A THICK-THIN DECOMPOSITION OF J-HOLOMORPHIC CURVES

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Contents

Abstract. We show the existence of a thick thin decomposition of the domain of a pseudo holomorphic curve with boundary. The geometry of the thick part is bounded uniformly in the energy. Furthermore, in the thick part, there is a uniform bound on the differential which is exponential in the energy. The thin part consists of annuli of small energy the number of which is at most linear in the energy and genus. The decomposition can be seen as a quantitative version of Gromov compactness which applies before passing to the limit.

1. Introduction

A basic tool in the study of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ is the thick thin decomposition of hyperbolic structures. Namely, for $\Sigma \in \mathcal{M}_g$, let h be the unique conformal metric h of constant curvature -1. For $x \in \Sigma$ denote by $\text{inj}(\Sigma, x; h)$ the radius of injectivity of (Σ, h) at x. Write

$$Thick(\Sigma; h) := \{x \in \Sigma | \, \text{inj}(\Sigma, x; h) \geq \sinh^{-1}(1)\}$$
$$Thin(\Sigma; h) := \{x \in \Sigma | \, \text{inj}(\Sigma, x; h) < \sinh^{-1}(1)\}.$$

Then $Thin$ consists of at most $3g-3$ disjoint cylinders and the components of $Thick$ have geometry that is bounded uniformly in g. Among other things, the thick thin decomposition provides an intuitive picture of the Deligne Mumford compactification of $\mathcal{M}_g$.

This paper is concerned with an analogous construction for the moduli spaces $\mathcal{M}_g(M, J; A)$ of J-holomorphic curves of genus g in a symplectic manifold (M, ω) representing $A \in H_2(M; \mathbb{Z})$ with J an ω-tame almost complex structure J. Namely, for

$$(u : \Sigma \to M) \in \mathcal{M}_g(M, J; A),$$

we construct a decomposition

$$\Sigma = Thick(\Sigma; u) \cup Thin(\Sigma; u).$$

$Thin(\Sigma; u)$ consists of disjoint annuli and cylinders whose number is proportional to $g + \int_\Sigma u^*\omega$. With respect to the standard cylindrical metric on $Thin(\Sigma; u)$, $|du|$ decays exponentially in the distance from $\partial Thin(\Sigma; u)$. The components of $Thick(\Sigma; u)$, once properly normalized, have uniformly bounded geometry with the bounds exponential in the energy of u. Furthermore, on $Thick$ there is a bound on $|du|$ which is exponential in the energy. We construct an analogous decomposition for bordered J-holomorphic curves with boundary in a Lagrangian submanifold L. This time, it is the complex double of the domain which is decomposed. Our thick thin decomposition is related to Gromov compactness in the same way the hyperbolic thick thin decomposition of Riemann surfaces is related to the Deligne-Mumford compactification.

1.1. **The main result.**

To formulate the result more precisely, we introduce the following definitions.

Definition 1.1. Let Σ be a closed Riemann surface and let h be a conformal metric of constant curvature on Σ. A **geodesic annulus** in Σ is a doubly connected subset of the form

$$A(r_1, r_2, p; h) = \{y \in \Sigma | r_1 < d_h(y, p) < r_2\},$$

for some $p \in \Sigma$, and $0 < r_1 < r_2 < \text{inj}(p; h)$. For a simple closed geodesic γ in Σ, let R_γ be the width of a geodesic tubular neighborhood of γ. Suppose

γ is oriented with unit normal v. A **geodesic cylinder** in Σ is a doubly connected subset of the form

$$C(r_1, r_2, \gamma; h) = \{y = \exp r v_p | p \in \gamma, r \in (r_1, r_2)\},$$

for some

$$-R_\gamma \leq r_1 < r_2 \leq R_\gamma.$$

A **bubble decomposition** of Σ is a collection of geodesic annuli and geodesic cylinders in Σ with pairwise disjoint closures. Write

$$Thin(\mathcal{B}) := \bigcup_{I \in \mathcal{B}} I,$$
$$Thick(\mathcal{B}) := \Sigma \setminus Thin(\mathcal{B}).$$

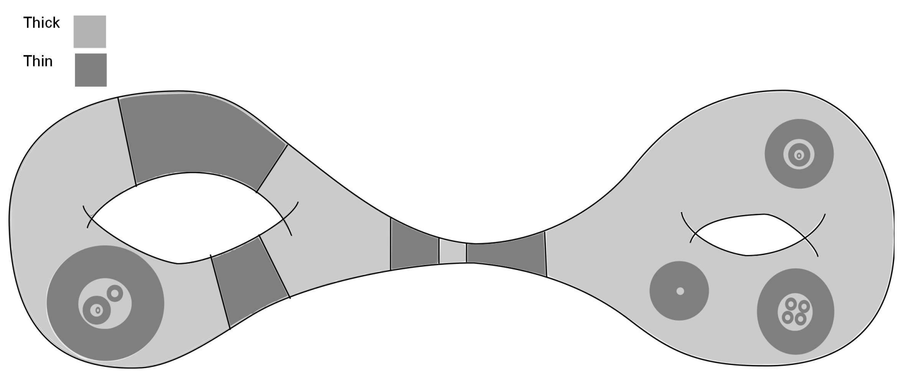

FIGURE 1.

For a bubble decomposition $\mathcal{B}$, let $V_{\mathcal{B}}$ denote a finite set with a bijection

$$V_{\mathcal{B}} \to \pi_0(Thick(\mathcal{B})), \qquad v \mapsto \Sigma_v.$$

Here, $\pi_0(\cdot)$ denoting the set of connected components.

Definition 1.2. Let Σ be a closed Riemann surface, let h be a conformal metric of constant curvature on Σ. Denote by ν_h the volume form on Σ. Let μ be a measure on Σ which is absolutely continuous with respect to any smooth volume form on Σ, and denote by

$$\frac{d\mu}{d\nu_h}$$

the Radon-Nikodym derivative of μ with respect to ν_h. Let $a, b, \delta > 0$. A (μ, h)**-adapted bubble decomposition** $\mathcal{B}$ with constants a, b, δ, is a bubble decomposition satisfying the following estimates.

(a) **Exponential decay in the thin part.** For any $I \in \mathcal{B}$, denote by $Mod(I)$ the modulus of I and by h_{st} the unique conformal metric such that (I, h_{st}) is isometric to $(0, Mod(I)) \times S^1$. Then for any $p \in I$,

$$\frac{d\mu}{d\nu_{h_{st}}}(p) \leq a e^{-b d_{h_{st}}(p, \partial I)}. \tag{1}$$

(b) **Bounded geometry and derivative.** For any $v \in V_{\mathcal{B}}$ let g_v be the genus of Σ_v and d_v its diameter with respect to h. Let $n_v := |\pi_0(\partial \Sigma_v)|$ and $\mu_v := \mu(\Sigma_v)$. Let

$$s_v := \frac{2(g_v + 1)}{d_v}$$

and define $h_v := s_v^2 h|_{\Sigma_v}$. Then the following hold.

(i)

$$\sup_{p \in \Sigma_v} \frac{d\mu}{d\nu_{h_v}}(p) \leq a e^{b(\mu_v + n_v)}. \tag{2}$$

(ii)

$$\inf_{p \in \Sigma_v} \operatorname{inj}(\Sigma_v, p; h_v) \geq a e^{-b(\mu_v + n_v)} \text{[1]}.$$

(iii) For any component γ of $\partial \Sigma_v$,

$$\ell(\gamma; h_v) > a e^{-b(\mu_v + n_v)}.$$

(iv) For any two distinct components γ_1 and γ_2 of $\partial \Sigma_v$,

$$d_{h_v}(\gamma_1, \gamma_2) \geq a e^{-b(\mu_v + n_v)}$$

(c) **Stability.** For any $v \in V_{\mathcal{B}}$ we have either

$$\mu_v \geq \delta,$$

or

$$2genus(\Sigma_v) + |\pi_0(\partial \Sigma_v)| \geq 3.$$

Remark 1.3. Note that because of the restriction to constant curvature metrics, only in the genus 0 case does the property of (μ, h)-adaptedness depend on h. In the other cases it would be more proper to talk of μ-adaptedness.

Remark 1.4. Note that the stability condition implies

$$|\pi_0(Thick(\mathcal{B}))| \leq 2g + 2\frac{\mu(\Sigma_v)}{\delta} - 3,$$

and a similar estimate for $|\pi_0(Thin(\mathcal{B}))|$.

[1] See Remark 6.4 below for the definition of inj for surfaces with boundary.

Remark 1.5. Fix an $E > 0$ and a $g \in \mathbb{N}$. The bounds (ii)-(iv) of Definition 1.2(b) imply that in the space of Riemannian manifolds with boundary equipped with the C^∞ topology, there exists a compact subset $K = K(g, E, a, b, \delta)$ with the following significance. For all measured Riemann surfaces (Σ, μ) with $genus(\Sigma) \leq g$ and $\mu(\Sigma) \leq E$, any constant curvature metric h on Σ, and any (μ, h)-adapted bubble decomposition $\mathcal{B}$ of Σ, the components of $Thick(\mathcal{B})$ belong to K. This follows from Theorem 3.3.1 in [1].

We now state the main result. Let (M, ω) be a compact symplectic manifold and J an ω-tame almost complex structure. For a Riemann surface Σ and a J-holomorphic curve

$$u : \Sigma \to M,$$

and for any subset $U \subset \Sigma$, write

$$\mu_u(U) := \int_U u^*\omega.$$

Theorem 1.6. *Let M be compact. Let $\mathcal{F}$ be the family of closed non-constant J-holomorphic curves in M. Then for every $(\Sigma, u) \in \mathcal{F}$ there is a conformal metric h of constant curvature on Σ and a (μ_u, h)-adapted bubble decomposition $\mathcal{B}_u$ of Σ with constants depending on $\mathcal{F}$ only.*

Remark 1.7. If we were to allow an arbitrary constant curvature metric in the genus 0 case, a simple counterexample to the theorem could be obtained as follows. Let h be the standard metric on $S^2 = \mathbb{C} \cup \{\infty\}$, let $u : S^2 \to M$ be a non-constant J-holomorphic curve, let $\psi_n : S^2 \to S^2$ be given by $\psi(z) = nz$ for any $z \in \mathbb{C} \subset S^2$ and let $u_n = u_n \circ \psi_n$. Then there are no uniformly (μ_u, h)-adapted bubble decompositions for this sequence.

