OPTIMAL STRATEGIES FOR A LONG-TERM STATIC INVESTOR

LINGJIONG ZHU

ABSTRACT. The optimal strategies for a long-term static investor are studied. Given a portfolio of a stock and a bond, we derive the optimal allocation of the capitols to maximize the expected long-term growth rate of a utility function of the wealth. When the bond has constant interest rate, three models for the underlying stock price processes are studied: Heston model, 3/2 model and jump diffusion model. We also study the optimal strategies for a portfolio in which the stock price process follows a Black-Scholes model and the bond process has a Vasicek interest rate that is correlated to the stock price.

1. INTRODUCTION

In this article, we are interested in the long-term optimal strategies for a static investor. The investor starts with a known initial wealth $V_0 > 0$ and the wealth at time t is denoted by V_t. The investor decides what fraction of wealth α_t to invest in a stock S_t and the remaining $1 - \alpha_t$ in a bond r_t, i.e.

$$(1.1) \qquad \frac{dV_t}{V_t} = \alpha_t \frac{dS_t}{S_t} + (1 - \alpha_t) r_t dt.$$

For a static investor, we assume that $\alpha_t \equiv \alpha$ is a constant between 0 and 1, i.e. $\alpha \in [0, 1]$.

We consider a hyperbolic absolute risk aversion (HARA) utility function $u(c)$ with constant relative risk aversion coefficient $\gamma \in (0, 1)$, i.e.

$$(1.2) \qquad u(c) = \frac{c^{1-\gamma}}{1 - \gamma}, \quad 0 < \gamma < 1.$$

We are interested in the optimal strategy to maximize the long-term growth rate, i.e.

$$(1.3) \qquad \max_{0 \le \alpha \le 1} \lim_{t \to \infty} \frac{1}{t} \log E[u(V_t)] = \max_{0 \le \alpha \le 1} \Lambda(\alpha),$$

if the long-term growth rate $\Lambda(\alpha) := \lim_{t \to \infty} \frac{1}{t} \log E[u(V_t)]$ exists for any $0 \le \alpha \le 1$. For the convenience of notation, let $\theta := 1 - \gamma \in (0, 1)$ and thus we are interested in

$$(1.4) \qquad \max_{0 \le \alpha \le 1} \lim_{t \to \infty} \frac{1}{t} \log E[(V_t)^\theta].$$

The optimal long-term growth rate of expected utility of wealth has been well studied in the literatures. Usually, the optimal strategy is taken to be dynamic and

some dynamic programming equations are studied, see e.g. Fleming and Sheu [8]. In this article, we only concentrate on the static strategies for the simplicity. This set-up allows us to gain analytical tractability for some more sophisticated models like Heston model and 3/2 model.

The problem of maximizing the long-term expected utility is closely related to maximizing the probability that the wealth exeeds a given benchmark for large time horizon, i.e. $\max_{0 \leq \alpha \leq 1} \lim_{t \to \infty} \frac{1}{t} \log P(V_t \geq V_0 e^{xt})$, where x is a given benchmark. In a static framework, an asymptotic outperformance criterion was for example considered in Stutzer [13]. An asymptotic dynamic version of the outperformance managment criterion was developed by Pham [11]. To find the optimal strategy for the long-term growth rate, the first step is to compute the limit $\lim_{t \to \infty} \frac{1}{t} \log E[u(V_t)]$. Under most of the standard models, the wealth process V_t has exponential growth rate and the existence of a logarithmic moment generating function plus some additional conditions can be used to obtain a large deviation principle for the probability that the wealth process outperformances a given benchmark, which is exponentially small. The connection is provided by Gärtner-Ellis theorem, see e.g. Dembo and Zeitouni [5]. For a survey on the applications of large deviations to finance, we refer to Pham [12].

In this article, we study in detail the optimal strategies for a static investor investing in stocks of two stochastic volatility models, i.e. the Heston model (Section 2), the 3/2 model (Section 3)

The Heston model, introduced by Heston [9] is a widely used stochastic volatility model. The volatility process is itself a Cox-Ross-Ingersoll process, which is an affine model and has great analytical tractability. The 3/2 model is another popular model of stochastic volatility. It has been applied to interest rate modeling, e.g. Ahn and Gao [1]. Carr and Sun [3] used the 3/2 model to price variance swaps and Drimus [6] used it to price options on realized variance.

Next, we study the optimal long-term static investment strategies when the underlying stock process follows a jump diffusion model (Section 4) assuming the alternative investment bond has constant short-rate. Finally, we study the case when the stock follows a classical Black-Scholes model while the bond has a Vasicek interest rate (Section 5).

As an illustration, let us first consider a toy model. Assume that the stock price follows a geometric Brownian motion with constant drift $\mu > 0$ and constant volatility $\sigma > 0$ and the bond has constant short-rate $r > 0$. We can write down a stochastic differential equation for the wealth process V_t,

$$(1.5) \qquad\qquad dV_t = \alpha \mu V_t dt + \alpha \sigma V_t dB_t + (1 - \alpha) r V_t dt,$$

where B_t is a standard Brownian motion starting at 0 at time 0 and therefore

$$(1.6) \qquad V_t = V_0 \exp\left\{ \left(\alpha\mu + (1-\alpha)r - \frac{1}{2}\alpha^2\sigma^2 \right) t + \alpha\sigma B_t \right\}.$$

Hence, we can compute that

$$(1.7) \qquad\qquad E[(V_t)^\theta] = V_0^\theta e^{\theta(\alpha\mu + (1-\alpha)r - \frac{1}{2}\alpha^2\sigma^2)t} e^{\frac{1}{2}\theta^2\alpha^2\sigma^2 t}.$$

Therefore, we are interested to maximize

$$(1.8) \qquad \Lambda(\alpha) = \theta \left[\alpha\mu + (1-\alpha)r - \frac{1}{2}\alpha^2\sigma^2 \right] + \frac{1}{2}\theta^2\alpha^2\sigma^2, \quad 0 \leq \alpha \leq 1.$$

It is easy to compute that

(1.9) $$\Lambda'(\alpha) = (\theta\mu - \theta r) - (\theta - \theta^2)\sigma^2\alpha.$$

Hence, $\Lambda'(\alpha) = 0$ if $\alpha = \frac{\mu-r}{(1-\theta)\sigma^2}$. Therefore, the optimal α^* is given by

(1.10) $$\alpha^* = \begin{cases} 0 & \text{if } \mu \leq r, \\ \frac{\mu-r}{(1-\theta)\sigma^2} & \text{if } 0 < \frac{\mu-r}{(1-\theta)\sigma^2} < 1, \\ 1 & \text{if } \mu - r \geq (1-\theta)\sigma^2. \end{cases}$$

The financial interpretation is clear. When $\mu \leq r$, it is optimal to invest in the bond only because the yield of the bond r exceeds the mean return of the stock. When $\mu > r$, it is not always optimal in only invest in stocks. The reason is although the mean return of the stock exceeds the yield of the bond, stocks are volatile and a large volatility can decrease the expected utility of the portfolio. This is consistent with the mean-variance analysis, which says that given the mean, the investor has the incentive to minimize the variance.

