The Existence of Strong Solutions to the $3D$ Zakharov-Kuznestov Equation in a Bounded Domain

Chuntian Wang

Department of Mathematics and The Institute for Scientific Computing and Applied Mathematics
Indiana University, Bloomington, IN 47405
email: wang211@umail.iu.edu

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Abstract

We consider the Zakharov-Kuznestov (ZK) equation posed in a limited domain $\mathcal{M} = (0,1)_x \times (-\pi/2, \pi/2)^d$, $d = 1, 2$ supplemented with suitable boundary conditions. We prove that there exists a solution $u \in \mathcal{C}([0,T]; H^1(\mathcal{M}))$ to the initial and boundary value problem for the ZK equation in both dimensions 2 and 3 for every $T > 0$. To the best of our knowledge, this is the first result of the global existence of strong solutions for the ZK equation in $3D$.

More importantly, the idea behind the application of anisotropic estimation to cancel the nonlinear term, we believe, is not only suited for this model but can also be applied to other nonlinear equations with similar structures.

At the same time, the uniqueness of solutions is still open in $2D$ and $3D$ due to the partially hyperbolic feature of the model.

Keywords: Zakharov-Kuznetsov equation, Korteweg-de Vries equation

1 Introduction

The Zakharov-Kuznestov (ZK) equation

$$\frac{\partial u}{\partial t} + \Delta \frac{\partial u}{\partial x} + c\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = f, \tag{1.1}$$

where $u = u(x, x^\perp, t)$, $x^\perp = y$ or $x^\perp = (y, z)$, describes the propagation of nonlinear ionic-sonic waves in a plasma submitted to a magnetic field directed along the x-axis. Here $c > 0$ is the sound velocity. It has been derived formally in a long wave, weakly nonlinear regime from the Euler-Poisson system in [ZK74] and [LS82]. A rigorous derivation is provided in [LLS13]. For more general physical references, see [BPS81] and [BPS83]. When u depends only on x and t, (1.1) reduces to the classical Korteweg-de Vries (KdV) equation.

Recently the ZK equation has caught much attention, not only because it is closely related with the physical phenomena but also because it is the start to explore more general problems that are partly hyperbolic (such as the inviscid primitive equations).

Concerning the initial and boundary value problems of the Korteweg-de Vries equation posed on a bounded interval $(0, L)$, we refer the interested readers to e.g. [BSZ03], [CG01a], [QT12] and [CG01b].

The initial and boundary value problem associated with (1.1) has been studied in the half space $\{(x, y) : x > 0\}$ ([Fam06]), on a strip like $\{(x, y) : x \in \mathbb{R}, 0 < y < L\}$ ([BF13]) or $\{(x, x^\perp) : 0 < x < 1, x^\perp \in \mathbb{R}^d, d = 1, 2\}$ ([Fam08] and [ST10]), and in a rectangle $\{(x, x^\perp) : 0 < x < 1, x^\perp \in (-\pi/2, \pi/2)^d, d = 1, 2\}$ ([STW12]). Specifically in [STW12], the authors have established, for arbitrary large initial data, the existence of global weak solutions in space dimensions 2 and 3 ($d = 1$ and 2 respectively) and a result of uniqueness of such solutions in the two-dimensional case.

As for the existence of strong solutions, the global existence in space dimension 2 has been proven in a half strip $\{(x, y) : x > 0, y \in (0, L)\}$ in [LT13]. The existence and exponential decay of regular solutions to the linearized ZK equation in a rectangle $\{(x, y) : x \in (0, L), y \in (0, B)\}$

has been studied in [DL14]. The local existence of strong solutions in space dimensions 2 and 3 is established in [Wan]. In these previous works, the boundary conditions on $x = 0, 1$ are assumed to be $u\big|_{x=0} = u\big|_{x=1} = u_x\big|_{x=1} = 0$; however here we suppose different boundary conditions to serve our purposes.

To the best of our knowledge, the global existence and uniqueness of regular solutions in $3D$ is still an open problem. In this article, we prove that there exists a global solution $u \in \mathcal{C}([0,T]; L^2(\mathcal{M}))$ for the initial and boundary value problem of the ZK equation in both $2D$ and $3D$, which we believe, will lead to the global well-posedness of strong of solutions in $3D$ eventually. It is interesting to observe that, for the $3D$ ZK equation, the nonlinear term has the same structure as the nonlinear term in the $3D$ Navier-Stkoes equations and that the basic a priori estimates ($L^\infty(0,T; L^2(\mathcal{M}))$ and $L^2(0,T; H^1(\mathcal{M}))$) are the same, although the structure of the linear operator is totally different (e.g. not coercive as in (3.11) below).

For the proof we use the parabolic regularization as in [ST10], [STW12] and [Wan]. There are four main difficulties. Firstly, as in the case of $3D$ Navier-Stokes equation, the nonlinear term will pose a problem when we apply the Sobolev imbedding in $3D$. Secondly, since the linear operator is not coercive, the L^p estimations (see e.g. [CT07]) does not work. Thirdly, some assumption on the trace $u_{xx}\big|_{x=0}^{x=1}$ is necessary for the estimate of $\nabla u \in L^\infty(0,T; L^2(\mathcal{M}))$. Finally, to pass to the limit on the boundary conditions, the methods in [ST10] and [STW12] are not applicable any more because of the change of the boundary conditions.

To overcome these difficulties, firstly we utilize the anisotropic resonance of the term u_{xxx} and the nonlinear term uu_x to cancel uu_x, which leads to a bound of the H^1 norm over $(0,T)$ for u. This step of canceling the nonlinear term may also be applied to other nonlinear equations with similar structures. Next, we suppose periodic boundary conditions of u and u_{x^j} at $x = 0$, 1, $j = 1, 2$, so that the trace $u_{xx}\big|_{x=0}^{x=1}$ now vanishes. Finally, we investigate a bound independent of ϵ for u^ϵ_{xxx} in $L^{3/2}(I_x; Y)$, with Y a Banach space in $x^\perp$ and t, which facilitates the passage to the limit on the traces of u_{x^j} at $x = 0$, 1, $j = 1, 2$.