1.2. **Curves with boundary.**

Definition 1.8. For any Riemann surface $\Sigma = (\Sigma, j)$, write $\overline{\Sigma} := (\Sigma, -j)$. The **complex double** is the Riemann surface

$$\Sigma_{\mathbb{C}} := \Sigma \cup \overline{\Sigma},$$

where the surfaces are glued together along the boundary by the identity. The complex structure on $\Sigma_{\mathbb{C}}$ is the unique one which coincides with j and with $-j$ when restricted suitably. $\Sigma_{\mathbb{C}}$ is endowed with a natural anti-holomorphic involution and for any $z \in \Sigma_{\mathbb{C}}$ we denote by $\overline{z}$ the image of z under this involution.

Definition 1.9. Let Σ be a connected Riemann surface. A subset $S \subset \Sigma_{\mathbb{C}}$ is said to be **clean** if either $S = \overline{S}$ or $S \cap \overline{S} = \emptyset$.

Definition 1.10. A bubble decomposition of $\Sigma_{\mathbb{C}}$ is said to be conjugation invariant if and only if all $I \in \mathcal{B}$ are clean and

$$I \in \mathcal{B} \qquad \Rightarrow \qquad \overline{I} \in \mathcal{B}.$$

Theorem 1.11. *Let $\mathcal{F}$ be the family of non-constant J-holomorphic curves in M with boundary in a compact Lagrangian submanifold L. Then for every $(\Sigma, u) \in \mathcal{F}$ there is a conjugation invariant conformal constant curvature metric h on $\Sigma_{\mathbb{C}}$ and a conjugation invariant (μ_u, h)-adapted bubble decomposition $\mathcal{B}_u$ of $\Sigma_{\mathbb{C}}$ with constants depending on $\mathcal{F}$ only.*

1.3. **The non-compact setting.**

Definition 1.12. For any Riemannian manifold X with sub-manifold Y and $\epsilon > 0$, we say that Y is ϵ-Lipschitz if

$$\frac{d_X(x,y)}{\min\{1, d_Y(x,y)\}} \geq \epsilon \qquad \forall x \neq y \in Y.$$

We say that Y is Lipschitz if there is an ϵ such that Y is ϵ-Lipschitz.

Denote by g_J the symmetrization of the positive definite form $\omega(\cdot, J\cdot)$. Denote by R the curvature of g_J, by B the second fundamental form of L with respect to g_J and for any tensor T on M or L let $\|T\|_n$ denote the C^n norm of T with respect to g_J.

Definition 1.13. Let S be a family of compact Riemann surfaces, possibly with boundary. We say that the data of S together with (M, ω, L, J) comprise a **bounded setting** if M and L are complete with respect to g_J and one of the following holds.

(a) $L = \emptyset$ and

$$\max\left\{\|R\|, \|J\|_2, \frac{1}{\mathrm{inj}(M; g_J)}\right\} < \infty.$$

(b) L is Lipschitz and

$$\max\left\{\|R\|_2, \|J\|_2, \|B\|_2, \frac{1}{\mathrm{inj}(M; g_J)}\right\} < \infty.$$

(c) Each connected component L' of L is Lipschitz and

$$\max\left\{\|R\|_2, \|J\|_2, \|B\|_2, \frac{1}{\mathrm{inj}(M; g_J)}\right\} < \infty.$$

Furthermore, there is an $\epsilon > 0$ such that for each $(u, \Sigma) \in \mathcal{F}$, there is a conformal metric h of constant curvature $0, \pm 1$, of unit area in case of zero curvature, such that $\partial\Sigma$ is totally geodesic and ϵ-Lipschitz.

Theorem 1.14. *Let $\mathcal{F}$ be the family of non-constant J-holomorphic curves in M with boundary in L and domain in a set S of Riemann surfaces such that S and (M, ω, J, L) comprise a bounded setting. Then for every $(\Sigma, u) \in \mathcal{F}$, there is a conjugation invariant conformal constant curvature metric h on $\Sigma_{\mathbb{C}}$ and a conjugation invariant (μ_u, h)-adapted bubble decomposition $\mathcal{B}_u$ of $\Sigma_{\mathbb{C}}$ with constants depending on $\mathcal{F}$ only.*

1.4. **Relation to Gromov compactness.** Fix an $E > 0$ and a $g \in \mathbb{N}$. Then for all u such that $genus(\Sigma) \le g$ and $\mu_u(\Sigma) \le E$, the components of $Thick(\mathcal{B}_u)$ are elements of K, where K is as in Remark 1.5. Furthermore, by Remark 1.4, $|\pi_0(Thick(\mathcal{B}_u)|$ is bounded uniformly in the set of all such u. By conformality we have that $|du|_h^2 = \frac{d\mu_u}{d\nu_h}$. Together with estimate (2) and elliptic regularity, we obtain C^∞ compactness of the restriction of J-holomorphic curves to their thick parts.

To see what happens in the thin part, let us elaborate on the geometric meaning of Definition 1.2(a). Let I be an open cylinder, let $u : I \to M$ be J-holomorphic and Let

$$\psi : I_L := (-L, L) \times S^1 \to I$$

be a biholomorphism. Let

$$h_{cone} = \sqrt{a} e^{-\frac{b}{2}(L-|r|)} h_{st}.$$

Then for $r \neq 0$, h_{cone} is a conformal metric on I_L whose shape is as an approximate cone as in the left of Figure 2. By inequality (1),

$$\frac{d\mu_{u\circ\psi}}{d\nu_{h_{cone}}} = \frac{d\mu_{u\circ\psi}}{d\nu_{h_{st}}} \frac{d\nu_{h_{st}}}{d\nu_{h_{cone}}} \le 1. \tag{3}$$

As $L \to \infty$ the approximate cones converge to an actual cone. See Figure 2. Gromov's compactness theorem is a consequence of this discussion, of elliptic

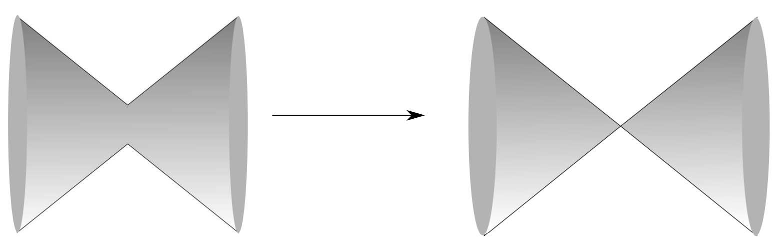

FIGURE 2.

regularity and of removal of singularities. Use of convergence theory of Riemannian manifolds in the context of Gromov compactness appears also in [5] and [6].

1.5. **The thin part.** The specification of the thin part of a J-holomorphic curve $u : \Sigma \to M$ is more involved then that of a hyperbolic surface. Furthermore, as a subset of Σ it appears to involve some choices which have to be made for each u. However, the combinatorial structure of the thick thin decomposition, e.g. the number of components of $Thin$, is independent of any such choices. For simplicity we describe the thin part of a closed J-holomorphic curve $u : \Sigma \to M$.

We recall the cylinder inequality [4, Lemma 4.7.3]. Let $I_a := [-a, a] \times S^1$. The cylinder inequality states that there are constants δ and c such that for any J-holomorphic map $u : I_a \to M$ we have

(4) $$\mu_u(I_a) \leq \delta \qquad \Rightarrow \qquad \mu_u(I_{a-t}) \leq e^{-ct}\mu_u(I_a),$$

for $t \in [\log 2, a]$.

Definition 1.15. An L-long neck is a geodesic cylinder or annulus $I \subset \Sigma$ such that $Mod(I) \geq 4L$, $\mu_u(I) \leq \delta/6$ and each component A of $\Sigma \setminus I$ is stable in the sense that one of the following conditions holds:

(a) $\mu(A) \geq \delta$;
(b) $2\,genus(A) + |\pi_0(\partial A)| \geq 3$.

For L large enough we define an equivalence relation on the set LN of L-long necks as follows. Suppose $I_1, I_2 \in LN$. Then $I_1 \sim I_2$ if and only if there exists an annulus I, not necessarily geodesic, such that I_1 and I_2 are nontrivially embedded in I and $\mu(I) \leq \delta/2$. That $\sim$ is indeed equivalence relation for L large enough follows from the cylinder inequality and the stability condition. See Lemma 5.5 below. Furthermore, each equivalence class is shown to contain an element of maximal modulus.

Pick an element A_c of maximal modulus from each $\sim$-equivalence class c and let $L_c = \frac{1}{2}Mod(A_c)$. There is a biholomorphism

$$f_c : A_c \to I_{\frac{1}{2}Mod(A_c)} = [-L_c, L_c] \times S^1,$$

unique up to automorphisms of the cylinder. The components of the thin part are the annuli $f_c^{-1}(I_{L_c - L})$. These are shown in the text to be disjoint for L large enough but chosen independently of the curve. There does not appear to be a unique maximal element in each equivalence class. Hence the choices referred to at the beginning of the subsection.

1.6. **Idea of the proof.** Let $\mathcal{B}$ be the set of components of the thin part as outlined in the previous subsection. In the text we show that $Thick(\mathcal{B})$ contains no long necks. It turns out that when $genus(\Sigma) > 0$, this implies that $\mathcal{B}$ is (μ_u, h)-adapted. Let us sketch for example how to obtain the derivative estimate in $Thick(\mathcal{B})$.