In general, for any wealth process V_t, assume $\Lambda(\alpha)$ exists and is smooth and strictly concave, If $\Lambda'(0) \leq 0$, then the optimal α^* is given by $\alpha^* = 0$. Otherwise, $\Lambda(\alpha)$ achieves a unique maximum at some $\alpha^\dagger \in (0, \infty)$. Then the optimal α^* is given by

(1.11) $$\alpha^* = \begin{cases} 1 & \text{if } \alpha^\dagger \geq 1, \\ \alpha^\dagger & \text{if } \alpha^\dagger \in (0, 1). \end{cases}$$

This is the general method behind analyzing all the models in this article.

2. HESTON MODEL

Let us assume that the stock price follows a Heston model, namely, the stock price has a stochastic volatility which follows a Cox-Ingersoll-Ross process,

(2.1) $$\begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dB_t, \\ d\nu_t = \kappa(\gamma - \nu_t)dt + \delta\sqrt{\nu_t}dW_t, \end{cases}$$

where W_t and B_t are two standard Brownian motions and $\langle W, B \rangle_t = \rho t$, where $-1 \leq \rho \leq 1$ is the correlation. Assume that $\mu, \kappa, \gamma, \delta > 0$. The volatility process ν_t is a Cox-Ingersoll-Ross process, introduced by Cox et al. [4]. We assume the Feller condition $2\kappa\gamma > \delta^2$ holds so that ν_t is always positive, see e.g. Feller [7].

The wealth process satisfies

(2.2) $$\begin{cases} dV_t = \alpha\mu V_t dt + \alpha\sqrt{\nu_t}V_t dB_t + (1-\alpha)rV_t dt, \\ d\nu_t = \kappa(\gamma - \nu_t)dt + \delta\sqrt{\nu_t}dW_t. \end{cases}$$

Then, we have

(2.3) $$V_t = V_0 \exp\left\{ \alpha\mu t - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds + (1-\alpha)rt + \alpha \int_0^t \sqrt{\nu_s}dB_s \right\}.$$

Hence, we get

(2.4) $$E[u(V_t)] = \frac{1}{\theta}V_0^\theta e^{\theta(\alpha\mu + (1-\alpha)r)t} E\left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s}dB_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)} \right].$$

Lemma 1. *For any $\nu > 0$,*

(2.5)
$$\lim_{t \to \infty} \frac{1}{t} \log E_{\nu_0 = \nu} \left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s} dB_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)} \right]$$
$$= \frac{\kappa^2 \gamma}{\delta^2} - \frac{\kappa\gamma}{\delta^2} \sqrt{\kappa^2 - \delta^2\theta^2\alpha^2(1-\rho^2) + \delta^2\theta\alpha^2 - 2\delta\kappa\alpha\rho\theta} - \frac{\theta\alpha\rho\kappa}{\delta}\gamma.$$

Proof. Write $B_t = \rho W_t + \sqrt{1-\rho^2} Z_t$, where Z_t is a standard Brownian motion independent of W_t. Let $\mathcal{F}_t^{\nu} := \sigma(\nu_s, 0 \le s \le t)$ be the natural sigma field of the volatility process up to time t. It is easy to compute that

(2.6)
$$E_{\nu_0=\nu} \left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s} dB_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)} \right]$$
$$= E_{\nu_0=\nu} \left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s}\rho dW_s + \alpha \int_0^t \sqrt{\nu_s}\sqrt{1-\rho^2} dZ_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)} \right]$$
$$= E_{\nu_0=\nu} \left[E \left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s}\rho dW_s + \alpha \int_0^t \sqrt{\nu_s}\sqrt{1-\rho^2} dZ_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)} \middle| \mathcal{F}_t^{\nu} \right] \right]$$
$$= E_{\nu_0=\nu} \left[e^{\theta\alpha \int_0^t \sqrt{\nu_s}\rho dW_s + \frac{1}{2}\theta^2\alpha^2(1-\rho^2) \int_0^t \nu_s ds - \frac{1}{2}\theta\alpha^2 \int_0^t \nu_s ds} \right]$$
$$= E_{\nu_0=\nu} \left[e^{\frac{\theta\alpha\rho}{\delta}\nu_t + (\frac{\theta^2\alpha^2(1-\rho^2)-\theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta) \int_0^t \nu_s ds} \right] e^{-\frac{\theta\alpha\rho}{\delta}\nu - \frac{\theta\alpha\rho\kappa}{\delta}\gamma t},$$

where the last step was due to the fact that $\nu_t - \nu_0 = \int_0^t \kappa(\gamma - \nu_s)ds + \int_0^t \delta\sqrt{\nu_s}dW_s$.

Let $u(t, \nu) := E_{\nu_0=\nu} \left[e^{\frac{\theta\alpha\rho}{\delta}\nu_t + (\frac{\theta^2\alpha^2(1-\rho^2)-\theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta) \int_0^t \nu_s ds} \right]$. Feynman-Kac formula implies that $u(t, \nu)$ satisfies the following partial differential equation,

(2.7)
$$\begin{cases} \frac{\partial u}{\partial t} = \kappa(\gamma - \nu)\frac{\partial u}{\partial \nu} + \frac{1}{2}\delta^2\nu\frac{\partial^2 u}{\partial \nu^2} + \left(\frac{\theta^2\alpha^2(1-\rho^2)-\theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta \right)\nu u, \\ u(0, \nu) = e^{\frac{\theta\alpha\rho}{\delta}\nu}. \end{cases}$$

Let us try $u(t, \nu) = e^{A(t)+B(t)\nu}$ and it is easy to see that $A(t), B(t)$ satisfy the following system of ordinary differential equations,

(2.8)
$$\begin{cases} A'(t) = \kappa\gamma B(t), \\ B'(t) = -\kappa B(t) + \frac{1}{2}\delta^2 B(t)^2 + \left(\frac{\theta^2\alpha^2(1-\rho^2)-\theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta \right), \\ A(0) = 0, \quad B(0) = \frac{\theta\alpha\rho}{\delta}. \end{cases}$$

We claim that there are two distinct solutions to the quadratic equation

(2.9)
$$\frac{1}{2}\delta^2 x^2 - \kappa x + \left(\frac{\theta^2\alpha^2(1-\rho^2) - \theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta \right) = 0,$$

and $B(t)$ converges to the smaller solution of (2.9).

