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REVERSE AND DUAL LOOMIS-WHITNEY-TYPE INEQUALITIES

STEFANO CAMPI, RICHARD J. GARDNER, AND PAOLO GRONCHI

Abstract. Various results are proved giving lower bounds for the mth intrinsic volume $V_m(K)$, $m = 1, \dots, n-1$, of a compact convex set K in $\mathbb{R}^n$, in terms of the mth intrinsic volumes of its projections on the coordinate hyperplanes (or its intersections with the coordinate hyperplanes). The bounds are sharp when $m = 1$ and $m = n-1$. These are reverse (or dual, respectively) forms of the Loomis-Whitney inequality and versions of it that apply to intrinsic volumes. For the intrinsic volume $V_1(K)$, which corresponds to mean width, the inequality obtained confirms a conjecture of Betke and McMullen made in 1983.

1. Introduction

The Loomis-Whitney inequality states that for a Borel set A in $\mathbb{R}^n$,

$$\mathcal{H}^n(A)^{n-1} \le \prod_{i=1}^{n} \mathcal{H}^{n-1}\left(A|e_i^\perp\right), \tag{1}$$

where $A|e_i^\perp$ denotes the orthogonal projection of A on the ith coordinate hyperplane $e_i^\perp$, and where equality holds when A is a coordinate box. (See Section 2 for unexplained notation and terminology.) First proved by Loomis and Whitney [19] in 1949, it is one of the fundamental inequalities in mathematics, included in many texts; see, for example, [5, Theorem 11.3.1], [10, p. 383], [11, Corollary 5.7.2], [15, Section 4.4.2], and [25, Lemma 12.1.4]. Since the present article is to some extent a sequel to [7], we refer to that paper for numerous references to geometrical, discrete, and analytical versions and generalizations of (1) and applications to Sobolev inequalities and embedding, stereology, geochemistry, data processing, and compressed sensing. In addition one may mention Balister and Bollobás [1] and Gyarmati, Matolcsi, and Ruzsa [14], where the Loomis-Whitney inequality finds use in combinatorics and the theory of sum sets, the former paper also citing Han [16], who proved an analogue of the Loomis-Whitney inequality for the entropy of a finite set of random variables; the observation of Bennett, Carbery, and Tao [4] that the multilinear Kakeya conjecture may be viewed as a generalization of the Loomis-Whitney inequality; and applications to group theory by Gromov [12], graph theory by Madras, Sumners, and Whittington [22], and data complexity by Ngo, Porat, Ré, and Rudra [24].

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The focus here is on reverse forms of the Loomis-Whitney inequality for compact convex sets—where lower bounds instead of upper bounds are obtained in terms of projections on coordinate subspaces—or on dual forms, where lower bounds are given in terms of intersections with coordinate subspaces. An example of the latter type is an inequality due to Meyer [23], which states that if K is a convex body in $\mathbb{R}^n$, then

$$V(K)^{n-1} \geq \frac{(n-1)!}{n^{n-1}} \prod_{i=1}^{n} V_{n-1}(K \cap e_i^{\perp}), \tag{2}$$

with equality if and only if K is a coordinate cross-polytope. Here V denotes volume, so that whereas (1) provides, in particular, an upper bound for the volume of a convex body in terms of volumes of its projections on coordinate hyperplanes, Meyer's inequality gives a lower bound in terms of the volumes of its sections by coordinate hyperplanes.

In [7], variants of the Loomis-Whitney inequality were found in which an upper bound for $V(K)$ is replaced by an upper bound for the intrinsic volume $V_m(K)$, for some $m \in \{1, \ldots, n-1\}$. Our interest is in doing the same for lower bounds. Intrinsic volumes include as special cases surface area and mean width, corresponding to the cases $m = n-1$ and $m = 1$, respectively, up to constant factors depending only on the dimension n.

Upper and lower bounds of this sort were first obtained in the pioneering study of Betke and McMullen [3]. Their motivation was somewhat different, but, as was noted in [7], it is a consequence of their results that

$$V_{n-1}(K) \leq \sum_{i=1}^{n} V_{n-1}\left(K|e_i^{\perp}\right), \tag{3}$$

with equality if and only if K is a (possibly lower-dimensional) coordinate box. (Here and for the remainder of the introduction, K is always a compact convex set in $\mathbb{R}^n$.) Similarly, we observe that it follows from [3, Theorem 2] that

$$V_{n-1}(K)^2 \geq \sum_{i=1}^{n} V_{n-1}(K|e_i^{\perp})^2, \tag{4}$$

with equality if and only if either $\dim K \leq n-1$ or $\dim K = n$ and K is a coordinate cross-polytope. Since each section is contained in the corresponding projection, the same inequality holds with $V_{n-1}(K|e_i^{\perp})$ replaced by $V_{n-1}(K \cap e_i^{\perp})$, though the equality condition is then slightly different. (See Theorem 3.5; note that (2) is false when sections are replaced by projections, since the left-hand side can then be zero when the right-hand side is positive.) For this reason we concentrate on lower bounds involving projections for the rest of the introduction.

Campi and Gronchi [7] conjectured a generalization of (3), namely,

$$V_m(K) \leq \frac{1}{n-m} \sum_{i=1}^{n} V_m\left(K|e_i^{\perp}\right), \tag{5}$$

where $m = 1, \ldots, n-1$, with equality if and only if K is a (possibly lower-dimensional) coordinate box. They proved (5) when $m = 1$ and when K is a zonoid, but for $m \in \{2, \ldots, n-$

2} and general K, it remains an open problem. (Though it is not mentioned in [7], inequality (5) for $m = 1$ confirms a conjecture of Betke and McMullen, namely, the case $r = 1$ and $s = d - 1$ of [3, Conjecture 3(a), p. 537].) This naturally suggests a corresponding generalization of (4):

$$V_m(K)^2 \geq \frac{1}{n-m} \sum_{i=1}^{n} V_m(K|e_i^\perp)^2. \tag{6}$$

In Theorem 5.1, we show that (6) holds when K is a zonoid. By a generalized Pythagorean theorem proved in Proposition 2.2, equality holds in (6) when K is contained in an m-dimensional plane. Exact equality conditions for (6) are complicated to interpret, but we provide a clear geometric description when $m = 1$.

It turns out that if $m < n-1$, (6) is false for general K, as we prove at the end of Section 5 for $n = 3$ and $m = 1$.

A lower bound for the mean width of K in terms of the mean widths of its projections on coordinate hyperplanes was also conjectured by Betke and McMullen in 1983 (the case $r = 1$ and $s = d - 1$ of [3, Conjecture 3(b), p. 537]). We show in Theorem 4.1 that their conjecture is true by proving the existence of a constant $c_0 = c_0(n)$, $n \geq 2$, such that

$$V_1(K) \geq c_0 \sum_{i=1}^{n} V_1(K|e_i^\perp),$$

with equality if and only if K is either a singleton or a regular coordinate cross-polytope.

Sharp reverse inequalities of the isoperimetric type are relatively rare and hard to prove. Examples can be found in [10, Remark 9.2.10(ii)] and in [20], [21], [26], and the references given in these papers.

Occasionally we take a more general viewpoint, considering estimates of the jth intrinsic volume $V_j(K)$ of K in terms of the mth intrinsic volumes of its projections on or intersections with coordinate hyperplanes. However, for the most part, the difficulty of finding sharp bounds forces us back to the case when $j = m$.

The paper is organized as follows. After the preliminary Section 2, lower bounds for $V_{n-1}(K)$ are presented in Section 3. The main argument and the case $n = 3$ of the Betke-McMullen conjecture is proved in Section 4; the long and somewhat technical case $n \geq 4$ is deferred to an appendix in order to maintain the flow of the paper. Results for zonoids are the topic of Section 5. In Section 6, we gather several supplementary results, some involving upper bounds as well as lower bounds. The final Section 7 lists some problems for future research.

2. Preliminaries

2.1. **General notation and basic facts.** As usual, S^{n-1} denotes the unit sphere and o the origin in Euclidean n-space $\mathbb{R}^n$. We assume throughout that $n \geq 2$. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$. If $x, y \in \mathbb{R}^n$, then $x \cdot y$ is the inner product of x and y and $[x, y]$ is the line segment with endpoints x and y. The unit ball in $\mathbb{R}^n$ is $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$. We write $e_1, \ldots, e_n$ for the standard orthonormal basis for $\mathbb{R}^n$. We will write $\operatorname{int} A$ and $\operatorname{conv} A$ for

the interior and convex hull, respectively, of a set $A \subset \mathbb{R}^n$. The *dimension* $\dim A$ of A is the dimension of the affine hull of A. The indicator function of A will be denoted by 1_A. The (orthogonal) projection of A on a plane H is denoted by $A|H$. If $u \in S^{n-1}$, then $u^\perp$ is the $(n-1)$-dimensional subspace orthogonal to u.

A set is *o-symmetric* if it is centrally symmetric, with center at the origin, and 1-*unconditional* if it is symmetric with respect to the coordinate hyperplanes.

We write $\mathcal{H}^k$ for k-dimensional Hausdorff measure in $\mathbb{R}^n$, where $k \in \{1, \dots, n\}$. The notation dz will always mean $d\mathcal{H}^k(z)$ for the appropriate $k = 1, \dots, n$.

We now collect some basic material concerning compact convex sets. Standard references are the books [10], [13], and [27].

Let K be a compact convex set in $\mathbb{R}^n$. Then $V(K)$ denotes its *volume*, that is, $\mathcal{H}^k(K)$, where $\dim K = k$. We write $\kappa_n = V(B^n) = \pi^{n/2}/\Gamma(n/2+1)$ for the volume of the unit ball B^n.

A *convex body* is a compact convex set with a nonempty interior.

A *coordinate box* is a (possibly degenerate) rectangular parallelepiped whose facets are parallel to the coordinate hyperplanes. A *cross-polytope* in $\mathbb{R}^n$ is the convex hull of k mutually orthogonal line segments with a point in common, for some $k \in \{1, \dots, n\}$; it is a *coordinate cross-polytope* if these line segments are parallel to the coordinate axes. The adjective *regular* for a polytope is used in the traditional sense, so that in particular, a regular cross-polytope in $\mathbb{R}^n$ has dimension n. We shall write $C^n = \operatorname{conv}\{\pm e_1, \dots, \pm e_n\}$ for the standard regular o-symmetric coordinate cross polytope in $\mathbb{R}^n$ and $Q^n = \prod_{i=1}^n [-e_i, e_i]$ for the o-symmetric coordinate cube in $\mathbb{R}^n$ with side length 2.

If $m \in \{1, \dots, n-1\}$, the *mth area measure* of K is denoted by $S_m(K, \cdot)$. When $m = n-1$, $S_{n-1}(K, \cdot) = S(K, \cdot)$ is the *surface area measure* of K. The quantity $S(K) = S(K, S^{n-1})$ is the *surface area* of K.

If K is a nonempty compact convex set in $\mathbb{R}^n$, then

$$h_K(x) = \sup\{x \cdot y : y \in K\},$$

for $x \in \mathbb{R}^n$, defines the *support function* h_K of K. Since it is positively homogeneous of degree 1, we shall sometimes regard h_K as a function on S^{n-1}.

We collect a few facts and formulas concerning mixed and intrinsic volumes. A *mixed volume* $V(K_{i_1}, \dots, K_{i_n})$ is a coefficient in the expansion of $V(t_1 K_1 + \cdots + t_n K_n)$ as a homogeneous polynomial of degree n in the parameters $t_1, \dots, t_n \geq 0$, where $K_1, \dots, K_n$ are compact convex sets in $\mathbb{R}^n$. The notation $V(K, i; L, n-i)$, for example, means that there are i copies of K and $n-i$ copies of L. The quantity

$$V_i(K) = \frac{1}{c_{n,i}} V\left(K, i; B^n, n-i\right), \tag{7}$$

where $i \in \{0, \dots, n\}$ and $c_{n,i} = \kappa_{n-i}/\binom{n}{i}$, is called an *intrinsic volume* since it is independent of the ambient space containing the compact convex set K. Then $V_n(K) = V(K)$, $V_{n-1}(K) =$

$S(K)/2$, and

$$V_1(K) = \frac{n\kappa_n}{2\kappa_{n-1}}(\text{mean width of } K). \tag{8}$$

See [10, Sections A.3 and A.6]. From (8), we obtain

$$V_1(K) = \frac{1}{\kappa_{n-1}} \int_{S^{n-1}} h_K(u)\, du. \tag{9}$$

For the reader's convenience, we state the following form of Minkowski's integral inequality [17, (6.13.8), p. 148]. If μ is a measure on a set X and f_i, $i = 1, \dots, k$, are μ-measurable functions on X, then for $p \geq 1$,

$$\int_X \left(\sum_{i=1}^k f_i(x)^p \right)^{1/p} d\mu(x) \geq \left(\sum_{i=1}^k \left(\int_X f_i(x)\, d\mu(x) \right)^p \right)^{1/p}, \tag{10}$$

with equality if and only if the functions f_i are *essentially proportional*. The latter term means that $f_i(x) = b_i g(x)$ for μ-almost all x, all $i = 1, \dots, k$, and some μ-measurable function g on X and constants b_i, $i = 1, \dots, k$.