For this, recall the gradient inequality [4, Lemma 4.3.1] which says that there is a constant $\delta' > \delta$ such that for any ball $B_r(p) \subset \Sigma$ we have

$$\mu_u(B_r(p)) \leq \delta' \qquad \Rightarrow \qquad \frac{d\mu_u}{d\nu_h}(p) \leq \frac{1}{r^2}\mu_u(B_r(p)).$$

Let $v \in V_{\mathcal{B}}$. Suppose for concreteness that Σ_v is a geodesic disk $D = B_1(z; h_v) \subset \Sigma$. In this paragraph all quantities are measured with respect to h_v, so we omit it from the notation. Let $p \in \Sigma_v$ be a point where the derivative obtains its maximum. Using the gradient inequality and the construction of $\mathcal{B}$ there is an a priori bound on the derivative in the annulus $B_1(z) \setminus B_{1/2}(z)$. Suppose $p \in B_{1/2}(z)$ and let $d = \frac{d\mu_u}{d\nu_{h_v}}(p)$. Suppose $d > 4$ and consider the annulus $A = B_{1/2}(p) \setminus B_{1/d}(p)$. Then $\Sigma \setminus A$ is stable in the sense appearing in Definition 1.15. Indeed, the gradient inequality implies

$$\mu_u(B_{1/d}(p)) \geq \delta'. \tag{5}$$

The component $\Sigma \setminus B_{1/2}(p)$ is clearly stable by the assumption on the genus. Since D is free of long necks, for any $\frac{1}{d} < r_1 < r_2 < \frac{1}{2}$ such that

$$\log r_2/r_1 > L,$$

we must have

$$\mu_u\left(B_{r_2}(p) \setminus B_{r_1}(p)\right) > \delta/6.$$

In particular,

$$\log d \leq \frac{6L}{\delta}\mu_u(A) \leq \frac{6L}{\delta}\mu_u(\Sigma_v),$$

which is just inequality (2).

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2. Preliminaries

2.1. Annuli.

Definition 2.1. A **standard annulus** I is a surface of the form $K \times S^1$ with $K \subset \mathbb{R}$ an interval which may be open, closed or half closed. We denote by h_{st} the product metric on I which assigns to S^1 the length 2π. We let j_{st} be the complex structure induced on I by h_{st} and the product orientation on I. We take $Mod(I) := |K|$, where $|\cdot|$ denote the Lebesgue measure. An **Annulus** (I, j) is a doubly connected surface with complex structure j. Up to translation it is bi-holomorphic to a unique standard annulus I_{st}. We define $Mod(I, j) := Mod(I_{st})$. When the complex structure is clear from the context we omit it.

Let (I, j) be an annulus and let h be a conformal Riemannian metric on I. We call global cylindrical coordinates (ρ, θ) on I, with

$$a \leq \rho \leq b, \qquad 0 \leq \theta < 2\pi,$$

axially symmetric if

$$h = d\rho^2 + h_\theta(\rho)^2 d\theta^2. \tag{6}$$

We say h is **axially symmetric** if I has axially symmetric coordinates. In this case, the conformal length of I is given by

$$Mod(I, j) = \int_a^b \frac{1}{h_\theta(\rho)} d\rho. \tag{7}$$

Definition 2.2. Let I be an annulus and let $L = Mod(I)$. Suppose $L < \infty$. Then there is a biholomorphism $f : K \times S^1 \to I$ with K an interval whose infimum is the origin. The map f is unique up to a rotation and a holomorphic reflection. A **sub-cylinder** of I is a subset of the form

$$f(K' \times S^1),$$

with $K' \subset K$ an interval. For $a \leq b \in K$ we write

$$S(a, b; I) := f([a, b] \times S^1) \subset I.$$

We also define

$$C(a, b; I) := S(a, L - b; I),$$

for a, b in the appropriate range. Note that composing f with a holomorphic reflection of $K \times S^1$ replaces $S(a, b)$ with $S(L-b, L-a)$. When applying the above notations we shall be careful to remove this ambiguity. On the other hand, the notation $C(a, a; I)$ is well defined. Denote by K^c the closure of K and by I^c the closure of I. It is convenient to extend the above definitions to $a, b \in K^c$ by defining $S(a, b; I) := S(a, b; I^c) \cap I$ and $C(a, b; I^c) := C(a, b; I) \cap I^c$.

Definition 2.3. Let U be a Riemann surface biholomorphic to the unit disk D_1. Let h be a conformal metric on U and let $z \in U$. Then there is a biholomorphism $\phi : U \to D_1$ with $\phi(z) = 0$, unique up to rotation. The **conformal radius of U viewed from z** is defined to be

$$r_{conf}(U, z; h) := 1/\|d\phi(z)\|_h.$$

Note that $r_{conf}(U, z; h)$ is not conformally invariant, since it depends on the metric at z. However, let ν_h denote the volume form of h, let μ be an absolutely continuous measure on U. Then the expression $\frac{d\mu(z)}{d\nu_h} r^2_{conf}(z)$ is conformally invariant.

The cases of interest for us will be conformal radii of geodesic disks with metrics of constant curvature K, viewed from their center. In these cases, the metric can be written in polar coordinates as

$$h = d\rho^2 + h_\theta^2(\rho) d\theta^2, \tag{8}$$

where

$$h_\theta(\rho) = \begin{cases} \sinh(\rho), & K = -1, \\ \rho, & K = 0, \\ \sin(\rho), & K = 1. \end{cases} \tag{9}$$

So, the conformal radius of $B_r(p)$ viewed from p is given by

$$r_{conf} = \exp(f(r))$$

where f is the function defined by

$$f'(r) = \frac{1}{h_\theta(r)}, \qquad f(r) = \log(r) + O(r) \text{ as } r \to 0.$$

More explicitly,

$$f(r) = \log(r) + \int_0^r \left(\frac{1}{h_\theta(\rho)} - \frac{1}{\rho} \right) d\rho. \tag{10}$$

It follows from equation (10) that

$$r_{conf} \geq r, \qquad K = 0, 1, \tag{11}$$

and for any κ there exists a constant $c > 0$ such that

$$r_{conf} \geq cr, \qquad K = -1, \ r < \kappa. \tag{12}$$

2.2. **Collars.** For later reference we include a statement of the thick thin decomposition for surfaces of genus $g > 1$. In the following we assume the surfaces are endowed with their unique metric h of constant curvature -1.

Theorem 2.4. [2, 4.1.1] *Let Σ be a compact Riemann surface of genus $g \geq 2$, and let $\gamma_1, ..., \gamma_m$ be pairwise disjoint simple closed geodesics on Σ. Then the following hold:*

(a) $m \leq 3g - 3$.

(b) There exist simple closed geodesics $\gamma_{m+1}, ..., \gamma_{3g-3}$, which, together with $\gamma_1, ..., \gamma_m$, decompose Σ into pairs of pants.

(c) The collars

$$\mathcal{C}(\gamma_i) = \{p \in \Sigma | dist(p, \gamma_i) \leq w(\gamma_i)\}$$

of widths

$$w(\gamma_i) = \sinh^{-1} \left(1/\sinh \left(\frac{1}{2} \ell(\gamma_i) \right) \right)$$

are pairwise disjoint for $i = 1, ..., 3g - 3$.

(d) Each $\mathcal{C}(\gamma_i)$ is isometric to the cylinder $[-w(\gamma_i), w(\gamma_i)] \times S^1$ with the Riemannian metric

$$d\rho^2 + \frac{\ell^2(\gamma_i) \cosh^2(\rho)}{4\pi^2} d\theta^2.$$

Denote by $\mathrm{inj}(p; h)$ the radius of injectivity of Σ at $p \in \Sigma$, i.e. the supremum of all r such that $B_r(p)$ is an embedded disk.

Theorem 2.5. [2, 4.1.6] *Let $\beta_1, ..., \beta_k$ be the set of all simple closed geodesics of length $\leq \sinh^{-1} 1$ on Σ. Then $k \leq 3g - 3$ and the following hold.*

(a) The geodesics $\beta_1, ..., \beta_k$ are pairwise disjoint.

(b) $\mathrm{inj}(p; h) > \sinh^{-1} 1$ *for all* $p \in \Sigma - (\mathcal{C}(\beta_1) \cup ... \cup \mathcal{C}(\beta_k))$.

(c) If $p \in \mathcal{C}(\beta_i)$, and $d = dist(p, \partial\mathcal{C}(\beta_i))$, then

$$\sinh(\mathrm{inj}(p; h)) = \cosh \frac{1}{2} \ell(\beta_i) \cosh d - \sinh d. \tag{13}$$

Definition 2.6. Let Σ be a closed Riemann surface. Let h be a metric of constant curvature $0, \pm 1$ on Σ. Let $\gamma \subset \Sigma$ be a simple closed geodesic in Σ. In Theorem 2.4, $\mathcal{C}(\gamma)$ was defined when $genus(\Sigma) > 1$. We extend the definition to the case $genus(\Sigma) \leq 1$ by letting

$$\rho_{max} = \max_{\{p \in \Sigma\}} d(p, \gamma),$$

and

$$\mathcal{C}(\gamma) = \{p \in \Sigma | d(p, \gamma) < \rho_{max}\}.$$

When $genus(\Sigma) = 0$, this is the sphere with two antipodes removed. It is also easy to verify that when $genus(\Sigma) = 1$, this is a torus with a geodesic parallel to γ removed. Global cylindrical coordinates ρ and θ are defined on $\mathcal{C}(\gamma)$ in the same way as for $genus(\Sigma) > 1$. Namely, $\rho(p) = d(p, \gamma; h)$ for any $p \in \mathcal{C}(\gamma)$ and θ maps lines of constant ρ to S^1 isometrically up to multiplication with an overall constant.

2.3. Cleanness.

Lemma 2.7. *Let Σ be a Riemann surface and let I_1 and I_2 be clean subsets of $\Sigma_{\mathbb{C}}$. Then $I_1 \cup I_2$ is clean if and only if at least one of the following holds:*

- *(a) I_1 and I_2 are conjugation invariant.*
- *(b) $I_i \cap \overline{I}_j = \emptyset$ for $1 \leq i, j \leq 2$.*
- *(c) $I_1 = \overline{I}_2$.*
- *(d) $I_1 \subset I_2$ or $I_2 \subset I_1$.*

Proof. If condition (a) holds then $I_1 \cup I_2$ is conjugation invariant and therefore clean. If condition (b) holds then

$$I_1 \cup I_2 \cap \overline{I_1 \cup I_2} = \bigcup_{1 \leq i,j \leq 2} I_i \cap \overline{I}_j = \emptyset.$$

So, $I_1 \cup I_2$ is again clean. That conditions (c) and (d) imply cleanness of $I_1 \cup I_2$ is obvious.