We can compute that

(2.10)
$$\Delta := \kappa^2 - 2\delta^2 \left(\frac{\theta^2\alpha^2(1-\rho^2) - \theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta \right).$$

If $\alpha = 0$, then $\Delta = \kappa^2 > 0$. If $\alpha \ne 0$, then,

(2.11)
$$\Delta = (\kappa^2 + \delta^2\theta^2\alpha^2\rho^2 - 2\delta\kappa\alpha\rho\theta) + \delta^2\alpha^2(\theta - \theta^2)$$
$$= (\kappa - \delta\theta\alpha\rho)^2 + \delta^2\alpha^2(\theta - \theta^2) > 0,$$

since $\theta \in (0, 1)$. Hence (2.9) has two distinct solutions. $B'(t)$ is positive when $B(t)$ is smaller than the smaller solution of (2.9) or larger than the larger solution of

(2.9). $B'(t)$ is negative if $B(t)$ lies between the two solutions of (2.9). Therefore, $B(t)$ converges to the smaller solution of (2.9) if $B(0) = \frac{\theta\alpha\rho}{\delta}$ is less than the larger solution of (2.9). When $\alpha = 0$, $B(0) = 0$ and the large solution of (2.9) equals to $\frac{2\kappa}{\delta^2} > 0$. Hence, we can assume that $\alpha > 0$. Let

$$(2.12) \qquad H(x) := \frac{1}{2}\delta^2 x^2 - \kappa x + \left(\frac{\theta^2\alpha^2(1-\rho^2) - \theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta\right).$$

It is easy to check that

$$(2.13) \qquad H\left(\frac{\theta\alpha\rho}{\delta}\right) = \frac{(\theta^2 - \theta)\alpha^2}{2} < 0,$$

since $\theta \in (0,1)$ and $\alpha > 0$. Therefore, we conclude that $B(0)$ is less than the larger solution of (2.9) and

$$(2.14) \qquad B(t) \to \frac{\kappa}{\delta^2} - \frac{1}{\delta^2}\sqrt{\kappa^2 - 2\delta^2\left(\frac{\theta^2\alpha^2(1-\rho^2) - \theta\alpha^2}{2} + \frac{\kappa\alpha\rho}{\delta}\theta\right)},$$

as $t \to \infty$ and hence
$$(2.15)$$
$$\frac{A(t)}{t} = \frac{1}{t}\kappa\gamma\int_0^t B(s)ds \to \frac{\kappa^2\gamma}{\delta^2} - \frac{\kappa\gamma}{\delta^2}\sqrt{\kappa^2 - \delta^2\theta^2\alpha^2(1-\rho^2) + \delta^2\theta\alpha^2 - 2\delta\kappa\alpha\rho\theta},$$

as $t \to \infty$. Recall that $E_{\nu_0=\nu}\left[e^{\theta(\alpha\int_0^t \sqrt{\nu_s}dB_s - \frac{1}{2}\alpha^2\int_0^t \nu_s ds)}\right] = u(t,\nu)e^{-\frac{\theta\alpha\rho}{\delta}\nu - \frac{\theta\alpha\rho\kappa}{\delta}\gamma t}$. Hence, we conclude that

$$(2.16) \qquad \lim_{t\to\infty}\frac{1}{t}\log E_{\nu_0=\nu}\left[e^{\theta(\alpha\int_0^t \sqrt{\nu_s}dB_s - \frac{1}{2}\alpha^2\int_0^t \nu_s ds)}\right]$$
$$= \frac{\kappa^2\gamma}{\delta^2} - \frac{\kappa\gamma}{\delta^2}\sqrt{\kappa^2 - \delta^2\theta^2\alpha^2(1-\rho^2) + \delta^2\theta\alpha^2 - 2\delta\kappa\alpha\rho\theta} - \frac{\theta\alpha\rho\kappa}{\delta}\gamma.$$

$\square$

Theorem 2.

$$(2.17) \qquad \Lambda(\alpha) = \lim_{t\to\infty}\frac{1}{t}\log E[u(V_t)]$$
$$= \frac{\kappa^2\gamma}{\delta^2} - \frac{\kappa\gamma}{\delta^2}\sqrt{\kappa^2 - \delta^2\theta^2\alpha^2(1-\rho^2) + \delta^2\theta\alpha^2 - 2\delta\kappa\alpha\rho\theta}$$
$$- \frac{\theta\alpha\rho\kappa}{\delta}\gamma + \theta\alpha\mu + \theta(1-\alpha)r.$$

Let us define

$$(2.18) \qquad \begin{cases} C_0 := \frac{\kappa^2\gamma}{\delta^2} + \theta r, \\ C_1 := \frac{\kappa^2\gamma^2}{\delta^4}(\delta^2\theta - \delta^2\theta^2(1-\rho^2)), \\ C_2 := \delta\kappa\rho\theta \cdot \frac{\kappa^2\gamma^2}{\delta^4}, \\ C_3 := \frac{\kappa^4\gamma^2}{\delta^4}, \\ C_4 := -\frac{\theta\rho\kappa}{\delta}\gamma + \theta(\mu - r). \end{cases}$$

When $C_4 + \frac{C_2}{\sqrt{C_3}} \leq 0$, the optimal $\alpha^* = 0$ and when $C_4 \geq \sqrt{C_1}$, the optimal $\alpha^* = 1$. Finally, if $-\frac{C_2}{\sqrt{C_3}} < C_4 < \sqrt{C_1}$,

$$(2.19) \qquad \alpha^* = \begin{cases} \alpha^\dagger & \text{if } \alpha^\dagger < 1, \\ 1 & \text{otherwise}, \end{cases}$$

where

$$(2.20) \qquad \alpha^\dagger = \frac{1}{C_1} \left[C_2 + C_4 \sqrt{\frac{C_1 C_3 - C_2^2}{C_1 - C_4^2}} \right].$$