However the uniqueness of solutions is still open in both $2D$ and $3D$, even with such a regularity and all the periodic boundary conditions satisfied. In particular, the methods in [ST10] and [STW12] can not be adapted to our case due to the lack of the boundary condition $u_x = 0$ at $x = 1$.

The article is organized as follows. Firstly we introduce the basic settings of the equation in Section 2. Secondly we introduce the parabolic regularization as in [ST10] and [STW12] (Section 3.1). Then we derive the estimates independent of ϵ for u^ϵ in $L^\infty(0,T; L^2(\mathcal{M}))$ (Section 3.2.1), ∇u^ϵ in $L^\infty(0,T; L^2(\mathcal{M}))$ (Section 3.2.2) and for u^ϵ_{xxx} in $L^{3/2}(I_x; H^{-1}_t(0,T;\ H^{-4}(I_{x^\perp})))$ (Section 3.2.3). Eventually we can pass to the limit on the parabolic regularization and the traces and deduce the global existence of solutions $u \in \mathcal{C}([0,T];\ H^1(\mathcal{M}))$ (Section 3.3). Finally, we discuss about the difficulties in the attempt of proving the uniqueness of solutions (Section 4).

2 ZK equation in a rectangle in dimensions 2 and 3

We aim to study the ZK equation:

$$\frac{\partial u}{\partial t} + \Delta \frac{\partial u}{\partial x} + c\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = f, \tag{2.1}$$

in a rectangle or parallelepiped domain in $\mathbb{R}^n$ with $n = 2$ or 3, denoted as $\mathcal{M} = (0,1)_x \times (-\pi/2, \pi/2)^d$, with $d = 1$ or 2, $\Delta u = u_{xx} + \Delta^\perp u$, $\Delta^\perp u = u_{yy}$ or $u_{yy} + u_{zz}$ depending on the dimension. In the sequel we will use the notations $I_x = (0,1)_x$, $I_y = (-\pi/2, \pi/2)_y$, $I_z = (-\pi/2, \pi/2)_z$, and $I_{x^\perp} = I_y$ or $I_y \times I_z$. We assume the boundary conditions of u, u_x and u_{xx} on $x = 0, 1$ to be periodic:

$$u(0,\, x^\perp,\, t) = u(1,\, x^\perp,\, t), \tag{2.2}$$

$$u_x(0,\, x^\perp,\, t) = u_x(1,\, x^\perp,\, t), \quad u_{xx}(0,\, x^\perp,\, t) = u_{xx}(1,\, x^\perp,\, t). \tag{2.3}$$

For the boundary conditions in the y and z directions, we will choose either the Dirichlet boundary conditions

$$u = 0 \text{ at } y = \pm\frac{\pi}{2} \text{ and } z = \pm\frac{\pi}{2}, \tag{2.4}$$

or the periodic boundary conditions

$$\begin{aligned} u\big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = u_y\big|_{y=-\frac{\pi}{2}}^{y=\frac{\pi}{2}} = 0, \\ u\big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = u_z\big|_{z=-\frac{\pi}{2}}^{z=\frac{\pi}{2}} = 0. \end{aligned} \tag{2.5}$$

The initial condition reads:

$$u(x,\, x^\perp,\, 0) = u_0(x,\, x^\perp). \tag{2.6}$$

We study the initial and boundary value problem (2.1)-(2.3) and (2.6) supplemented with the boundary condition (2.4), that is, the Dirichlet case on the $x^\perp$ boundaries, and we will make some remarks on the extension to the periodic boundary condition case.

We denote by $|\cdot|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(\mathcal{M})$, and by $[\cdot]_2$ the following seminorm which will be useful in the sequel:

$$\left(\int_{\mathcal{M}} u_{xx}^2 + u_{yy}^2 + u_{yy}^2 \, d\,\mathcal{M} \right)^{1/2} =: [u]_2, \quad u \in H^2(\mathcal{M}). \tag{2.7}$$

3 Existence of solutions $u \in \mathcal{C}([0,T]; H^1(\mathcal{M}))$ in dimensions 2 and 3

To prove this result, we use the parabolic regularization as in [STW12], but with different boundary conditions. For the sake of simplicity we only treat the more complicated case when $d = 2$.

3.1 Parabolic regularization

To begin with, we recall the parabolic regularization introduced in [ST10] and [STW12], that is, for $\epsilon > 0$ "small", we consider the parabolic equation,

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t} + \Delta \dfrac{\partial u^\epsilon}{\partial x} + c\dfrac{\partial u^\epsilon}{\partial x} + u^\epsilon \dfrac{\partial u^\epsilon}{\partial x} + \epsilon L u^\epsilon = f, \\ u^\epsilon(0) = u_0, \end{cases} \tag{3.1}$$

where

$$Lu^\epsilon := \frac{\partial^4 u^\epsilon}{\partial x^4} + \frac{\partial^4 u^\epsilon}{\partial y^4} + \frac{\partial^4 u^\epsilon}{\partial z^4},$$

supplemented with the boundary conditions (2.2)-(2.4) and the additional boundary conditions

$$u^\epsilon_{xxx}(0,\, x^\perp,\, t) = u^\epsilon_{xxx}(1,\, x^\perp,\, t), \tag{3.2}$$

$$u^\epsilon_{yy} = 0 \text{ at } y = \pm\frac{\pi}{2},\;\; u^\epsilon_{zz} = 0 \text{ at } z = \pm\frac{\pi}{2}. \tag{3.3}$$

Note that from (2.3) and (3.2) we infer

$$u^\epsilon_{x^j}(0,\, x^\perp,\, t) = u^\epsilon_{x^j}(1,\, x^\perp,\, t), \quad j = 1,\, 2,\; 3. \tag{3.4}$$

We also note that since $u^\epsilon_{yy}\big|_{x=0}^{x=1} = u^\epsilon_{zz}\big|_{x=0}^{x=1} = 0$, (3.4) is equivalent to

$$\Delta u^\epsilon\big|_{x=0}^{x=1} = 0. \tag{3.5}$$

It is a classical result (see e.g. [Lio69], [LSU68] or also [STW12]) that there exists a unique solution to the parabolic problem which is sufficiently regular for all the subsequent calculations to be valid; in particular, we have

$$u^\epsilon \in L^2(0,T;H^4(\mathcal{M})) \cap \mathcal{C}^1([0,T];H^2(\mathcal{M})). \tag{3.6}$$

3.2 Estimates independent of ϵ

We establish the estimates independent of ϵ for various norms of the solutions.