Conversely, suppose $I_1 \cup I_2$ is clean. We divide into the case where I_1 is conjugation invariant and the case where it is not. If I_1 is conjugation invariant then in particular $I_1 \cup I_2 \cap \overline{I_1 \cup I_2} \neq \emptyset$. So, by cleanness, $I_1 \cup I_2 = \overline{I_1 \cup I_2}$. Now, if condition (d) holds we are done. So we may assume that $I_2 \setminus I_1 \neq \emptyset$. Let $p \in I_2 \setminus I_1$. Then, since $I_1 = \overline{I_1}$, $p \in \overline{I_1 \cup I_2} \setminus \overline{I}_1 \subset \overline{I}_2$. In particular, $I_2 \cap \overline{I}_2 \neq \emptyset$. By cleanness of I_2 this implies I_2 is also conjugation invariant, so condition (a) holds.

Next we consider the case where I_1 is not conjugation invariant. If I_2 is conjugation invariant, exchanging the roles of I_1 and I_2 in the previous paragraph we deduce that condition (d) holds and we are done. Suppose now that I_2 is not conjugation invariant and consider $I_1 \cup I_2$. If $I_1 \cup I_2$ is conjugation invariant, cleanness and non conjugation invariance of I_1 and I_2 imply that

$$I_1 \subset \overline{I}_2 \setminus \overline{I_1} \subset \overline{I}_2$$

and, similarly, $I_2 \subset \overline{I}_1$. By conjugation invariance of the inclusion of sets, this implies Condition (c). If, on the other hand, $I_1 \cup I_2$ is not conjugation invariant, cleanness implies that $I_1 \cup I_2 \cap \overline{I_1 \cup I_2} = \emptyset$. So Condition (b) holds. □

Lemma 2.8. *Let Σ be a connected Riemann surface with non-empty boundary. Let $g = genus(\Sigma_{\mathbb{C}})$ and let $\gamma \subset \Sigma_{\mathbb{C}}$ be a simple closed geodesic in $\Sigma_{\mathbb{C}}$.*

(a) For $g \geq 2$, $\mathcal{C}(\gamma)$[2] is clean if $\ell(\gamma) < 2\sinh^{-1}(1)$.

(b) For $g \leq 1$, $\mathcal{C}(\gamma)$ is clean if and only if either $\gamma \subset \partial\Sigma$, or

$$\gamma \cap \partial\Sigma \neq \emptyset$$

and $\gamma \perp \partial\Sigma$.

Proof. (a) Since conjugation is an isometry we have that $\overline{\gamma}$ is also a simple closed geodesic, $\ell(\gamma) = \ell(\overline{\gamma})$ and $\overline{\mathcal{C}(\gamma)} = \mathcal{C}(\overline{\gamma})$. From Theorem 2.4(c) it therefore follows that $\mathcal{C}(\gamma) \cap \overline{\mathcal{C}(\gamma)} \neq \emptyset$ if and only if $\gamma \cap \overline{\gamma} \neq \emptyset$. Thus it suffices to prove that γ is clean for γ short enough. Suppose $\gamma \cap \overline{\gamma} \neq \emptyset$. If $\gamma \neq \overline{\gamma}$ then γ intersects $\overline{\gamma}$ transversally. Therefore, by [2, 4.1.2], $\ell(\gamma) \geq 2\sinh^{-1}(1)$.

(b) By definition of $\mathcal{C}(\gamma)$ for this case, it is open and dense in Σ. Therefore, we always have $\mathcal{C}(\gamma) \cap \overline{\mathcal{C}(\gamma)} \neq \emptyset$. Thus, $\mathcal{C}(\gamma)$ is clean if and only if $\mathcal{C}(\gamma)$ is conjugation invariant. That is, since conjugation is an isometry, if and only if $\gamma = \overline{\gamma}$. We claim that this is equivalent to the condition of the Lemma. For this it suffices to show that if $\gamma \not\subset \partial\Sigma$, then $\gamma = \overline{\gamma}$ if and only if $\gamma \cap \partial\Sigma \neq \emptyset$ and $\gamma \perp \partial\Sigma$.

Indeed, if $\gamma \cap \partial\Sigma = \emptyset$, then since γ is connected it is contained in one component $\Sigma \setminus \partial\Sigma$ and is thus not conjugation invariant. So we assume $\gamma \cap \partial\Sigma \neq \emptyset$. Let $p \in \gamma \cap \partial\Sigma$ and let v be a vector tangent to γ at p. Since p is fixed under conjugation and since both γ and $\overline{\gamma}$ are geodesics, we have that $\gamma = \overline{\gamma}$ if and only if $\overline{v}$ is also tangent to γ at p. But γ intersects $\partial\Sigma$ transversally since they are distinct geodesics. Therefore $v \neq \overline{v}$. Since $T_p\gamma$ is one dimensional it follows that $\overline{v}$ is tangent to γ if and only if $\overline{v} = -v$. That is, $\overline{v}$ is tangent to γ if and only if v points in the direction of the imaginary axis in $T_p\Sigma_{\mathbb{C}}$. Since $T_p\partial\Sigma$ is the real axis, the claim follows. □

Lemma 2.9. *Let Σ be a Riemann surface. Let I_1 and I_2 be doubly connected and clean subsets of $\Sigma_{\mathbb{C}}$ which do not contain a component of $\partial\Sigma$. Suppose $I_1 \hookrightarrow I_2$ homologically nontrivially. Then I_2 is conjugation invariant if and only if I_1 is conjugation invariant.*

Proof. First we claim that I_1 is conjugation invariant if and only if each component of $I_1 \setminus \partial\Sigma$ is simply connected. Assume I_1 is not conjugation invariant. Then since I_1 is clean, we have $I_1 \setminus \partial\Sigma = I_1$. So, I_1 is the only

[2]See Definition 2.6.

connected component and is not simply connected. Conversely, assume I_1 is conjugation invariant. Suppose by contradiction that for one component A of $\Sigma_{\mathbb{C}} \setminus \partial\Sigma$, there is a component of $I_1 \cap A$ that is not simply connected. Since $I_1 \cap A$ is isometric to $I_1 \cap \overline{A}$, it is homotopy equivalent to it. Since I_1 does not contain any component of $\partial\Sigma$, each component of $I_1 \cap \partial\Sigma$ is contractible. Thus the Mayer Vietoris sequence implies that I_1 is at least two connected. This is a contradiction.

Now we prove the lemma. Assume I_1 is not conjugation invariant. Then since I_1 is clean, we have $I_1 \cap \overline{I}_1 = \emptyset$. Since $I_2 \subset I_1$, this implies $I_2 \cap \overline{I}_2 = \emptyset$. Conversely, assume by contradiction that I_1 is conjugation invariant and I_2 is not. Let A be the connected component of $\Sigma_{\mathbb{C}}$ containing I_2. I_2 is then contained in $I_1 \cap A$ which by the previous paragraph is simply connected. This contradicts the fact that I_2 is embedded non-trivially in I_1. □

3. Thick thin measure

For the rest of the discussion, fix constants $c_1, c_2, c_3, \delta_1, \delta_2 > 0$ such that $c_3 \leq 1$ and that $\delta_2 < \frac{1}{2}\delta_1$.

Definition 3.1. Let (Σ, j) be a Riemann surface, possibly bordered. Let μ be a finite measure on Σ and extend μ to a measure on $\Sigma_{\mathbb{C}}$ by reflection. That is,

$$\mu(U) := \mu(\overline{U}),$$

for $U \subset \overline{\Sigma}$ a measurable set. Suppose further that μ is absolutely continuous and has a continuous density $\frac{d\mu}{d\nu_h}$, where h is any Riemannian metric on $\Sigma_{\mathbb{C}}$.

The measure μ will be called **thick thin** if it satisfies the following two conditions.

(a) **gradient inequality**. Let $U \subset \Sigma_{\mathbb{C}}$ be biholomorphic to the unit disk such that $U \cap \partial\Sigma$ is connected, and let $z \in U$. Then for any conformal metric h on $(\Sigma_{\mathbb{C}}, j)$,

$$\mu(U) < \delta_1 \quad \Rightarrow \quad \frac{d\mu}{d\nu_h}(z) \leq c_1 \frac{\mu(U)}{r_{conf}^2}, \tag{14}$$

where $r_{conf} = r_{conf}(U, z; h)$.

(b) **cylinder inequality**. Let $I \subset \Sigma_{\mathbb{C}}$ be clean and doubly connected such that $Mod(I) > 2c_2$. Then for all $t \in \left(c_2, \frac{1}{2}Mod(I)\right)$ we have,

$$\mu(I) < \delta_2 \qquad \Rightarrow \qquad \mu(C(t, t; I)) \leq e^{-c_3 t}\mu(I).$$

A family of measured Riemann surfaces which are thick thin with respect to given constants c_i, δ_i will be referred to as a uniformly thick thin family.

Remark 3.2. Let μ be a thick thin measure on Σ, and let h be a conformal metric of constant curvature $K = 0, \pm 1$ on $\Sigma_{\mathbb{C}}$. By inequalities (11) and (12), there is a constant c_1' depending linearly on c_1 such that for any $z \in \Sigma$ and

$r \in (0, \min(\sinh^{-1}(1), \mathrm{inj}(\Sigma; h, z)))$,

$$\mu(B_r(z;h)) < \delta_1 \Rightarrow \frac{d\mu}{d\nu_h}(z) \leq c_1' \frac{\mu(B_r(z;h))}{r^2}. \tag{15}$$

Let Σ, μ and h be as in Remark 3.2. For any point $z \in \Sigma_{\mathbb{C}}$, let $d = \frac{d\mu}{d\nu_h}(z)$ and let

$$r_d := \sqrt{\frac{c_1'\delta_1}{d}}. \tag{16}$$

Lemma 3.3. *Suppose*

$$r_d \in (0, \min(\sinh^{-1}(1), \mathrm{inj}(\Sigma; h, z))).$$

Then

$$\mu(B_{r_d}(z;h)) \geq \delta_1.$$

Moreover, we have

$$\frac{d\mu}{d\nu_h}(z) \leq c_1' \frac{\mu(B_r(z;h))}{r^2}, \qquad r \leq r_d.$$

Proof. This is immediate from the gradient inequality. □

To simplify our formulas, we scale μ so that $c_1'\delta_1 = 1$.

