)))$, e.g. we write $P_1(E_{14}^+, E_{14}^+)$ for $P_1(\text{NF}(E_{14}^+(a)), \text{NF}(E_{14}^+(a)))$.

For the parabolic singularity types X_9 and J_{10} , the sets P_1 , P_2 , and P_3 can be described in terms of the following definition.

DEFINITION 8. For $\Omega \subset \mathbb{C}$, let $(f_i : \Omega \rightarrow \mathbb{C})_{i \in I}$ be a family of complex-valued functions on Ω . We define the joint graph of $(f_i)_{i \in I}$ over Ω as

$$\Gamma_{\Omega}((f_i)_{i \in I}) := \{(a, f_i(a)) \in \Omega \times \mathbb{C} \mid a \in \Omega, i \in I\}.$$

It turns out that for the hyperbolic and exceptional unimodal singularities, P_1 , P_2 and P_3 are just unions of sets of the form $(a, ra)_{a \in \mathbb{K}}$ for some $r \in \mathbb{K}$. For those cases we use the following notations.

DEFINITION 9. For any polynomial $p(X) \in \mathbb{C}[X]$, we define the sets $C_0(p(X))$ and $R_0(p(X))$ as

$$\begin{aligned} C_0(p(X)) &:= \{(a, ra) \in \mathbb{C}^2 \mid a, r \in \mathbb{C}, p(r) = 0\}, \\ R_0(p(X)) &:= \{(a, ra) \in \mathbb{R}^2 \mid a, r \in \mathbb{R}, p(r) = 0\}. \end{aligned}$$

Additionally, we define $C(p(X))$ and $R(p(X))$ as

$$\begin{aligned} C(p(X)) &:= C_0(p(X)) \setminus \{(0, 0)\}, \\ R(p(X)) &:= R_0(p(X)) \setminus \{(0, 0)\}. \end{aligned}$$

REMARK 10. We occasionally use the notation $R(X^l - s)$ with $l \in \mathbb{N} \setminus \{0\}$ and $s \in \{-1, +1\}$, e.g. in Tables 5 and 6. Of course, this could be written in a more explicit way for many values of l and s ; for instance, we could write \emptyset instead of $R(X^4 + 1)$. But distinguishing between different cases would spoil the symmetries of those tables and we therefore stick to the shorthand notation.

3. Weighted Jets and Filtrations of Power Series and Transformations

We briefly introduce the concepts of (piecewise) weighted jets and filtrations. For background regarding the definitions in this section, we refer to Arnold (1974). We assume that the reader is familiar with the notions of weighted degrees, quasihomogeneous polynomials, and Newton polygons.

REMARK 11. Let w be a weight on the variables (x_1, \dots, x_n) . Throughout this paper we always assume that the weighted degree of x_i , denoted by $w\text{-deg}(x_i)$, is a natural number for each $i = 1, \dots, n$.

DEFINITION 12. Let $w_0 := (w_1, \dots, w_s) \in (\mathbb{N}^n)^s$ be a finite family of weights on the variables (x_1, \dots, x_n) . For any term $t \in \mathbb{K}[x_1, \dots, x_n]$, we define the piecewise weight of t w.r.t. w_0 as

$$w_0\text{-deg}(t) := \min_{i=1, \dots, s} w_i\text{-deg}(t).$$

A polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ is called piecewise quasihomogeneous of degree d w.r.t. w_0 if $w_0\text{-deg}(t) = d$ for any term t of f .

DEFINITION 13. Let w be a (piecewise) weight on the variables (x_1, \dots, x_n) .

- (1) Let $f = \sum_{i=0}^{\infty} f_i$ be the decomposition of $f \in \mathbb{K}[[x_1, \dots, x_n]]$ into weighted homogeneous parts f_i of w -degree i . We denote the weighted j -jet of f w.r.t. w by

$$w\text{-jet}(f, j) := \sum_{i=0}^j f_i.$$

- (2) A power series in $\mathbb{K}[[x_1, \dots, x_n]]$ has filtration $d \in \mathbb{N}$ if all its terms are of weighted degree d or higher. The power series of filtration d form a vector space $E_d^w \subset \mathbb{K}[[x_1, \dots, x_n]]$.

REMARK 14. Note that $d < d'$ implies $E_{d'}^w \subseteq E_d^w$. Since the filtration of the product $E_{d'}^w \cdot E_d^w$ is $d' + d$, it follows that E_d^w is an ideal in the ring of power series. We denote the ideal consisting of power series of filtration strictly greater than d by $E_{>d}^w$. If the weight of each variable is 1, we simply write E_d and $E_{>d}$, respectively.

There are also similar concepts for coordinate transformations:

DEFINITION 15. Let ϕ be a \mathbb{K} -algebra automorphism of $\mathbb{K}[[x_1, \dots, x_n]]$ and let w be a (piecewise) weight on the variables.

- (1) For $j > 0$ we define the weighted j -jet of ϕ w.r.t. w , denoted by ϕ_j^w , to be the map given by

$$\phi_j^w(x_i) := w\text{-jet}(\phi(x_i), w\text{-deg}(x_i) + j) \quad \forall i = 1, \dots, n.$$

If the weight of each variable is 1, i.e. $w = (1, \dots, 1)$, we simply write ϕ_j for ϕ_j^w .

- (2) ϕ has filtration d if, for all $\lambda \in \mathbb{N}$,

$$(\phi - \text{id})E_\lambda^w \subset E_{\lambda+d}^w.$$

REMARK 16. Let ϕ be a \mathbb{K} -algebra automorphism of $\mathbb{K}[[x_1, \dots, x_n]]$.

- (1) Note that $\phi_0(x_i) = \text{jet}(\phi(x_i), 1)$ for all $i = 1, \dots, n$. Furthermore note that ϕ_0^w may have filtration less than or equal to 0 for any weight w .
- (2) Let $w_0 = (w_1, \dots, w_s) \in (\mathbb{N}^n)^s$ be a piecewise weight on (x_1, \dots, x_n) , let $f_0 \in \mathbb{K}[x_1, \dots, x_n]$ be piecewise quasihomogeneous of degree d_0 w.r.t. w_0 and $f_1 \in \mathbb{K}[x_1, \dots, x_n]$ quasihomogeneous of degree d_1 w.r.t. w_1 . For any $\delta \geq 0$, we always have $(\phi - \phi_\delta^{w_1})(f_1) \in E_{>d_1+\delta}^{w_1}$, but the analogon for w_0 does not hold in general: To give a counterexample, let us consider the

case $n = s = 2$, $w_0 = ((1, 4), (4, 1))$, $f_0 = x_1x_2$, and let ϕ be given by $\phi(x_1) := x_1 + x_2^2$, $\phi(x_2) := x_2 + x_2^2$. Then f_0 is of degree $d_0 = 5$, but $(\phi - \phi_0^{w_0})(f_0) = x_2^4$ is of degree 4 w.r.t. w_0 and thus not an element of $E_5^{w_0} = E_{d_0+0}^{w_0}$.

4. Sufficient Sets of Transformations

The results in this section considerably narrow down the transformations we need to consider between specific unimodal normal forms of the same main type in order to check if they are equivalent or not. In fact these results are in many cases the main step for determining the structure of the equivalence classes of the unimodal singularities up to corank 2.

DEFINITION 17. *Let f and g be elements in $\mathbb{C}(a)[[x_1, \dots, x_n]]$ and let S be a subset of $\text{Aut}_{\mathbb{C}}(\mathbb{C}[[x_1, \dots, x_n]])$. We call S a sufficient set of coordinate transformations for the pair (f, g) if*

$$\forall u, v \in \mathbb{C} : (\mathbb{T}_{\mathbb{C}}(f(u), g(v)) \neq \emptyset \Leftrightarrow \mathbb{T}_{\mathbb{C}}^S(f(u), g(v)) \neq \emptyset).$$

The sufficient sets which we consider here can be described using the following notation.

DEFINITION 18. *Let M_x and M_y be sets of monomials in $\mathbb{C}[[x, y]]$ and let $\mathbb{C}M_x$ and $\mathbb{C}M_y$ be the \mathbb{C} -vector spaces spanned by these sets, i.e. $\mathbb{C}M_x := \bigoplus_{m \in M_x} \mathbb{C}m$ and analogously for $\mathbb{C}M_y$. We define the set of coordinate transformations spanned by M_x and M_y as*

$$S(M_x, M_y) := \{\phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]]) \mid \phi(x) \in \mathbb{C}M_x, \phi(y) \in \mathbb{C}M_y\}.$$

THEOREM 19. *Let T be one of the main singularity types listed in Table 2, let S be the corresponding set of automorphisms, and let T_1 and T_2 be subtypes of T . Then S is a sufficient set of coordinate transformations for $(\text{NF}(T_1(a)), \text{NF}(T_2(a)))$.*

TABLE 2. Sufficient sets for unimodal singularities of corank 2

	T	S
$P.$	X_9	$S(\{x, y\}, \{x, y\})$
	J_{10}	$S(\{x, y^2\}, \{y\})$
$Hyperbolic$	J_{10+k}	$S(\{x\}, \{y\})$
	X_{9+k}	$S(\{x\}, \{y\})$
	$Y_{r,s}$	$r \neq s$
$r = s$		$S(\{x\}, \{y\}) \cup S(\{y\}, \{x\})$
$Except.$	E_{12}, E_{13}, E_{14}	$S(\{x\}, \{y\})$
	Z_{11}, Z_{12}, Z_{13}	$S(\{x\}, \{y\})$
	W_{12}, W_{13}	$S(\{x\}, \{y\})$

PROOF OF THEOREM 19. We give different proofs for the parabolic, the hyperbolic, and the exceptional cases as indicated in Table 2.