3.2.1 L^2 estimate independent of ϵ

We first show a bound independent of ϵ for u^ϵ in $L^\infty(0,T;L^2(\mathcal{M}))$.

Lemma 3.1. *We assume that*

$$u_0 \in L^2(\mathcal{M}), \tag{3.7}$$

$$f \in L^2(0,T;L^2(\mathcal{M})). \tag{3.8}$$

Then for every $T > 0$ the following estimates independent of ϵ hold:

$$u^\epsilon \textit{ is bounded in } L^\infty(0,T;\, L^2(\mathcal{M})), \tag{3.9}$$

$$\sqrt{\epsilon}\, u^\epsilon \textit{ is bounded in } L^2(0,T;\, H^2(\mathcal{M})). \tag{3.10}$$

Proof. As in [STW12], we multiply (3.1) with u, integrate over $\mathcal{M}$ and integrate by parts, dropping the superscript ϵ for the moment we find:

- $\displaystyle\int_{\mathcal{M}} \frac{\partial u}{\partial t} u \, d\mathcal{M} = \frac{1}{2}\frac{d}{dt}|u|^2,$

- $\int_{\mathcal{M}} \Delta u_x \, u \, d\mathcal{M} + \int_{\mathcal{M}} c u_x \, u \, d\mathcal{M} =$ (thanks to (2.2))

$$
\begin{aligned}
&= -\int_{\mathcal{M}} \nabla u_x \, \nabla u \, d\mathcal{M} + \frac{c}{2} \int_{I_{x^\perp}} u^2 \big|_{x=0}^{x=1} \, dx^\perp \\
&= -\frac{1}{2} \int_{I_{x^\perp}} (\nabla u)^2 \big|_{x=0}^{x=1} \, dx^\perp + \frac{c}{2} \int_{I_{x^\perp}} u^2 \big|_{x=0}^{x=1} \, dx^\perp \\
&= \text{(thanks to (2.2) and (3.4))} = 0,
\end{aligned}
\tag{3.11}
$$

- $\int_{\mathcal{M}} u u_x \, u \, d\mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial x} \left(\frac{u^3}{3} \right) d\mathcal{M} =$ (thanks to (2.2)) $= 0$,
- $\epsilon \int_{\mathcal{M}} u_{xxxx} \, u \, d\mathcal{M} =$ (thanks to (2.2) and (3.4))

$$
= -\epsilon \int_{\mathcal{M}} u_{xxx} \, u_x \, d\mathcal{M} = \text{(thanks to (3.4))} = \epsilon \int_{\mathcal{M}} u_{xx}^2 \, d\mathcal{M},
$$

- $\epsilon \int_{\mathcal{M}} (u_{xxxx} + u_{yyyy} + u_{zzzz}) \, u \, d\mathcal{M} = \epsilon \int_{\mathcal{M}} u_{xx}^2 + u_{yy}^2 + u_{yy}^2 \, d\mathcal{M}$

$$
= \text{(thanks to (2.7))} = \epsilon [u]_2^2,
$$

- $\int_{\mathcal{M}} f u \, d\mathcal{M} \leq \frac{1}{2} |f|^2 + \frac{1}{2} |u|^2.$

Hence we find

$$
\frac{d}{dt} |u^\epsilon(t)|^2 + 2\epsilon [u^\epsilon]_2^2 \leq |f|^2 + |u^\epsilon|^2. \tag{3.12}
$$

Using the Gronwall lemma we classically infer

$$
\sup_{t \in (0,T)} |u^\epsilon(t)|^2 + \epsilon \int_0^T [u^\epsilon]_2^2 \, dt \leq const := \mu_1, \tag{3.13}
$$

where μ_i indicates a constant depending only on the data u_0, f, etc, whereas C' below is an absolute constant. These constants may be different at each occurrence. Let us admit for the moment the following:

Lemma 3.2.

$$
|u^\epsilon|_{H^2(\mathcal{M})}^2 \leq C' \left([u^\epsilon]_2^2 + |u^\epsilon|^2 \right). \tag{3.14}
$$

By the previous lemma, we have

$$
\begin{aligned}
\epsilon \int_0^T |u^\epsilon|_{H^2(\mathcal{M})}^2 \, dt &\leq C' \left(\epsilon \int_0^T [u^\epsilon]_2^2 \, dt + \epsilon \int_0^T |u^\epsilon|^2 \, dt \right) \\
&\leq C' \left(\epsilon \int_0^T [u^\epsilon]_2^2 \, dt + \epsilon \, T \sup_{t \in (0,T)} |u^\epsilon(t)|^2 \right) \\
&\leq \text{(thanks to (3.13))} \\
&\leq const := \mu_2,
\end{aligned}
$$

which implies (3.10). Thus Lemma 3.1 is proven once we have proven Lemma (3.2).