Proof. Recall that $E[u(V_t)] = \frac{1}{\theta} V_0^\theta e^{\theta(\alpha\mu + (1-\alpha)r)t} E\left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s} dB_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)} \right]$. By Lemma 1, we have

$$(2.21) \qquad \Lambda(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log E[u(V_t)]$$

$$= \frac{\kappa^2 \gamma}{\delta^2} - \frac{\kappa\gamma}{\delta^2} \sqrt{\kappa^2 - \delta^2\theta^2\alpha^2(1-\rho^2) + \delta^2\theta\alpha^2 - 2\delta\kappa\alpha\rho\theta}$$

$$- \frac{\theta\alpha\rho\kappa}{\delta}\gamma + \theta\alpha\mu + \theta(1-\alpha)r$$

$$= -\sqrt{C_1\alpha^2 - 2C_2\alpha + C_3} + C_4\alpha + C_0.$$

From the definition in (2.18), it is clear that $C_0, C_1, C_2, C_3 > 0$. But C_4 may or may not be positive.

It is easy to compute that

$$(2.22) \qquad \Lambda'(\alpha) = C_4 - \frac{C_1\alpha - C_2}{\sqrt{C_1\alpha^2 - 2C_2\alpha + C_3}},$$

and

$$(2.23) \qquad \Lambda''(\alpha) = -\frac{C_1\sqrt{C_1}(C_1 C_3 - C_2^2)}{((C_1\alpha - C_2)^2 + (C_1 C_3 - C_2^2))^{3/2}}.$$

On the other hand, since $\theta \in (0, 1)$, we have

$$(2.24) \qquad C_1 C_3 - C_2^2 = \frac{\kappa^2\gamma^2}{\delta^4}\frac{\kappa^4\gamma^2}{\delta^4}(\delta^2\theta - \delta^2\theta^2(1-\rho^2)) - \delta^2\kappa^2\rho^2\theta^2\frac{\kappa^4\gamma^4}{\delta^8}$$

$$= \frac{\kappa^6\gamma^4}{\delta^8}(\delta^2\theta - \delta^2\theta^2(1-\rho^2) - \delta^2\rho^2\theta^2)$$

$$= \frac{\kappa^6\gamma^4}{\delta^6}(\theta - \theta^2) > 0.$$

Hence, we conclude that $\Lambda''(\alpha) < 0$ for any α, i.e. $\Lambda(\alpha)$ is strictly concave in α.

Note that $\Lambda'(0) = C_4 + \frac{C_2}{\sqrt{C_3}}$. If $C_4 + \frac{C_2}{\sqrt{C_3}} \leq 0$, since $\Lambda(\alpha)$ is strictly concave, the maximum must be achieved at $\alpha^* = 0$. Now assume that $C_4 + \frac{C_2}{\sqrt{C_3}} > 0$. When $C_4 > \sqrt{C_1}$, it is easy to check that $\Lambda'(\alpha) \sim (C_4 - \sqrt{C_1})$ as $\alpha \to \infty$, and since $\Lambda(\alpha)$ is strictly concave, it yields that $\Lambda(\alpha)$ is increasing in $\alpha \geq 0$ and the maximum is

achieved at $\alpha^\dagger = 1$. If $C_4 = \sqrt{C_1}$,

$$(2.25) \qquad \Lambda'(\alpha) = \frac{C_4\sqrt{C_1\alpha^2 - 2C_2\alpha + C_3} - C_1\alpha + C_2}{\sqrt{C_1\alpha^2 - 2C_2\alpha + C_3}}$$

$$= \frac{\frac{C_4}{C_1}\sqrt{(C_1\alpha - C_2)^2 + (C_3C_1 - C_2^2)} - C_1\alpha + C_2}{\sqrt{C_1\alpha^2 - 2C_2\alpha + C_3}}$$

$$= \frac{\sqrt{(C_1\alpha - C_2)^2 + (C_3C_1 - C_2^2)} - (C_1\alpha - C_2)}{\sqrt{C_1\alpha^2 - 2C_2\alpha + C_3}} > 0,$$

since $C_3C_1 - C_2^2 > 0$. Thus, $\alpha^* = 1$ when $C_4 = \sqrt{C_1}$.

Now assume that $-\frac{C_2}{\sqrt{C_3}} < C_4 < \sqrt{C_1}$. Then $\Lambda'(0) = C_4 + \frac{C_2}{\sqrt{C_3}} > 0$ and $\Lambda(\alpha) \to -\infty$ as $\alpha \to \infty$. Thus, there exists a unique global maximum on $(0, \infty)$, given by $\alpha^\dagger$. So that

$$(2.26) \qquad \Lambda'(\alpha^\dagger) = \sqrt{C_1}\left[\frac{C_4}{\sqrt{C_1}} - \frac{(C_1\alpha^\dagger - C_2)}{\sqrt{(C_1\alpha^\dagger - C_2)^2 + C_1C_3 - C_2^2}}\right] = 0.$$

$C_1\alpha^\dagger - C_2$ has the same sign as C_4 which is positive. Hence, we can solve for $\alpha^\dagger$ and get

$$(2.27) \qquad \alpha^\dagger = \frac{1}{C_1}\left[C_2 + C_4\sqrt{\frac{C_1C_3 - C_2^2}{C_1 - C_4^2}}\right].$$

The optimal α^* is given by

$$(2.28) \qquad \alpha^* = \begin{cases} \alpha^\dagger & \text{if } \alpha^\dagger < 1, \\ 1 & \text{otherwise.} \end{cases}$$

$\square$

3. 3/2 MODEL

Let us assume that the stock price follows a 3/2 model, namely,

$$(3.1) \qquad \begin{cases} dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dB_t, \\ d\nu_t = \kappa \nu_t(\gamma - \nu_t) + \delta \nu_t^{3/2} dW_t, \end{cases}$$

where B_t and W_t are two standard Brownian motions, which are assumed to be independent for simplicity.