2.2. **Projections and sections.** Let K be a compact convex set in $\mathbb{R}^n$ and let $m \in \{1, \dots, n-1\}$. Define $\mathcal{P}(K, m)$ to be the class of compact convex sets L in $\mathbb{R}^n$ such that

$$V_m(L|e_i^\perp) = V_m(K|e_i^\perp),$$

for $i = 1, \dots, n$. Similarly, we let $\mathcal{S}(K, m)$ be the class of compact convex sets L in $\mathbb{R}^n$ such that

$$V_m(L \cap e_i^\perp) = V_m(K \cap e_i^\perp),$$

for $i = 1, \dots, n$.

If $j, m \in \{1, \dots, n\}$ and K is any compact convex set in $\mathbb{R}^n$, the jth intrinsic volume of bodies in $\mathcal{S}(K, m)$ is unbounded, so there are no bodies in $\mathcal{S}(K, m)$ of maximal jth intrinsic volume. To see this, for each $i = 1, \dots, n$, let D_i be a (possibly degenerate) $(n-1)$-dimensional ball in the part of $e_i^\perp$ belonging to the positive orthant, such that $o \notin D_i$ and $V_m(D_i) = V_m(K \cap e_i^\perp)$. Then for any x in the interior of the positive orthant, we have $L_x = \operatorname{conv}\{D_1 \dots, D_n, x\} \in \mathcal{S}(K, m)$ and $V_j(L_x) \to \infty$ as $|x| \to \infty$, for each $j = 1, \dots, n$. This fact motivates us to focus on sets of minimal jth intrinsic volume in $\mathcal{S}(K, m)$.

Lemma 2.1. *If K is a compact convex set in $\mathbb{R}^n$ and $j, m \in \{1, \dots, n\}$, then there exists a minimizer of V_j in $\mathcal{S}(K, m)$. If $j \geq m + 2$, then the minimum of V_j in $\mathcal{S}(K, m)$ is zero.*

Proof. The second statement in the lemma and the case $j \geq m+2$ of the first statement follow from the existence of an $(m+1)$-dimensional set in $\mathcal{S}(K, m)$. To see this, choose an $(m+1)$-dimensional plane in $\mathbb{R}^n$ whose intersection H with the positive orthant has dimension $m+1$ and satisfies $V_m(H \cap e_i^\perp) > V_m(K \cap e_i^\perp)$, for $i = 1, \dots, n$. For each $i = 1, \dots, n$, let $L_i \subset H \cap e_i^\perp$ be such that $V_m(L_i) = V_m(K \cap e_i^\perp)$ and $L_i \cap e_k^\perp = \emptyset$ if $k \neq i$. Then $\operatorname{conv}\{L_1, \dots, L_n\}$ is an $(m+1)$-dimensional set in $\mathcal{S}(K, m)$.

Suppose that $j \le m+1$ and let $a = \min_{1\le i\le n} V_m(K \cap e_i^\perp)$. Let $c \ge 0$ and let

$$\mathcal{M} = \mathcal{M}(c) = \{L \in \mathcal{S}(K,m) : V_j(L) \le c\}.$$

By the inequality [27, (6.4.7), p. 334] between the intrinsic volumes V_j and V_k for $k \ge j$, there is a constant d such that $V_k(L) \le d$ for each $L \in \mathcal{M}$ and $j \le k \le n$. Let $w > (m+1)d/a$. We claim that if $L \in \mathcal{M}$, then $L \subset [-w,w]^n$. Indeed, if this is not true, then there is an $x = (x_1,\dots,x_n) \in L$ and $i_0 \in \{1,\dots,n\}$ such that $|x_{i_0}| > w$. If $J = \operatorname{conv}\{x, L \cap e_{i_0}^\perp\}$, then $J \subset L$. Since J is a cone, the formula [27, (4.5.35), p. 255] from translative integral geometry, with E_k replaced by $e_{i_0}^\perp$, yields

$$V_{m+1}(J) = V_m(L \cap e_{i_0}^\perp)|x_{i_0}|/(m+1).$$

From these facts, we obtain

$$V_{m+1}(L) \ge V_{m+1}(J) > V_m(L \cap e_{i_0}^\perp)w/(m+1) \ge aw/(m+1) > d,$$

contradicting the definition of d. This proves the claim. As a consequence, the class $\mathcal{M}$ is compact in the Hausdorff metric and the existence of the minimizer follows. □

Since intrinsic volumes are monotonic (see, for example, [10, (A.18), p. 399]) and $K \cap e_i^\perp \subset K$ for $i = 1,\dots,n$, we have the trivial lower bound

$$V_m(K) \ge \max_{1\le i\le n} V_m(K \cap e_i^\perp), \tag{11}$$

for $m = 1,\dots,n-1$. It is also true, but not trivial, that

$$V_m(K) \ge \max_{1\le i\le n} V_m(K|e_i^\perp), \tag{12}$$

for $m = 1,\dots,n-1$. This follows from the observation in [7, p. 556] (where it is stated for $u = e_i$), that

$$V_m(K) \ge V_m(K|u^\perp), \tag{13}$$

for all $u \in S^{n-1}$. Of course, (12) is stronger than (11) because the obvious containment $K \cap e_i^\perp \subset K|e_i^\perp$ implies that

$$V_m\left(K|e_i^\perp\right) \ge V_m\left(K \cap e_i^\perp\right),$$

for $i, m = 1,\dots,n-1$. The easy bounds (11) and (12) imply that for all $p > 0$,

$$V_m(K) \ge \left(\frac{1}{n}\sum_{i=1}^n V_m(K|e_i^\perp)^p\right)^{1/p} \ge \left(\frac{1}{n}\sum_{i=1}^n V_m(K \cap e_i^\perp)^p\right)^{1/p}. \tag{14}$$

The so-called Pythagorean inequalities state that for a compact convex set K in $\mathbb{R}^n$ and $m \in \{1,\dots,n-1\}$,

$$V_m(K|u^\perp)^2 \le \sum_{i=1}^n V_m(K|e_i^\perp)^2, \tag{15}$$

for all $u \in S^{n-1}$. These were proved by Firey [9] (see also [5, (3), p. 153] and [10, Theorem 9.3.8 and Note 9.6]).

We are not aware of an explicit statement and proof of the following result in the literature.

Proposition 2.2. *If $m \in \{1, \dots, n-1\}$ and A is a Borel set contained in an m-dimensional plane in $\mathbb{R}^n$, then*

$$\mathcal{H}^m(A)^2 = \frac{1}{n-m} \sum_{i=1}^{n} \mathcal{H}^m(A|e_i^\perp)^2.$$

Proof. It is a well-known consequence of the Cauchy-Binet theorem that the following generalized Pythagorean theorem holds (see, for example, [8]):

$$\mathcal{H}^m(A)^2 = \sum \left\{ \mathcal{H}^m(A|S)^2 : S \text{ is an } m\text{-dimensional coordinate subspace} \right\}. \tag{16}$$

Note that if $1 \le i_1 < i_2 < \cdots < i_{n-m} \le n$ and S is the m-dimensional subspace that is the orthogonal complement of the subspace spanned by $e_{i_1}, e_{i_2}, \dots, e_{i_{n-m}}$, then

$$A|S = \left(\cdots \left((A|e_{i_1}^\perp)|e_{i_2}^\perp \right) | \cdots \right) |e_{i_{n-m}}^\perp.$$

Here, the order of the successive projections of A on the $e_{i_k}^\perp$'s can be changed arbitrarily. Using this and (16) (twice, the second time with A replaced by $A|e_i^\perp$), we obtain

$$\begin{aligned}
\mathcal{H}^m(A)^2 &= \sum_{1 \le i_1 < i_2 < \cdots < i_{n-m} \le n} \mathcal{H}^m \left(\left(\cdots \left((A|e_{i_1}^\perp)|e_{i_2}^\perp \right) | \cdots \right) |e_{i_{n-m}}^\perp \right)^2 \\
&= \frac{1}{n-m} \sum_{i=1}^{n} \sum_{1 \le i_1 < i_2 < \cdots < i_{n-m-1} \le n,\ i \ne i_k} \mathcal{H}^m \left(\left(\cdots \left((A|e_i^\perp)|e_{i_1}^\perp \right) | \cdots \right) |e_{i_{n-m-1}}^\perp \right)^2 \\
&= \frac{1}{n-m} \sum_{i=1}^{n} \mathcal{H}^m(A|e_i^\perp)^2,
\end{aligned}$$

since the double sum in the second equation counts each summand in the first sum $n-m$ times. □

3. The case $m = n-1$

After the following lemma, we will apply Meyer's inequality (2) to deal with the problem of minimum volume in $\mathcal{S}(K, n-1)$.

Lemma 3.1. *If $s_1, \dots, s_n$ are positive real numbers, there is a unique n-dimensional o-symmetric coordinate cross-polytope C in $\mathbb{R}^n$ such that*

$$V_{n-1}(C \cap e_i^\perp) = s_i,$$

for $i = 1, 2, \dots, n$.

Proof. Let C be the o-symmetric coordinate cross-polytope defined by

$$C = \operatorname{conv} \{ [-t_i e_i, t_i e_i] : i = 1, \dots, n \},$$

where $t_i > 0$, $i = 1, \dots, n$. Then we require that

$$V_{n-1}(C \cap e_i^\perp) = \frac{2^{n-1}}{(n-1)!} \prod_{k \neq i} t_k = s_i,$$

for $i = 1, \dots, n$. It is easily checked that the unique solution of this system is given by

$$t_i = \frac{1}{2s_i} \left((n-1)! \prod_{k=1}^{n} s_k \right)^{1/(n-1)},$$

for $i = 1, \dots, n$. Since $t_i > 0$, $i = 1, \dots, n$, we have $\dim C = n$. □

Corollary 3.2. *If K is a convex body in $\mathbb{R}^n$ containing the origin in its interior, then there is a unique n-dimensional o-symmetric coordinate cross-polytope C in $\mathcal{S}(K, n-1)$ and C is the unique volume minimizer in $\mathcal{S}(K, n-1)$.*

Proof. Let $s_i = V_{n-1}(K \cap e_i^\perp)$, $i = 1, \dots, n$, and note that $s_i > 0$ for all i since $o \in \operatorname{int} K$. By Lemma 3.1, there is a unique n-dimensional o-symmetric coordinate cross-polytope C in $\mathcal{S}(K, n-1)$. Furthermore, (2) implies that C has minimal volume in the class $\mathcal{S}(K, n-1)$ and that C is the unique volume minimizer. □

The previous result is clearly false in general if $o \notin \operatorname{int} K$. For example, if K is a ball containing the origin and supported by the hyperplane $e_n^\perp$, then $V_{n-1}(K \cap e_i^\perp) > 0$ for $i = 1, \dots, n-1$ and $V_{n-1}(K \cap e_n^\perp) = 0$, so no o-symmetric coordinate cross-polytope exists in $\mathcal{S}(K, n-1)$.

The following result was proved by Betke and McMullen [3, Theorem 2]. Their motivation and notation were different to ours. To obtain the proposition as we state it, in [3, Theorem 2] take $d = n$, $\alpha_i = a_i$, and $u_i = e_i$, $i = 1, \dots, n$, and note that the zonotope $Z(\mathcal{L})$ is then the coordinate box $\sum_{i=1}^n a_i[-e_i, e_i]$.

Proposition 3.3. *Let K be a compact convex set in $\mathbb{R}^n$ and let $a_1, \dots, a_n$ be positive real numbers. Then*

$$\min\{a_i : i = 1, \dots, n\} \leq \frac{1}{V_{n-1}(K)} \sum_{i=1}^{n} a_i V_{n-1}(K|e_i^\perp) \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2}.$$

Equality holds on the left (or on the right) if and only if the support of $S(K, \cdot)$ is contained in the set of directions of the contact points of the coordinate box $\sum_{i=1}^n a_i[-e_i, e_i]$ with its inscribed (or circumscribed, respectively) ball.

Corollary 3.4. *If K is a compact convex set in $\mathbb{R}^n$, then*

$$V_{n-1}(K) \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_{n-1}(K|e_i^\perp) \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_{n-1}(K \cap e_i^\perp). \tag{17}$$

Equality holds in the left-hand inequality if and only if either $\dim K < n-1$, or $\dim K = n-1$ and K is orthogonal to a diagonal of a coordinate cube, or $\dim K = n$ and K is a regular

coordinate cross-polytope. Equality holds in the right-hand inequality involving $V_{n-1}(K)$ *if and only if either* $\dim K < n-1$ *or* $\dim K = n$ *and* K *is an o-symmetric regular coordinate cross-polytope.*

Proof. The right-hand inequality in Proposition 3.3, with $a_i = 1$ for $i = 1, \dots, n$, immediately gives the left-hand inequality in (17). If $\dim K < n-1$, both sides of the inequality are zero, and if $\dim K \geq n-1$, the equality condition follows easily from that of Proposition 3.3.

Suppose that equality holds in the right-hand inequality involving $V_{n-1}(K)$. Then the equality condition for the left-hand inequality applies. If $\dim K = n-1$ and K is orthogonal to a diagonal of a coordinate cube, then $V_{n-1}(K \cap e_i^\perp) = 0$ for $i = 1, \dots, n$, so this possibility is eliminated. Suppose that $\dim K = n$ and K is a regular coordinate cross-polytope. Since equality holds throughout (17) and $V_{n-1}(K|e_i^\perp) \geq V_{n-1}(K \cap e_i^\perp)$ for $i = 1, \dots, n$, we have $V_{n-1}(K|e_i^\perp) = V_{n-1}(K \cap e_i^\perp)$ for $i = 1, \dots, n$, and it follows easily that K is o-symmetric. □

The lower bounds in (17) are not always better than the corresponding lower bounds in (11) and (12) for $m = n-1$. In fact, the following considerably stronger result can also be obtained from Proposition 3.3.