In each case, let T_1 and T_2 be subtypes of the same main type T , and for $u \in \mathbb{C}$ let $\phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]])$ be a coordinate transformation which takes $f := \text{NF}(T_1(u))$ to $\text{NF}(T_2(v))$ for some $v \in \mathbb{C}$.

Parabolic cases. The normal forms of both X_9 and J_{10} are quasihomogeneous with weights $w := (1, 1)$ and $w := (2, 1)$, respectively. Let us first consider the case $T = X_9$. We have

$$\phi(f) = \phi_0^w(f) + (\phi - \phi_0^w)(f) = \phi_0^w(f) + R$$

with $R \in E_{>4}^w$. This implies $\phi(f) = \phi_0^w(f)$ because $\phi(f) = \text{NF}(T_2(v))$ is homogeneous of degree 4 w.r.t. the weight w . So any possible value of v which can be reached via some $\phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]])$ can also be obtained by $\phi_0^w \in \text{S}(\{x, y\}, \{x, y\})$, i.e., $\text{S}(\{x, y\}, \{x, y\})$ is a sufficient set of coordinate transformations for the pair $(\text{NF}(T_1(a)), \text{NF}(T_2(a)))$.

Let us now consider the case $T = J_{10}$. Again we have $\phi(f) = \phi_0^w(f)$, but in this case ϕ_0^w is of the form $\phi_0^w(x) = \alpha x + \beta y + \gamma y^2$, $\phi_0^w(y) = \delta y$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Comparing the coefficients of $\phi(f) = \text{NF}(T_2(v))$ and $\phi_0^w(f) = \beta^3 y^3 + (\text{other terms})$ yields $\beta = 0$ and therefore $\phi_0^w \in \text{S}(\{x, y^2\}, \{y\})$ as expected.

Hyperbolic cases. We present a proof for the main type $T = J_{10+k}$, the proofs for X_{9+k} and $Y_{r,s}$ are similar. For $Y_{r,s}$ with $r = s$, we have to take the special shape of S into account, cf. Table 2.

It does not matter for the arguments below whether we assume $T_1 = J_{10+k}^+$ or $T_1 = J_{10+k}^-$, the same holds for T_2 . We write \pm whenever the sign can be either plus or minus in order to prove all cases at once.

The Newton polygon of $f = \text{NF}(T_1(u)) = x^3 \pm x^2 y^2 + u y^{6+k}$ has two faces defined by $f_1 := x^3 \pm x^2 y^2$ and $f_2 := \pm x^2 y^2 + u y^{6+k}$. Let w_0 be the piecewise weight given by the two weights $w_1 := (12 + 2k, 6 + k)$ and $w_2 := (12 + 3k, 6)$. Then f is piecewise quasihomogeneous of degree $d := 36 + 6k$ w.r.t. w_0 .

We now proceed in three steps: In the first two, we show $\phi_0^{w_1} \in \text{S}(\{x\}, \{y\})$ and $\phi_0^{w_2} \in \text{S}(\{x\}, \{y\})$. Finally we conclude that $\phi(f)$ is equal to $\phi_0^{w_0}(f)$ and that $\phi_0^{w_0}$ is an element of $\text{S}(\{x\}, \{y\})$ which proves the claim.

The transformation $\phi_0^{w_1}$ is generically of the form

$$\begin{aligned} \phi_0^{w_1}(x) &= \alpha x + \beta_1 y + \beta_2 y^2, \\ \phi_0^{w_1}(y) &= \gamma y \end{aligned}$$

with coefficients $\alpha, \beta_1, \beta_2, \gamma \in \mathbb{C}$. With these notations we have

$$\begin{aligned} \phi(f) &= \phi_0^{w_1}(f) + (\phi - \phi_0^{w_1})(f) \\ &= \beta_1^3 y^3 + (3\alpha\beta_2^2 \pm 2\alpha\beta_2\gamma^2)xy^4 + (\beta_2^3 \pm \beta_2^2\gamma^2)y^6 + (\text{other terms}) \end{aligned}$$

on the one hand and

$$\phi(f) = \text{NF}(T_1(v)) = x^3 \pm x^2 y^2 + v y^{6+k}$$

on the other hand. This implies (in this order) $\beta_1 = 0$, $\alpha \neq 0$, $\beta_2 = 0$ and hence $\phi_0^{w_1} \in \text{S}(\{x\}, \{y\})$.

The second step is a proof by contradiction. Let m be the largest integer which is not greater than $\frac{k}{2} + 2$. Then similar as above, the automorphism $\phi_0^{w_2}$ is of the form

$$\begin{aligned} \phi_0^{w_2}(x) &= \alpha x + \beta_1 y + \beta_2 y^2 + \dots + \beta_m y^m, \\ \phi_0^{w_2}(y) &= \gamma y \end{aligned}$$

with $\alpha, \beta_1, \dots, \beta_m, \gamma \in \mathbb{C}$ and $\alpha, \gamma \neq 0$. We have already shown $\beta_1 = \beta_2 = 0$. Assume $\beta_s \neq 0$ for some $s \in \{3, \dots, m\}$ and let s be minimal with this property. Then the coefficient of xy^{s+2} in $\phi(f) = \phi_0^{w_2}(f) + (\phi - \phi_0^{w_2})(f)$ is $\pm 2\alpha\beta_s\gamma^2$ which implies $\beta_s = 0$ in contradiction to the assumption. Hence $\beta_3 = \dots = \beta_m = 0$ and $\phi_0^{w_2} \in \text{S}(\{x\}, \{y\})$.

For the last step, we consider the following equations:

$$\begin{aligned}\phi(f) &= \phi_0^{w_1}(f) + \underbrace{(\phi - \phi_0^{w_1})(f)}_{=: R_1 \in E_{>d}^{w_1}} \\ \phi(f) &= \phi_0^{w_2}(f) + \underbrace{(\phi - \phi_0^{w_2})(f)}_{=: R_2 \in E_{>d}^{w_2}} \\ \phi(f) &= \phi_0^{w_0}(f) + \underbrace{(\phi - \phi_0^{w_0})(f)}_{=: R_0}\end{aligned}$$

Note that it is not a priori clear that R_0 lies in $E_{>d}^{w_0}$ if we only consider these equations, cf. Remark 16(2). Nevertheless, this can be shown if we take into account the results of the two previous steps: By definition of the piecewise weight w_0 , any term in $\phi_0^{w_0}(x)$ also appears in $\phi_0^{w_1}(x)$ or $\phi_0^{w_2}(x)$ (or both), analogously for $\phi_0^{w_0}(y)$. Therefore we have $\phi_0^{w_0}(x) = \alpha x$ and $\phi_0^{w_0}(y) = \gamma y$, hence $\phi_0^{w_0} = \phi_0^{w_1} = \phi_0^{w_2}$ and $\phi_0^{w_0} \in S(\{x\}, \{y\})$. This implies

$$R_0 = R_1 = R_2 \in E_{>d}^{w_1} \cap E_{>d}^{w_2} = E_{>d}^{w_0}.$$

Since $\phi(f) = \text{NF}(T_2(v))$ is piecewise quasihomogeneous of degree d w.r.t. w_0 , we finally get $R_0 = 0$ and $\phi(f) = \phi_0^{w_0}(f)$. This proves the claim.

Exceptional cases. The normal forms of all the exceptional cases in Table 2 are semi-quasihomogeneous polynomials, i.e., in these cases $f = \text{NF}(T_1(u))$ is of the form $f = f_0 + f_1$ where f_0 is quasihomogeneous of degree $d \in \mathbb{N}$ w.r.t. some weight $w = (w_x, w_y)$, f_1 has weighted degree $d + \delta > d$, and the Milnor number $\mu(f_0)$ of f_0 is finite (for the definition of the Milnor number, see Marais and Steenpaß, 2015). In all the cases, f_1 consists of the term which contains the parameter and we have

$$\begin{aligned}\phi(f) &= \phi_\delta^w(f_0) + \phi_0^w(f_1) + (\phi - \phi_\delta^w)(f_0) + (\phi - \phi_0^w)(f_1) \\ &= w\text{-jet}(\phi_\delta^w(f_0), d + \delta) + \phi_0^w(f_1) + R\end{aligned}$$

with $R \in E_{>d+\delta}^w$. As above, $\phi(f) = \text{NF}(T_2(v))$ implies $R = 0$. If we show

$$(*) \quad \phi_\delta^w \in S(\{x\}, \{y\}),$$

then it follows that ϕ_δ^w is equal to ϕ_0^w and therefore

$$\begin{aligned}\phi(f) &= w\text{-jet}(\phi_0^w(f_0), d + \delta) + \phi_0^w(f_1) \\ &= \phi_0^w(f_0) + \phi_0^w(f_1) \\ &= \phi_0^w(f).\end{aligned}$$

This, together with (*), proves the claim.