Therefore, the wealth process satisfies

$$(3.2) \qquad \begin{cases} dV_t = \alpha \mu V_t dt + \alpha \sqrt{\nu_t} V_t dB_t + (1-\alpha)rV_t dt, \\ d\nu_t = \kappa(\gamma - \nu_t)dt + \delta \nu_t^{3/2} dW_t, \end{cases}$$

Then, we have

$$(3.3) \qquad V_t = V_0 \exp\left\{\alpha\mu t - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds + (1-\alpha)rt + \alpha \int_0^t \sqrt{\nu_s} dB_s\right\}.$$

Hence, we get

$$(3.4) \qquad E[u(V_t)] = \frac{1}{\theta}V_0^\theta e^{\theta(\alpha\mu + (1-\alpha)r)t} E\left[e^{\theta(\alpha \int_0^t \sqrt{\nu_s}dB_s - \frac{1}{2}\alpha^2 \int_0^t \nu_s ds)}\right]$$

$$= \frac{1}{\theta}V_0^\theta e^{\theta(\alpha\mu + (1-\alpha)r)t} E\left[e^{-\frac{1}{2}\alpha^2(\theta - \theta^2)\int_0^t \nu_s ds}\right].$$

The volatility process ν_t is not an affine process but it is still analytically tractable. The Laplace tranform of $\int_0^t \nu_s ds$ is known, see e.g. Lewis [10].

$$(3.5) \quad E_{\nu_0}\left[e^{-\lambda \int_0^t \nu_s ds}\right] = \frac{\Gamma(b-a)}{\Gamma(b)} \left(\frac{2\kappa\gamma}{\delta^2 \nu_0 (e^{\kappa\gamma t}-1)}\right)^a M\left(a, b, -\frac{2\kappa\gamma}{\delta^2 \nu_0 (e^{\kappa\gamma t}-1)}\right),$$

where

$$(3.6) \quad \begin{cases} a := -\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right)^2 + \frac{2\lambda}{\delta^2}}, \\ b := 1 + 2\sqrt{\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right)^2 + \frac{2\lambda}{\delta^2}}, \end{cases}$$

and $\Gamma(\cdot)$ is the standard Gamma function and $M(a,b,z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$ is the confluent hypergeometric function, also known as Kummer's function (see e.g. Abramowitz and Stegun [2]), where $(c)_0 := 1$ and $(c)_n := c(c+1)\cdots(c+n-1)$ for $n \geq 1$. In our case,

$$(3.7) \quad \lambda := \frac{1}{2}\alpha^2(\theta - \theta^2) > 0,$$

since $\theta \in (0,1)$. We are interested in the asymptotic behavior of the Laplace transform as $t \to \infty$. As $t \to \infty$, $-\frac{2\kappa\gamma}{\delta^2 \nu_0 (e^{\kappa\gamma t}-1)} \to 0$ and $M(a,b,-\frac{2\kappa\gamma}{\delta^2 \nu_0 (e^{\kappa\gamma t}-1)}) \to 1$. Then, it is easy to see that

$$(3.8)$$

$$\lim_{t\to\infty} \frac{1}{t} \log E\left[e^{-\frac{1}{2}\alpha^2(\theta-\theta^2)\int_0^t \nu_s ds}\right] = -a\kappa\gamma$$

$$= \kappa\gamma\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) - \kappa\gamma\sqrt{\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) + \frac{\alpha^2(\theta - \theta^2)}{\delta^2}}.$$

Hence, we conclude that

$$(3.9) \quad \Lambda(\alpha) = \lim_{t\to\infty} \frac{1}{t} \log E[u(V_t)]$$

$$= \theta\alpha\mu + \theta(1-\alpha)r + \kappa\gamma\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) - \kappa\gamma\sqrt{\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) + \frac{\alpha^2(\theta - \theta^2)}{\delta^2}}.$$

It is straightforward to check that $\Lambda''(\alpha) < 0$ and it is easy to compute that

$$(3.10) \quad \Lambda'(\alpha) = \theta(\mu - r) - \kappa\gamma\frac{(\theta - \theta^2)}{\delta^2}\frac{\alpha}{\sqrt{\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) + \frac{\alpha^2(\theta-\theta^2)}{\delta^2}}}.$$

When $\mu - r \leq 0$, since $\Lambda(\alpha)$ is strictly concave, $\Lambda(\alpha)$ is decreasing for $\alpha \geq 0$ and thus the optimal α^* is achieved at $\alpha^* = 0$. Now, assume that $\mu - r > 0$. When $\theta(\mu - r) - \frac{\kappa\gamma}{\delta}\sqrt{\theta - \theta^2} \geq 0$, $\Lambda'(\alpha) \geq 0$ for any α and the optimal α^* is achieved at $\alpha^* = 1$. When $\theta(\mu - r) - \frac{\kappa\gamma}{\delta}\sqrt{\theta - \theta^2} < 0$, there exists a unique global maximum of $\Lambda(\alpha)$ achieved at $\alpha^\dagger \in (0, \infty)$ so that $\Lambda'(\alpha^\dagger) = 0$. Observe that if $\Lambda'(\alpha) = 0$ in (3.10), then α has the same sign as $\mu - r > 0$. After some algebraic manipulations, we get

$$(3.11) \quad \alpha^\dagger = \frac{\theta(\mu - r)\delta^2}{\sqrt{\kappa^2\gamma^2(\theta - \theta^2) - \theta^2(\mu - r)^2\delta^2}\sqrt{\theta - \theta^2}}\sqrt{\frac{1}{2} + \frac{\kappa}{\delta^2}}.$$

Thus, $\alpha^* = \alpha^\dagger$ if $\alpha^\dagger < 1$ and $\alpha^* = 1$ otherwise.

We summarise our results in the following theorem.