Theorem 3.5. *If* K *is a compact convex set in* $\mathbb{R}^n$*, then*

$$V_{n-1}(K)^2 \geq \sum_{i=1}^{n} V_{n-1}(K|e_i^\perp)^2 \geq \sum_{i=1}^{n} V_{n-1}(K \cap e_i^\perp)^2. \tag{18}$$

Equality holds in the left-hand inequality if and only if either $\dim K \leq n-1$ *or* $\dim K = n$ *and* K *is a coordinate cross-polytope. Equality holds in the right-hand inequality involving* $V_{n-1}(K)$ *if and only if either* $\dim K < n-1$*, or* $\dim K = n-1$ *and* K *is contained in a coordinate hyperplane, or* $\dim K = n$ *and* K *is an o-symmetric coordinate cross-polytope.*

Proof. The right-hand inequality in Proposition 3.3, with $a_i = V(K|e_i^\perp)$ for $i = 1, \dots, n$, immediately gives the left-hand inequality in (18). When $\dim K < n-1$, both sides are zero, and when $\dim K = n-1$, equality holds by Proposition 2.2 with $m = n-1$. Otherwise, if $\dim K = n$, the equality condition follows easily from that of Proposition 3.3. The right-hand inequality involving $V_{n-1}(K)$ and its equality condition is then straightforward. □

It is easy to check, by partial differentiation with respect to a_i, $i = 1, \dots, n$, that the choice of the a_i's in the previous proof is optimal. In particular, we have

$$\sum_{i=1}^{n} V_{n-1}(K|e_i^\perp)^2 \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} V_{n-1}(K|e_i^\perp),$$

an inequality that follows from the fact that the 2-mean of a finite set of nonnegative numbers is greater than or equal to the 1-mean (average). Moreover, the bound involving projections in (18) is always at least as good as the easy lower bound (12), since the maximum of a finite set of nonnegative numbers is less than or equal to their pth sum for any $p > 0$ and in particular when $p = 2$; see, for example, [10, (B.6), p. 414].

The Pythagorean inequality (15) for $m = n-1$ and (18) split the inequality (13), since together they imply that

$$V_{n-1}(K|u^\perp)^2 \le \sum_{i=1}^n V_{n-1}(K|e_i^\perp)^2 \le V_{n-1}(K)^2,$$

for all $u \in S^{n-1}$.

The following result follows directly from Lemma 3.1 and Theorem 3.5 (compare the proof of Corollary 3.2).

Corollary 3.6. *If K is a convex body in $\mathbb{R}^n$ containing the origin in its interior, then the unique surface area minimizer in $\mathcal{S}(K, n-1)$ is the unique n-dimensional o-symmetric coordinate cross-polytope in $\mathcal{S}(K, n-1)$.*

Now we consider the problem of finding lower bounds for the jth intrinsic volume of sets in $\mathcal{S}(K, n-1)$ when $j < n-1$. In view of Corollaries 3.2 and 3.6, it would be reasonable to conjecture that for any convex body K in $\mathbb{R}^n$ containing the origin in its interior and any $j \in \{1, \dots, n-1\}$, the minimizer of the jth intrinsic volume in $\mathcal{S}(K, n-1)$ is the unique n-dimensional o-symmetric coordinate cross-polytope in $\mathcal{S}(K, n-1)$. However, it turns out that this is not true in general. Indeed, when $n = 3$ and $j = 1$, a counterexample is given by

$$K_1 = \operatorname{conv}\{B^3 \cap e_1^\perp, B^3 \cap e_2^\perp, B^3 \cap e_3^\perp\}.$$

To see this, note that by (9), we have

$$V_1(K_1) = \frac{1}{\pi} \int_{S^2} h_{K_1}(u)\, du.$$

The computation of the latter integral is somewhat tedious, so we just provide a sketch. We consider the part of S^2 lying in the positive octant for which $h_K(u)$ equals the support function of the unit disk in the yz-plane. Using spherical polar coordinate angles (θ, φ), $0 \le \theta \le 2\pi$, $0 \le \varphi \le \pi$, we find that

$$h_{K_1}(\theta, \varphi) = \left(\sin^2\theta \sin^2\varphi + \cos^2\varphi\right)^{1/2},$$

for $\pi/4 \le \theta \le \pi/2$ and $0 \le \varphi \le \arctan(\csc\theta)$. By symmetry, we then have

$$\int_{S^2} h_{K_1}(u)\, du = 48 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\arctan(\csc\theta)} \left(\sin^2\theta \sin^2\varphi + \cos^2\varphi\right)^{1/2} \sin\varphi \, d\varphi \, d\theta.$$

Using standard substitutions, the inner integral evaluates to

$$\frac{1}{2} - \frac{\sin^2\theta}{\sqrt{2}(1+\sin^2\theta)} - \frac{\sin^2\theta}{2\cos\theta} \log\left(\frac{(\sqrt{2}+\cos\theta)\sin\theta}{(\cos\theta+1)\sqrt{1+\sin^2\theta}}\right).$$

Numerical integration then yields $V_1(K_1) = 3.8663...$.

Now recall the definition of the standard o-symmetric regular coordinate cross-polytope C^n in $\mathbb{R}^n$ and let $K_2 = \sqrt{\pi/2}C^3$. Since $C^3|e_i^\perp$ is a square of side length $\sqrt{2}$, we have

$$V_2(K_2 \cap e_i^\perp) = \pi = V_2(K_1 \cap e_i^\perp),$$

for $i = 1, 2, 3$. Using the formula [10, (A.36), p. 405] for the ith intrinsic volume of a convex polytope, with $n = 3$, $i = 1$, and $P = C^3$, we see that

$$V_1(C^3) = \sum_{F \in \mathcal{F}_1(C^3)} \gamma(F, C^3) V_1(F),$$

where $\mathcal{F}_1(C^3)$ is the set of edges of C^3 and $\gamma(F, C^3)$ is the normalized exterior angle of C^3 at the edge F. For each of the 12 edges of C^3 we have $V_1(F) = \sqrt{2}$ and it is easy to calculate that $\gamma(F, C^3) = \arccos(1/3)/2\pi$. Therefore

$$V_1(C^3) = 12\sqrt{2}\frac{\arccos(1/3)}{2\pi}, \tag{19}$$

so

$$V_1(K_2) = \sqrt{\pi/2} V_1(C^3) = \frac{6 \arccos(1/3)}{\sqrt{\pi}} = 4.1669... > V_1(K_1).$$

The minimizer of the jth intrinsic volume in $\mathcal{S}(K, n-1)$ must clearly be equal to the convex hull of its intersections with the coordinate hyperplanes, but it is not obvious why it should be 1-unconditional. One can attempt to prove this by considering the 1-unconditional convex body K^s that results from performing successive Steiner symmetrizations on K with respect to each of the coordinate hyperplanes (i.e., n symmetrizations, in any order). Then $V_j(K^s) \leq V_j(K)$, since Shephard [28] proved that jth intrinsic volumes do not increase under Steiner symmetrization. (Shephard does not state this fact explicitly, but it follows from [28, (6), p. 232], with $K_1 = \cdots = K_j = K$ and $K_{j+1} = \cdots = K_n = B^n$, and [28, (17), p. 264].) Also, $V_{n-1}(K \cap e_i^\perp)$ does not change under Steiner symmetrization with respect to $e_k^\perp$ if $k \neq i$, but if $k = i$ the intersection with $e_k^\perp$ may increase.

4. The case $m = 1$: The Betke-McMullen conjecture

The following result confirms a conjecture of Betke and McMullen (the case $r = 1$ and $s = d - 1$ of [3, Conjecture 3(b), p. 537]).

Theorem 4.1. *Let $n \geq 3$. For each compact convex set K in $\mathbb{R}^n$ with $\dim K \geq 1$, define*

$$F(K) = \frac{V_1(K)}{\sum_{i=1}^n V_1(K|e_i^\perp)}. \tag{20}$$

Then $F(K)$ is minimal if and only if K is a regular coordinate cross-polytope.

Note that the previous theorem also holds when $n = 2$, by Corollary 3.4, but then $F(K)$ is also minimal when K is a line segment orthogonal to a diagonal of a coordinate square. This explains why we prefer to state Theorem 4.1 for $n \geq 3$ and assume throughout the proof that this restriction holds.

We shall present the main argument in a series of lemmas. This will include the complete proof for $n = 3$. The case $n \geq 4$ of Lemma 4.6 required for the full result is contained in the appendix.

Lemma 4.2. *The functional F defined by* (20) *attains its minimum in the class of convex bodies K in $\mathbb{R}^n$ with the symmetries of the coordinate cube Q^n (or, equivalently, the regular coordinate cross-polytope C^n) satisfying $C^n \subset K \subset Q^n$ and such that K is the convex hull of its intersections with coordinate hyperplanes.*

Proof. The fact that F attains its minimum in the class of compact convex sets in $\mathbb{R}^n$ follows from a standard compactness argument.

Let G be the group of symmetries of $Q^n = \sum_{i=1}^n [-e_i, e_i]$ and let $|G|$ denote its cardinality. For every compact convex set K in $\mathbb{R}^n$ and $g \in G$, let gK be the image of K under g. Then the G-symmetral of K is the set

$$K^G = \frac{1}{|G|} \sum_{g \in G} gK. \tag{21}$$

Using the rigid motion covariance of Minkowski addition, we have, for every $h \in G$,

$$hK^G = h\left(\frac{1}{|G|} \sum_{g \in G} gK\right) = \frac{1}{|G|} \sum_{g \in G} h(gK) = \frac{1}{|G|} \sum_{g \in G} (hg)K = \frac{1}{|G|} \sum_{g' \in G} g'K = K^G.$$

It follows that K^G has the symmetries of Q^n.

We claim that $F(K^G) = F(K)$. To see this, note that by (7) with $i = 1$, we have

$$V_1(K) = \frac{n}{\kappa_{n-1}} V\left(K, 1; B^n, n-1\right). \tag{22}$$

Also, by [10, (A.43), p. 407] with $i = n-1$, $K_1 = K$, $K_2 = \cdots = K_{n-1} = B^n$, and $u_1 = e_i$, we have

$$V_1\left(K|e_i^\perp\right) = \frac{1}{n} V\left(K, 1; [0, e_i], 1; B^n, n-2\right). \tag{23}$$

Summing (23) over i, taking into account the multilinearity of mixed volumes (see, for example, [10, (A.16), p. 399]), and using the resulting equation and (22), we obtain

$$F(K) = c_1 \frac{V\left(K, 1; B^n, n-1\right)}{V\left(K, 1; Q^n, 1; B^n, n-2\right)}, \tag{24}$$

for some constant $c_1 = c_1(n)$. For every $g \in G$, we have $gB^n = B^n$, so by the invariance of a mixed volume under a rigid motion of its arguments (see, for example, [10, (A.17), p. 399]), we have

$$V\left(gK, 1; B^n, n-1\right) = V\left(K, 1; g^{-1}B^n, n-1\right) = V\left(K, 1; B^n, n-1\right). \tag{25}$$

Similarly, using $gQ^n = Q^n$, we get

$$V\left(gK, 1; Q^n, 1; B^n, n-2\right) = V\left(K, 1; Q^n, 1; B^n, n-2\right). \tag{26}$$

Substituting (25) and (26) into (24), we obtain $F(gK) = F(K)$. Finally, $F(K^G) = F(K)$ follows from this, the multilinearity of mixed volumes, and the definition (21) of K^G. This proves the statement in the lemma regarding symmetries.

Suppose that F attains its minimum at K. The function F is invariant under dilatations, so if K has the symmetries of Q^n, we can assume that it contains the points $\pm e_i$, $i = 1, \dots, n$, in its boundary. It is then clear from the symmetries of K that $C^n \subset K \subset Q^n$. Since K must also be 1-unconditional, we have $K|e_i^\perp = K \cap e_i^\perp$ for $i = 1, \dots, n$, and then clearly we may also assume that $K = \operatorname{conv}\{K \cap e_i^\perp : i = 1, \dots, n\}$. □

Lemma 4.3. *Let $C^n \subset K \subset Q^n$ be a convex body in $\mathbb{R}^n$ with the symmetries of Q^n and such that K is the convex hull of its intersections with the coordinate hyperplanes. Then*

(27)
$$F(K) = \frac{2\kappa_{n-2}}{\kappa_{n-1}} \frac{\int_0^{\frac{1}{\sqrt{n-1}}\sqrt{\frac{1-x_{n-1}^2}{n-2}}} \int_{x_{n-1}}^{\sqrt{\frac{1-x_3^2-\cdots-x_{n-1}^2}{2}}} \cdots \int_{x_3}^{} h_K(x_1, \dots, x_{n-1}, 0) p(x_{n-1})/x_1 \, dx_2 \cdots dx_{n-1}}{\int_0^{\frac{1}{\sqrt{n-1}}\sqrt{\frac{1-x_{n-1}^2}{n-2}}} \int_{x_{n-1}}^{\sqrt{\frac{1-x_3^2-\cdots-x_{n-1}^2}{2}}} \cdots \int_{x_3}^{} h_K(x_1, \dots, x_{n-1}, 0)/x_1 \, dx_2 \cdots dx_{n-1}},$$

where $x_1 = \sqrt{1 - x_2^2 - \cdots - x_{n-1}^2}$ and p is a nonnegative, increasing, continuous function.