We denote by $\mathcal{M} = \mathcal{M}(c_1, c_2, c_3, \delta_1, \delta_2)$ the family of measured Riemann surfaces (Σ, j, μ) such that μ is thick-thin.

Lemma 3.4. *There is a constant a with the following significance. Let $(\Sigma, \mu) \in \mathcal{M}$. Let $I \subset \Sigma_{\mathbb{C}}$ be clean and doubly connected, and let $h = h_{st}$. Suppose $\mu(I) < \delta_2$. Let $z \in C(c_2 + \pi, c_2 + \pi; I)$ be a point with cylindrical coordinates*

$$(\rho, \theta) \in \left[-\frac{1}{2}Mod(I) + c_2 + \pi, \frac{1}{2}Mod(I) - c_2 - \pi\right] \times S^1.$$

Then,

$$\frac{d\mu}{d\nu_h}(z) < a e^{-c_3(\frac{1}{2}Mod(I) - |\rho|)}\mu(I). \tag{17}$$

Proof. Combining the gradient inequality and the cylinder inequality,

$$\begin{aligned} \frac{d\mu}{d\nu_h}(z) &\leq \frac{c_1}{\pi^2}\mu\left([\rho - \pi, \rho + \pi] \times S^1\right) \\ &\leq \frac{c_1}{\pi^2}\mu\left([-|\rho| - \pi, |\rho| + \pi] \times S^1\right) \\ &\leq \frac{c_1}{\pi^2}e^{-c_3(\frac{1}{2}Mod(I) - \pi - |\rho|)}\mu(I). \end{aligned} \tag{18}$$

□

Definition 3.5. Let $(\Sigma, \mu) \in \mathcal{M}$. A connected compact sub-manifold with boundary $A \subset \Sigma_{\mathbb{C}}$ is said to be ***μ*-stable** if one of the following holds:

(a) $\mu(A) \geq \delta_1/2$;
(b) $\#\pi_0(\partial A) \geq 2$ and $\mu(A) \geq \delta_2/6$;

(c) $2\,genus(A) + \#\pi_0(\partial A) \geq 3$.

A compact sub-manifold with boundary $A \subset \Sigma_C$ is said to be μ-stable if each of its connected components is μ-stable.

4. Preparation

4.1. **Choice of spherical metric.** As explained in Remark 1.7, the genus 0 case requires a non-trivial choice of Fubini-Study metric. This is done in the following lemma.

Lemma 4.1. *There is a constant K_0 with the following property. For each $(\Sigma, \mu) \in \mathcal{M}$ with $genus(\Sigma_{\mathbb{C}}) = 0$ and $\mu(\Sigma_{\mathbb{C}}) \neq 0$, there exists a conjugation invariant unit curvature Fubini-Study metric, h, on $\Sigma_{\mathbb{C}}$ such that one of the following conditions holds.*

(a)

$$\sup_{\Sigma_{\mathbb{C}}} \frac{d\mu}{d\nu_h} \leq K_0.$$

(b) There is a point $q \in \Sigma$ such that

$$\mu(B_{\pi/2}(q; h)) \geq \delta_1$$

and

$$\left.\frac{d\mu}{d\nu_h}\right|_{B_{\pi/2}(q;h)} \leq K_0.$$

Furthermore, if $\partial\Sigma \neq \emptyset$, then $q \in \partial\Sigma$.

(c) $\partial\Sigma \neq \emptyset$. Use h to identify $\Sigma_{\mathbb{C}}$ with the standard sphere in such a way that $\partial\Sigma$ is identified with the equator and let q be the north pole. Letting $d := \frac{d\mu}{d\nu_h}(q)$, we have $r_d \leq \pi/4$, and

$$\sup_{x \in B_{r_d}(q)} \frac{d\mu}{d\nu_h}(x) \leq K_0 d.$$

Furthermore, for any disc $D \subset \Sigma_{\mathbb{C}}$ of radius $\min\left\{\sqrt{\frac{\delta_1/2}{\pi K_0}}, \pi/4\right\}$, the complement $\Sigma_{\mathbb{C}} \backslash D$ is stable.

Remark 4.2. When $\partial\Sigma = \emptyset$ we have that $\Sigma_{\mathbb{C}}$ has two components: Σ and $\overline{\Sigma}$. The conjugate component is if no interest. In the sequel we shall avoid talking about $\Sigma_{\mathbb{C}}$ in the closed context.

Remark 4.3. The three cases are correspond to the quantitative counterparts of the possible behaviors of the Gromov limit of a sequence of genus 0 curves:

(a) No bubbling.
(b) Bubbling off of spheres or disks without the boundary degenerating.
(c) Bubbling in which the boundary degenerates to a point.

Proof. Choose initially any conjugation invariant unit curvature metric h_0 on $\Sigma_{\mathbb{C}}$. Let $p \in \Sigma$ be a point where the maximum of $\frac{d\mu}{d\nu_{h_0}}$ is obtained, and let

$$d_0 := \frac{d\mu}{d\nu_{h_0}}(p).$$

If $r_{d_0} > \pi/6$, then the estimate

$$\frac{d\mu}{d\nu_{h_0}} \leq \frac{36}{\pi^2},$$

holds globally. Thus, condition (a) holds with $K_0 = \frac{36}{\pi^2}$.

Assume

$$r_{d_0} \leq \pi/6. \tag{19}$$

If $\partial\Sigma = \emptyset$, let $p_1 := p$. Otherwise, let p_1 be the midpoint of a length minimizing geodesic which connects p and $\overline{p}$. Let p_2 be the antipode of p_1 with respect to h_0. With respect to h_0, let

$$(\rho, \theta) : \Sigma_{\mathbb{C}} \setminus \{p_2\} \to \mathbb{C}$$

be geodesic polar coordinates centered at p_1. In case $\partial\Sigma = \emptyset$, assume further that $\partial\Sigma \setminus \{p_2\}$ is given by $\{\theta = 0\}$. Let $\phi : \Sigma_{\mathbb{C}} \setminus \{p_2\} \to \mathbb{C}$ be stereographic projection. Explicitly, in polar coordinates ϕ is given by

$$(\rho, \theta) \mapsto \tan\frac{\rho}{2}e^{i\theta}. \tag{20}$$

Note that p and $\overline{p}$ are mapped by ϕ to the imaginary axis.

Let $r = d(p, p_1; h_0)$. Suppose first that

$$r < 2r_{d_0}. \tag{21}$$

We prove that condition (b) holds. Let $\chi : \mathbb{C} \to \mathbb{C}$ be the map

$$z \mapsto \frac{z}{\tan\left(\frac{r + r_{d_0}}{2}\right)}.$$

Let $\psi : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ be the holomorphic map defined by

$$\psi\big|_{\Sigma_{\mathbb{C}} \setminus p_2} = \phi^{-1} \circ \chi \circ \phi,$$

and let $h_1 := \psi^* h_0$. Note that the change of metric from h_0 to h_1 scales the disc of radius $r + r_{d_0}$ around p_1 to become the hemisphere centered at p_1. In particular, $B_{r_{d_0}}(p; h_0) \subset B_{\pi/2}(p_1; h_1)$. So, by Lemma 3.3,

$$\mu(B_{\pi/2}(p_1; h_1)) \geq \delta_1.$$

We show now that the energy density is bounded on the hemisphere centered at p_1, uniformly in $\mathcal{M}$. First note that for any $z \in \Sigma_{\mathbb{C}}$,

$$\frac{d\mu}{d\nu_{h_1}}(z) = \frac{d\mu}{d\nu_{h_0}}(z)\frac{d\nu_{h_0}}{d\nu_{h_1}}(z) = \|d\psi\|_{h_0}^{-2}(z)\frac{d\mu}{d\nu_{h_0}}(z). \tag{22}$$

A computation gives

$$\|d\psi\|_{h_0}(x) = \frac{\tan\frac{r+r_{d_0}}{2}}{\cos^2(\rho(x)/2)\tan^2\frac{r+r_{d_0}}{2} + \sin^2(\rho(x)/2)}. \tag{23}$$

Assumptions (19) and (21) imply that $\|d\psi\|_{h_0}^{-1}$ increases with distance from p_1 on the ball $B_{\pi/2}(p_1;h_1)$. In particular,

$$\sup_{B_{\pi/2}(p_1;h_1)} \|d\psi\|_{h_0}^{-1} = \sin(r + r_{d_0}). \tag{24}$$

Using equations (21), (22), and the definition of r_{d_0}, we get

$$\sup_{B_{\pi/2}(p_1;h_1)} \frac{d\mu}{d\nu_{h_1}} \leq K_0,$$

for an appropriate constant K_0 which is independent of μ. This is condition (b) with $h = h_1$ and $q = p_1$.

Now suppose

$$r \geq 2r_{d_0}. \tag{25}$$

Let $\chi : \mathbb{C} \to \mathbb{C}$ be the map

$$z \mapsto \frac{z}{\tan(\frac{r}{2})}.$$

Let $\psi : \Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ be the holomorphic map defined by

$$\psi\big|_{\Sigma_{\mathbb{C}}\setminus\{p_2\}} = \phi^{-1} \circ \chi \circ \phi,$$

and let $h_1 := \psi^* h_0$. Write $d_2 := \frac{d\mu}{d\nu_{h_1}}(p)$, $A := B_{r_{d_2}}(p;h_1)$, and

$$C := \frac{\|d\psi\|_{h_0}^2(p)}{\inf_{w\in A}\|d\psi\|_{h_0}^2(w)}.$$

Then we have the bound

$$\frac{d\mu}{d\nu_{h_1}}\bigg|_A \leq C d_2.$$

Note that $\|d\psi\|_{h_0}$ is obtained by substituting r in place of $r + r_{d_0}$ in equation (24), and that $r \leq \pi/2$. Therefore, $\|d\psi\|_{h_0}$ is decreasing for $\rho(x) \in [0,\pi]$. Let x_0 be the point which maximizes $\rho(x)$ on A. One computes that

$$C = \frac{\cos^2(\rho(x_0)/2)}{2\cos^2 r/2} + \frac{\sin^2(\rho(x_0)/2)}{2\sin^2(r/2)}.$$

To bound C it suffices to bound the ratio $\frac{\rho(x_0)}{r}$. By direct computation,

$$\rho(x_0) = 2\tan^{-1}\left(\tan\frac{r}{2}\tan\frac{\pi/2 + r_{d_2}}{2}\right).$$

Note now that $r_{d_2} = \frac{r_{d_0}}{\sin r}$. Using assumption (25) and the fact the function $r \mapsto \frac{r}{2\sin r}$ is monotone increasing for $0 < r < \pi$ and that $r \leq \pi/2$, we conclude that $r_{d_2} \leq \pi/4$. Thus, $\frac{\rho(x_0)}{r} \leq C'$ for some uniform constant

C'. There is therefore an a priori constant K_0 bounding C. This gives condition (c) with $h = h_1$ and $q = p$.