Theorem 3.

$$(3.12) \quad \Lambda(\alpha) = \lim_{t \to \infty} \frac{1}{t} \log E[u(V_t)]$$

$$= \theta\alpha\mu + \theta(1-\alpha)r + \kappa\gamma\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) - \kappa\gamma\sqrt{\left(\frac{1}{2} + \frac{\kappa}{\delta^2}\right) + \frac{\alpha^2(\theta - \theta^2)}{\delta^2}}.$$

The optimal α^* is given by $\alpha^* = 0$ if $\mu - r \leq 0$ and $\alpha^* = 1$ if $\theta(\mu-r) - \frac{\kappa\gamma}{\delta}\sqrt{\theta - \theta^2} \geq 0$ and if $\mu > r$ and $\theta(\mu - r) - \frac{\kappa\gamma}{\delta}\sqrt{\theta - \theta^2} < 0$, the optimal α^* is given by

$$(3.13) \qquad\qquad \alpha^* = \begin{cases} \alpha^\dagger & \text{if } \alpha^\dagger < 1, \\ 1 & \text{otherwise,} \end{cases}$$

where

$$(3.14) \qquad \alpha^\dagger = \frac{\theta(\mu - r)\delta^2}{\sqrt{\kappa^2\gamma^2(\theta - \theta^2) - \theta^2(\mu - r)^2\delta^2}\sqrt{\theta - \theta^2}}\sqrt{\frac{1}{2} + \frac{\kappa}{\delta^2}}.$$

Remark 4. *For simplicity, we only considered the case when $\rho = 0$. Indeed, for general $-1 \leq \rho \leq 1$, the joint Fourier-Laplace transform of the logarithm of the spot price, i.e. $\log(S_t/S_0)$ and the total integrated variance, i.e. $\int_0^t \nu_s ds$ is also known in the close-form, see e.g. Carr and Sun [3] and our methods can still be applied to obtain the optimal strategy α^*. But the computations would be more involved.*

4. JUMP DIFFUSION MODEL

Let us assume that the stock price follows a jump diffusion model. More precisely,

$$(4.1) \qquad\qquad dS_t = \mu S_{t-} dt + \sigma S_{t-} dB_t + S_{t-} dJ_t,$$

where $J_t = \sum_{i=1}^{N_t}(Y_i - 1)$, where Y_i are i.i.d. random variables distributed on $(0, \infty)$ with a smooth and bounded probability density function and Y_i are independent of N_t which is a standard Poisson process with intensity $\lambda > 0$. We further assume that $E[Y_1] < \infty$.

The wealth process satisfies

$$(4.2) \qquad dV_t = \alpha\mu V_{t-} dt + \alpha\sigma V_{t-} dB_t + \alpha V_{t-} dJ_t + (1 - \alpha)rV_{t-} dt.$$

Then, we have

$$(4.3) \qquad V_t = V_0 e^{\alpha\mu t + (1-\alpha)rt + \alpha\sigma B_t - \frac{1}{2}\alpha^2\sigma^2 t + \sum_{i=1}^{N_t} \log(\alpha(Y_i - 1) + 1)}.$$

Therefore,

$$(4.4) \quad \Lambda(\alpha) = \theta[\alpha\mu + (1 - \alpha)r] + \frac{1}{2}(\theta^2 - \theta)\alpha^2\sigma^2 + \lambda(E[(\alpha(Y_1 - 1) + 1)^\theta] - 1).$$

Remark 5. *(i) If $Y_1 \equiv y$ is a positive constant, then*

$$(4.5) \qquad \Lambda(\alpha) = \theta[\alpha\mu + (1 - \alpha)r] + \frac{1}{2}(\theta^2 - \theta)\alpha^2\sigma^2 + \lambda((\alpha(y - 1) + 1)^\theta - 1).$$

(ii) If Y_1 is exponentially distributed with parameter $\rho > 0$, then

$$(4.6) \qquad E[(\alpha(Y_1 - 1) + 1)^\theta] = \int_0^\infty (\alpha y + 1 - \alpha)^\theta e^{-\rho y}\rho dy$$

$$= e^{\rho(\frac{1}{\alpha} - 1)}(\alpha/\rho)^\theta \Gamma\left(\theta + 1, \rho\left(\frac{1}{\alpha} - 1\right)\right),$$

where $\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt$ is an upper incomplete Gamma function.

It is easy to compute that

$$(4.7) \qquad \Lambda'(\alpha) = \theta(\mu - r) + (\theta^2 - \theta)\sigma^2 \alpha + \lambda\theta E\left[(\alpha(Y_1 - 1) + 1)^{\theta-1}(Y_1 - 1)\right],$$

and

$$(4.8) \qquad \Lambda''(\alpha) = (\theta^2 - \theta)\sigma^2 + \lambda\theta(\theta - 1)E\left[(\alpha(Y_1 - 1) + 1)^{\theta-2}(Y_1 - 1)^2\right] < 0,$$

since $\theta \in (0, 1)$ and $\alpha \in [0, 1]$.

In the expression of $\Lambda(\alpha)$,

$$(4.9) \qquad \begin{aligned} |E[(\alpha(Y_1 - 1) + 1)^\theta]| &\leq E[(\alpha|Y_1 - 1| + 1)^\theta] \\ &\leq E[(\alpha(Y_1 + 1) + 1)^\theta] \\ &\leq E[(\alpha(Y_1 + 1) + 1)] \\ &= \alpha(E[Y_1] + 1) + 1, \end{aligned}$$

since $\theta \in (0, 1)$ and $\alpha(Y_1 + 1) + 1 \geq 1$ a.s. The coefficient of α^2 term in $\Lambda(\alpha)$ is $\frac{1}{2}(\theta^2 - \theta)\sigma^2$ which is negative. Thus, $\Lambda(\alpha) \to -\infty$ as $\alpha \to \infty$. Recall that $\Lambda(\alpha)$ is strictly concave. Therefore, if $\Lambda'(0) = \theta(\mu - r) + \lambda\theta(E[Y_1] - 1) \leq 0$, then, the optimal α^* is achieved at $\alpha^* = 0$. If $\Lambda'(0) = \theta(\mu - r) + \lambda\theta(E[Y_1] - 1) > 0$, then, there exists a unique $\alpha^\dagger \in (0, \infty)$ so that $\Lambda'(\alpha^\dagger) = 0$. In this case, the optimal α^* is given by

$$(4.10) \qquad \alpha^* = \begin{cases} 1 & \text{if } \alpha^\dagger \geq 1, \\ \alpha^\dagger & \text{if } \alpha^\dagger \in (0, 1). \end{cases}$$

We summarize our conclusions in the following theorem.