Proof. Since the assumptions on K force it to be 1-unconditional, we have $K|e_i^\perp = K \cap e_i^\perp$, $i = 1, \dots, n$. Hence, K is the convex hull of its projections on the coordinate hyperplanes, from which we obtain

$$h_K(x) = \max_{1 \le i \le n} h_{K|e_i^\perp}(x) = \max_{1 \le i \le n} h_K(x|e_i^\perp), \tag{28}$$

for all $x \in \mathbb{R}^n$.

In view of the symmetries of K, $V_1\left(K|e_i^\perp\right)$ is the same for $i = 1, \dots, n$, so identifying $e_i^\perp$ with $\mathbb{R}^{n-1}$ and using first (9) with n replaced by $n-1$ and then the symmetries of K again, we have

$$\sum_{i=1}^n V_1(K|e_i^\perp) = \frac{n}{\kappa_{n-2}} \int_{S^{n-1} \cap e_n^\perp} h_K(u) \, du = \frac{n! \, 2^{n-1}}{\kappa_{n-2}} \int_{\Omega \cap e_n^\perp} h_K(u) \, du, \tag{29}$$

where

$$\Omega = S^{n-1} \cap \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \ge x_2 \ge \cdots \ge x_n \ge 0\}. \tag{30}$$

By (9) and (28), we obtain

$$V_1(K) = \frac{1}{\kappa_{n-1}} \int_{S^{n-1}} \max_{1 \le i \le n} h_K(u|e_i^\perp) \, du = \frac{n! \, 2^n}{\kappa_{n-1}} \int_\Omega h_K(u|e_n^\perp) \, du. \tag{31}$$

If $u \in \Omega$, then $u = \sin\varphi \, v + \cos\varphi \, e_n$ for some $v \in \Omega \cap e_n^\perp$ and $0 \le \varphi \le \pi/2$. Then by (30), we have

$$\sin\varphi (v \cdot e_{n-1}) = u \cdot e_{n-1} = u_{n-1} \ge u_n = \cos\varphi \ge 0,$$

from which it follows that $\varphi \in [\pi/2 - \arctan(v\cdot e_{n-1}), \pi/2]$. Since h_K is positively homogeneous of degree 1, (31) becomes

$$\begin{aligned} V_1(K) &= \frac{n!\,2^n}{\kappa_{n-1}} \int_{\Omega\cap e_n^\perp} \int_{\pi/2-\arctan(v\cdot e_{n-1})}^{\pi/2} h_K(\sin\varphi\, v) \sin^{n-2}\varphi \, d\varphi \, dv \\ &= \frac{n!\,2^n}{\kappa_{n-1}} \int_{\Omega\cap e_n^\perp} h_K(v) \int_{\pi/2-\arctan(v\cdot e_{n-1})}^{\pi/2} \sin^{n-1}\varphi \, d\varphi \, dv \\ &= \frac{n!\,2^n}{\kappa_{n-1}} \int_{\Omega\cap e_n^\perp} h_K(v) p(v\cdot e_{n-1})\, dv, \end{aligned} \tag{32}$$

where

$$p(t) = \int_{\pi/2-\arctan t}^{\pi/2} \sin^{n-1}\varphi \, d\varphi,$$

a nonnegative, increasing, continuous function of t. Substituting (29) and (32) into (20), we obtain

$$F(K) = \frac{2\kappa_{n-2}}{\kappa_{n-1}} \frac{\int_{\Omega\cap e_n^\perp} h_K(u) p(u\cdot e_{n-1})\, du}{\int_{\Omega\cap e_n^\perp} h_K(u)\, du}. \tag{33}$$

We rewrite the integrals in (33) as integrals over the graph of the function

$$x_1 = f(x_2, \ldots, x_{n-1}) = \sqrt{1 - x_2^2 - \cdots - x_{n-1}^2}. \tag{34}$$

The required formula for a surface integral is given explicitly in [2, Equation (24)]. Here, the Jacobian is $\sqrt{1+|\nabla f|^2} = 1/x_1$. Let Ω_1 be the projection of $\Omega \cap e_n^\perp$ onto $e_1^\perp$. Then Ω_1 is determined by the inequalities

$$\begin{aligned} 0 \le \quad x_{n-1} \quad &\le \frac{1}{\sqrt{n-1}}, \\ x_{n-1} \le \quad x_{n-2} \quad &\le \sqrt{\frac{1-x_{n-1}^2}{n-2}}, \\ x_{n-2} \le \quad x_{n-3} \quad &\le \sqrt{\frac{1-x_{n-2}^2-x_{n-1}^2}{n-3}}, \\ &\vdots \\ x_3 \le \quad x_2 \quad &\le \sqrt{\frac{1-x_3^2-x_4^2-\cdots-x_{n-1}^2}{2}}. \end{aligned} \tag{35}$$

The equation (27) results immediately. □

The following result is an inequality of the Chebyshev type; see, for example, [17, Theorem 236, p. 168].

Lemma 4.4. *Let $f : [a,b] \to \mathbb{R}$ be continuous and with zero average over $[a,b]$, and suppose that there exists a $c \in [a,b]$ such that $f \le 0$ on $[a,c]$ and $f \ge 0$ on $[c,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, increasing, and continuous, then $\int_a^b f(t)g(t)\,dt \ge 0$.*

Proof. Since g is nonnegative and increasing on $[a,b]$, we have $0 \le g(t) \le g(c)$ on $[a,c]$ and $g(t) \ge g(c) \ge 0$ on $[c,b]$. The assumptions on f and g imply that

$$\begin{aligned}\int_a^b f(t)g(t)\,dt &= \int_a^c f(t)g(t)\,dt + \int_c^b f(t)g(t)\,dt \\ &\ge g(c)\int_a^c f(t)\,dt + g(c)\int_c^b f(t)\,dt = g(c)\int_a^b f(t)\,dt = 0.\end{aligned}$$

□

Recall that Ω_1 is the region defined by (35) and define $S(t) = \Omega_1 \cap \{x \in \mathbb{R}^n : x \cdot e_{n-1} = t\}$. For $n \ge 4$, define

$$J_K(x_{n-1}) = \begin{cases} \frac{\int_{S(x_{n-1})} h_K(x_1,\dots,x_{n-1},0)/x_1\,dx_2\cdots dx_{n-2}}{\int_{S(x_{n-1})} dx_2\cdots dx_{n-2}}, & \text{if } 0 \le x_{n-1} < 1/\sqrt{n-1}, \\ h_K(1,\dots,1,0), & \text{if } x_{n-1} = 1/\sqrt{n-1}, \end{cases} \tag{36}$$

where x_1 is defined by (34). For $n = 3$, let

$$J_K(x_2) = h_K(x_1, x_2, 0)/x_1, \tag{37}$$

for $0 \le x_2 \le 1/\sqrt{2}$. The function J_K is continuous on $[0, 1/\sqrt{n-1}]$. To see this, note that for $0 \le x_{n-1} < 1/\sqrt{n-1}$, $J_K(x_{n-1})$ is the average of h_K/x_1 over $S(x_{n-1})$, while the value of h_K/x_1 at the singleton $S(1/\sqrt{n-1}) = \{(1/\sqrt{n-1},\dots,1/\sqrt{n-1},0)\}$ is $h_K(1,\dots,1,0)$.

Lemma 4.5. *Let K be as in Lemma 4.3. If the function J_K be defined by (36) and (37) is increasing, then $F(K) \ge F(C^n)$, with equality if and only if $K = C^n$.*

Proof. Note firstly that $h_{C^n}(x) = x_1$ for all $x = (x_1,\dots,x_n)$ such that $0 = x_n \le x_{n-1} \le \cdots \le x_1 \le 1$ and hence $h_{C^n}(x) = x_1$ on Ω_1. For $n \ge 4$, let

$$G_K(x_{n-1}) = \int_{S(x_{n-1})} \left(\frac{h_K(x_1,\dots,x_{n-1},0)/x_1}{\int_{\Omega_1} h_K(x_1,\dots,x_{n-1},0)/x_1\,dx_2\cdots dx_{n-1}} - \frac{1}{\int_{\Omega_1} dx_2\cdots dx_{n-1}} \right) dx_2\cdots dx_{n-2}.$$

For $n = 3$, let

$$G_K(x_2) = \frac{h_K(x_1,x_2,0)/x_1}{\int_{\Omega_1} h_K(x_1,x_2,0)/x_1\,dx_2} - \frac{1}{\int_{\Omega_1} dx_2}.$$

Suppose that $n \ge 3$. In view of (27), the inequality $F(K) \ge F(C^n)$ is equivalent to

$$\int_0^{1/\sqrt{n-1}} G_K(x_{n-1})p(x_{n-1})\,dx_{n-1} \ge 0. \tag{38}$$

From (36) and the definition of G_K, we see that $G_K(x_{n-1}) \le 0$ or $G_K(x_{n-1}) \ge 0$ according as $J_K(x_{n-1}) \le I$ or $J_K(x_{n-1}) \ge I$, respectively, where

$$I = \frac{\int_{\Omega_1} h_K(x_1, \dots, x_{n-1}, 0)/x_1 \, dx_2 \cdots dx_{n-1}}{\int_{\Omega_1} dx_2 \cdots dx_{n-1}}. \tag{39}$$

Now G_K is continuous and its definition implies that its average over $[0, 1/\sqrt{n-1}]$ is zero. Therefore if J_K is increasing, then for some $0 \le t_0 \le 1/\sqrt{n-1}$, we have $G_K \le 0$ on $[0, t_0]$ and $G_K \ge 0$ on $[t_0, 1/\sqrt{n-1}]$. Since p is nonnegative, increasing, and continuous, (38) follows from Lemma 4.4 with f and g replaced by G_K and p, respectively.

If $F(K) = F(C^n)$, then by (27),

$$\int_0^{1/\sqrt{n-1}} G_K(x_{n-1}) p(x_{n-1}) \, dx_{n-1} = 0.$$

Since $G_K \le 0$ on $[0, t_0]$, $G_K \ge 0$ on $[t_0, 1/\sqrt{n-1}]$, and G_K has zero average over $[0, 1/\sqrt{n-1}]$, the fact that p is nonnegative and strictly increasing implies that the inequality in (38) is strict, yielding a contradiction unless G_K vanishes on $[0, 1/\sqrt{n-1}]$. It follows that $J_K(x_{n-1}) = I$, for $0 \le x_{n-1} \le 1/\sqrt{n-1}$, where I is as in (39). In particular, $J_K(0) = J_K(1/\sqrt{n-1}) = h_K(1, \dots, 1, 0)$. Now K has the same symmetries as Q^n, so for each $i \in \{1, \dots, n-1\}$, h_K is convex and even as a function of x_i and hence increases with $x_i \in [0, 1]$. Therefore $h_K(x_1, ..., x_{n-1}, 0)/x_1 \le h_K(x_1, x_1, ..., x_1, 0)/x_1 = h_K(1, 1, ..., 1, 0)$. Since $J_K(0)$ is the average of h_K/x_1 on $S(0)$, it follows that h_K/x_1 coincides with $h_K(1, \dots, 1, 0)$ on $S(0)$. In particular, $h_K(1, \dots, 1, 0) = h_K(1, 0, \dots, 0) = 1$. This means that the face of C^n orthogonal to $(1, \dots, 1, 0)$ supports K. The assumptions on K inherited from those in Lemma 4.3 imply that $K = C^n$. □

Lemma 4.6. *Let K be as in Lemma 4.3. Then the function J_K defined by* (36) *is increasing.*

Proof. As we observed in the proof of the previous lemma, the fact that K has the same symmetries as Q^n implies that for each $i \in \{1, \dots, n-1\}$, h_K is convex and even as a function of x_i and hence increases with $x_i \in [0, 1]$.

Let $n = 3$. By Lemma 4.5, it suffices to show that $J_K(x_2) = h_K(x_1, x_2, 0)/x_1$ is increasing for $x_2 \in [0, 1/\sqrt{2}]$, where $h_K(x_1, x_2, 0)/x_1 = h_K(1, x_2/\sqrt{1 - x_2^2}, 0)$. This is true because h_K is an increasing function of its second argument in $[0, 1]$ and $x_2/\sqrt{1 - x_2^2}$ is increasing for $x_2 \in [0, 1)$.

The case $n \ge 4$ is proved in the appendix. □

Proof of Theorem 4.1. Let K be a compact convex set in $\mathbb{R}^n$ with $\dim K \ge 1$. By Lemma 4.2, we may assume that K satisfies the hypotheses of Lemma 4.3. Then Lemmas 4.5 and 4.6 imply that $F(K) \ge F(C^n)$, with equality if and only if $K = C^n$. □

Corollary 4.7. *Let $n \geq 3$. If K is a compact convex set in $\mathbb{R}^n$, then there is a constant $c_0 = c_0(n)$ such that*

$$V_1(K) \geq c_0 \sum_{i=1}^{n} V_1(K|e_i^{\perp}) \geq c_0 \sum_{i=1}^{n} V_1(K \cap e_i^{\perp}), \tag{40}$$

with equality in either inequality involving $V_1(K)$ if and only if either $\dim K = 0$ or $\dim K = n$ and K is an o-symmetric regular coordinate cross-polytope (or one of its translates, in the case of the left-hand inequality).