The statement (*) can be shown separately for each of the eight cases by some easy computations. We carry out the proof for W_{13} , the other cases follow similarly. The normal forms of the subtypes of W_{13} are $\pm x^4 + xy^4 + ay^6$, so in this case we have $w = (4, 3)$, $d = 16$, and $\delta = 2$. The \pm -sign does not matter for the computations which follow, but we carry it along in order to prove all subcases at once. The transformation ϕ_δ^w is generically of the form

$$\begin{aligned}\phi_\delta^w(x) &= \alpha x + \beta y + \gamma y^2, \\ \phi_\delta^w(y) &= \varepsilon x + \zeta y\end{aligned}$$

with $\alpha, \beta, \gamma, \varepsilon, \zeta \in \mathbb{C}$ because any other term would raise the weighted degree by more than δ . With these notations, we now successively compare the coefficients of $\phi_\delta^w(f)$ and $\phi(f) = \text{NF}(T_2(v)) = \pm x^4 + xy^4 + vy^6$. The coefficient of y^4 in $\phi_\delta^w(f)$ is $\pm\beta^4$, therefore we have $\beta = 0$. The remaining coefficients of xy^4 , x^2y^3 , and x^3y^2 are now $\alpha\zeta^4$, $4\alpha\varepsilon\zeta^3$, and $\pm 4\alpha^3\gamma + 6\alpha\varepsilon^2\zeta^2$, respectively, which shows that (in this

order) $\alpha\zeta \neq 0$, $\varepsilon = 0$, and $\gamma = 0$. Hence ϕ_δ^w is in fact of the form $\phi_\delta^w(x) = \alpha x$, $\phi_\delta^w(y) = \zeta y$ which proves (*) for $T_1, T_2 \in \{W_{13}^+, W_{13}^-\}$.

□

5. On the Computation of the Results

Based on the previous section, the results presented in Section 6 can be computed using SINGULAR for all those singularity types which are covered by Theorem 19. The main tools for these computations are elimination, Gröbner covers, and primary decomposition. For details on these topics, we refer to Greuel and Pfister (2008). For each pair of singularity subtypes T_1, T_2 , the computation follows the same structure: One can first compute the set $P_1(T_1, T_2)$ using elimination and factorization. The set $P_2(T_1, T_2)$ can then be derived from this as the intersection of $P_1(T_1, T_2)$ with $\mathbb{R} \times \mathbb{R}$. In order to determine $P_3(T_1, T_2)$, one finally has to check for each point or branch in $P_2(T_1, T_2)$ whether or not there is a real transformation which changes the parameter in such a way. Gröbner covers and primary decomposition are convenient tools to simplify the often complicated ideals which occur in this last step.

Although our approach is almost algorithmic, we do not present it as an algorithm here because each case requires slightly different means depending on the intermediate results. Especially the computation of $P_3(T_1, T_2)$ is rather straightforward in some cases whereas it requires careful considerations in other cases.

However, writing down every detail of the computations for each case is beyond the scope of this section. Instead, we present the general framework and give explicit SINGULAR commands for $T_1 = T_2 = X_9^{++}$ which is one of the more complicated cases (cf. Theorem 29).

The singularity type \tilde{Y}_r does not appear in Table 2 and thus needs special care. The structure of the equivalence classes of this type can be computed on the basis of the data for the type $Y_{r,s}$, cf. Section 5.4.

5.1. How to Compute $P_1(T_1, T_2)$. We denote the parameter occurring in $\text{NF}(T_1)$ by a and the one in $\text{NF}(T_2)$ by b . The computation is done in four steps:

Step 1. Set up a generic transformation using Theorem 19:

Let $S = S(M_x, M_y) \subset \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]])$ be the sufficient set of $\mathbb{C}[[x, y]]$ -automorphisms for $(\text{NF}(T_1(a)), \text{NF}(T_2(a)))$ given in Theorem 19. Let t_1, \dots, t_r be coefficients for the monomials in M_x and M_y and let ϕ be a generic element of S with these coefficients, i.e. let ϕ be of the form $\phi(x) = t_1 \cdot x + (\text{other terms})$ (or of the form $\phi(x) = t_1 \cdot y + (\text{other terms})$ in case T_1 and T_2 are subtypes of $Y_{r,s}$ with $r = s$).

Step 2. Set up a system of equations for the parameters:

By comparing the coefficients in $\phi(\text{NF}(T_1(a)))$ and $\text{NF}(T_2(b))$, we get a set of equations in a, b, t_1, \dots, t_r which is equivalent to $\phi(\text{NF}(T_1(a))) = \text{NF}(T_2(b))$. Let $I \subset \mathbb{C}[a, b, t_1, \dots, t_r]$ be the ideal generated by these equations. Then the vanishing set $V(I)$ describes completely which transformations take $\text{NF}(T_1(a))$ to $\text{NF}(T_2(b))$ for which values of a and b .

Step 3. Use elimination to obtain an equation in a and b only:

We can now eliminate the variables t_1, \dots, t_r from I and thus obtain an ideal $I' \subset \mathbb{C}[a, b]$ which is in all cases generated by one polynomial g . This elimination geometrically corresponds to the projection $\mathbb{A}_{\mathbb{C}}^{2+r} \supset V(I) \mapsto V(I') \subset \mathbb{A}_{\mathbb{C}}^2$.

Step 4. Compute the zeros of this equation:

After factorizing $g \in \mathbb{C}[a, b]$ into irreducible factors g_1, \dots, g_s , we compute the roots in b of each factor (over $\mathbb{C}(a)$ or suitable extensions thereof if necessary). We thus get roots of the form $b = f(a)$ where $f(a)$ can be considered as a function in a .

These functions explicitly determine the possible values of b for each given a and their joint graph is exactly $P_1(T_1, T_2)$.

EXAMPLE 20. We compute $P_1(X_9^{++}, X_9^{++})$ with Singular, following the steps above.

Step 1. Set up a generic transformation using Theorem 19:

For convenience we work over $\mathbb{Q}(a, b, t_1, t_2, t_3, t_4)[x, y]$:

```
> ring R = (0,a,b,t1,t2,t3,t4), (x,y), dp;
> poly f = x^4+a*x^2*y^2+y^4;
```

According to Theorem 19,

$$S = \{ \phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]]) \mid \phi(x) = t_1x + t_2y, \phi(y) = t_3x + t_4y, t_1, \dots, t_4 \in \mathbb{C} \}$$

is a sufficient set of automorphisms for X_9 :

```
> map phi = R, t1*x+t2*y, t3*x+t4*y;
```

Step 2. Set up a system of equations for the parameters:

```
> matrix C = coef(phi(f), xy);
> print(C);
x^4, x^3*y, x^2*y^2, x*y^3, y^4,
C[2,1], C[2,2], C[2,3], C[2,4], C[2,5]
> C[2,1];
(a*t1^2*t3^2+t1^4+t3^4)
```

Now the second row of the matrix C contains the coefficients of $\text{phi}(X_9^{++}(a))$, $C[2, 1]$ for instance is the one belonging to x^4 . Using the corresponding coefficients of $X_9^{++}(b) = x^4 + b \cdot x^2y^2 + y^4$, we can define the ideal I as above:

```
> matrix D[1][5] = 1, 0, b, 0, 1;
> ideal I = C[2,1..5]-D[1,1..5];
```

Step 3. Use elimination to obtain an equation in a and b only:

As the next step, we map this ideal to $\mathbb{Q}(a)[b, t_1, t_2, t_3, t_4]$ and eliminate the variables t_i :

```
> ring S = (0,a), (b,t1,t2,t3,t4), dp;
> ideal I = imap(R, I);
> ideal g = eliminate(I, t1*t2*t3*t4);
> g;
g[1]=(a^4-8*a^2+16)*b^6+(-a^6-720*a^2-1152)*b^4
+(8*a^6+720*a^4+20736)*b^2+(-16*a^6+1152*a^4-20736*a^2)
```

Step 4. Compute the zeros of this equation:

Factorizing the single generator of this ideal finally yields the functions $f_1^{1,1}, \dots, f_6^{1,1}$ defined in Theorem 29. Note that $a^2 \neq 4$.

```
> factorize(g[1]);
[1]:
_ [1]=1
_ [2]=b+(-a)
_ [3]=b+(a)
_ [4]=(a-2)*b+(-2*a-12)
_ [5]=(a+2)*b+(-2*a+12)
_ [6]=(a+2)*b+(2*a-12)
_ [7]=(a-2)*b+(2*a+12)
[2]:
1,1,1,1,1,1,1
```

5.2. How to Compute $P_2(T_1, T_2)$. Given $P_1(T_1, T_2)$, it is easy to compute $P_2(T_1, T_2)$ even “by hand” because we have

$$P_2(T_1, T_2) = P_1(T_1, T_2) \cap (\mathbb{R} \times \mathbb{R}).$$

EXAMPLE 21. Continuing the example above, the values of $f_1^{1,1}(a), \dots, f_6^{1,1}(a)$ are clearly real for $a \in \mathbb{R}$, cf. Theorem 29. The set $P_2(X_9^{++}, X_9^{++})$ is thus the joint graph of these functions over $\mathbb{R} \setminus \{-2, 2\}$.