We prove the last part of the claim. Suppose condition (a) holds. As is well known, the gradient inequality implies

$$\mu(\Sigma_{\mathbb{C}}) \neq 0 \Rightarrow \mu(\Sigma_{\mathbb{C}}) \geq \delta_1.$$

It is straightforward to verify that $\mu(D) \leq \delta_1/2$, implying the claim. If condition (b) holds, one similarly verifies that

$$\mu(D \cap B_{\pi/2}(q; h)) \leq \delta_1/2.$$

So, for each component[3] Σ' of $\Sigma_{\mathbb{C}}$,

$$\mu(\Sigma' \setminus D) \geq \mu(B_{\pi/2}(q; h) \setminus D) \geq \delta_1/2.$$

Suppose now that condition (c) holds. Then D meets at most one of the discs $B_1 = B_{r_d}(q; h)$ and $B_2 = B_{r_d}(\overline{q}; h)$. By Lemma 3.3, $\mu(B_i) \geq \delta_1$ for $i = 1, 2$. □

4.2. **Admissible annuli.** From now to Section 6, we fix a $(\Sigma, \mu) \in \mathcal{M}$. However, all constants are that appear in the sequel are independent of Σ and μ. Let

$$\Sigma' = \begin{cases} \Sigma, & \partial\Sigma = \emptyset, \\ \Sigma_{\mathbb{C}}, & \partial\Sigma \neq \emptyset. \end{cases}$$

If $genus(\Sigma') \geq 2$, let h be the unique conformal metric of constant curvature -1 on Σ'. If $genus(\Sigma') = 1$, let h be the unique conformal metric of constant curvature 0 and of unit area. Finally, if $genus(\Sigma') = 0$, let h be a conformal metric of constant curvature 1 which satisfies the property of Lemma 4.1.

Our goal in the following three sections is to construct a (μ, h)-adapted bubble decomposition of $\Sigma_{\mathbb{C}}$. We make the following assumption

Assumption 4.3.1. (Σ, μ) satisfies one of the following:

(a) $genus(\Sigma') = 0$ and h does not satisfy condition (a) in Lemma 4.1.

(b)
$$genus(\Sigma') = 1, \ \textit{and} \ \mu(\Sigma') > \delta_2.$$

(c)
$$genus(\Sigma') > 1.$$

The cases not covered Assumption 4.3.1 are referred to as the **trivial cases**. The genus 0 trivial case automatically admits a (μ, h)-adapted bubble decomposition and requires no treatment. The trivial genus 1 case will be treated separately in the proof of Theorem 1.6.

We need to partially break the symmetry in the cases of genus 0 and 1 in Definition 4.4 below. For this we introduce some notation. Suppose $genus(\Sigma') = 0$. If in Lemma 4.1 condition (b) holds for h, let q be the point

[3]See remark 4.2.

given there and let $\tilde{\Sigma} := \Sigma' \setminus \{q\}$. If, instead, condition (c) holds, let q be the point given there and let $\tilde{\Sigma} := \Sigma' \setminus \{q, \overline{q}\}$. For $genus(\Sigma') \geq 1$, let $\tilde{\Sigma} := \Sigma'$.

Suppose now $genus(\tilde{\Sigma}) = 1$. Our normalization of h implies there is at most one element of $H_1(\tilde{\Sigma}; \mathbb{Z})$ which is represented by a simple geodesic of length less than 1. Suppose such a class exists, and denote it by A. Pick closed geodesics α_0 and α_1 representing A as follows. If $\partial\Sigma \neq \emptyset$ and the components of $\partial\Sigma$ represent A, let α_0 and α_1 be the components of $\partial\Sigma$. Otherwise, let I_0 be a sub-cylinder maximizing the modulus among all the subcylinders I such that $\mu(I) = \delta_2/6$ and each component of ∂I represents A. Fix a biholomorphism

$$f : I \to \left[-\frac{1}{2}Mod(I_0), \frac{1}{2}Mod(I_0)\right],$$

and let $\alpha_0 := f^{-1}(\{0\} \times S^1)$. Let α_1 be the geodesic whose image is $\tilde{\Sigma} \setminus \mathcal{C}(\alpha_0)$. I_0 will play a role in the construction of the bubble decomposition. We therefore define it also when $\partial\Sigma \neq \emptyset$. In this case, define $I_0 \subset \mathcal{C}(\alpha_0)$ to be a conjugation invariant sub-cylinder containing α_0 and satisfying $\mu(I_0) = \delta_2/6$.

Definition 4.4. An admissible annulus is a doubly connected clean open $I \subset \tilde{\Sigma}$ of one of the following forms:

(a) There is a point $z \in \tilde{\Sigma}$ and positive reals $r \in (0, \frac{1}{3}\operatorname{inj}(\tilde{\Sigma}; h, z)]$ and $r' \in (0, \frac{1}{5}r]$ such that $I = A(r, r', z; h)$. Furthermore, I is contractible in $\tilde{\Sigma}$[4].

(b) In case $genus(\tilde{\Sigma}) = 1$ assume α_1 is defined. In case $genus(\tilde{\Sigma}) = 0$ assume $\partial\Sigma \neq \emptyset$. There is a simple closed geodesic $\gamma \subset \tilde{\Sigma}$ satisfying

$$\begin{cases} \ell(\gamma) < 2\sinh^{-1}(1), & genus(\tilde{\Sigma}) > 1, \\ \gamma = \alpha_1, & genus(\tilde{\Sigma}) = 1, \\ \gamma = \partial\Sigma, & genus(\tilde{\Sigma}) = 0 \end{cases}$$

such that I is an open sub-cylinder[5] of $\mathcal{C}(\gamma)$[6].

If I is of the type (a) it will be referred to as a trivial admissible annulus. Otherwise, it will be referred to a non trivial admissible annulus. We will also use the term admissible cylinder for nontrivial annuli. We denote by $\mathcal{A}_h$ the collection of admissible annuli both trivial and non trivial.

When $genus(\tilde{\Sigma}) = 1$ and α_0 is defined, we will also use the notation $\hat{\mathcal{A}}_h$ for the union of $\mathcal{A}_h$ with the set of sub-cylinders of $\mathcal{C}(\alpha_0)$. In all other cases, $\hat{\mathcal{A}}_h := \mathcal{A}_h$.

Remark 4.5. Note that an admissible trivial annulus is uniquely representable as the difference between two discs in $\tilde{\Sigma}$. Henceforth, whenever

[4]This is of course redundant when $genus(\tilde{\Sigma}) > 0$.

[5]See Definition 2.2

[6]See Definition 2.6

we represent an admissible annulus I as the difference $I = B \setminus B'$, it is intended that $B' \subset \tilde{\Sigma}$.

Remark 4.6. Recall our notation $C(a, b; I)$ for an annulus I and reals a, b. When $a \neq b$ this notation is well defined only up to a holomorphic reflection since it depends on the choice of holomorphic parametrization

$$(\rho, \theta) : [0, Mod(I)] \times S^1 \to I.$$

We adopt the convention that for a trivial annulus $I = B \setminus B'$, ρ increases as the distance to the center of B_1 increases. For nontrivial annuli we assume that for each simple closed geodesic we fixed a choice of holomorphic parametrization of $\mathcal{C}(\gamma)$ by

$$[0, Mod(\mathcal{C}(\gamma))] \times S^1$$

once and for all. This induces a choice for all the admissible nontrivial annuli.

4.3. Topological relatedness.

Definition 4.7. Let $I_1, I_2 \in \mathcal{A}_h$. We say that I_1 and I_2 are **topologically related** if there exists a doubly connected clean $I \subset \tilde{\Sigma}$ such that both I_1 and I_2 are nontrivially embedded in I.

Theorem 4.8. *Let $I_1, I_2 \in \mathcal{A}_h$. I_1 and I_2 are topologically related if and only if one of the following holds:*

(a) There is a simple closed geodesic γ such that $\mathcal{C}(\gamma)$ is clean and both I_1 and I_2 are sub-cylinders of $\mathcal{C}(\gamma)$. Furthermore, when $genus(\tilde{\Sigma}) = 1$, we have that α_1 is defined and that $\gamma = \alpha_1$. When $genus(\tilde{\Sigma}) = 0$, we have that $\partial\Sigma \neq \emptyset$ and $\gamma = \partial\Sigma$.

(b) There are concentric geodesic discs $B'_i \subset B_i \subset \tilde{\Sigma}$ such that $I_i = B_i \backslash B'_i$, for $i = 1, 2$. Furthermore,
  - *(i) $B'_1 \cap B'_2 \neq \emptyset$,*
  - *(ii) $I_1 \cup I_2$ is clean.*

To prove Theorem 4.8 we first prove the following Lemmas some of which will also be used later.

Lemma 4.9. *Let $I = B \setminus B'$ be an admissible trivial annulus. Then both B and B' are clean. Furthermore, I is conjugation invariant if and only if both B and B' are.*

Proof. If $I \cap \overline{I} = \emptyset$ there is a component A of $\tilde{\Sigma} \setminus \partial\Sigma$ such that $I \subset A$. Suppose by contradiction that $B \not\subset A$ then $B \cap \partial\Sigma \neq \emptyset$. Since $\partial B \subset \partial I$ this implies there is a component of $\partial\Sigma$ contained in B. When $genus(\tilde{\Sigma}) > 0$ this is a contradiction since B is contractible whereas each component of $\partial\Sigma$ is a closed geodesic. When $genus(\tilde{\Sigma}) = 0$ it is straightforward to verify by definition that I cannot be admissible. Thus $B \cap \overline{B} = \emptyset$. Since $B' \subset B$ the same is true for B'.