Proof. Theorem 4.1 gives the left-hand inequality in (40) and its equality condition. The right-hand inequality involving $V_1(K)$ and its equality condition follow trivially since $V_1(K|e_i^{\perp}) \geq V_1(K \cap e_i^{\perp})$ for $i = 1, \ldots, n$. □

With modified equality conditions, the previous corollary also holds when $n = 2$, by Corollary 3.4.

5. The case $m < n - 1$: Results for zonoids

Theorem 5.1. *Let K be a zonoid in $\mathbb{R}^n$ and let $m \in \{1, \ldots, n-2\}$. Then*

$$V_m(K)^2 \geq \frac{1}{n-m} \sum_{i=1}^{n} V_m(K|e_i^{\perp})^2. \tag{41}$$

Proof. Suppose that $m \in \{1, \ldots, n-2\}$ and that K is a zonoid with generating measure μ_K in S^{n-1} (see [10, p. 149]). We use the formula [27, Theorem 5.3.3] for the mth intrinsic volume of a zonoid (twice, once for K and once for the zonoid $K|e_i^{\perp}$), Proposition 2.2 (with

$A = \sum_{k=1}^m [o, \omega_k]$), and Minkowski's integral inequality (10) (with $p = 2$ and $k = n$), to obtain

$$\begin{aligned}
&V_m(K)\\
&= \frac{2^m}{m!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} V_m \left(\sum_{k=1}^m [o, \omega_k] \right) \, d\mu_K(\omega_1) \cdots d\mu_K(\omega_m)\\
&= \frac{2^m}{m!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} \left(\frac{1}{n-m} \sum_{i=1}^n V_m \left(\left(\sum_{k=1}^m [o, \omega_k] \right) |e_i^\perp| \right)^2 \right)^{1/2} \, d\mu_K(\omega_1) \cdots d\mu_K(\omega_m)\\
&\geq \left(\frac{1}{n-m} \sum_{i=1}^n \left(\frac{2^m}{m!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} V_m \left(\left(\sum_{k=1}^m [o, \omega_k] \right) |e_i^\perp| \right) d\mu_K(\omega_1) \cdots d\mu_K(\omega_m) \right)^2 \right)^{1/2}\\
&= \left(\frac{1}{n-m} \sum_{i=1}^n \left(\frac{2^m}{m!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} V_m \left(\sum_{k=1}^m [o, \omega_k | e_i^\perp] \right) \, d\mu_K(\omega_1) \cdots d\mu_K(\omega_m) \right)^2 \right)^{1/2}\\
&= \left(\frac{1}{n-m} \sum_{i=1}^n \left(\frac{2^m}{m!} \int_{S^{n-1}} \cdots \int_{S^{n-1}} V_m \left(\sum_{k=1}^m \left[o, \frac{\omega_k | e_i^\perp}{|\omega_k | e_i^\perp|} \right] \right) \prod_{k=1}^m |\omega_k | e_i^\perp| \, d\mu_K(\omega_1) \cdots d\mu_K(\omega_m) \right)^2 \right)^{1/2}\\
&= \left(\frac{1}{n-m} \sum_{i=1}^n \left(\frac{2^m}{m!} \int_{S^{n-1} \cap e_i^\perp} \cdots \int_{S^{n-1} \cap e_i^\perp} V_m \left(\sum_{k=1}^m [o, \theta_k] \right) \, d\mu_{K|e_i^\perp}(\theta_1) \cdots d\mu_{K|e_i^\perp}(\theta_m) \right)^2 \right)^{1/2}\\
&= \left(\frac{1}{n-m} \sum_{i=1}^n V_m(K|e_i^\perp)^2 \right)^{1/2}.
\end{aligned}$$

Here $\mu_{K|e_i^\perp}$ denotes the generating measure in $S^{n-1} \cap e_i^\perp$ of the zonoid $K|e_i^\perp$. The penultimate equality in the previous display is a consequence of the formula (see, for example, [18, (10)])

$$\mu_{K|e_i^\perp}(A) = \int_{S^{n-1} \setminus \{\pm e_i\}} 1_A \left(\frac{u|e_i^\perp}{|u|e_i^\perp|} \right) |u|e_i^\perp| \, d\mu_K(u),$$

where 1_A is the characteristic function of an arbitrary Borel set A in S^{n-1}. This proves the inequality (41). □

By Proposition 2.2, equality holds in (41) when $\dim K \leq m$. Otherwise, if equality holds, then equality holds in the previous displayed inequality. A direct consequence of the equality condition for Minkowski's integral inequality (10) is that equality in (41) holds if and only if there are constants b_i, $i = 1, \ldots, n$, and a function $g(\omega_1, \ldots, \omega_m)$ which is measurable with

respect to the product measure $\nu_K = \mu_K \times \cdots \times \mu_K$ on $(S^{n-1})^m$, such that

$$V_m\left(\left(\sum_{k=1}^m [o,\omega_k]\right)|e_i^\perp\right) = b_i g(\omega_1,\dots,\omega_m), \tag{42}$$

for all $i = 1,\dots,n$ and ν_K-almost all $(\omega_1,\dots,\omega_m) \in (S^{n-1})^m$. When $m = 1$, this condition assumes the following more explicit form.

Corollary 5.2. *Let K be a zonoid in $\mathbb{R}^n$ with $\dim K \geq 1$ and let $m = 1$. Then (41) holds with equality if and only if either $\dim K = 1$ or $\dim K > 1$ and K is a zonotope of the form*

$$K = \sum_{r=1}^{2^n} [o, a_r(\varepsilon_1 u_1,\dots,\varepsilon_n u_n)], \tag{43}$$

where $u = (u_1,\dots,u_n) \in S^{n-1}$, $a_r \geq 0$, and $\varepsilon_i = \pm 1$, $i = 1,\dots,n$.

Proof. If $m = 1$ and $\dim K > 1$, the equality condition (42) states that there are constants b_i, $i = 1,\dots,n$, and a μ_K-measurable function $g(\omega)$ on S^{n-1}, such that

$$V_1\left([o,\omega]|e_i^\perp\right) = b_i g(\omega), \tag{44}$$

for $i = 1,\dots,n$ and ν_K-almost all $\omega \in S^{n-1}$. Let $\omega_0 = (\alpha_1,\dots,\alpha_n)$ and $\omega_1 = (\beta_1,\dots,\beta_n)$ be two points in S^{n-1} for which (44) holds. We may assume that $g(\omega_1) \neq 0$, for otherwise $g(\omega) = 0$ for all ω for which (44) holds and this implies that $K = \{o\}$. Let $c = g(\omega_0)/g(\omega_1)$. Then from (44) with $\omega = \omega_0$ and $\omega = \omega_1$, we obtain

$$\sum_{1\leq k\leq n,\ k\neq i} \alpha_k^2 = c^2 \sum_{1\leq k\leq n,\ k\neq i} \beta_k^2,$$

that is, $1-\alpha_i^2 = c^2(1-\beta_i^2)$, for $i = 1,\dots,n$. Adding these equations gives $c^2 = 1$ and then we conclude that $\alpha_i = \pm\beta_i$, for $i = 1,\dots,n$. This shows that the support of μ_K must be a subset of the 2^n points in S^{n-1} of the form $(\varepsilon_1\alpha_1,\dots,\varepsilon_n\alpha_n)$, where $\varepsilon_i = \pm 1$, $i = 1,\dots,n$. Hence K is a zonotope given by (43). □

Inequality (41) is not generally true for arbitrary convex bodies. To see this, let $n = 3$ and $m = 1$, and recall that C^3 denotes the standard o-symmetric regular coordinate cross-polytope in $\mathbb{R}^3$. Then $C^3|e_i^\perp$ is a square of side length $\sqrt{2}$, so $V_1(C^3|e_i^\perp) = 4\sqrt{2}/2 = 2\sqrt{2}$, for $i = 1,2,3$. Using (19), we see that (41) is false for C^3 if and only if

$$\frac{\left(12\sqrt{2}\arccos(1/3)/2\pi\right)^2}{3(2\sqrt{2})^2} < 1/2.$$

Computation shows that the left-hand side of the previous inequality is 0.46058..., so (41) is indeed false for C^3.

6. The case $m < n-1$: Other results

The problem of finding a sharp inequality of the form (41) that holds for general compact convex sets when $m < n-1$ appears to be difficult (see Problem 7.3). We can prove a weaker result, for which we need a definition. Let K be a convex body in $\mathbb{R}^n$ and let $m \in \{1, \dots, n-2\}$. Then the mth area measure $S_m(K, \cdot)$ of K satisfies the hypotheses of Minkowski's existence theorem (see, for example, [10, (A.20), p. 399]) and hence is the surface area measure of a unique convex body $B_m K$ with centroid at the origin; in other words, $S(B_m K, \cdot) = S_m(K, \cdot)$. In fact, it is enough to assume that $\dim K > m$ in order to conclude that a compact convex set $B_m K$ satisfying $S(B_m K, \cdot) = S_m(K, \cdot)$ exists. To see this, assume firstly that K is a polytope, and suppose that the support of $S_m(K, \cdot)$ is contained in $u^\perp$ for some $u \in S^{n-1}$. If $v \in S^{n-1} \setminus u^\perp$, the supporting set to K in the direction v must have dimension less than m. Hence the union of all such supporting sets also has dimension less than m. But this union contains the boundary of K except the shadow boundary of K in the direction u, and therefore has the same dimension as $K|u^\perp$. However, if $\dim K > m$, then $\dim(K|u^\perp) \geq m$, a contradiction. We conclude that $S_m(K, \cdot)$ is not contained in $u^\perp$ for any $u \in S^{n-1}$, so it satisfies the hypotheses of Minkowski's existence theorem. By approximation, the same conclusion is reached for arbitrary compact convex K with $\dim K > m$.

The following lemma is stated with the assumption $\dim K > m$. This is natural, since if $\dim K < m$, both sides of (45) are zero, while if $\dim K = m$, we have the equality provided by Proposition 2.2.

Theorem 6.1. *Let $m \in \{1, \dots, n-2\}$ and K be a compact convex set in $\mathbb{R}^n$ with $\dim K > m$. Then*

$$V_m(K)^2 \geq \frac{1}{\pi} \left(\frac{\Gamma(\frac{n-m}{2})}{\Gamma(\frac{n-m+1}{2})} \right)^2 \sum_{i=1}^{n} V_m(K|e_i^\perp)^2. \tag{45}$$

Proof. As was noted above, the assumption $\dim K > m$ guarantees that $B_m K$ exists. By [10, (A.34), p. 405], we have

$$V_m(K) = \frac{\binom{n}{m}}{n\kappa_{n-m}} S_m(K, S^{n-1}).$$

This and the fact that $S_m(K, S^{n-1}) = S(B_m K, S^{n-1})$ yields

$$V_m(K) = \frac{2\binom{n}{m}}{n\kappa_{n-m}} V_{n-1}(B_m K). \tag{46}$$

By the generalized Cauchy projection formula [10, (A.45), p. 408],

$$V_m(K|u^\perp) = \frac{\binom{n-1}{m}}{2\kappa_{n-m-1}} \int_{S^{n-1}} |u \cdot v| \, dS_m(K, v),$$

for all $u \in S^{n-1}$. This and $S_m(K, \cdot) = S(B_m K, \cdot)$ imply that

$$V_m(K|u^\perp) = \frac{\binom{n-1}{m}}{\kappa_{n-m-1}} V_{n-1}(B_m K|u^\perp), \tag{47}$$

for all $u \in S^{n-1}$. Now using (46), (18) with K replaced by $B_m K$, and (47) with $u = e_i^\perp$, we obtain

$$\begin{aligned} V_m(K)^2 &= \left(\frac{2\binom{n}{m}}{n\kappa_{n-m}}\right)^2 V_{n-1}(B_m K)^2 \geq \left(\frac{2\binom{n}{m}}{n\kappa_{n-m}}\right)^2 \sum_{i=1}^n V_{n-1}(B_m K|e_i^\perp)^2 \\ &= \left(\frac{2\kappa_{n-m-1}\binom{n}{m}}{n\kappa_{n-m}\binom{n-1}{m}}\right)^2 \sum_{i=1}^n V_m(K|e_i^\perp)^2 = \frac{1}{\pi}\left(\frac{\Gamma(\frac{n-m}{2})}{\Gamma(\frac{n-m+1}{2})}\right)^2 \sum_{i=1}^n V_m(K|e_i^\perp)^2. \end{aligned}$$

□

The previous bound is not optimal. Indeed, the proof of Theorem 6.1 shows that equality in (45) would imply that equality holds in the left-hand inequality in (18) when K is replaced by $B_m K$. The equality condition for (18) then yields that either $\dim B_m K \leq n-1$ or $B_m K$ is a coordinate cross-polytope. In either case, the surface area measure of $B_m K$, which is just the mth area measure of K, would have atoms unless it is the zero measure. This contradicts [27, Theorem 4.6.5], which states that an mth area measure cannot be positive on sets whose Hausdorff dimension is less than $n-m-1$.