To give another example, for $T_1 = T_2 = X_9^{+-}$ the set $P_1(T_1, T_2)$ is the joint graph of $f_1^{i,i}, \dots, f_6^{i,i}$ over $\mathbb{C} \setminus \{-2i, 2i\}$. The values of $f_1^{i,i}(a)$ and $f_2^{i,i}(a)$ are clearly real for $a \in \mathbb{R}$, but those of $f_3^{i,i}(a), \dots, f_6^{i,i}(a)$ are not except at some exceptional points which are already covered by $f_1^{i,i}$ and $f_2^{i,i}$. So in this case we have

$$P_2(X_9^{+-}, X_9^{+-}) = \Gamma_{\mathbb{R}}(f_1^{i,i}, f_2^{i,i}) = \Gamma_{\mathbb{R}}(f_1^{1,1}, f_2^{1,1}).$$

5.3. How to Compute $P_3(T_1, T_2)$. We do the computation in two steps:

Step 1. Reduce the problem to a finite number of branches and exceptional points:

Since $P_3(T_1, T_2) \subset P_2(T_1, T_2)$ by definition, we can determine $P_3(T_1, T_2)$ by checking for each pair $(a, b) \in P_2(T_1, T_2)$ whether or not there is a real coordinate transformation $\phi \in \text{Aut}_{\mathbb{R}}(\mathbb{R}[[x, y]])$ which takes $\text{NF}(T_1(a))$ to $\text{NF}(T_2(b))$. This can be reduced to a finite problem as follows: Let $g_j, j \in \{1, \dots, s\}$, be the irreducible factors of the polynomial g as in Section 5.1. Then in all the cases, $P_2(T_1, T_2)$ is a finite union of “branches” of the form $V(g_j)$ and some exceptional points. We can check whether a branch $V(g_j)$ or an exceptional point (q_a, q_b) in $P_2(T_1, T_2)$ belongs $P_3(T_1, T_2)$ by simply adding appropriate relations to the ideal I and looking at the real solutions of the resulting ideal. In other words, we define $J := I + \langle g_j \rangle$ or $J := I + \langle a - q_a, b - q_b \rangle$, respectively, and investigate $V_{\mathbb{R}}(J)$. Note that we have $I \subset \mathbb{R}[a, b, t_1, \dots, t_r]$ and $g_j \in \mathbb{R}[a, b]$ and thus $J \subset \mathbb{R}[a, b, t_1, \dots, t_r]$ in all the cases.

Step 2. For each branch and each exceptional point, check if a real transformation exists:

$P_3(T_1, T_2)$ is the image of $V_{\mathbb{R}}(J) \subset \mathbb{A}_{\mathbb{R}}^{2+r}$ under the projection $\mathbb{A}_{\mathbb{R}}^{2+r} \rightarrow \mathbb{A}_{\mathbb{R}}^2$, i.e. we have $(p_a, p_b) \in P_3(T_1, T_2)$ if and only if there is a coordinate transformation with real coefficients $(p_{t_1}, \dots, p_{t_r})$ such that $(p_a, p_b, p_{t_1}, \dots, p_{t_r})$ is an element of $V_{\mathbb{R}}(J) \subset \mathbb{A}_{\mathbb{R}}^{2+r}$.

It turns out that the ideal J is quite complicated in some cases and that it can be difficult to determine $V_{\mathbb{R}}(J)$ by just computing a Gröbner basis of J . One way out is then to consider J as a parametric ideal $J \subset \mathbb{R}(a)[b, t_1, \dots, t_r]$ and to compute a Gröbner cover thereof by using the SINGULAR library `grobcov.lib` (Montes and Schönemann, 2015). A Gröbner cover completely describes the possible shapes of Gröbner bases of J for different values of a . It contains a generic Gröbner basis of J , i.e. one which is a Gröbner basis except for finitely many exceptional values of a , and additionally Gröbner bases of J for each of these exceptional values. The ideals in a Gröbner cover of J typically have a much easier structure than J itself. We can thus treat them one by one and determine their real solutions. We will often find generators such as $(t_j)^4 + 1$, indicating that the vanishing set over \mathbb{R} of this ideal is empty.

If any of the ideals in the Gröbner cover of J are still too complicated and if their vanishing set over \mathbb{R} cannot be easily read off, another trick is to compute a primary decomposition of these ideals with the SINGULAR library `primdec.lib` (Decker et al., 2015b). Typically, it is then easy to see that some of the primary components have no solutions over \mathbb{R} whereas the real solutions of the remaining components can be easily determined.

EXAMPLE 22.

Step 1. Reduce the problem to a finite number of branches and exceptional points:

We have already seen in Example 21 that $P_2(X_9^{++}, X_9^{++})$ is the joint graph of $f_1^{1,1}, \dots, f_6^{1,1}$ over $\mathbb{R} \setminus \{-2, 2\}$. We now have to check for each of these functions whether their graph is also contained in $P_3(X_9^{++}, X_9^{++})$.

Step 2a. Check if a real transformation exists for $f_3^{1,1}$:

This is clearly the case for $f_1^{1,1} = \text{id}$. To check this for $f_3^{1,1}$, we continue the SINGULAR session from Example 20, add the corresponding relation to the ideal I and compute a Gröbner cover of the resulting ideal J :

```
> ideal J = I, (a-2)*b+(-2*a-12);
> LIB "grobcov.lib";
> grobcov(J);
```

The output of the last command is too long to be printed here. We will find that the Gröbner basis of J for generic a contains the generators $(t_2)^2 + (t_4)^2$ and $(t_3)^2 + (t_4)^2$ which imply $t_2 = t_3 = t_4 = 0$ for any real solution of this ideal. But this is a contradiction to $\phi \in \text{Aut}_{\mathbb{R}}(\mathbb{R}[[x, y]])$. The exceptional cases for the parameter a are $a + 2 = 0$, $a - 2 = 0$, $a^2 + 12 = 0$, $a + 6 = 0$, $a - 6 = 0$, and $a = 0$. The first two cases are excluded by the definition of the singularity type X_9^{++} , $a^2 + 12 = 0$ would imply $a \notin \mathbb{R}$, for $a + 6 = 0$ and $a = 0$ the corresponding Gröbner bases of J contain generators similar to those mentioned above, and finally $a - 6 = 0$ implies $b - 6 = 0$ such that this case is already covered by the graph of $f_1^{1,1}$.

Step 2b. Check if a real transformation exists for $f_5^{1,1}$:

To give one more example, let us consider $f_5^{1,1}$:

```
> J = I, (a+2)*b+(2*a-12);
> grobcov(J);
```

The crucial generator of the Gröbner basis of J for generic a is now the polynomial $(a + 2)(t_4)^4 - 1$ which has a real root if and only if $a > -2$. Considering the other generators, it is easy to see that given $t_4 \in \mathbb{R}$, $t_1 = t_2 = t_3 = -t_4$ is a real solution. The exceptional values of a in this case are the same as above and again, we do not have to consider $a + 2 = 0$, $a - 2 = 0$, and $a^2 + 12 = 0$. The relation $a + 6 = 0$ implies $b + 6 = 0$ which is already covered by $f_1^{1,1}$. Finally, $t_1 = t_2 = t_3 = \frac{1}{\sqrt{2}}$, $t_4 = -\frac{1}{\sqrt{2}}$ and $t_1 = t_2 = t_3 = \frac{1}{\sqrt{8}}$, $t_4 = -\frac{1}{\sqrt{8}}$ are real solutions for the cases $a = 0$ and $a - 6 = 0$, respectively. To sum up, the graph of $f_5^{1,1}$ over $\mathbb{R}^{>-2}$ belongs to $P_3(X_9^{++}, X_9^{++})$, but not the part over $\mathbb{R}^{<-2}$.

Step 2c. Check if a real transformation exists for the other branches:

Continuing in this manner, one can show that $f_2^{1,1}$, $f_4^{1,1}$ and $f_6^{1,1}$ do not contribute any additional points, so we get

$$P_3(X_9^{++}, X_9^{++}) = \Gamma_{\mathbb{R}'}(f_1^{1,1}) \cup \Gamma_{\mathbb{R}^{>-2}}(f_5^{1,1})$$

where $\mathbb{R}' := \mathbb{R} \setminus \{-2, 2\}$.