Proof of Lemma 3.2. We first observe that using the generalized Poincaré inequality (see [Tem97]) we have

$$|u_x^\epsilon - \int_0^1 u_x^\epsilon \, dx|_{L^2(I_x)} \le C'|u_{xx}^\epsilon|_{L^2(I_x)}. \tag{3.15}$$

Thanks to (2.2), we have $\int_0^1 u_x^\epsilon \, dx = u^\epsilon|_{x=0}^{x=1} = 0$, and hence (3.15) implies

$$|u_x^\epsilon|_{L^2(I_x)} \le C'|u_{xx}^\epsilon|_{L^2(I_x)}.$$

Squaring both sides and integrating both sides on $I_{x^\perp}$, we find

$$|u_x^\epsilon| \le C'|u_{xx}^\epsilon|. \tag{3.16}$$

Similarly we can show that $|u_y^\epsilon| \le C'|u_{yy}^\epsilon|$ and $|u_z^\epsilon| \le C'|u_{zz}^\epsilon|$, which implies

$$|\nabla u^\epsilon| \le C'[u^\epsilon]_2. \tag{3.17}$$

Next we see that, for smooth functions

$$\begin{aligned} |u_{xy}^\epsilon|^2 &= (\text{thanks to (2.2) and (3.4)}) \\ &= -\int_{\mathcal{M}} u_y^\epsilon u_{xxy}^\epsilon d\mathcal{M} \\ &= (\text{thanks to (2.4)}) \\ &= \int_{\mathcal{M}} u_{yy}^\epsilon u_{xx}^\epsilon d\mathcal{M} \\ &\le |u_{xx}^\epsilon|^2 + |u_{yy}^\epsilon|^2 \le [u^\epsilon]_2^2. \end{aligned} \tag{3.18}$$

Similarly we can prove that $|u_{xz}^\epsilon| \le [u^\epsilon]_2$ and $|u_{zy}^\epsilon| \le [u^\epsilon]_2$, and hence

$$|u_{xy}^\epsilon|^2 + |u_{xz}^\epsilon|^2 + |u_{yz}^\epsilon|^2 \le C'[u^\epsilon]_2^2. \tag{3.19}$$

Then inequality (3.18) and (3.19) extend by continuity to all H^2 function periodic in x and satisfying (2.4) and (3.3). Finally from (3.19) and (3.17) we deduce (3.14). □

3.2.2 H^1 estimate independent of ϵ

Now we establish the key observation, a bound independent of ϵ for ∇u^ϵ in $L^\infty(0,T;L^2(\mathcal{M}))$.

Proposition 3.1. *Under the same assumptions as in Lemma 3.1, we further suppose that*

$$u_0 \in H^1(\mathcal{M}) \cap L^3(\mathcal{M}), \tag{3.20}$$

$$f \in L^2(0,T;H^2(I_x;\, H^2 \cap H_0^1(I_{x^\perp}))) \cap L^2(0,T;L^\infty(\mathcal{M})), \tag{3.21}$$

and f and f_x assume the periodic boundary conditions on $x = 0,\, 1$. Then for every $T > 0$, the following estimates independent of ϵ hold:

$$u^\epsilon \textit{ is bounded in } L^\infty(0,T;\, H^1(\mathcal{M})), \tag{3.22}$$

$$\sqrt{\epsilon}\,\nabla u_{xx}^\epsilon,\ \sqrt{\epsilon}\,\nabla u_{yy}^\epsilon,\ \sqrt{\epsilon}\,\nabla u_{zz}^\epsilon \textit{ are bounded in } L^2(0,T;L^2(\mathcal{M})). \tag{3.23}$$

Proof. We multiply (3.1) with $-\Delta u^\epsilon - \frac{1}{2}\,(u^\epsilon)^2$, integrate over $\mathcal{M}$ and integrate by parts. Firstly we show the calculation details of the multiplication by Δu^ϵ, integration over $\mathcal{M}$ and integration by parts (dropping the super index of ϵ for the moment):

- $$\int_{\mathcal{M}} u_t\,\Delta u\,d\mathcal{M} = -\int_{\mathcal{M}} \nabla u_t\,\nabla u\,d\mathcal{M} + \int_{\partial\mathcal{M}} u_t\,\frac{\partial u}{\partial n}\,d\,\partial\mathcal{M}$$
$$= \text{(thanks to (2.2) and (3.4))}$$
$$= -\int_{\mathcal{M}} \nabla u_t\,\nabla u\,d\mathcal{M} = -\frac{1}{2}\frac{d}{dt}\,|\nabla u|^2\,,$$
- $$\int_{\mathcal{M}} \Delta u_x\,\Delta u\,d\mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{(\Delta u)^2}{2}\right)\,d\mathcal{M} = \frac{1}{2}\int_{I_{x^\perp}} (\Delta u)^2\big|_{x=0}^{x=1}\,d\,I_{x^\perp} = \text{(thanks to (3.5))} = 0,$$
- $$c\int_{\mathcal{M}} u_x\,\Delta u\,d\mathcal{M} = c\int_{\mathcal{M}} u_x\,u_{xx} + u_x\,\Delta^\perp u\,d\mathcal{M}$$
$$= \text{(thanks to (2.2))} = c\int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{(u_x)^2}{2}\right)\,d\mathcal{M} - c\int_{\mathcal{M}} \nabla^\perp u_x\,\nabla^\perp u\,d\mathcal{M}$$
$$= c\int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{(u_x)^2}{2}\right)\,d\mathcal{M} - c\int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{(\nabla^\perp u)^2}{2}\right)\,d\mathcal{M}$$
$$= \text{(thanks to (3.4) and (2.2))} = 0,$$

- $$\int_{\mathcal{M}} u_{xxxx}\,u_{xx}\,d\,\mathcal{M} = \text{(thanks to (3.4))} = -\int_{\mathcal{M}} u_{xxx}^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} u_{xxxx}\,u_{yy}\,d\,\mathcal{M} = \text{(thanks to (2.2)-(2.4) and (3.4))} = -\int_{\mathcal{M}} u_{xxy}^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} u_{xxxx}\,u_{zz}\,d\,\mathcal{M} = \text{(thanks to (2.2)-(2.4) and (3.4))} = -\int_{\mathcal{M}} u_{xxz}^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} u_{yyyy}\,\Delta u\,d\,\mathcal{M} = \text{(thanks to (2.4) and (3.3))} = -\int_{\mathcal{M}} u_{yyy}\,\Delta u_y\,d\,\mathcal{M}$$
$$= \text{(thanks to (3.3))} = \int_{\mathcal{M}} u_{yy}\,\Delta u_{yy}\,d\,\mathcal{M}$$
$$= \text{(thanks to (3.3))} = -\int_{\mathcal{M}} (\nabla u_{yy})^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} u_{zzzz}\,\Delta u\,d\,\mathcal{M} = -\int_{\mathcal{M}} (\nabla u_{zz})^2\,d\,\mathcal{M},$$
- $$\int_{\mathcal{M}} f\,\Delta u\,d\mathcal{M} = \text{(thanks to (3.21))} = \int_{\mathcal{M}} \Delta f\,u\,d\mathcal{M},$$