Theorem 6.

$$(4.11) \quad \Lambda(\alpha) = \theta[\alpha\mu + (1 - \alpha)r] + \frac{1}{2}(\theta^2 - \theta)\alpha^2\sigma^2 + \lambda(E[(\alpha(Y_1 - 1) + 1)^\theta] - 1).$$

When $\theta(\mu - r) + \lambda\theta(E[Y_1] - 1) \leq 0$, $\alpha^* = 0$. Otherwise,

$$(4.12) \qquad \alpha^* = \begin{cases} 1 & \text{if } \alpha^\dagger \geq 1, \\ \alpha^\dagger & \text{if } \alpha^\dagger \in (0, 1), \end{cases}$$

where $\alpha^\dagger$ is the unique positive solution to

$$(4.13) \qquad \theta(\mu - r) + (\theta^2 - \theta)\sigma^2 \alpha + \lambda\theta E\left[(\alpha(Y_1 - 1) + 1)^{\theta-1}(Y_1 - 1)\right] = 0.$$

5. BLACK-SCHOLES MODEL WITH VASICEK INTEREST RATE

Let us assume that the stock price follows a Black-Scholes model with constant drift μ and volatility σ and the interest rate r_t follows a Vasicek model. The Vasicek model is a standard interest rate model, introduced by Vasicek [14]. The wealth process satisfies the following stochastic differential equation.

$$(5.1) \qquad dV_t = \alpha\mu V_t dt + \alpha\sigma V_t dB_t + (1 - \alpha)r_t V_t dt,$$

where B_t is a standard Brownian motion starting at 0 at time 0 and

$$(5.2) \qquad dr_t = \kappa(\gamma - r_t)dt + \delta dW_t,$$

where W_t is a standard Brownian motion so that $\langle W, B \rangle_t = \rho t$, where $-1 \leq \rho \leq 1$ is the correlation.

Therefore, the wealth process is given by

$$(5.3) \qquad V_t = V_0 e^{\alpha \mu t + \alpha \sigma B_t - \frac{1}{2}\alpha^2 \sigma^2 t + (1-\alpha)\int_0^t r_s ds}.$$

Lemma 7. *For any $r_0 = r > 0$,*

$$(5.4) \qquad \lim_{t \to \infty} \frac{1}{t} \log E_{r_0 = r}\left[e^{\theta(\alpha\sigma B_t + (1-\alpha)\int_0^t r_s ds)} \right]$$

$$= \kappa\gamma \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right) + \frac{\delta^2}{2}\left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right)^2$$

$$- \frac{\theta\alpha\sigma\kappa\gamma\rho}{\delta} + \frac{1}{2}\theta^2\alpha^2\sigma^2(1-\rho^2).$$

Proof. Write $B_t = \rho W_t + \sqrt{1-\rho^2}Z_t$, where Z_t is a standard Brownian motion independent of B_t and W_t. Therefore, we have

$$(5.5)$$

$$E_{r_0=r}\left[e^{\theta(\alpha\sigma B_t + (1-\alpha)\int_0^t r_s ds)} \right]$$

$$= E_{r_0=r}\left[e^{\theta(\alpha\sigma\rho W_t + \alpha\sigma\sqrt{1-\rho^2}Z_t + (1-\alpha)\int_0^t r_s ds)} \right]$$

$$= E_{r_0=r}\left[e^{\frac{1}{2}\theta^2\alpha^2\sigma^2(1-\rho^2)t + \theta(\alpha\sigma\rho W_t + (1-\alpha)\int_0^t r_s ds)} \right]$$

$$= E_{r_0=r}\left[e^{\theta(1-\alpha)\int_0^t r_s ds + \frac{\theta\alpha\sigma\rho}{\delta}r_t + \frac{\theta\alpha\sigma\kappa\rho}{\delta}\int_0^t r_s ds} \right] e^{-\frac{\theta\alpha\sigma\rho r_0}{\delta} - \frac{\theta\alpha\sigma\kappa\gamma\rho}{\delta}t + \frac{1}{2}\theta^2\alpha^2\sigma^2(1-\rho^2)t},$$

where the last line uses the fact that $W_t = \frac{r_t - r_0}{\delta} - \frac{\kappa\gamma}{\delta}t + \frac{\kappa}{\delta}\int_0^t r_s ds$. $\qquad \square$

Let $u(t,r) := E_{r_0=r}\left[e^{\theta(1-\alpha)\int_0^t r_s ds + \frac{\theta\alpha\sigma\rho}{\delta}r_t + \frac{\theta\alpha\sigma\kappa\rho}{\delta}\int_0^t r_s ds} \right]$. Then, $u(t,r)$ satisfies the following partial differential equation,

$$(5.6) \qquad \begin{cases} \frac{\partial u}{\partial t} = \kappa(\gamma - r)\frac{\partial u}{\partial r} + \frac{1}{2}\delta^2\frac{\partial u^2}{\partial r^2} + \left(\theta(1-\alpha) + \frac{\theta\alpha\sigma\kappa\rho}{\delta}\right)ru = 0, \\ u(0,r) = e^{\frac{\theta\alpha\sigma\rho}{\delta}r}. \end{cases}$$

Let us try $u(t,r) = e^{A(t) + B(t)r}$. Then, we get

$$(5.7) \qquad \begin{cases} A'(t) = \kappa\gamma B(t) + \frac{1}{2}\delta^2 B(t)^2, \\ B'(t) = -\kappa B(t) + \theta(1-\alpha) + \frac{\theta\alpha\sigma\kappa\rho}{\delta}, \\ A(0) = 0, \quad B(0) = \frac{\theta\alpha\sigma\rho}{\delta}. \end{cases}$$

It is not hard to see that $B(t) \to \frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta}$ as $t \to \infty$ and therefore

$$(5.8) \qquad \frac{A(t)}{t} = \frac{\kappa\gamma}{t}\int_0^t B_s ds + \frac{\delta^2}{2t}\int_0^t B(s)^2 ds$$

$$\to \kappa\gamma\left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta}\right) + \frac{\delta^2}{2}\left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta}\right)^2,$$

as $t \to \infty$. Hence, we conclude that

$$(5.9) \qquad \lim_{t \to \infty} \frac{1}{t} \log E_{r_0=r} \left[e^{\theta(\alpha\sigma B_t + (1-\alpha)\int_0^t r_s ds)} \right]$$

$$= \kappa\gamma \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right) + \frac{\delta^2}{2} \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right)^2$$

$$- \frac{\theta\alpha\sigma\kappa\gamma\rho}{\delta} + \frac{1}{2}\theta^2\alpha^2\sigma^2(1-\rho^2).$$

Theorem 8.