REMARK 23. With the above notations, the irreducible factors g_j , $j = 1, \dots, s$, of the polynomial g are luckily of degree 1 in b in almost all cases. If one of those factors, say g_1 , has degree in b greater than 1, and if additionally the corresponding ideal $J = I + \langle g_1 \rangle$ has both real and complex solutions, then an extra calculation is needed: Let $f_1(a), \dots, f_k(a)$ be the roots of g_1 in b as above, i.e. $g_1 = (b - f_1(a)) \dots (b - f_k(a))$ (over $\mathbb{C}(a)$ or over a suitable extension thereof if necessary). Then we have to check which of these roots $f_1(a), \dots, f_k(a)$ belong to the real solutions of J and which of them can only be reached via complex transformations.

This is especially crucial for the singularities of type J_{10} in order to distinguish between $f_3^{\sigma,\rho}$, $f_4^{\sigma,\rho}$, $f_5^{\sigma,\rho}$, and $f_6^{\sigma,\rho}$, cf. Theorem 30.

REMARK 24. The hyperbolic singularity types listed in Table 1 are actually infinite series of types. One might argue that the computations described in Sections 5.1 to 5.3 must be carried out for each single $k > 0$ (for J_{10+k} and X_{9+k}) and for each pair $r, s > 4$ (for $Y_{r,s}$) in order to check the results presented in Theorems 32 to 34. This is, of course, impossible in practice. But it turns out that the results are periodic in k and r, s , respectively. Hence it suffices to carry these computations out for sufficiently many values of k and r, s . If we closely examine the intermediate steps, then we can easily check that the results are indeed periodic.

5.4. The Special Type \tilde{Y}_r . Theorem 19 does not give any sufficient set for subtypes of \tilde{Y}_r and indeed it turns out that there is no degree-bounded sufficient set for this case, cf. Remark 28.

But since \tilde{Y}_r is \mathbb{C} -equivalent to $Y_{r,r}$, we can use the structure of the equivalence classes of $Y_{r,r}$ (cf. Theorem 34) to determine $P_1(T_1, T_2)$, $P_2(T_1, T_2)$, and $P_3(T_1, T_2)$ for $T_1, T_2 \in \{\tilde{Y}_r^+, \tilde{Y}_r^-\}$. To do so, let us first define the principal part of a power series.

DEFINITION 25. Let $f \in \mathbb{K}[[x_1, \dots, x_n]]$ be a power series, let Γ_f be its Newton polygon, and let f_0 be the sum of those terms of f which lie on Γ_f . Then we call f_0 the principal part of f .

The following result is due to Arnold (1974, Corollary 9.9).

LEMMA 26. Let $f \in \mathbb{C}[[x, y]]$ be a power series whose principal part is of the form $f_0 = x^a + \lambda x^2 y^2 + y^b$, where $0 \neq \lambda \in \mathbb{C}$, $a \geq 4$, and $b \geq 5$. Then f and its principal part f_0 are \mathbb{C} -equivalent, i.e. $f \stackrel{\mathbb{C}}{\sim} f_0$.

Based upon this lemma, we can now specify an explicit equivalence between the normal forms of \tilde{Y}_r and $Y_{r,r}$.

LEMMA 27. For any $r > 4$ and any $a \in \mathbb{C} \setminus \{0\}$, we have

$$\left(a, \left(\frac{1}{4}\right)^r a^2\right) \in P_1\left(\tilde{Y}_r^+, Y_{r,r}^{++}\right) \cap P_1\left(\tilde{Y}_r^-, Y_{r,r}^{--}\right).$$

PROOF. Let $\phi \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]])$ be the coordinate transformation defined by $\phi(x) := \frac{1}{2}(x+y)$ and $\phi(y) := \frac{1}{2}i(x-y)$. Then the principal parts of $\phi(\text{NF}(\tilde{Y}_r^+(a)))$ and $\phi(\text{NF}(\tilde{Y}_r^-(a)))$ are of the form $\left(\frac{1}{2}\right)^r a \cdot x^r + \lambda x^2 y^2 + \left(\frac{1}{2}\right)^r a \cdot y^r$ with $\lambda = 1$ and $\lambda = -1$, respectively, so the result follows from Lemma 26. \square

Section 5.1 tells us how to compute $P_1(T_1, T_2)$ for $T_1, T_2 \in \{Y_{r,r}^{++}, Y_{r,r}^{--}\}$, cf. Theorem 34. We can use this data and the above lemma to compute $P_1(T_1, T_2)$ for $T_1, T_2 \in \{\tilde{Y}_r^+, \tilde{Y}_r^-\}$. Let us consider the case $P_1(\tilde{Y}_r^+, \tilde{Y}_r^+)$, the other cases follow similarly. According to Lemma 27, $\text{NF}(\tilde{Y}_r^+(a))$ is \mathbb{C} -equivalent to $\text{NF}(Y_{r,r}^{++}(ca^2))$ with $c := \left(\frac{1}{4}\right)^r$ for any $r > 4$ and any $a \in \mathbb{C} \setminus \{0\}$. This in turn is \mathbb{C} -equivalent to $\text{NF}(Y_{r,r}^{++}(\zeta ca^2))$ for any ζ satisfying $\zeta^l - 1 = 0$ where $l = \text{gcd}(2, r+1)$, cf. Theorem 34. Applying Lemma 27 again leads to $\text{NF}(Y_{r,r}^{++}(\zeta ca^2)) \stackrel{\mathbb{C}}{\sim} \text{NF}(\tilde{Y}_r^+(\pm\sqrt{\zeta}a))$, and we thus get the diagram shown in Figure 1.

This proves $\text{NF}(\tilde{Y}_r^+(a)) \stackrel{\mathbb{C}}{\sim} \text{NF}(\tilde{Y}_r^+(\pm\sqrt{\zeta}a))$ for ζ as above, and since the diagram is commutative, there are no equivalences for other values of the parameters than these. Hence

$$P_1\left(\tilde{Y}_r^+, \tilde{Y}_r^+\right) = C(X^{2l} - 1)$$

FIGURE 1. Equivalences between $\text{NF}(\tilde{Y}_r^+)$ and $\text{NF}(Y_{r,r}^{++})$ ($c := (\frac{1}{4})^r$)

$$\begin{array}{ccc}
 \text{NF}(\tilde{Y}_r^+(a)) & \longleftrightarrow & \text{NF}(\tilde{Y}_r^+(\pm\sqrt{\zeta}a)) \\
 \uparrow & \circlearrowleft & \uparrow \\
 \text{NF}(Y_{r,r}^{++}(ca^2)) & \longleftrightarrow & \text{NF}(Y_{r,r}^{++}(\zeta ca^2))
 \end{array}$$

with l as above. The set $P_2(\tilde{Y}_r^+, \tilde{Y}_r^+)$ can now be determined as in Section 5.2. In fact it is easy to see that

$$P_2(\tilde{Y}_r^+, \tilde{Y}_r^+) = R(X^2 - 1).$$

We clearly have $(a, a) \in P_3(\tilde{Y}_r^+, \tilde{Y}_r^+)$ for $a \in \mathbb{R} \setminus \{0\}$, and also $(a, -a) \in P_3(\tilde{Y}_r^+, \tilde{Y}_r^+)$ if r is odd. For the case where r is even, let us consider $\text{NF}(\tilde{Y}_r^+(a))$ as a function in x and y over \mathbb{R}^2 and let the parameter a be positive. In this case the function $\text{NF}(\tilde{Y}_r^+(a)) = (x^2 + y^2)^2 + ax^r$ takes only non-negative values whereas $\text{NF}(\tilde{Y}_r^+(-a)) = (x^2 + y^2)^2 - ax^r$ attains also negative values. Hence there is no real coordinate transformation which takes $\text{NF}(\tilde{Y}_r^+(a))$ to $\text{NF}(\tilde{Y}_r^+(-a))$. The argument is similar for $a < 0$. To sum up, we have

$$P_3(\tilde{Y}_r^+, \tilde{Y}_r^+) = \begin{cases} R(X^2 - 1), & \text{if } r \text{ is odd,} \\ R(X - 1), & \text{if } r \text{ is even.} \end{cases}$$

REMARK 28. Let $r \geq 8$ be a multiple of 4 and let $\phi_r \in \text{Aut}_{\mathbb{C}}(\mathbb{C}[[x, y]])$ be a coordinate transformation which takes $f := \text{NF}(\tilde{Y}_r^+(a))$ to $\text{NF}(\tilde{Y}_r^+(-a))$. Assume that the degree of both $\phi_r(x)$ and $\phi_r(y)$ is less than $\frac{r}{4}$ and let $f = f_0 + f_1$ be decomposed into its principal part $f_0 := (x^2 + y^2)^2$ and $f_1 = ax^r$. Then we have

$$\phi(f) = \phi(f_0) + \phi(f_1) = (x^2 + y^2)^2 - ax^r$$

where the degree of $\phi(f_0)$ is less than r . Therefore $\phi(f_0) = \phi_0(f_0) = (x^2 + y^2)^2$ and $\phi(f_1) = \phi_0(f_1) = -ax^r$. If ϕ_0 is given by $\phi_0(x) = \alpha x + \beta y$, $\phi_0(y) = \gamma x + \delta y$ with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the second of these two equations implies $\beta = 0$ and $\alpha^r = -1$, but the first one in turn implies $\gamma = 0$ and $\alpha^4 = 1$ which is a contradiction.