Hence we find after changing the sign,

$$\frac{1}{2}\frac{d}{dt}\,|\nabla u^\epsilon|^2 - \int_{\mathcal{M}} u^\epsilon u_x^\epsilon\,\Delta u^\epsilon\,d\mathcal{M} + \epsilon[\nabla u^\epsilon]_2^2 = -\int_{\mathcal{M}} \Delta f\,u^\epsilon\,d\mathcal{M}. \tag{3.24}$$

Next we show the calculation details of the multiplication by $(u^\epsilon)^2$, integrating over $\mathcal{M}$ and integrating by parts:

- $\displaystyle\int_{\mathcal{M}} u_t u^2 \, d\,\mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial t}\left(\frac{u^3}{3}\right) \, d\,\mathcal{M} = \frac{1}{3}\frac{d}{dt}\left(\int_{\mathcal{M}} u^3 \, d\mathcal{M}\right),$
- $\displaystyle\int_{\mathcal{M}} \Delta u_x \, u^2 \, d\mathcal{M} = -2\int_{\mathcal{M}} \Delta u \, u u_x \, d\mathcal{M} + \int_{I_{x^\perp}} \Delta u \, u^2\big|_{x=0}^{x=1} d\, I_{x^\perp}$

$$= \text{(thanks to (3.5) and (2.2))} = -2\int_{\mathcal{M}} \Delta u \, u u_x \, d\mathcal{M},$$

- $\displaystyle c\int_{\mathcal{M}} u_x \, u^2 \, d\mathcal{M} = c\int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{u^3}{3}\right) d\mathcal{M} = \frac{c}{3}\int_{I_{x^\perp}} u^3\big|_{x=0}^{x=1} d\, I_{x^\perp} = \text{(thanks to (2.2))} = 0,$
- $\displaystyle\int_{\mathcal{M}} u u_x \, u^2 \, \mathcal{M} = \int_{\mathcal{M}} \frac{\partial}{\partial x}\left(\frac{u^4}{4}\right) d\mathcal{M} = \frac{1}{4}\int_{I_{x^\perp}} u^4\big|_{x=0}^{x=1} d\, I_{x^\perp} = \text{(thanks to (2.2))} = 0,$
- $\displaystyle\int_{\mathcal{M}} u_{xxxx} \, u^2 \, d\,\mathcal{M} = \text{(thanks to (2.2) and (3.4))} = -2\int_{\mathcal{M}} u_{xxx} \, u_x \, u \, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{yyyy} \, u^2 \, d\,\mathcal{M} = \text{(thanks to (2.4))} = -2\int_{\mathcal{M}} u_{yyy} \, u_y \, u \, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{zzzz} \, u^2 \, d\,\mathcal{M} = \text{(thanks to (2.4))} = -2\int_{\mathcal{M}} u_{zzz} \, u_z \, u \, d\,\mathcal{M}.$

Hence we find

$$\begin{aligned} \frac{1}{3}\frac{d}{dt}\left(\int_{\mathcal{M}} (u^\epsilon)^3 \, d\mathcal{M}\right) - 2\int_{\mathcal{M}} \Delta u^\epsilon \, u^\epsilon u_x^\epsilon \, d\mathcal{M} = \\ 2\epsilon\int_{\mathcal{M}} u_{xxx}^\epsilon \, u_x^\epsilon \, u^\epsilon + u_{yyy}^\epsilon \, u_y^\epsilon \, u^\epsilon + u_{zzz}^\epsilon \, u_z^\epsilon \, u^\epsilon \, d\,\mathcal{M} + \int_{\mathcal{M}} f\,(u^\epsilon)^2 \, d\mathcal{M}. \end{aligned} \tag{3.25}$$

Adding (3.24) to (3.25) multiplied by $-1/2$, we observe that the terms $\int_{\mathcal{M}} \Delta u^\epsilon \, u^\epsilon u_x^\epsilon \, d\mathcal{M}$ get canceled, which yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|\nabla u^\epsilon\right|^2 + \epsilon[\nabla u^\epsilon]_2^2 = & \frac{1}{6}\frac{d}{dt}\left(\int_{\mathcal{M}} (u^\epsilon)^3 \, d\mathcal{M}\right) \\ & - \epsilon\int_{\mathcal{M}} u_{xxx}^\epsilon \, u_x^\epsilon \, u^\epsilon + u_{yyy}^\epsilon \, u_y^\epsilon \, u^\epsilon + u_{zzz}^\epsilon \, u_z^\epsilon \, u^\epsilon \, d\,\mathcal{M} \\ & - \int_{\mathcal{M}} \Delta f u^\epsilon \, d\mathcal{M} - \frac{1}{2}\int_{\mathcal{M}} f\,(u^\epsilon)^2 \, d\mathcal{M}. \end{aligned}$$

Integrating both sides in time from 0 to t, we obtain for every $t \in (0, T)$,

$$\begin{aligned} \frac{1}{2}\left|\nabla u^\epsilon(t)\right|^2 + \epsilon\int_0^t [\nabla u^\epsilon]_2^2 \, ds = & \frac{1}{6}\int_{\mathcal{M}} (u^\epsilon(t))^3 \, d\mathcal{M} + \kappa_0 \\ & - \epsilon\int_0^t\int_{\mathcal{M}} u_{xxx}^\epsilon \, u_x^\epsilon \, u^\epsilon + u_{yyy}^\epsilon \, u_y^\epsilon \, u^\epsilon + u_{zzz}^\epsilon \, u_z^\epsilon \, u^\epsilon \, d\mathcal{M} \, ds \\ & - \int_0^t\int_{\mathcal{M}} \Delta f u^\epsilon \, d\mathcal{M} \, ds - \frac{1}{2}\int_0^t\int_{\mathcal{M}} f\,(u^\epsilon)^2 \, d\mathcal{M} \, ds, \end{aligned} \tag{3.26}$$

where

$$\kappa_0 := \frac{1}{2}\left|\nabla u_0\right|^2 - \frac{1}{6}\int_{\mathcal{M}} u_0^3 \, d\mathcal{M}.$$