If $I = \overline{I}$ then ∂I is conjugation invariant. Thus, either each component of ∂I is conjugation invariant, or each component of ∂I is contained in different component of $\tilde{\Sigma} \backslash \partial \Sigma$. But this latter case is ruled as in the previous paragraph. In particular we get that ∂B and $\partial B'$ are each conjugation invariant. So, the same is true for B and B'. Thus we have proven the first part of the lemma and one direction of the second part. The other direction is obvious. $\square$

Lemma 4.10. *Let $p_i \in \tilde{\Sigma}$ and $r_i \in (0, \frac{1}{3}\operatorname{inj}(\tilde{\Sigma}; h, p_i))$ and write $B_i = B_{r_i}(p_i; h)$ for $i = 1, 2$. If $B_1 \cap B_2 \neq \emptyset$, then*

(a) $B_1 \cup B_2$ is contained in a geodesic disc.
(b) The closure of $B_1 \cap B_2$ is homeomorphic to the closed disc.
(c) The closure of $B_1 \cup B_2$ is homeomorphic to the closed disc.

Proof. Let

$$r = \operatorname{inj}(\Sigma; h, p_1),$$

and without loss of generality assume

$$r \geq \operatorname{inj}(\Sigma; h, p_2).$$

Then $r_2 < \frac{1}{3}r$ and so, since $d(p_1, p_2) \leq r_1 + r_2 < \frac{2}{3}r$, we have

$$B_1 \cup B_2 \subset B_r(p_1).$$

This gives part (a). Since the curvature is constant, the sizes of the r_i imply that the balls B_i are geodesically convex. Thus, $B_1 \cap B_2$ is geodesically convex and therefore simply connected. It follows from Van Kampen's theorem that $B_1 \cup B_2$ is also simply connected. Clearly, the closures of $B_1 \cup B_2$ and $B_1 \cap B_2$ are topological surfaces with boundary. Parts (b) and (c) follow. $\square$

Lemma 4.11. *Let $I_i = A(r_i, r_i'; p_i) \in \mathcal{A}_h$ for $i = 1, 2$. Suppose*

$$B^c_{r_1'}(p_1) \cap B^c_{r_2'}(p_2) \neq \emptyset.$$

Let

$$I = B_{r_1}(p_1) \cup B_{r_2}(p_2) \setminus B^c_{r_1'}(p_1) \cap B^c_{r_2'}(p_2).$$

Then I is doubly connected. Furthermore, if $I_1 \cup I_2$ is clean, so is I.

Proof. Write $B_i = B_{r_i}(p_i)$ and $B_i' = B^c_{r_i'}(p_i)$ for $i = 1, 2$. By Lemma 4.10 parts (b) and (c), $B_1' \cap B_2'$ is homeomorphic to the closed disc and $B_1 \cup B_2$ is homeomorphic to the open disc. Denoting by I^c the closure of I, the Mayer Vietoris sequence implies that $H_1(I^c; \mathbb{Z}) = \mathbb{Z}$. The only orientable surface with boundary satisfying this is the annulus.

To see that I is clean if $I_1 \cup I_2$ is, distinguish between the possibilities for I_1 and I_2 according to Lemma 2.7.

(a) I_i is conjugation invariant for $i = 1, 2$. By Lemma 4.9, so are B_i and B_i' and therefore, so is I.

(b) $I_i \cap \overline{I_j} = \emptyset$ for $i, j = 1, 2$. By Lemma 4.9, $B_i \cap \overline{B_i} = \emptyset$. Since $B_1 \cap B_2 \neq \emptyset$, B_1 and B_2 belong to the same component of $\tilde{\Sigma} \setminus \partial\Sigma$. Therefore, $I \subset B_1 \cup B_2$ belongs to one component of $\tilde{\Sigma} \setminus \partial\Sigma$. In particular, $I \cap \overline{I} = \emptyset$.

(c) $I_1 = \overline{I}_2$. We claim that I_1 is conjugation invariant and so $I = I_1 = I_2$. Indeed if I_1 is not conjugation invariant, then Lemma 4.9 implies B_1' is clean and not conjugation invariant. So, $B_1' \cap \overline{B}_1' = \emptyset$. But since $I_1 = \overline{I}_2$, $\overline{B}_1' = B_2'$. We thus get a contradiction to the assumption $B_1' \cap B_2' \neq \emptyset$.

(d) Without loss of generality $I_1 \subset I_2$. In this case it is clear that $B_2' \subset B_1'$, so $I = I_2$.

□

Proof of Theorem 4.8. Assume Condition (a) holds. Then $\mathcal{C}(\gamma)$ plays the role of I in Definition 4.7. Assume Condition (b), holds and let

$$I = B_1 \cup B_2 \setminus (B_1' \cap B_2').$$

By Lemma 4.11, I is clean and doubly connected. Clearly, I_i is nontrivially embedded in I for $i = 1, 2$. Thus I_1 and I_2 are topologically related.

Conversely, let I_1 and I_2 be embedded nontrivially in a clean $I \subset \tilde{\Sigma}$. I_1 and I_2 are homologous in I to a homology generator of I. This implies that I_1 and I_2 are either both trivially embedded or both nontrivially embedded in $\tilde{\Sigma}$. These correspond to the cases where I is embedded trivially and nontrivially respectively.

In the first case, if $genus(\tilde{\Sigma}) \leq 1$ the only non-trivial annuli are the sub-cylinders of $\mathcal{C}(\alpha_1)$ and $\mathcal{C}(\partial\Sigma)$ which are clean. If $genus(\tilde{\Sigma}) > 1$, there are simple closed geodesics γ_i, for $i = 1, 2$, such that I_i is a sub-cylinder of $\mathcal{C}(\gamma_i)$. We have that γ_i is freely homotopic to any component of ∂I_i which in turn is freely homotopic to any component of ∂I. So γ_1 is freely homotopic to γ_2. Since there is a unique simple closed geodesic in each free homotopy class, this implies $\gamma_1 = \gamma_2$. Clearly, $\ell(\gamma) < 2\sinh^{-1}(1)$ since $\mathcal{C}(\gamma)$ contains admissible cylinders as sub cylinders. Therefore, by Lemma 2.8, $\mathcal{C}(\gamma)$ is clean.

Now consider the case where I_i are trivial for $i = 1, 2$. Then I must be trivially embedded in $\tilde{\Sigma}$. Since $\tilde{\Sigma}$ is not a sphere, $\tilde{\Sigma} \setminus I$ has exactly one component A with the topology of a disc. Clearly, $A \subset B_1' \cap B_2'$. This gives the first part of (b). Now, if $I \cap \overline{I} = \emptyset$ then clearly $I_1 \cup I_2 \subset I$ is clean. If I is conjugation invariant then by Lemma 2.9, so are I_1 and I_2. This implies that so is $I_1 \cup I_2$, giving the second part of (b). □

Let I_1 and I_2 be topologically related. We associate with I_1 and I_2 two clean doubly connected sub-surfaces $M(I_1, I_2)$ and $m(I_1, I_2)$ in which both are non-trivially embedded. One should think of $M(I_1, I_2)$ as the minimal annulus in $\tilde{\Sigma}$ in which I_1 and I_2 are nontrivially embedded. On the other hand, $m(I_1, I_2)$ should be thought of as the maximal *admissible* annulus which is nontrivially embedded in $M(I_1, I_2)$.

Formally, the definitions are as follows. When I_1 an I_2 are sub-cylinders of $\mathcal{C}(\gamma)$ for a simple closed geodesic γ, suppose that I_i is given in (ρ, θ) coordinates[7] by

$$I_i := \{z \in \mathcal{C}(\gamma) | \rho_{0,i} < \rho(z) < \rho_{1,i}\}.$$

Let $\rho_0 = \min\{\rho_{0,1}, \rho_{0,2}\}$, $\rho_1 = \max\{\rho_{1,1}, \rho_{1,2}\}$, $\rho = \max\{|\rho_0|, |\rho_1|\}$. Define

$$M(I_1, I_2) := \begin{cases} \{z \in \mathcal{C}(\gamma) | -\rho < \rho(z) < \rho\}, & \gamma \subset \partial\Sigma \text{ and } 0 \in [\rho_0, \rho_1], \\ \{z \in \mathcal{C}(\gamma) | \rho_0 < \rho(z) < \rho_1\}, & \text{otherwise.} \end{cases}$$

When I_1 and I_2 are trivial, write $I_i = B_i \setminus B'_i$ and take

$$M(I_1, I_2) := B_1 \cup B_2 \setminus B'_1 \cap B'_2.$$

We now define $m(I_1, I_2)$. If I_1 and I_2 are nontrivial take $m(I_1, I_2) := M(I_1, I_2)$. Otherwise, suppose $I_i = B_{r_i}(p_i) \setminus B^c_{r'_i}(p_i)$ and assume without loss of generality that $r_2 \leq r_1$. Then define

$$m(I_1, I_2) := A\left(d(p_2, \partial B_1; h), r'_2, p_2; h\right).$$

Lemma 4.12. *$M(I_1, I_2)$ and $m(I_1, I_2)$ are clean and doubly connected.*

Proof. That $M(I_1, I_2)$ is clean and doubly connected follows from the definition, from Theorem 4.8, and from Lemma 4.11. To prove the same for $m(I_1, I_2)$ we may assume I_1 and I_2 are trivial as otherwise $m(I_1, I_2) = M(I_1, I_2)$. Clearly, $m(I_1, I_2)$ is doubly connected. We show that it is clean. If $m(I_1, I_2) \cap \overline{m(I_1, I_2)} = \emptyset$, we are done. Otherwise, let $I_i = B_i \setminus B'_i$ for $i = 1, 2$. We claim that B_1 and B_2 are each conjugation invariant. Indeed, since $m(I_1, I_2) \subset B_1 \cup B_2$, we have that $B_1 \cup B_2$ meets $\partial\Sigma$. Without loss of generality, B_1 meets $\partial\Sigma$. Since by Lemma 4.9 B_1 is clean, it must be conjugation invariant. Thus, I_1 meets $\partial\Sigma$. So, I_1 is conjugation invariant. By definition of topological relatedness and by Lemma 2.9, I_2 is also conjugation invariant. By Lemma 4.9 again, B_2 is conjugation invariant. In particular, the centers p_1, p_2 of B_1 and B_2 lie on $\partial\Sigma$. It is now clear by construction that $m(I_1, I_2)$ is conjugation invariant. □

Lemma 4.13. *Suppose the pairs (I_1, I_2) and (I_2, I_3) are topologically related.*

(a) If I_i is trivial for $i = 1, 2, 3$, then

$$M(I_1, I_3) \subset M(I_1, I_2) \cup M(I_2, I_3).$$

(b) If I_i is nontrivial for $i = 1, 2, 3$, then one of the following holds.
- *(i) $M(I_1, I_3) \subset M(I_1, I_2) \cup M(I_2, I_3)$,*
- *(ii) $M(I_1, I_3) \subset M(I_1, \overline{I_1})$ and $M(I_2, I_3) = \overline{M(I_2, I_3)}$.*
- *(iii) $M(I_1, I_3) \subset M(I_3, \overline{I_3})$ and $M(I_1, I_2) = \overline{M(I_1, I_2)}$.*

Proof. (a) This is straightforward set theory.