So the degree of either $\phi_r(x)$ or $\phi_r(y)$ must at least $\frac{r}{4}$. This shows that a degree-bounded sufficient set of coordinate transformations for $(\text{NF}(\tilde{Y}_r^+(a)), \text{NF}(\tilde{Y}_r^+(-a)))$ and for arbitrarily high r does not exist.

6. Results

In this section we present the sets P_1, P_2, P_3 in table form for every unimodal real singularity type up to corank 2.

THEOREM 29. *The structure of the equivalence classes of the X_9 singularities is as shown in Table 3 where for $j = 1, \dots, 6$ and $\rho, \sigma \in \{1, i\}$, the function $f_j^{\rho, \sigma}$ is defined as follows:*

$$\begin{aligned}
 f_1^{\rho, \sigma}(a) &:= +\rho\sigma \cdot a, & f_3^{\rho, \sigma}(a) &:= \frac{+2\sigma a + 12\rho\sigma}{a - 2\rho}, & f_5^{\rho, \sigma}(a) &:= \frac{-2\sigma a + 12\rho\sigma}{a + 2\rho}, \\
 f_2^{\rho, \sigma}(a) &:= -\rho\sigma \cdot a, & f_4^{\rho, \sigma}(a) &:= \frac{+2\sigma a - 12\rho\sigma}{a + 2\rho}, & f_6^{\rho, \sigma}(a) &:= \frac{-2\sigma a - 12\rho\sigma}{a - 2\rho}.
 \end{aligned}$$

Furthermore, we use the following notations:

$$\mathbb{C}' := \mathbb{C} \setminus \{-2, 2\}, \quad \mathbb{R}' := \mathbb{R} \setminus \{-2, 2\}, \quad \mathbb{C}'' := \mathbb{C} \setminus \{-2i, 2i\}.$$

TABLE 3. P_1 , P_2 and P_3 for the X_9 singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
X_9^{++}	X_9^{++}	$\Gamma_{\mathbb{C}'}(f_1^{1,1}, \dots, f_6^{1,1})$	$\Gamma_{\mathbb{R}'}(f_1^{1,1}, \dots, f_6^{1,1})$	$\Gamma_{\mathbb{R}'}(f_1^{1,1})$ $\cup \Gamma_{\mathbb{R}'>-2}(f_5^{1,1})$
X_9^{--}	X_9^{--}			$\Gamma_{\mathbb{R}'}(f_1^{1,1})$ $\cup \Gamma_{\mathbb{R}'<+2}(f_3^{1,1})$
X_9^{++}	X_9^{--}			$\Gamma_{\mathbb{R}'<-2}(f_4^{1,1})$
X_9^{--}	X_9^{++}			$\Gamma_{\mathbb{R}'>+2}(f_6^{1,1})$
X_9^{+-}	X_9^{+-}	$\Gamma_{\mathbb{C}''}(f_1^{i,i}, \dots, f_6^{i,i})$	$\Gamma_{\mathbb{R}}(f_1^{1,1}, f_2^{1,1})$	$\Gamma_{\mathbb{R}}(f_1^{1,1})$
X_9^{-+}	X_9^{-+}			
X_9^{+-}	X_9^{-+}			
X_9^{-+}	X_9^{+-}			
X_9^{++}	X_9^{+-}	$\Gamma_{\mathbb{C}'}(f_1^{1,i}, \dots, f_6^{1,i})$	$\{(-6, 0), (0, 0), (6, 0)\}$	\emptyset
X_9^{++}	X_9^{-+}			
X_9^{--}	X_9^{+-}			
X_9^{--}	X_9^{-+}			
X_9^{+-}	X_9^{++}	$\Gamma_{\mathbb{C}''}(f_1^{i,1}, \dots, f_6^{i,1})$	$\{(0, -6), (0, 0), (0, 6)\}$	\emptyset
X_9^{-+}	X_9^{++}			
X_9^{+-}	X_9^{--}			
X_9^{-+}	X_9^{--}			

THEOREM 30. *The structure of the equivalence classes of the J_{10} singularities is as shown in Table 4 where for $j = 1, \dots, 6$ and $\rho, \sigma \in \{-1, +1\}$, the function $f_j^{\rho, \sigma}$ is defined as follows:*

$$\begin{aligned} f_1^{\rho, \sigma}(a) &:= +\sqrt{\rho\sigma} \cdot a, \\ f_2^{\rho, \sigma}(a) &:= -\sqrt{\rho\sigma} \cdot a, \\ f_3^{\rho, \sigma}(a) &:= +\sqrt{\frac{-\rho\sigma(a^2 - \rho \cdot 4)(a^2 - \rho \cdot 9) + a(a^2 - \rho \cdot 3)\sqrt{a^2 - \rho \cdot 4}}{2(a^2 - \rho \cdot 4)}}, \\ f_4^{\rho, \sigma}(a) &:= -\sqrt{\frac{-\rho\sigma(a^2 - \rho \cdot 4)(a^2 - \rho \cdot 9) + a(a^2 - \rho \cdot 3)\sqrt{a^2 - \rho \cdot 4}}{2(a^2 - \rho \cdot 4)}}, \\ f_5^{\rho, \sigma}(a) &:= +\sqrt{\frac{-\rho\sigma(a^2 - \rho \cdot 4)(a^2 - \rho \cdot 9) - a(a^2 - \rho \cdot 3)\sqrt{a^2 - \rho \cdot 4}}{2(a^2 - \rho \cdot 4)}}, \\ f_6^{\rho, \sigma}(a) &:= -\sqrt{\frac{-\rho\sigma(a^2 - \rho \cdot 4)(a^2 - \rho \cdot 9) - a(a^2 - \rho \cdot 3)\sqrt{a^2 - \rho \cdot 4}}{2(a^2 - \rho \cdot 4)}}. \end{aligned}$$

In each case, ρ and σ are given by

$$\rho := \begin{cases} +1, & \text{if } T_1 = J_{10}^+, \\ -1, & \text{if } T_1 = J_{10}^-, \end{cases} \quad \sigma := \begin{cases} +1, & \text{if } T_2 = J_{10}^+, \\ -1, & \text{if } T_2 = J_{10}^-. \end{cases}$$

Furthermore, we use the following notations:

$$\begin{aligned} \xi &:= \frac{3}{\sqrt{2}}, & I_1 &:=]-\infty, -\xi[\subset \mathbb{R}, \\ \mathbb{C}' &:= \mathbb{C} \setminus \{-2, 2\}, & I_2 &:=]-\xi, -2[\subset \mathbb{R}, \\ \mathbb{R}' &:= \mathbb{R} \setminus \{-2, 2\}, & I_3 &:=]+2, +\xi[\subset \mathbb{R}, \\ \mathbb{C}'' &:= \mathbb{C} \setminus \{-2i, 2i\}, & I_4 &:=]+\xi, +\infty[\subset \mathbb{R}. \end{aligned}$$