$$(5.10) \qquad \Lambda(\alpha) = \lim_{t \to \infty} \frac{1}{t} \log E[u(V_t)]$$

$$= \kappa\gamma \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right) + \frac{\delta^2}{2} \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right)^2$$

$$- \frac{\theta\alpha\sigma\kappa\gamma\rho}{\delta} + \frac{1}{2}\theta^2\alpha^2\sigma^2(1-\rho^2) + \theta\alpha\mu - \frac{1}{2}\theta\alpha^2\sigma^2.$$

When $\frac{\delta^2\theta}{2\kappa^2} - \frac{\delta\theta\sigma\rho}{\kappa} + \frac{\sigma^2\theta}{2} - \frac{\sigma^2}{2} \geq 0$, the optimal α^* is given by

$$(5.11) \qquad \alpha^* = \begin{cases} 0 & \text{if } \gamma + \frac{\delta^2\theta}{2\kappa^2} \geq \frac{1}{2}(\theta-1)\sigma^2 + \mu, \\ 1 & \text{if } \gamma + \frac{\delta^2\theta}{2\kappa^2} < \frac{1}{2}(\theta-1)\sigma^2 + \mu. \end{cases}$$

Otherwise, the optimal α^* is given by

$$(5.12) \qquad \alpha^* = \begin{cases} 1 & \text{if } \alpha^\dagger \geq 1, \\ \alpha^\dagger & \text{if } \alpha^\dagger \in (0,1), \\ 0 & \text{if } \alpha^\dagger \leq 0, \end{cases}$$

where

$$(5.13) \qquad \alpha^\dagger = \frac{-\gamma\theta + \theta\mu + \frac{\delta\theta^2\sigma\rho}{\kappa} - \frac{\delta^2\theta^2}{\kappa^2}}{2\theta \left[-\frac{\delta^2\theta}{2\kappa^2} + \frac{\delta\theta\sigma\rho}{\kappa} - \frac{\sigma^2\theta}{2} + \frac{\sigma^2}{2} \right]}.$$

Proof. Recall that $V_t = V_0 e^{\alpha\mu t + \alpha\sigma B_t - \frac{1}{2}\alpha^2\sigma^2 t + (1-\alpha)\int_0^t r_s ds}$. Applying Lemma 7, we have

$$(5.14) \qquad \Lambda(\alpha) = \lim_{t \to \infty} \frac{1}{t} \log E[u(V_t)]$$

$$= \kappa\gamma \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right) + \frac{\delta^2}{2} \left(\frac{\theta(1-\alpha)}{\kappa} + \frac{\theta\alpha\sigma\rho}{\delta} \right)^2$$

$$- \frac{\theta\alpha\sigma\kappa\gamma\rho}{\delta} + \frac{1}{2}\theta^2\alpha^2\sigma^2(1-\rho^2) + \theta\alpha\mu - \frac{1}{2}\theta\alpha^2\sigma^2.$$

Thus, $\Lambda(\alpha)$ is a quadratic function of α. If the coefficient of α^2 in $\Lambda(\alpha)$ is non-negative, i.e.

$$(5.15) \qquad \theta \left[\frac{\delta^2\theta}{2\kappa^2} - \frac{\delta\theta\sigma\rho}{\kappa} + \frac{\sigma^2\theta}{2} - \frac{\sigma^2}{2} \right] \geq 0,$$

then $\Lambda(\alpha)$ is convex in α and the optimal α is achieved at either $\alpha = 0$ or $\alpha = 1$. Indeed, one can compute that

$$(5.16) \qquad \Lambda(0) = \gamma\theta + \frac{\delta^2\theta^2}{2\kappa^2},$$

and

$$\Lambda(1) = \frac{1}{2}(\theta^2 - \theta)\sigma^2 + \theta\mu. \tag{5.17}$$

Therefore, when $\frac{\delta^2\theta}{2\kappa^2} - \frac{\delta\theta\sigma\rho}{\kappa} + \frac{\sigma^2\theta}{2} - \frac{\sigma^2}{2} \geq 0$,

$$\alpha^* = \begin{cases} 0 & \text{if } \gamma + \frac{\delta^2\theta}{2\kappa^2} \geq \frac{1}{2}(\theta - 1)\sigma^2 + \mu, \\ 1 & \text{if } \gamma + \frac{\delta^2\theta}{2\kappa^2} < \frac{1}{2}(\theta - 1)\sigma^2 + \mu. \end{cases} \tag{5.18}$$

If the coefficient of α^2 in $\Lambda(\alpha)$ is negative, the function $\Lambda(\alpha)$ has a unique maximum at some $\alpha^\dagger \in (0, \infty)$ and

$$\alpha^* = \begin{cases} 1 & \text{if } \alpha^\dagger \geq 1, \\ \alpha^\dagger & \text{if } \alpha^\dagger \in (0, 1), \\ 0 & \text{if } \alpha^\dagger \leq 0, \end{cases} \tag{5.19}$$

where

$$\alpha^\dagger = \frac{-\gamma\theta + \theta\mu + \frac{\delta\theta^2\sigma\rho}{\kappa} - \frac{\delta^2\theta^2}{\kappa^2}}{2\theta\left[-\frac{\delta^2\theta}{2\kappa^2} + \frac{\delta\theta\sigma\rho}{\kappa} - \frac{\sigma^2\theta}{2} + \frac{\sigma^2}{2}\right]}. \tag{5.20}$$

$\square$

6. Concluding Remarks

In this article, we studied the optimal long-term strategy for a static investor for the Heston model, the 3/2 model, the jump diffusion model and the Black-Scholes model with Vasicek interest rate. It will be interesting to generalize our results to the multivariate case, i.e. when the investor can invest in a basket of stocks $S_t^{(i)}$, $1 \leq i \leq d$, and the wealth process is given by

$$\frac{dV_t}{V_t} = \sum_{i=1}^{d} \alpha_i \frac{dS_t^{(i)}}{S_t^{(i)}} + \left(1 - \sum_{i=1}^{d} \alpha_i\right) r_t dt. \tag{6.1}$$

One can also study the Heston model, the 3/2 model, and the jump diffusion model with stochastic interest rate. In Section 5, the interest rate is assumed to follow the Vasicek model. A drawback of the Vasicek model is that the process can go negative with positive probability. Our analysis in Section 5 cannot be directly applied to the Cox-Ingersoll-Ross interest rate unless one assumes that $\rho = 0$. This can be left for the future investigations.

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Courant Institute of Mathematical Sciences
New York University
251 Mercer Street
New York, NY-10012
United States of America
E-mail address: ling@cims.nyu.edu