For example, when $n = 3$ and $m = 1$, the constant in (45) is

$$\frac{1}{\pi}\left(\frac{\Gamma(1)}{\Gamma(\frac{3}{2})}\right)^2 = \frac{4}{\pi^2} = 0.40528...,$$

lower than the probable best bound 0.46058... for the regular coordinate cross-polytope. (Note that it is higher than constant $1/3$ in the easy bound (14).)

The hypothesis (48) in the following lemma was shown in [7, Theorem 4.1] to be equivalent to the existence of a coordinate box Z such that $V_m(Z|e_i^\perp) = V_m(K|e_i^\perp)$, for $i = 1, \ldots, n$. Inequality (48) is true when $m = 1$ or $m = n-1$; this follows from (12) and [7, Theorem 3.1] or the left-hand inequality in Theorem 3.3 with $a_i = 1$ for $i = 1, \ldots, n$, respectively. We prove below in Lemma 6.3 that (48) is also true when $m = n-2$. For $m \in \{2, \ldots, n-3\}$, (48) remains a conjecture.

Lemma 6.2. *Let K be a compact convex set in $\mathbb{R}^n$ and let $m \in \{1, \ldots, n-2\}$. If*

$$V_m(K|e_k^\perp) \leq \frac{1}{n-m}\sum_{i=1}^n V_m(K|e_i^\perp), \tag{48}$$

for $k = 1, \ldots, n$, then

$$\sum_{i=1}^n V_m(K|e_i^\perp)^2 \leq \frac{1}{n-m}\left(\sum_{i=1}^n V_m(K|e_i^\perp)\right)^2. \tag{49}$$

Proof. Let $V_m(K|e_i^\perp) = c_i$, for $i = 1, \ldots, n$. By homogeneity, we may assume without loss of generality that $\sum_{i=1}^n c_i = 1$. Using (48), we obtain the additional constraints $0 \leq c_k \leq 1/(n-m)$, for $k = 1, \ldots, n$.

The set of $(c_1,\dots,c_n)$ satisfying these constraints is an $(n-1)$-dimensional convex polytope P contained in the hyperplane $\{x=(x_1,\dots,x_n)\in\mathbb{R}^n : x_1+\cdots+x_n=1\}$. The maximum distance d from the origin to a point in P is attained at a vertex of P. At such a vertex, we have either $c_k=0$ or $c_k=1/(n-m)$ for at least $n-1$ of the k's. If less than m of these c_k's are zero, we would have $\sum_{k=1}^n c_k>1$, a contradiction. Therefore at least m of the c_k's are zero. Consequently,

$$\begin{aligned} d &= \left(\sum_{i=1}^n c_i^2\right)^{1/2} \le \left((n-m)\max_{1\le k\le n} c_k^2\right)^{1/2} \\ &\le \left((n-m)\frac{1}{(n-m)^2}\right)^{1/2} = \frac{1}{\sqrt{n-m}} = \frac{1}{\sqrt{n-m}}\sum_{i=1}^n c_i, \end{aligned}$$

which yields (49). □

Lemma 6.3. *Let K be a compact convex set in $\mathbb{R}^n$. Then (48) holds when $m=n-2$.*

Proof. Without loss of generality, let $k=1$. We identify $e_1^\perp$ with $\mathbb{R}^{n-1}$ and apply the left-hand inequality in Proposition 3.3 with $a_i=1$ for each i and K and n replaced by $L=K|e_1^\perp$ and $n-1$, respectively, to obtain

$$V_{n-2}(L)\le\sum_{i=2}^n V_{n-2}(L|e_i^\perp).$$

By (13) with m, K, and u replaced by $n-2$, $K|e_i^\perp$, and e_1, respectively, we have

$$V_{n-2}(L|e_i^\perp)=V_{n-2}\left((K|e_1^\perp)|e_i^\perp\right)=V_{n-2}\left((K|e_i^\perp)|e_1^\perp\right)\le V_{n-2}(K|e_i^\perp),$$

for $i=2,\dots,n$. It follows that

$$V_{n-2}(K|e_1^\perp)=V_{n-2}(L)\le\sum_{i=2}^n V_{n-2}(L|e_i^\perp)\le\sum_{i=2}^n V_{n-2}(K|e_i^\perp).$$

Adding $V_{n-2}(K|e_1^\perp)$ to both sides, we obtain (48) with $m=n-2$. □

When $m=1$ or $m=n-2$, the following result establishes a relationship between the lower bound for $V_m(K)$ for a zonoid K from (41) and the upper bound for $V_m(K)$ from [7, Theorem 3.1] (for $m=1$) or Lemma 6.3 (for $m=n-2$). It represents a reverse Cauchy-Schwarz inequality for the numbers $V_m(K|e_i^\perp)$, $i=1,\dots,n$. We do not know if the result holds for $m\in\{2,\dots,n-3\}$; see Problem 7.1.

Theorem 6.4. *Let K be a compact convex set in $\mathbb{R}^n$ and let $m=1$ or $m=n-2$. Then*

$$\sum_{i=1}^n V_m(K|e_i^\perp)^2\le\frac{1}{\sqrt{n-m}}\left(\sum_{i=1}^n V_m(K|e_i^\perp)\right)^2. \tag{50}$$

Proof. When $m = 1$, (48) holds, by (12) for $m = 1$ and [7, Theorem 3.1], and when $m = n-2$, (48) holds by Lemma 6.3. Then (50) with $m = 1$ or $m = n - 2$ follows directly from Lemma 6.2. □

We end this section with a counterpart to [7, Theorem 4.1].

Theorem 6.5. *Let K be a compact convex set in $\mathbb{R}^n$. There is a line segment L such that*

$$V_1(L|e_i^\perp) = V_1(K|e_i^\perp), \tag{51}$$

for $i = 1, \dots, n$, if and only if

$$V_1(K|e_i^\perp)^2 \le \frac{1}{n-1}\sum_{k=1}^{n} V_1(K|e_k^\perp)^2, \tag{52}$$

for $i = 1, \dots, n$.

Proof. Let $V_1(K|e_i^\perp) = a_i$, for $i = 1, \dots, n$. If $L = [-x/2, x/2]$ and $x = (x_1, \dots, x_n)$, then (51) is equivalent to

$$\sum_{1\le k\le n,\ k\ne i} x_k^2 = a_i^2, \tag{53}$$

for $i = 1, \dots, n$. Summing the equations in (53) over i, we obtain

$$(n-1)\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} a_i^2.$$

Subtracting $(n-1)$ times the ith equation in (53), we get

$$(n-1)x_i^2 = \sum_{k=1}^{n} a_k^2 - (n-1)a_i^2. \tag{54}$$

The left-hand side is nonnegative, so the right-hand side is also, and this is equivalent to (52).

Conversely, assuming that (52) holds, we know that the right-hand side of (54) is nonnegative and hence (54) can be solved for x_i. The values of x_i thus obtained also satisfy (53), so (51) holds for $L = [-x/2, x/2]$ when $x = (x_1, \dots, x_n)$. □

There are convex bodies for which (52) does not hold. To see this, let P be the coordinate box defined by $P = \sum_{k=1}^{n} s_k[-e_k/2, e_k/2]$. Suppose that $s_k > 0$ for $k = 1, \dots, n-1$ and $s_n = 0$. If we take $i = n$ and $K = P$, then (52) becomes

$$(n-1)\left(\sum_{k=1}^{n-1} s_k\right)^2 \le \sum_{l=1}^{n}\left(\sum_{1\le k\le n-1,\ k\ne l} s_k\right)^2,$$

which yields

$$\sum_{1\le k\le n-1,\ k\ne l} s_k s_l \le 0.$$

This is false, so (52) does not hold for P under the assumptions above. By continuity, it is also false for the n-dimensional coordinate box $P = \sum_{k=1}^{n} s_k[-e_k/2, e_k/2]$ when $s_n > 0$ is sufficiently small.

The fact that an $(n-1)$-dimensional coordinate box such as P does not satisfy (52) can also be seen as follows. Since $s_n = 0$, we have $V_1(P) = V_1(P|e_n^\perp)$. By Corollary 5.2, we know that strict inequality holds in (41) when $K = P$. This contradicts (52) when $K = P$ and $i = n$.

Note that if K does satisfy (52), Theorem 6.5 states that there is a line segment $L \in \mathcal{P}(K, 1)$. Since equality holds in Theorem 5.1 when $K = L$, we see by that theorem that L is a zonoid of minimal mean width in $\mathcal{P}(K, 1)$.

7. Open problems

Problem 7.1. Does (50) hold when $m \in \{2, \dots, n-3\}$?

Problem 7.2. Is there a version of Theorem 6.5 for $m \in \{2, \dots, n-2\}$?

Problem 7.3. Let K be a compact convex set in $\mathbb{R}^n$ and let $m \in \{1, \dots, n-2\}$. Then is there a constant $c_2 = c_2(n, m)$ such that

$$V_m(K)^2 \geq c_2 \sum_{i=1}^{n} V_m(K|e_i^\perp)^2 \geq c_2 \sum_{i=1}^{n} V_m(K \cap e_i^\perp)^2,$$

with equality in either inequality involving $V_m(K)$ when $\dim K = n$ if and only if K is an o-symmetric regular coordinate cross-polytope (or one of its translates, in the case of the left-hand inequality)?

Problem 7.4. Let K be a convex body in $\mathbb{R}^n$ and let $m \in \{1, \dots, n-2\}$. Then is there a constant $c_3 = c_3(n, m)$ such that

$$V_{m+1}(K)^{mn} \geq c_3 \prod_{i=1}^{n} V_m(K \cap e_i^\perp)^{m+1},$$

with equality if and only if K is an o-symmetric coordinate cross-polytope?

Problem 7.5. Let K be a convex body in $\mathbb{R}^3$. Is

$$V_2(K)^2 \geq \frac{1}{16} \left(\sum_{i=1}^{3} V_1(K \cap e_i^\perp)^2 \right)^2 - \frac{1}{8} \sum_{i=1}^{3} V_1(K \cap e_i^\perp)^4,$$

with equality if and only if K is an o-symmetric coordinate cross-polytope?

An upper bound for $V_2(K)$ analogous to the lower bound in the previous problem was obtained in [7, Theorem 4.6]. The proposed lower bound clearly relates to Heron's formula; in one version, this states that a triangle with sides of length a, b, and c has area

$$\frac{1}{4} \sqrt{(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)}.$$

Appendix: Proof of the case $n \geq 4$ of Lemma 4.6

This appendix is devoted to proving that if K is as in Lemma 4.3 and $n \geq 4$, then the function J_K defined by (36) is increasing. It has already been observed in the proof of Lemma 4.6 that since K has the same symmetries as Q^n, we have that for each $i \in \{1, \dots, n-1\}$, h_K is convex and even as a function of x_i and hence increases with x_i for $x_i \in [0,1]$. We recall that by (34),

$$x_1^2 + \cdots + x_{n-1}^2 = 1. \tag{55}$$

The plan is to consider the cases $n = 4$ and $n = 5$ separately and then dispose of the remaining case $n \geq 6$ by means of an induction argument.

Let $n = 4$. By (36),

$$J_K(x_3) = \frac{\int_{x_3}^{\sqrt{\frac{1-x_3^2}{2}}} h_K\left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, 0\right) dx_2}{\sqrt{\frac{1-x_3^2}{2}} - x_3} = \int_0^1 h_K\left(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, 0\right) dt,$$

where

$$x_2 = x_3 + t\left(\sqrt{\frac{1-x_3^2}{2}} - x_3\right) \tag{56}$$

and where x_2 and hence x_1 are now functions of x_3 and t. It therefore suffices to show that x_2/x_1 and x_3/x_1 are increasing functions of x_3 for any fixed t. To this end, using (55), (56), and straightforward but quite tedious calculation, we find that

$$x_1^3 \frac{\partial(x_2/x_1)}{\partial x_3} = 1 - t \geq 0,$$

showing that x_2/x_1 increases with x_3 for fixed t. Similarly, further calculations yield

$$x_1^3 \frac{\partial(x_3/x_1)}{\partial x_3} = \frac{(2-t^2)\sqrt{(1-x_3^2)/2} - t(1-t)x_3}{\sqrt{2}\sqrt{1-x_3^2}} \geq \frac{(2-t)x_3}{\sqrt{2}\sqrt{1-x_3^2}} \geq 0,$$

where we used $\sqrt{(1-x_3^2)/2} \geq x_3$. This shows that x_3/x_1 increases with x_3 for fixed t and completes the proof for $n = 4$.