TABLE 4. P_1 , P_2 and P_3 for the J_{10} singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
J_{10}^+	J_{10}^+	$\Gamma_{\mathbb{C}'}(f_1^{\rho, \sigma}, \dots, f_6^{\rho, \sigma})$	$\begin{aligned} &\Gamma_{\mathbb{R}'}(f_1^{\rho, \sigma}, f_2^{\rho, \sigma}) \\ &\cup \Gamma_{\mathbb{R}^{>+2}}(f_3^{\rho, \sigma}, f_4^{\rho, \sigma}) \\ &\cup \Gamma_{\mathbb{R}^{<-2}}(f_5^{\rho, \sigma}, f_6^{\rho, \sigma}) \\ &\cup \{(0, -\xi), (0, +\xi)\} \\ &\cup \{(-\xi, 0), (+\xi, 0)\} \end{aligned}$	$\begin{aligned} &\Gamma_{\mathbb{R}'}(f_1^{\rho, \sigma}) \\ &\cup \Gamma_{\mathbb{R}^{>+2}}(f_4^{\rho, \sigma}) \\ &\cup \Gamma_{\mathbb{R}^{<-2}}(f_5^{\rho, \sigma}) \end{aligned}$
J_{10}^-	J_{10}^-	$\Gamma_{\mathbb{C}''}(f_1^{\rho, \sigma}, \dots, f_6^{\rho, \sigma})$	$\Gamma_{\mathbb{R}}(f_1^{\rho, \sigma}, f_2^{\rho, \sigma})$	$\Gamma_{\mathbb{R}}(f_1^{\rho, \sigma})$
J_{10}^+	J_{10}^-	$\Gamma_{\mathbb{C}'}(f_1^{\rho, \sigma}, \dots, f_6^{\rho, \sigma})$	$\begin{aligned} &\{(0, 0)\} \\ &\cup \Gamma_{\mathbb{R}^{>+2}}(f_3^{\rho, \sigma}, f_4^{\rho, \sigma}) \\ &\cup \Gamma_{\mathbb{R}^{<-2}}(f_5^{\rho, \sigma}, f_6^{\rho, \sigma}) \end{aligned}$	$\begin{aligned} &\Gamma_{I_4}(f_3^{\rho, \sigma}) \cup \Gamma_{I_3}(f_4^{\rho, \sigma}) \\ &\cup \Gamma_{I_2}(f_5^{\rho, \sigma}) \cup \Gamma_{I_1}(f_6^{\rho, \sigma}) \\ &\cup \{(-\xi, 0), (+\xi, 0)\} \end{aligned}$
J_{10}^-	J_{10}^+	$\Gamma_{\mathbb{C}''}(f_1^{\rho, \sigma}, \dots, f_6^{\rho, \sigma})$	$\begin{aligned} &\{(0, 0)\} \\ &\cup \Gamma_{\mathbb{R}}(f_3^{\rho, \sigma}, \dots, f_6^{\rho, \sigma}) \end{aligned}$	$\Gamma_{\mathbb{R}}(f_3^{\rho, \sigma}, f_6^{\rho, \sigma})$

REMARK 31. In Theorem 30, the definitions of $f_1^{\rho, \sigma}, \dots, f_6^{\rho, \sigma}$ involve square roots of possibly complex values. These square roots are defined as follows: For any complex number $z = re^{i\phi} \in \mathbb{C}$ with $r, \phi \in \mathbb{R}$, $r > 0$, and $0 \leq \phi < 2\pi$, we set

$$\sqrt{z} := \sqrt{r}e^{i\frac{\phi}{2}}.$$

In particular, $\text{Im}(\sqrt{z}) > 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}^{>0}$ and $\sqrt{z} \geq 0$ for all $z \in \mathbb{R}^{>0}$.

THEOREM 32. *The structure of the equivalence classes of the J_{10+k} singularities is as shown in Table 5 where in each case, l and s are given by*

$$l := \frac{6}{\gcd(6, k)}, \text{ and}$$

$$s := \begin{cases} +1, & \text{if } k \equiv 2 \pmod{4}, \\ -1, & \text{else.} \end{cases}$$

TABLE 5. P_1 , P_2 and P_3 for the J_{10+k} singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
J_{10+k}^+	J_{10+k}^+	$C(X^l - 1)$	$R(X^l - 1)$	$R(X^l - 1)$
J_{10+k}^-	J_{10+k}^-			
J_{10+k}^+	J_{10+k}^-	$C(X^l - s)$	$R(X^l - s)$	\emptyset
J_{10+k}^-	J_{10+k}^+			

THEOREM 33. *The structure of the equivalence classes of the X_{9+k} singularities is as shown in Table 6 where in each case, l and s are given by*

$$l := \frac{4}{\gcd(4, k)}, \text{ and}$$

$$s := \begin{cases} +1, & \text{if } k \equiv 4 \pmod{8}, \\ -1, & \text{else.} \end{cases}$$

TABLE 6. P_1 , P_2 and P_3 for the X_{9+k} singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
X_{9+k}^{++}	X_{9+k}^{++}	$C(X^l - 1)$	$R(X^l - 1)$	$R(X^{k+1} - 1)$
X_{9+k}^{+-}	X_{9+k}^{+-}			
X_{9+k}^{-+}	X_{9+k}^{-+}			
X_{9+k}^{--}	X_{9+k}^{--}			
X_{9+k}^{++}	X_{9+k}^{+-}	$C(X^l - 1)$	$R(X^l - 1)$	\emptyset
X_{9+k}^{+-}	X_{9+k}^{++}			
X_{9+k}^{-+}	X_{9+k}^{--}			
X_{9+k}^{--}	X_{9+k}^{-+}			
X_{9+k}^{++}	X_{9+k}^{-+}	$C(X^l - s)$	$R(X^l - s)$	\emptyset
X_{9+k}^{+-}	X_{9+k}^{--}			
X_{9+k}^{-+}	X_{9+k}^{++}			
X_{9+k}^{--}	X_{9+k}^{+-}			
X_{9+k}^{++}	X_{9+k}^{--}	$C(X^l - s)$	$R(X^l - s)$	\emptyset
X_{9+k}^{+-}	X_{9+k}^{-+}			
X_{9+k}^{-+}	X_{9+k}^{+-}			
X_{9+k}^{--}	X_{9+k}^{++}			

THEOREM 34. *The structure of the equivalence classes of the $Y_{r,s}$ singularities is as shown in Table 7 where in each case, l , s_1 and s_2 are given by*

$$l := \frac{r}{\gcd(r, s)} \cdot \gcd(2, r + 1, s + 1),$$

$$s_1 := \begin{cases} +1, & \text{if } r \equiv 0 \pmod{4} \text{ or } s \equiv 0 \pmod{4}, \\ -1, & \text{else,} \end{cases}$$

$$s_2 := \begin{cases} +1, & \text{if } r \not\equiv 0 \pmod{2} \text{ or } \frac{s}{\gcd(r,s)} \equiv 0 \pmod{2}, \\ -1, & \text{else.} \end{cases}$$

In the special case where $r = s$, additional equivalences occur. They are listed in Table 8.

 TABLE 7. P_1 , P_2 and P_3 for the $Y_{r,s}$ singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
$Y_{r,s}^{++}$	$Y_{r,s}^{++}$	$C(X^l - 1)$	$R(X^l - 1)$	$R(X^{s+1} - 1)$
$Y_{r,s}^{-+}$	$Y_{r,s}^{-+}$			
$Y_{r,s}^{+-}$	$Y_{r,s}^{+-}$			
$Y_{r,s}^{--}$	$Y_{r,s}^{--}$			
$Y_{r,s}^{++}$	$Y_{r,s}^{-+}$	$C(X^l - s_1)$	$R(X^l - s_1)$	\emptyset
$Y_{r,s}^{-+}$	$Y_{r,s}^{++}$			
$Y_{r,s}^{+-}$	$Y_{r,s}^{--}$			
$Y_{r,s}^{--}$	$Y_{r,s}^{+-}$			
$Y_{r,s}^{++}$	$Y_{r,s}^{+-}$	$C(X^l - s_2)$	$R(X^l - s_2)$	$\begin{cases} R(X^l - s_2), & \text{if } r \not\equiv 0 \pmod{2} \\ \emptyset, & \text{if } r \equiv 0 \pmod{2} \end{cases}$
$Y_{r,s}^{-+}$	$Y_{r,s}^{--}$			
$Y_{r,s}^{+-}$	$Y_{r,s}^{++}$			
$Y_{r,s}^{--}$	$Y_{r,s}^{-+}$			
$Y_{r,s}^{++}$	$Y_{r,s}^{--}$	$C(X^l - s_1 s_2)$	$R(X^l - s_1 s_2)$	\emptyset
$Y_{r,s}^{-+}$	$Y_{r,s}^{+-}$			
$Y_{r,s}^{+-}$	$Y_{r,s}^{-+}$			
$Y_{r,s}^{--}$	$Y_{r,s}^{++}$			

 TABLE 8. Additional equivalences for the $Y_{r,s}$ singularities in the special case $r = s$

T_1	T_2	Additional elements of $P_3(T_1, T_2)$
$Y_{r,s}^{++}$	$Y_{r,s}^{+-}$	$\begin{cases} R(X + 1), & \text{if } r \equiv 0 \pmod{2} \text{ and } a < 0 \\ \emptyset, & \text{else} \end{cases}$
$Y_{r,s}^{-+}$	$Y_{r,s}^{--}$	
$Y_{r,s}^{+-}$	$Y_{r,s}^{++}$	$\begin{cases} R(X + 1), & \text{if } r \equiv 0 \pmod{2} \text{ and } a > 0 \\ \emptyset, & \text{else} \end{cases}$
$Y_{r,s}^{--}$	$Y_{r,s}^{-+}$	

REMARK 35. Note that there are also equivalences between subtypes of $Y_{r,s}$ and subtypes of $Y_{s,r}$ which can be obtained by just swapping the variables x and y . For $r = s$ these are exactly the additional equivalences listed in Table 8. But equivalences of this kind also occur for $r \neq s$, e.g. we have $R(X - 1) \subset P_1(Y_{5,7}^{++}, Y_{7,5}^{++})$, but we do not consider those cases in Theorem 34.