We estimate each term on the right-hand-side of (3.26); we will use here the interpolation space $H^{1/2}(\mathcal{M})$ as defined in [LM72] where it is shown that $H^{1/2}(\mathcal{M}) \subset L^3(\mathcal{M})$ in dimension 3 with a continuous embedding. Dropping the superscript ϵ for the moment we then find:

$$\begin{aligned}
\left|\frac{1}{6}\int_{\mathcal{M}} u^3(t)\, d\mathcal{M}\right| &\le \frac{1}{6}|u(t)|^3_{L^3(\mathcal{M})} \\
&\le C'|u(t)|^3_{H^{1/2}(\mathcal{M})} \\
&\le C'|u(t)|^{3/2}\left|\nabla u(t)\right|^{3/2} \\
&\le C'|u(t)|^6 + \frac{1}{4}\left|\nabla u(t)\right|^2,
\end{aligned}$$

$$\begin{aligned}
\epsilon\left|\int_{\mathcal{M}} u_{xxx}\, u_x\, u\, d\,\mathcal{M}\right| &\le \epsilon\,|u_{xxx}|\,|u_x\, u| \\
&\le C'\epsilon\,|u_x\, u|^2 + \frac{\epsilon}{10}|u_{xxx}|^2 \\
&\le C'\epsilon|u|^2_{L^4(\mathcal{M})}|u_x|^2_{L^4(\mathcal{M})} + \frac{\epsilon}{10}|u_{xxx}|^2 \\
&\le (\text{by } H^{3/4}(\mathcal{M}) \subset L^4(\mathcal{M}) \text{ in 3D}) \\
&\le C'\epsilon|u|^{1/2}|\nabla u|^{3/2}|u_x|^{1/2}|u_x|^{3/2}_{H^1(\mathcal{M})} + \frac{\epsilon}{10}|u_{xxx}|^2 \\
&\le C'\epsilon|u|^{1/2}|\nabla u|^2|u|^{3/2}_{H^2(\mathcal{M})} + \frac{\epsilon}{10}|u_{xxx}|^2,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
\epsilon\left|\int_{\mathcal{M}} u_{yyy}\, x\, u_y\, u\, d\,\mathcal{M}\right| &\le (\text{by similar estimates as above}) \\
&\le C'\epsilon|u|^{1/2}|\nabla u|^2|u|^{3/2}_{H^2(\mathcal{M})} + \frac{\epsilon}{10}|u_{yyy}|^2,
\end{aligned}$$

$$\begin{aligned}
\epsilon\left|\int_{\mathcal{M}} u_{zzz}\, x\, u_z\, u\, d\,\mathcal{M}\right| &\le (\text{by similar estimates as above}) \\
&\le C'\epsilon|u|^{1/2}|\nabla u|^2|u|^{3/2}_{H^2(\mathcal{M})} + \frac{\epsilon}{10}|u_{zzz}|^2,
\end{aligned}$$

$$\left|\int_{\mathcal{M}} \Delta f\, u\, d\mathcal{M}\right| \le \left|\Delta f\right|^2 + \left|u\right|^2,$$

$$\left|\int_{\mathcal{M}} f u^2\, d\mathcal{M}\right| \le |f|_{L^\infty(\mathcal{M})}|u|^2 \le |f|^2_{L^\infty(\mathcal{M})} + |u|^4.$$

Collecting the above estimates, along with (3.26) we observe that the terms with third-order derivatives in the RHS of (3.27) and the following two inequalities can be canceled by a term on the LHS of (3.26). Thus (3.26) now yields

$$\begin{aligned}
\frac{1}{4}|\nabla u^\epsilon(t)|^2 &+ \frac{\epsilon}{10}\int_0^t [\nabla u^\epsilon]_2^2\,ds \\
&\le \int_0^t \left(1 + C'\epsilon |u^\epsilon|^{1/2}|u^\epsilon|^{3/2}_{H^2(\mathcal{M})}\right)|\nabla u^\epsilon(s)|^2\,ds + C'|u^\epsilon(t)|^6 + \kappa_0 \\
&\quad + \int_0^t |\Delta f|^2\,ds + \int_0^t |u^\epsilon|^2 + |u^\epsilon|^4\,ds + \int_0^t |f|^2_{L^\infty(\mathcal{M})}\,ds \\
&\le \text{(thanks to (3.13))} \\
&\le \int_0^t \left(1 + C'\epsilon\mu_1^{1/4}|u^\epsilon|^{3/2}_{H^2(\mathcal{M})}\right)|\nabla u^\epsilon(s)|^2\,ds + C'\mu_1^3 + \kappa_0 \\
&\quad + |f|^2_{L^2(0,T;H_0^2(\mathcal{M}))} + (\mu_1 + \mu_1^2)T + |f|^2_{L^2(0,T;L^\infty(\mathcal{M}))}.
\end{aligned} \tag{3.28}$$

In particular, setting $\sigma^\epsilon(t) := 1 + C'\epsilon\mu_1^{1/4}|u^\epsilon|^{3/2}_{H^2(\mathcal{M})}$, from (3.28) we deduce

$$\begin{aligned}
\frac{1}{4}|\nabla u^\epsilon(t)|^2 + \frac{\epsilon}{10}\int_0^t [\nabla u^\epsilon]_2^2\,ds \le \int_0^t &\sigma^\epsilon(s)|\nabla u^\epsilon(s)|^2\,ds \\
&+ C'\mu_1^3 + \kappa_0 + |f|^2_{L^2(0,T;H_0^2(\mathcal{M}))} \\
&+ (\mu_1 + \mu_1^2)T + |f|^2_{L^2(0,T;L^\infty(\mathcal{M}))}.
\end{aligned} \tag{3.29}$$