Henceforth we assume that $n \geq 5$ and make a change of variables by setting

$$y_i = x_i/x_1, \ i = 2, \dots, n-2, \quad \text{and} \quad y_{n-1} = x_{n-1}. \tag{57}$$

Then

$$\frac{\partial y_i}{\partial x_j} = \begin{cases} (x_1^2 + x_i^2)/x_1^3, & \text{if } i = j = 2, \dots, n-2, \\ x_i x_j / x_1^3, & \text{if } i, j = 2, \dots, n-2, \ j \neq i. \end{cases}$$

The Jacobian of this transformation may be calculated by first noticing that the determinant reduces to one of size $(n-3) \times (n-3)$, since apart from the $(n-2, n-2)$ entry, the last row and column have all entries equal to zero. Then, factor x_i/x_1^3 from the ith column and x_j from the jth row. Next, subtract the first column from all the others and multiply the ith row

by x_{i+1}^2. Finally, add each row to the first row. The result is $1/x_1^{3(n-3)}$ times the determinant of a lower triangular matrix with first diagonal entry $\sum_{i=1}^{n-2} x_i^2$ and all other diagonal entries equal to x_1^2. Therefore, using (55), the Jacobian reduces to

$$\frac{\partial(y_2,\dots,y_{n-1})}{\partial(x_2,\dots,x_{n-1})} = \frac{1-y_{n-1}^2}{x_1^{n-1}}.$$

It will be convenient to set, for $i = 2,\dots,n-2$,

$$Y_i = (1, y_2, \dots, y_i) \quad \Rightarrow \quad |Y_i|^2 = 1 + y_2^2 + \cdots + y_i^2. \tag{58}$$

Then, using (55) and (57) to get

$$x_1^2 = \frac{1-y_{n-1}^2}{1+y_2^2+\cdots+y_{n-2}^2} = \frac{1-y_{n-1}^2}{|Y_{n-2}|^2}, \tag{59}$$

we can rewrite the Jacobian as

$$\frac{\partial(y_2,\dots,y_{n-1})}{\partial(x_2,\dots,x_{n-1})} = \frac{(1+y_2^2+\cdots+y_{n-2}^2)^{(n-1)/2}}{(1-y_{n-1}^2)^{(n-3)/2}} = \frac{|Y_{n-2}|^{n-1}}{(1-y_{n-1}^2)^{(n-3)/2}}. \tag{60}$$

In order to describe the region of integration in the expression (36) for J_K, we begin by recalling that the general region $\Omega \subset S^{n-1}$ of interest is given by (30) and hence

$$0 \le x_{n-1} \le x_{n-2} \le \cdots \le x_1. \tag{61}$$

We already know from (35) and (57) that

$$0 \le y_{n-1} \le \frac{1}{\sqrt{n-1}}.$$

To bound $y_2 = x_2/x_1$, observe that by (55), y_2 is an increasing function of x_2. Therefore by (57) and (61), the maximum and minimum of y_2 occur when $x_2 = x_1$ and $x_2 = x_3 = \cdots = x_{n-2} = y_{n-1}$, respectively. By (55), this gives $L_2 \le y_2 \le 1$, where

$$L_2 = \frac{y_{n-1}}{(1-(n-2)y_{n-1}^2)^{1/2}}. \tag{62}$$

Similarly, if $i \in \{3,\dots,n-3\}$, once $y_{n-1}, y_2, \dots, y_{i-1}$ are fixed, the maximum and minimum of $y_i = x_i/x_1$ occur when $x_i = x_{i-1}$ and $x_i = x_{i+1} = \cdots = x_{n-2} = y_{n-1}$, respectively. Using (55) and (57) again, we find that $L_i \le y_i \le y_{i-1}$, where

$$L_i = \frac{y_{n-1}(1+y_2^2+\cdots+y_{i-1}^2)^{1/2}}{(1-(n-i)y_{n-1}^2)^{1/2}} = \frac{y_{n-1}|Y_{i-1}|}{(1-(n-i)y_{n-1}^2)^{1/2}}, \tag{63}$$

for $i = 3,\dots,n-2$. Consequently, (58), (59), (60), (62), and (63) allow the function J_K defined by (36) to be rewritten as

$$J_K(y_{n-1}) = \frac{\int_{L_2}^1 \int_{L_3}^{y_2} \cdots \int_{L_{n-2}}^{y_{n-3}} h_K\left(Y_{n-2}, \frac{y_{n-1}|Y_{n-2}|}{\sqrt{1-y_{n-1}^2}}, 0\right) |Y_{n-2}|^{1-n}\, dy_{n-2}\cdots dy_2}{\int_{L_2}^1 \int_{L_3}^{y_2} \cdots \int_{L_{n-2}}^{y_{n-3}} |Y_{n-2}|^{1-n}\, dy_{n-2}\cdots dy_2}. \tag{64}$$

(Note that the denominator of (60) does not depend on $y_2, \dots, y_{n-2}$ and so can be factored from the integrals in the numerator and denominator of J_K and then canceled.) Here, and in what follows, we abbreviate $h_K(1, y_2, \dots, y_i, z_{i+1}, \dots, z_{n-1}, 0)$, for $i = 2, \dots, n-2$, by writing $h_K(Y_i, z_{i+1} \dots, z_{n-1}, 0)$ instead.

If h_K is differentiable, then by its definition, J_K is also differentiable with respect to y_{n-1} on $(0, 1/\sqrt{n-1})$. Assuming this, we shall prove that the derivative is nonnegative and hence that J_K is increasing. In fact, we may assume without loss of generality that h_K is differentiable, or, equivalently (see [27, p. 107]) that K is strictly convex. Indeed, if this is not the case, we may choose a sequence $\{K_i\}$ of strictly convex bodies converging to K in the Hausdorff metric; see [27, p. 158–160] for even stronger results of this type. Then h_{K_i} converges uniformly on S^{n-1} to h_K (see [27, p. 54]) and hence J_{K_i} converges to J_K. Therefore, if each J_{K_i} is increasing, J_K is also increasing.

We proceed to differentiate J_K with respect to y_{n-1}. We shall use the fact that for $i = 2, \dots, n-3$,

$$y_i = L_i \quad \Rightarrow \quad L_{i+1} = L_i. \tag{65}$$

Indeed, (62) and (63) imply that both equations in (65) are equivalent to

$$1 + y_2^2 + \cdots + y_i^2 = (1 + y_2^2 + \cdots + y_{i-1}^2)\, \frac{1 - (n-i-1)y_{n-1}^2}{1 - (n-i)y_{n-1}^2}.$$

Let $J_K = N/D$, where $N = N(K)$ and D are the numerator and denominator in (64), and let $z = y_{n-1}|Y_{n-2}|/\sqrt{1 - y_{n-1}^2}$. Applying Leibniz's rule for differentiating the integrals N and D, we notice that the terms involving the derivatives of the limits L_2 and 1 of the integrals with respect to y_2 vanish, since $y_2 = L_2$ implies $L_2 = L_3$, in view of (65), and hence $y_2 = L_3$. Similarly, for $i = 3, \dots, n-3$, terms involving the derivatives of the limits L_i and y_{i-1} of the integrals with respect to y_i vanish, since (65) says that $y_i = L_i$ implies $L_i = L_{i+1}$, and then $y_i = L_{i+1}$. Consequently, $(dJ_K/dy_{n-1})D^2$ equals

$$D \int_{L_2}^{1} \int_{L_3}^{y_2} \cdots \int_{L_{n-3}}^{y_{n-4}} \left(-\frac{\partial L_{n-2}}{\partial y_{n-1}} \left(h_K(Y_{n-2}, z, 0)|Y_{n-2}|^{1-n} \right)\Big|_{y_{n-2}=L_{n-2}} + \int_{L_{n-2}}^{y_{n-3}} \frac{e_{n-1} \cdot \nabla h_K(Y_{n-2}, z, 0)}{(1 - y_{n-1}^2)^{3/2}|Y_{n-2}|^{n-2}} \right)$$
$$\times\, dy_{n-3} \cdots dy_2 - N \int_{L_2}^{1} \int_{L_3}^{y_2} \cdots \int_{L_{n-3}}^{y_{n-4}} \left(-\frac{\partial L_{n-2}}{\partial y_{n-1}} \, |Y_{n-2}|^{1-n}\Big|_{y_{n-2}=L_{n-2}} \right) dy_{n-3} \cdots dy_2.$$

Recalling that all components of ∇h_K are nonnegative, we conclude that the second term in the first integral is nonnegative. Substituting

$$\frac{\partial L_{n-2}}{\partial y_{n-1}} = \frac{(1 + y_2^2 + \cdots + y_{n-3}^2)^{1/2}}{(1 - 2y_{n-1}^2)^{3/2}} = \frac{|Y_{n-3}|}{(1 - 2y_{n-1}^2)^{3/2}},$$

we find that dJ_K/dy_{n-1} is at least a positive constant multiple of

$$-D\int_{L_2}^{1}\int_{L_3}^{y_2}\cdots\int_{L_{n-3}}^{y_{n-4}}\frac{(1-2y_{n-1}^2)^{\frac{n-4}{2}}}{|Y_{n-3}|^{n-2}(1-y_{n-1}^2)^{\frac{n-1}{2}}}\,h_K(Y_{n-2},z,0)|_{y_{n-2}=L_{n-2}}\,dy_{n-3}\cdots dy_2$$
$$+N\int_{L_2}^{1}\int_{L_3}^{y_2}\cdots\int_{L_{n-3}}^{y_{n-4}}\frac{(1-2y_{n-1}^2)^{\frac{n-4}{2}}}{|Y_{n-3}|^{n-2}(1-y_{n-1}^2)^{\frac{n-1}{2}}}\,dy_{n-3}\cdots dy_2.$$

The common expressions depending only on y_{n-1} can be factored and absorbed into the constant multiplying factor. Furthermore, the restriction $y_{n-2}\in[L_{n-2},y_{n-3}]$ and the fact that $h_K(Y_{n-2},z,0)$ is increasing with respect to y_{n-2} means that $h_K(Y_{n-2},z,0)$ has its minimum when $y_{n-2}=L_{n-2}$. Letting

$$R=R(y_2,\dots,y_{n-3},y_{n-1})=\int_{L_{n-2}}^{y_{n-3}}|Y_{n-2}|^{1-n}\,dy_{n-2}=\int_{L_{n-2}}^{y_{n-3}}\left(|Y_{n-3}|^2+t^2\right)^{(1-n)/2}\,dt$$

and using the expressions for N and D from (64), we see that dJ_K/dy_{n-1} is at least a positive constant multiple of

$$-\left(\int_{\Sigma_1}R\,dx\right)\left(\int_{\Sigma_1}|Y_{n-3}|^{2-n}\,h_K(Y_{n-2},z,0)|_{y_{n-2}=L_{n-2}}\,dx\right)$$
$$+\left(\int_{\Sigma_1}R\,h_K(Y_{n-2},z,0)|_{y_{n-2}=L_{n-2}}\,dx\right)\left(\int_{\Sigma_1}|Y_{n-3}|^{2-n}\,dx\right),$$

where $dx=dy_{n-3}\cdots dy_2$ and Σ_1 is the corresponding region of integration from the previous integrals. Note that when $y_{n-2}=L_{n-2}$, (63) with $i=n-2$ implies that $z=L_{n-2}$, so $h_K(Y_{n-2},z,0)=h_K(Y_{n-3},L_{n-2},L_{n-2},0)$. To prove that $dJ_K/dy_{n-1}\ge 0$, it will therefore suffice to show that

(66)
$$\int_{\Sigma_1}h_K(Y_{n-3},L_{n-2},L_{n-2},0)\left(\frac{\int_{L_{n-2}}^{y_{n-3}}(|Y_{n-3}|^2+t^2)^{\frac{1-n}{2}}\,dt}{\int_{\Sigma_1}\int_{L_{n-2}}^{y_{n-3}}(|Y_{n-3}|^2+t^2)^{\frac{1-n}{2}}\,dt\,dx}-\frac{|Y_{n-3}|^{2-n}}{\int_{\Sigma_1}|Y_{n-3}|^{2-n}\,dx}\right)dx\ge 0.$$

The substitution $t=|Y_{n-3}|u$ allows (66) to be rewritten in the form

$$\int_{\Sigma_1}h_K(Y_{n-3},L_{n-2},L_{n-2},0)\left(\frac{|Y_{n-3}|^{2-n}\,U}{\int_{\Sigma_1}|Y_{n-3}|^{2-n}\,U\,dx}-\frac{|Y_{n-3}|^{2-n}}{\int_{\Sigma_1}|Y_{n-3}|^{2-n}\,dx}\right)dx\ge 0, \tag{67}$$

where

$$U=\int_{y_{n-1}/(1-2y_{n-1}^2)^{1/2}}^{y_{n-3}/|Y_{n-3}|}(1+u^2)^{\frac{1-n}{2}}\,du. \tag{68}$$

Let $n=5$. Then $\Sigma_1=[L_2,1]=[y_4/\sqrt{1-3y_4^2},1]$ by (62), and $dx=dy_2$. In view of (63) with $i=3$ and the fact that h_K increases with respect to its arguments, $h_K(Y_2,L_3,L_3,0)$

is increasing with respect to y_2. The part of the integrand in (67) in parentheses, S say, is clearly continuous with zero average over Σ_1. The factor $|Y_{n-3}|^{2-n} = (1+y_2^2)^{-3/2}$ in S is nonnegative and the remaining factor is increasing with respect to y_2, since only the upper limit $y_2/\sqrt{1+y_2^2}$ in the integral expression for U depends on y_2. Therefore there is some $c = c(y_4) \in [y_4/\sqrt{1-3y_4^2}, 1]$ such that $S \le 0$ for $y_2 \in [y_4/\sqrt{1-3y_4^2}, c]$ and $S \ge 0$ for $y_2 \in [c,1]$. Applying Lemma 4.4 with $f = S$ and $g = h_K$, we obtain (67). This completes the proof for $n = 5$.