THEOREM 36. *The structure of the equivalence classes of the \tilde{Y}_r singularities is as shown in Table 9 where in each case, l and s are given by*

$$l := 2 \cdot \gcd(2, r + 1), \text{ and}$$

$$s := \begin{cases} +1, & \text{if } r \equiv 0 \pmod{4}, \\ -1, & \text{else.} \end{cases}$$

TABLE 9. P_1 , P_2 and P_3 for the \tilde{Y}_r singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
\tilde{Y}_r^+	\tilde{Y}_r^+	$C(X^l - 1)$	$R(X^2 - 1)$	$\begin{cases} R(X^2 - 1), & \text{if } r \equiv 1 \pmod{2} \\ R(X - 1), & \text{if } r \equiv 0 \pmod{2} \end{cases}$
\tilde{Y}_r^-	\tilde{Y}_r^-			
\tilde{Y}_r^+	\tilde{Y}_r^-	$C(X^l - s)$	$R(X^2 - s)$	\emptyset
\tilde{Y}_r^-	\tilde{Y}_r^+			

THEOREM 37. *The structure of the equivalence classes of the exceptional unimodal singularities is as shown in Table 10.*

TABLE 10. P_1 , P_2 and P_3 for the exceptional unimodal singularities

T_1	T_2	$P_1(T_1, T_2)$	$P_2(T_1, T_2)$	$P_3(T_1, T_2)$
E_{12}	E_{12}	$C_0(X^{21} - 1)$	$R_0(X - 1)$	$R_0(X - 1)$
E_{13}	E_{13}	$C_0(X^{15} - 1)$	$R_0(X - 1)$	$R_0(X - 1)$
E_{14}^+	E_{14}^+	$C_0(X^{12} - 1)$	$R_0(X^2 - 1)$	$R_0(X - 1)$
E_{14}^-	E_{14}^-	$C_0(X^{12} - 1)$	$R_0(X^2 - 1)$	$R_0(X - 1)$
E_{14}^+	E_{14}^-	$C_0(X^{12} + 1)$	\emptyset	\emptyset
E_{14}^-	E_{14}^+	$C_0(X^{12} + 1)$	\emptyset	\emptyset
Z_{11}	Z_{11}	$C_0(X^{15} - 1)$	$R_0(X - 1)$	$R_0(X - 1)$
Z_{12}	Z_{12}	$C_0(X^{11} - 1)$	$R_0(X - 1)$	$R_0(X - 1)$
Z_{13}^+	Z_{13}^+	$C_0(X^9 - 1)$	$R_0(X - 1)$	$R_0(X - 1)$
Z_{13}^-	Z_{13}^-	$C_0(X^9 - 1)$	$R_0(X - 1)$	$R_0(X - 1)$
Z_{13}^+	Z_{13}^-	$C_0(X^9 + 1)$	$R_0(X + 1)$	\emptyset
Z_{13}^-	Z_{13}^+	$C_0(X^9 + 1)$	$R_0(X + 1)$	\emptyset
W_{12}^+	W_{12}^+	$C_0(X^{10} - 1)$	$R_0(X^2 - 1)$	$R_0(X - 1)$
W_{12}^-	W_{12}^-	$C_0(X^{10} - 1)$	$R_0(X^2 - 1)$	$R_0(X - 1)$
W_{12}^+	W_{12}^-	$C_0(X^{10} + 1)$	\emptyset	\emptyset
W_{12}^-	W_{12}^+	$C_0(X^{10} + 1)$	\emptyset	\emptyset
W_{13}^+	W_{13}^+	$C_0(X^8 - 1)$	$R_0(X^2 - 1)$	$R_0(X - 1)$
W_{13}^-	W_{13}^-	$C_0(X^8 - 1)$	$R_0(X^2 - 1)$	$R_0(X - 1)$
W_{13}^+	W_{13}^-	$C_0(X^8 + 1)$	\emptyset	\emptyset
W_{13}^-	W_{13}^+	$C_0(X^8 + 1)$	\emptyset	\emptyset

7. Interpretation of the Results

Looking more closely at Theorem 19 and Section 6, it turns out that the structure of the equivalence classes is quite simple for some singularity types whereas it is very involved for others. To describe this in more detail, let us first consider the *sufficient sets* given in Theorem 19:

- It suffices to work with scalings of the form $\phi(x) = \alpha x$, $\phi(y) = \beta y$ as coordinate transformations to figure out the structure of the equivalence classes of the hyperbolic and exceptional unimodal singularities. (To be precise, for $Y_{r,s}$ with $r = s$ we also have to take into account transformations of the form $\phi(x) = \beta y$, $\phi(y) = \alpha x$ where the variables are swapped, cf. Table 2.)
- For the two parabolic types X_9 and J_{10} , scalings are not sufficient. Instead, we have to consider more complicated transformations which involve more terms.
- One can see from the proof of Theorem 19 that these differences reflect the different shapes of the normal forms: The normal forms of the hyperbolic singularity types listed in Table 2 are weighted quasihomogeneous, those of the exceptional singularity types are semi-quasihomogeneous. Both shapes turn out to be very restrictive w.r.t. possible coordinate transformations. In contrast to this, the normal forms of the parabolic types are quasihomogeneous and thus allow for more freedom in this regard.
- The singularity type \tilde{Y}_r is an exception. It is complex equivalent to $Y_{r,r}$, but it appears as a separate singularity type over \mathbb{R} . There is no degree-bounded sufficient set for the normal form of this type (cf. Remark 28), so the computational methods described in Sections 5.1 and 5.3 do not work. Instead, we have to use other methods, cf. Section 5.4.

As a consequence of the differences w.r.t. sufficient sets described above, there are two *general forms of equivalences* as presented in Section 6:

- For the hyperbolic and the exceptional singularities, the equivalences between different subtypes can be described by constant factors. More precisely, if T_1 and T_2 are subtypes of the same hyperbolic or exceptional main singularity type, then there exists a finite set of constants $r_1, \dots, r_m \in \mathbb{C}$ such that the equivalences between the normal forms of T_1 and T_2 are exactly those of the form $\text{NF}(T_1(a)) \sim \text{NF}(T_2(r_i a))$ with $a \in \mathbb{C}$ or $a \in \mathbb{R}$ as appropriate. Therefore we use the notations $C(p(X))$, $R(p(X))$ and $C_0(p(X))$, $R_0(p(X))$ with $p(X) \in \mathbb{C}[X]$ (see Definition 9) for the hyperbolic and the exceptional cases, respectively, cf. Theorems 32 to 37.
- The equivalences which occur among subtypes of the two parabolic singularity types X_9 and J_{10} are much more involved and cannot be written down in terms of constant factors. We describe them as joint graphs of certain functions, cf. Theorems 29 and 30.

The results presented in Section 6 have *consequences for the algorithmic classification of the unimodal singularities of corank 2 over \mathbb{R}* . They are indeed intended to be the first step in this direction. Once again, we can distinguish between different cases. Note that the following remarks apply to the classification over \mathbb{R} and therefore only deal with real coordinate transformations and real values of the involved parameters:

- In the exceptional cases, there are no equivalences between different subtypes of the same main type and the value of the parameter is uniquely determined, cf. Theorem 37.

- For the singularity types J_{10+k} , X_{9+k} , and \tilde{Y}_r , there are no equivalences between different subtypes of the same main type, but the parameter can in some cases change its sign within the same subtype, cf. Theorems 32, 33, and 36.
- For $Y_{r,s}$, there are equivalences even between different subtypes. Therefore the question which real subtype a given singularity of main type $Y_{r,s}$ belongs to is not always well-posed, e.g., a singularity can be both of type $Y_{5,7}^{++}$ and of type $Y_{5,7}^{+-}$. However, the first of the two signs is always uniquely determined. The parameter can change its sign, but its absolute value is uniquely determined, cf. Theorem 34.
- The structures of the equivalence classes of the two parabolic cases X_9 and J_{10} are the most complicated among all the cases discussed here. There are equivalences between different subtypes and the parameter may change in non-trivial ways. For X_9 , the possible values which a given parameter can be transformed to can be expressed as rational functions of this parameter (cf. Theorem 29), whereas the corresponding functions for J_{10} involve radical expressions (cf. Theorem 30). Note that there are, however, no equivalences between the subtypes X_9^{++} , X_9^{--} on the one hand and X_9^{+-} , X_9^{-+} on the other hand, i.e. the product of the two signs which occur in the subtypes of X_9 is uniquely determined.
- It is a remarkable result that for any value of $a \in \mathbb{R}$, the normal form of $J_{10}^-(a)$ is \mathbb{R} -equivalent to the normal form of $J_{10}^+(a')$ for some $a' \in \mathbb{R}$ while the converse is not true, cf. Theorem 30. In other words, the real subtype J_{10}^- is redundant whereas J_{10}^+ is not.

To sum up, the normal forms which are listed in the classifications of the unimodal singularities over \mathbb{C} and over \mathbb{R} by Arnold et al. (1985) cover the whole space of unimodal singularities, but some of them are equivalent. There are equivalences between normal forms for different values of the parameter and also between different subtypes, to an extent that the subtype J_{10}^- is even redundant.

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