Since $|u^\epsilon|^{3/2}_{H^2(\mathcal{M})} \le |u^\epsilon|^2_{H^2(\mathcal{M})} + C'$, we find

$$\begin{aligned}
\int_0^T \sigma^\epsilon(s)\,ds &\le T + C'\epsilon\mu_1^{1/4}\int_0^T \left(|u^\epsilon|^2_{H^2(\mathcal{M})} + C'\right)ds \\
&\le \text{(thanks to (3.10))} \\
&\le \text{const} := \mu_3.
\end{aligned}$$

We can then apply the Gronwall inequality to (3.29) to obtain

$$\sup_{t\in(0,T)} |\nabla u^\epsilon(t)|^2 + \frac{\epsilon}{10}\int_0^T [\nabla u^\epsilon]_2^2\,ds \le const := \mu_4. \tag{3.30}$$

This together with (3.9) implies (3.22) and (3.23). □

3.2.3 Estimates independent of ϵ for u^ϵ_{xxx} and $u^\epsilon u^\epsilon_x$

For the sake of the passage to the limit on the boundary conditions and the compactness argument, we now derive bounds independent of ϵ for u^ϵ_{xxx} and $u^\epsilon\, u^\epsilon_x$. In particular, to obtain the estimates for u^ϵ_{xxx}, we first deduce a bound independent of ϵ for $\epsilon\, u^\epsilon_{xxxx}$ in $L^2(0,T;L^2(\mathcal{M}))$.

Proposition 3.2. *Under the same assumptions as in Proposition 3.1, we further suppose that*

$$u_{0xx} \in L^2(\mathcal{M}), \tag{3.31}$$

$$f_{xxx} \in L^2(0,T;L^2(\mathcal{M})), \tag{3.32}$$

and f_{xx} assume the periodic boundary condition on $x = 0, 1$. Then we have the following bounds independent of ϵ,

$$\epsilon\,[u_{xx}]_2 \text{ is bounded in } L^2(0,T;L^2(\mathcal{M})), \tag{3.33}$$

$$u^\epsilon u^\epsilon_x \text{ is bounded in } L^\infty(0,T;L^{3/2}(\mathcal{M})). \tag{3.34}$$

$$u^\epsilon_{xxx} \text{ is bounded in } L^{3/2}(I_x;H_t^{-1}(0,T;\,H^{-4}(I_{x^\perp}))), \tag{3.35}$$

Proof. For notational simplicity, we will drop the super index ϵ in the calculations. Multiplying (3.1) by u^ϵ_{xxxx}, integrating over $\mathcal{M}$ and integrating by parts we find:

- $\displaystyle\int_{\mathcal{M}} u_t\, u_{xxxx}\, d\mathcal{M} = (\text{thanks to (2.2) and (3.4)}) = \frac{1}{2}\frac{d}{dt}|u_{xx}|^2,$
- $\displaystyle\int_{\mathcal{M}} \Delta u_x\, u_{xxxx}\, d\mathcal{M} = (\text{thanks to (2.2), (3.4) and (2.4)}) = 0,$
- $\displaystyle\int_{\mathcal{M}} u_x u_{xxxx}\, d\mathcal{M} = (\text{thanks to (2.2),(3.4) and (2.4)}) = 0,$
- $\displaystyle\int_{\mathcal{M}} uu_x\, u_{xxxx}\, d\mathcal{M} = -\int_{\mathcal{M}} u_x^2\, u_{xxx}\, d\mathcal{M} - \int_{\mathcal{M}} uu_{xx}\, u_{xxx}\, d\mathcal{M} = \frac{5}{2}\int_{\mathcal{M}} u_x\, u_{xx}^2\, d\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{yyyy}u_{xxxx}\, d\,\mathcal{M} = (\text{thanks to (2.2), (3.4) and (2.4)}) = \int_{\mathcal{M}} u_{xxyy}^2\, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} u_{zzzz}u_{xxxx}\, d\,\mathcal{M} = \int_{\mathcal{M}} u_{xxzz}^2\, d\,\mathcal{M},$
- $\displaystyle\int_{\mathcal{M}} f u_{xxxx}\, d\mathcal{M} = -\int_{\mathcal{M}} f_{xxx}\, u_x\, d\mathcal{M} \le |f_{xxx}|^2 + |u_x|^2.$

Hence we find

$$\frac{1}{2}\frac{d}{dt}|u^\epsilon_{xx}|^2 + \epsilon[u^\epsilon_{xx}]_2^2 \le \frac{5}{2}\int_{\mathcal{M}} u^\epsilon_x\ (u^\epsilon_{xx})^2\ d\mathcal{M} + |f_{xxx}|^2 + |u^\epsilon_x|^2.$$

Multiplying both sides by ϵ we obtain

$$\frac{\epsilon}{2}\frac{d}{dt}|u^\epsilon_{xx}|^2 + \epsilon^2[u^\epsilon_{xx}]_2^2 \le \frac{5\epsilon}{2}\int_{\mathcal{M}} u^\epsilon_x\ (u^\epsilon_{xx})^2\ d\mathcal{M} + \epsilon|f_{xxx}|^2 + \epsilon|u^\epsilon_x|^2. \tag{3.36}$$

We estimate the first term on the right-hand side of (3.36) and find

$$\begin{aligned}
\epsilon \left| \int_{\mathcal{M}} u_x \, u_{xx}^2 \, d\mathcal{M} \right| &\le \epsilon \, |u_x| |u_{xx}|^2_{L^4(\mathcal{M})} \\
&\le C' \epsilon \, |u_x| |u_{xx}|^{1/2} |\nabla u_{xx}|^{3/2} \\
&\le \text{(by the intermediate derivative theorem } \ |u_{xx}|^2 \le |u_x| \, |u_{xxx}|) \\
&\le C' \epsilon |u_x|^{5/4} |u_{xxx}|^{1/4} |\nabla u_{xx}|^{3/2} \\
&\le C' \epsilon |u_x|^{5/4} |\nabla u_{xx}|^{7/4} \\
&\le \text{(thanks to (3.30))} \\
&\le C' \epsilon \mu_4^{5/8} |\nabla u_{xx}|^{7/4}.
\end{aligned}$$