For the remainder of the proof we assume that $n \ge 6$. The proof will be by induction on n. Assume that the lemma holds for all dimensions less than n. We shall make two further changes of variables, the first of which is to let $v_{n-3} = y_{n-3}/|Y_{n-3}|$. Then it is easy to check that

$$\frac{\partial(y_2,\dots,y_{n-4},v_{n-3})}{\partial(y_2,\dots,y_{n-3})} = \frac{1+y_2^2+\cdots+y_{n-4}^2}{(1+y_2^2+\cdots+y_{n-3}^2)^{3/2}} = \frac{|Y_{n-4}|^2}{|Y_{n-3}|^3}. \tag{69}$$

We also have

$$y_{n-3} = \frac{v_{n-3}|Y_{n-4}|}{\sqrt{1-v_{n-3}^2}} \tag{70}$$

and

$$L_{n-2} = \frac{y_{n-1}|Y_{n-4}|}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-v_{n-3}^2}}. \tag{71}$$

Setting

$$T = T(v_{n-3}, y_{n-1}) = \frac{U}{\int_{\Sigma_1} |Y_{n-3}|^{2-n}\, U\, dx} - \frac{1}{\int_{\Sigma_1} |Y_{n-3}|^{2-n}\, dx} \tag{72}$$

and noting that by (70) and (71), we have

$$h_K = h_K\left(1, y_2, \dots, y_{n-4}, \frac{v_{n-3}|Y_{n-4}|}{\sqrt{1-v_{n-3}^2}}, \frac{y_{n-1}|Y_{n-4}|}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-v_{n-3}^2}}, \frac{y_{n-1}|Y_{n-4}|}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-v_{n-3}^2}}, 0\right),$$

we use (69) to rewrite (66) in the form

$$\int_{\Sigma_2} h_K |Y_{n-3}|^{2-n} T \frac{|Y_{n-3}|^3}{|Y_{n-4}|^2}\, dx = \int_{\Sigma_2} h_K \frac{(1-v_{n-3}^2)^{\frac{n-5}{2}}}{|Y_{n-4}|^{n-3}} T\, dx \ge 0. \tag{73}$$

Here Σ_2 is the new domain of integration obtained from Σ by the last change of variable, given explicitly by

$$\int_{\Sigma_2} dx = \int_{L_2}^1 \int_{L_3}^{y_2} \cdots \int_{L_{n-4}}^{y_{n-5}} \int_{\frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}}}^{\frac{y_{n-4}}{\sqrt{|Y_{n-4}|^2+y_{n-4}^2}}} dv_{n-3}\, dy_{n-4} \cdots dy_2,$$

where we used (63) with $i = n-3$ to obtain the lower limit for integration with respect to v_{n-3}. The sign of the integrand in (73) coincides with the sign of T. Also, by (68) and (72), T is increasing with respect to v_{n-3}. It follows that the sign of the integrand in (73) coincides with

that of $v_{n-3} - m(y_{n-1})$, for a suitable function m of y_{n-1}. Hence, since h_K is also increasing with respect to v_{n-3}, the integral in (73) is

$$\int_{L_2}^{1}\int_{L_3}^{y_2}\cdots\int_{L_{n-4}}^{y_{n-5}}\left(-\int_{\frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}}}^{m(y_{n-1})} h_K\frac{(1-v_{n-3}^2)^{\frac{n-5}{2}}}{|Y_{n-4}|^{n-3}}|T|\,dv_{n-3}\right.$$
$$\left.+\int_{m(y_{n-1})}^{\frac{y_{n-4}}{\sqrt{|Y_{n-4}|^2+y_{n-4}^2}}} h_K\frac{(1-v_{n-3}^2)^{\frac{n-5}{2}}}{|Y_{n-4}|^{n-3}}T\,dv_{n-3}\right)dy_{n-4}\cdots dy_2$$
$$\geq\int_{L_2}^{1}\int_{L_3}^{y_2}\cdots\int_{L_{n-4}}^{y_{n-5}} h_K(M)\left(\int_{\frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}}}^{\frac{y_{n-4}}{\sqrt{|Y_{n-4}|^2+y_{n-4}^2}}}\frac{(1-v_{n-3}^2)^{\frac{n-5}{2}}}{|Y_{n-4}|^{n-3}}T\,dv_{n-3}\right)dy_{n-4}\cdots dy_2, \tag{74}$$

where

$$M=\left(1,y_2,\dots,y_{n-4},\frac{m|Y_{n-4}|}{\sqrt{1-m^2}},\frac{y_{n-1}|Y_{n-4}|}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-m^2}},\frac{y_{n-1}|Y_{n-4}|}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-m^2}},0\right).$$

Our aim is to show that the previous integral is nonnegative. To this end, we introduce our second and final change of variables, by letting $v_i = y_i/|Y_{n-4}|$, for $i = 2,\dots,n-4$. Then

$$\frac{\partial v_i}{\partial y_j}=\begin{cases}(|Y_{n-4}|^2-y_i^2)/|Y_{n-4}|^3, & \text{if } i=j=2,\dots,n-4,\\ -y_iy_j/|Y_{n-4}|^3, & \text{if } i,j=2,\dots,n-4,\ j\neq i,\end{cases}$$

and by manipulations similar to those set out for the initial change of variables (57), we find that

$$\frac{\partial(v_2,\dots,v_{n-4})}{\partial(y_2,\dots,y_{n-4})}=|Y_{n-4}|^{3-n}.$$

It is easy to check that in (74), the upper limit of integration with respect to v_{n-3} becomes $v_{n-4}/\sqrt{1+v_{n-4}^2}$ in terms of the new variables. For the limits of integration with respect to the new variables, we first obtain

$$\frac{y_{n-1}}{\sqrt{1-3y_{n-1}^2}}\leq v_{n-4}\leq\frac{1}{\sqrt{n-4}}.$$

Here the lower bound follows directly from $L_{n-4}\leq y_{n-4}$ and (63) with $i=n-4$. To obtain the upper bound, we first note that $y_2\leq 1$, by (57) and (61), and that this is equivalent to $2y_2^2+y_3^2+\cdots+y_{n-4}^2\leq|Y_{n-4}|^2$. Expressing this inequality in terms of the new variables, we see that the region of integration is contained in the ellipsoid

$$2v_2^2+v_3^2+\cdots+v_{n-4}^2\leq 1, \tag{75}$$

from which the upper bound follows directly. Now, once $v_{n-4}, v_{n-5}, \dots, v_{i+1}$ have been fixed, we find that

$$v_{i+1} \le v_i \le \left(\frac{1 - v_{n-4}^2 - \cdots - v_{i+1}^2}{i} \right)^{1/2},$$

for $i = 2, \dots, n-3$. Here the lower bound results from (61) and the changes of variable via (57), while the upper bound is again a consequence of (75) and the fact that to maximize v_i, one must take $v_2 = v_3 = \cdots = v_{i+1} = v_i$ to reach the boundary of the ellipsoid. Thus the integral in (74) becomes

$$\int_{\frac{y_{n-1}}{\sqrt{1-3y_{n-1}^2}}}^{\frac{1}{\sqrt{n-4}}} Z_K V \, dv_{n-4}, \tag{76}$$

where

$$Z_K = Z_K(v_{n-4}, m, y_{n-1}) = \int_{v_{n-4}}^{\left(\frac{1-v_{n-4}^2}{n-5}\right)^{1/2}} \cdots \int_{v_3}^{\left(\frac{1-v_{n-4}^2-\cdots-v_3^2}{2}\right)^{1/2}} h_K(M) \, dv_2 \cdots dv_{n-5}, \tag{77}$$

$$M = \frac{\left((1 - v_2^2 - \cdots - v_{n-4}^2)^{1/2}, v_2, \dots, v_{n-4}, \frac{m}{\sqrt{1-m^2}}, \frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}\sqrt{1-m^2}}, \frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}\sqrt{1-m^2}}, 0 \right)}{(1 - v_2^2 - \cdots - v_{n-4}^2)^{1/2}},$$

and

$$V = V(v_{n-4}, y_{n-1}) = \int_{\frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}}}^{\frac{v_{n-4}}{\sqrt{1+v_{n-4}^2}}} (1 - v_{n-3}^2)^{\frac{n-5}{2}} T \, dv_{n-3}.$$

We claim that

$$\int_{\frac{y_{n-1}}{\sqrt{1-3y_{n-1}^2}}}^{\frac{1}{\sqrt{n-4}}} Z_{C^n} V \, dv_{n-4} = 0, \tag{78}$$

where C^n, as in Lemma 4.5, satisfies $h_{C^n}(x) = x_1$ for all $x = (x_1, \dots, x_n)$ such that $0 = x_n \le x_{n-1} \le \cdots \le x_1 \le 1$. Indeed, we have $h_{C^n}(Y_{n-3}, L_{n-2}, L_{n-2}, 0) = 1$ in view of (58) with $i = n-3$. From this, it is clear that the left-hand side of (67), and hence the integral in (76), vanishes when $K = C^n$. This proves the claim.

We know that $T \le 0$ for $v_{n-3} \le m(y_{n-1})$ and $T \ge 0$ for $v_{n-3} \ge m(y_{n-1})$. It follows that $V \le 0$ and is decreasing if $v_{n-4}/\sqrt{1 + v_{n-4}^2} \le m(y_{n-1})$, and hence when

$$\frac{y_{n-1}}{\sqrt{1 - 3y_{n-1}^2}} \le v_{n-4} \le \frac{m(y_{n-1})}{\sqrt{1 - m(y_{n-1})^2}}.$$

For larger values of v_{n-4}, V is increasing with respect to v_{n-4} and so must become positive in order to satisfy (78). Consequently, there exists a function $q = q(m, y_{n-1})$ such that $V \le 0$ if

$v_{n-4} \le q$ and $V > 0$ if $v_{n-4} \ge q$. Assuming that Z_K/Z_{C^n} is an increasing function of v_{n-4}, we could then write

$$\int_{\frac{y_{n-1}}{\sqrt{1-3y_{n-1}^2}}}^{\frac{1}{\sqrt{n-4}}} Z_K V \, dv_{n-4} = -\int_{\frac{y_{n-1}}{\sqrt{1-3y_{n-1}^2}}}^{q} Z_{C^n} \frac{Z_K}{Z_{C^n}} |V| \, dv_{n-4} + \int_q^{\frac{1}{\sqrt{n-4}}} Z_{C^n} \frac{Z_K}{Z_{C^n}} V \, dv_{n-4}$$

$$\ge \frac{Z_K(q, m, y_{n-1})}{Z_{C^n}(q, m, y_{n-1})} \int_{\frac{y_{n-1}}{\sqrt{1-3y_{n-1}^2}}}^{\frac{1}{\sqrt{n-4}}} Z_{C^n}(v_{n-4}, m, y_{n-1}) V \, dv_{n-4} = 0,$$

thus completing the proof of the lemma.

It remains to show that Z_K/Z_{C^n} is an increasing function of v_{n-4}, for $n \ge 6$. Let L be the $(n-3)$-dimensional convex body with support function

$$h_L(x_1, \dots, x_{n-3})$$
$$= h_K\left(x_1, \dots, x_{n-4}, \frac{m}{\sqrt{1-m^2}}, \frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-m^2}}, \frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-m^2}}, x_{n-3}\right).$$

Since h_L is defined by fixing three coordinates in h_K, it is invariant under exchanges of the other coordinates. Therefore L has the symmetries of the coordinate cube Q^{n-3}.

By (36), we have

$$J_L(x_{n-4}) = \frac{\int_{S(x_{n-4})} h_L\left(1, \frac{x_2}{x_1}, \dots, \frac{x_{n-4}}{x_1}, 0\right) dx_2 \cdots dx_{n-5}}{\int_{S(x_{n-4})} dx_2 \cdots dx_{n-5}}$$

$$= \frac{\int_{S(x_{n-4})} \frac{1}{x_1} h_K\left(x_1, x_2, \dots, x_{n-4}, \frac{m}{\sqrt{1-m^2}}, \frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-m^2}}, \frac{y_{n-1}}{\sqrt{1-2y_{n-1}^2}\,\sqrt{1-m^2}}, 0\right) dx_2 \cdots dx_{n-5}}{\int_{S(x_{n-4})} \frac{1}{x_1} x_1 \, dx_2 \cdots dx_{n-5}}$$

$$= Z_K(x_{n-4}, m, y_{n-1}) / Z_{C^n}(x_{n-4}, m, y_{n-1}),$$

where the previous equality follows from (77) on noting that the limits of integration there coincide with (35) with n replaced by $n-3$. Moreover, since L has the symmetries of Q^{n-3}, there is a $t > 0$ such that tL satisfies the hypotheses of Lemma 4.3 with n replaced by $n-3$. In view of the obvious facts that $J_{tK} = tJ_K$ and that since $n \ge 6$, we have $3 \le n-3 < n$, we can appeal to the inductive hypothesis to conclude that $J_L(x_{n-4})$ is an increasing function of x_{n-4}. It follows that $Z_K(v_{n-4}, m, y_{n-1})/Z_{C^n}(v_{n-4}, m, y_{n-1})$ is an increasing function of v_{n-4} and the lemma is proved. □

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Dipartimento di Ingegneria dell'Informazione e di Scienze Matematiche, Università degli Studi di Siena, Via Roma 56, 53100 Siena, Italy
E-mail address: campi@dii.unisi.it

Department of Mathematics, Western Washington University, Bellingham, WA 98225-9063
E-mail address: Richard.Gardner@wwu.edu

Dipartimento di Matematica "U. Dini", Università degli Studi di Firenze, Piazza Ghiberti 27, 50122 Firenze, Italy
E-mail address: paolo@fi.iac.cnr.it