This along with (3.36) implies

$$\frac{\epsilon}{2} \frac{d}{dt} |u^\epsilon_{xx}|^2 + \epsilon^2 [u^\epsilon_{xx}]^2_2 \le C' \epsilon \, \mu_4^{5/8} |\nabla u^\epsilon_{xx}|^{7/4} + \epsilon |f_{xxx}|^2 + \epsilon \mu_4.$$

Integrating both sides in t from 0 to T, we find

$$\epsilon^2 \int_0^T [u^\epsilon_{xx}]^2_2 \, dt \le \frac{\epsilon}{2} |u_{0xx}|^2 + C' \mu_4^{5/8} \int_0^T \epsilon |\nabla u^\epsilon_{xx}|^{7/4} \, dt + \epsilon |f_{xxx}|^2_{L^2(0,T;L^2(\mathcal{M}))} + \epsilon \mu_4 T. \tag{3.37}$$

From (3.23), we see that $\int_0^T \epsilon \, |\nabla u^\epsilon_{xx}|^{7/4} \ dt \le C' \int_0^T \epsilon \left(|\nabla u^\epsilon_{xx}|^2 + 1 \right) \ dt \le const := \mu_6$. This along with (3.37) implies (3.33).

Now since

$$\int_{\mathcal{M}} (u u_x)^{3/2} \, d\mathcal{M} \le C' |u|^{3/2}_{L^6(\mathcal{M})} |u_x|^{3/2} \le \text{(by } H^1(\mathcal{M}) \subset L^6(\mathcal{M}) \text{ in } 3D) \le C' |u|^3_{H^1},$$

this along with (3.22) implies (3.34), and hence

$$u^\epsilon u^\epsilon_x \text{ is bounded in } L^{3/2}(I_x; L^{3/2}((0,T) \times I_{x^\perp})). \tag{3.38}$$

Finally rewriting (3.1) we find

$$u^\epsilon_{xxx} = -u^\epsilon_t - \Delta^\perp u^\epsilon_x - c u^\epsilon_x - u^\epsilon \, u^\epsilon_x - \epsilon \, u^\epsilon_{xxxx} - \epsilon \, u^\epsilon_{yyyy} - \epsilon \, u^\epsilon_{zzzz}. \tag{3.39}$$

Thanks to (3.33), we see that $\epsilon u^\epsilon_{xxxx}$ remains bounded in $L^2(0,T; \, L^2(\mathcal{M}))$. Moreover since u^ϵ remains $L^\infty(0,T; H^1(\mathcal{M}))$, we find that each term on the right-hand side of (3.39) except for $u^\epsilon \, u^\epsilon_x$ remains bounded at least in $L^2(I_x; H^{-1}_t(0,T; \, H^{-4}(I_{x^\perp})))$. This together with (3.38) implies that each term on the right-hand side of (3.39) remains bounded at least in $L^{3/2}(I_x; H^{-1}_t(0,T; \, H^{-4}(I_{x^\perp})))$. Thus we obtain (3.35) from (3.39).

. □

3.3 The main result

Using a compactness argument, we can pass to the limit in (3.1) and obtain (2.1), with a function $u \in \mathcal{C}([0,T]; H^1(\mathcal{M})) \cap H^3(I_x; H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp})))$. Moreover, from (3.35) we see that u^ϵ_{xxx} converges weakly in $L^{3/2}(I_x; H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp})))$, hence by the trace theorem and Mazur's theorem, we deduce that $u^\epsilon_{x^j}(0, x^\perp, t)$ and $u^\epsilon_{x^j}(1, x^\perp, t)$ converge weakly in $H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp}))$, $j = 1,\ 2$. Thus from (3.4) we obtain (2.3).

Now we are ready to state the main result of the article by collecting all the previous estimates.

Theorem 3.1. *The assumptions are the same as in Proposition 3.2, that is (3.7), (3.8), (3.21), (3.20)), (3.32), (3.31), and f and f_{x^j} assume the periodic boundary conditions on $x = 0,\ 1$, $j = 1, 2$. Then the initial and boundary value problem for the ZK equation, that is, (2.1), (2.2)-(2.4) and (2.6), possesses at least a solution u:*

$$u \in \mathcal{C}([0,T]; H^1(\mathcal{M})) \cap W^{3,3/2}(I_x; H_t^{-1}(0,T;\, H^{-4}(I_{x^\perp}))). \tag{3.40}$$

Remark 3.1. *We can obtain stronger regularity for $\bar{u}(x^\perp, t) := \int_0^1 u(x, x^\perp, t)\, dx$. Integrating (2.1) in x from 0 to 1, we find by (2.2) and (2.3)*

$$\frac{\partial \bar{u}}{\partial t} = \bar{f}. \tag{3.41}$$

Thus $u = \bar{u} + v$, where $\bar{u}$ satisfies (3.41), and v satisfies $\bar{v} = 0$ and (3.40).

4 Discussions about the uniqueness of solutions.

Let u and v be two solutions of (2.1)-(2.4) and (2.6) and let $w = u - v$. Letting $\bar{w}(x^\perp, t) := \int_0^1 u(x, x^\perp, t)\, dx$, we see that $\dfrac{\partial \bar{w}}{\partial t} = 0$ and hence

$$\bar{w}(t) = 0, \quad \forall\ t \in [0,T]. \tag{4.1}$$

However, it is not clear if we can further prove that $w(t) = 0, \quad \forall\ t \in [0,T]$. Firstly, the ideas in the proof of existence can not be extended to prove the uniqueness because the structure of the nonlinear term is changed. Secondly, the methods in [ST10] and [STW12] are not applicable due to the lack of assumptions on the boundary condition u_x at $x = 1$. For the same reason, the proof of the local existence in [Wan] fails as well, which prevents us from using the methods in [CT07].

To conclude, the uniqueness of solutions in both dimensions 2 and 3 are still open due to the partially hyperbolic feature of this model.

Remark 4.1. *As for the periodic case, that is, (2.1) and the boundary and initial conditions (2.2), (2.3), (2.5) and (2.6), the results are exactly the same as in the Dirichlet case discussed above. The reasoning is totally the same and therefore we skip it.*

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