[7] See the discussion subsequent to Definition 2.6.

(b) By Theorem 4.8 there is a simple closed geodesic γ such that

$$I_i = \{z \in \mathcal{C}(\gamma) | \rho_{0,i} < \rho(z) < \rho_{1,i}\},$$

for $i = 1, 2, 3$. Suppose without loss of generality that

$$\rho_{0,1} \leq \rho_{0,3}.$$

Assume

$$M(I_1, I_3) \not\subset M(I_1, I_2) \cup M(I_2, I_3). \tag{26}$$

Considering the definition of $M(\cdot,\cdot)$, this assumption implies that $\gamma \subset \partial\Sigma$ and that

$$\rho_{0,1}\rho_{1,3} < 0. \tag{27}$$

We claim, further, that

$$\rho_{0,1} \leq \rho_{0,2} < \rho_{1,2} \leq \rho_{1,3}. \tag{28}$$

Indeed, if $\rho_{1,2} > \rho_{1,3}$ then combining the definition of $M(\cdot,\cdot)$ and inequality (27) it would follow that $M(I_1, I_3) \subset M(I_1, I_2)$. Similarly, if $\rho_{0,2} < \rho_{0,1}$ we would get that $M(I_1, I_3) \subset M(I_2, I_3)$. Now, if $|\rho_{0,1}| \geq |\rho_{1,3}|$, inequality (28) implies condition (ii). Indeed, the first part of condition (ii) is immediate. Furthermore, we must have in this case $\rho_{0,2}\rho_{1,3} < 0$ for otherwise we would have $M(I_1, I_2) = M(I_1, \overline{I_1})$. By the first part of condition (ii) this would contradict equation (26). Thus $M(I_2, I_3)$ meets $\partial\Sigma$. Since $M(I_2, I_3)$ is clean, this implies $M(I_2, I_3)$ is conjugation invariant. If $|\rho_{0,1}| \leq |\rho_{1,3}|$ we get condition (iii) by switching the roles of I_1 and I_3.

□

Lemma 4.14. *Suppose I_1 and I_2 are topologically related. Then I_1 is conjugation invariant if and only if I_2 is.*

Proof. If I_1 is conjugation invariant then by Lemmas 4.12 and 2.9 it follows that $M(I_1, I_2)$ is conjugation invariant. Again applying Lemma 2.9 it follows that I_2 is conjugation invariant. The converse is obtained by exchanging the roles of I_1 and I_2. □

Lemma 4.15. *For $i = 1, 2, 3$ let I_i be admissible annuli. Write $I_i = B_i \setminus B_i'$ where B_i' is a clean closed disc, B_i a clean open disc and B_i' is concentric with B_i. Suppose that the pairs (I_1, I_2) and (I_2, I_3) are topologically related. If $B_1' \not\subseteq M(I_2, I_3)$ and $B_3' \not\subseteq M(I_1, I_2)$ then I_1 and I_3 are topologically related.*

Proof. We show first that $I_1 \cup I_3$ is clean. If I_1 is conjugation invariant then by Lemma 4.14 so is I_3. If I_1 is not conjugation invariant then there is a component A of $\tilde{\Sigma} \setminus \partial\Sigma$ which contains I_1. By Theorem 4.8(b), $B_2' \cap B_1' \neq \emptyset$. Since I_2 is not conjugation invariant it follows that $I_2 \subset A$. By repeating this argument, $I_3 \subset A$. Thus $I_1 \cup I_3$ is clean.

We now wish show that $B_1' \cap B_3' \neq \emptyset$. For this we show first that either $B_1' \subset B_2 \cup B_3$ or $B_3' \subset B_1 \cup B_2$. By Theorem 4.8, $B_1' \cap B_2' \neq \emptyset$ and $B_2' \cap B_3' \neq \emptyset$. That is,

$$d(p_1, p_2) \leq r_1' + r_2'$$

and

$$d(p_2, p_3) \leq r_2' + r_3'.$$

Let now $x \in B_1'$. Then

$$d(x, p_2) \leq d(x, p_1) + d(p_1, p_2) \leq 2r_1' + r_2',$$

and

$$d(x, p_3) \leq d(x, p_1) + d(p_1, p_2) + d(p_2, p_3) \leq 2r_1' + 2r_2' + r_3'. \tag{29}$$

Suppose $x \notin B_2 \cup B_3$ then $2r_1' + r_2' > r_2 \geq 5r_2'$. This implies

$$r_2' < \frac{1}{2} r_1'. \tag{30}$$

Combining estimates (29) and (30) and the estimate $r_3 \geq 5r_3'$, we get

$$r_3' < \frac{3}{4} r_1'. \tag{31}$$

On the other hand, for *any* $x \in B_3'$ we have

$$d(x, p_1) \leq d(x, p_3) + d(p_1, p_3) \leq r_3' + d(p_1, p_3).$$

Therefore, combining estimates (30) and (31),

$$d(p_1, p_3) \leq r_1' + 2r_2' + r_3' < (2 + \frac{3}{4}) r_1' < r_1 - r_3'.$$

That is, $B_3' \subset B_1 \subset B_1 \cup B_2$ as claimed.

We use this to show that $B_1' \cap B_2' \cap B_3' \neq \emptyset$. Suppose by contradiction

$$B_1' \cap B_2' \cap B_3' = \emptyset. \tag{32}$$

Then in case $B_1' \subset B_2 \cup B_3$, assumption (32) implies $B_1' \subset M(I_2, I_3)$. Similarly in case $B_3' \subset B_1 \cup B_2$, assumption (32) implies $B_3' \subset M(I_1, I_2)$. In any case we get a contradiction to the assumptions of the lemma. □

4.4. **Essential disjointeness.**

Theorem 4.16. *There is a constant K_1 with the following significance. Let $I_1, I_2 \in \hat{\mathcal{A}}_h$. Suppose $C(K_1, K_1; I_1) \cap C(K_1, K_1; I_2) \neq \emptyset$. Then either $\tilde{\Sigma}$ is a torus covered by I_1 and I_2, or*

$$b_1(I_1 \cup I_2) \leq 1.$$

Here for a topological space X, $b_1(X)$ denotes the first Betti number of X.

The proof of Theorem 4.16 spans this subsection.

Definition 4.17. I_1 and I_2 are said to be **essentially disjoint** if

$$C(K_1, K_1; I_1) \cap C(K_1, K_1; I_2) = \emptyset,$$

where K_1 is a constant satisfying Theorem 4.16 that is fixed once and for all. In later uses it will be convenient to assume further that $K_1 \geq c_2 + \pi$.

Lemma 4.18. *Let I_1 and I_2 be admissible nontrivial annuli. If*

$$I_1 \cap I_2 \neq \emptyset$$

then each component[8] *of $I_1 \cap I_2$ is a sub-cylinder of I_i for $i = 1, 2$.*

Proof. By definition 4.4 there are simple closed geodesics γ_i such that I_i is a sub-cylinder of $\mathcal{C}(\gamma_i)$ for $i = 1, 2$. First assume $genus(\tilde{\Sigma}) > 1$. The assumption

$$I_1 \cap I_2 \neq \emptyset$$

and Theorem 2.4(c) imply that $\gamma_1 \cap \gamma_2 \neq \emptyset$. If γ_1 intersects γ_2 transversally in a nonempty set then by [2, 4.1.1] there is an $i \in \{1, 2\}$ such that $\ell(\gamma_i) \geq 2\sinh^{-1}(1)$. This is contrary to the definition of admissibility. Since γ_1 and γ_2 are geodesics which intersect non transversally, $\gamma_1 = \gamma_2$. The intersection of sub-cylinders of a given cylinder is a sub-cylinder. Thus the claim follows.

Assume now that $genus(\tilde{\Sigma}) = 0$. Then by definition $\partial\Sigma \neq \emptyset$ and I_i are both sub-cylinders of the $\mathcal{C}(\partial\Sigma)$. So, the claim follows as before. Finally, assume $genus(\tilde{\Sigma}) = 1$. We claim that γ_1 is parallel to γ_2. Indeed, the alternative is that γ_1 intersects γ_2 transversally. But then

$$\ell(\gamma_1)\ell(\gamma_2) > Area(\Sigma; h) = 1 > \sinh^{-1}(1),$$

contradicting the admissibility of I_1 and I_2. This implies the claim. □

Lemma 4.19. *Let I and B be a geodesic annulus and a geodesic disc, respectively, in the hyperbolic disc, in the Riemann sphere, or in the flat plane. Then $b_1(I \cup B) \leq 1$.*

Proof. Suppose by contradiction otherwise. Then, by the Meyer Vietoris sequence, $I \cap B$ has at least two components. In particular, there is a boundary component of $\gamma \subset \partial I$ such that $\gamma \cap \partial B$ consists of at least four points. On the other hand, any two geodesic circles are also circles with respect to the flat metric on the disc. Any two such circles intersect in at must two points. A contradiction. □

Lemma 4.20. *There is a constant K with the following significance. Let I_1 and I_2 be admissible trivial annuli. Assume that $b_1(I_1 \cup I_2) \geq 2$ Then*

$$C(K, K; I_1) \cap C(K, K; I_2) = \emptyset.$$

Proof. Write $I_i = A(r_i, r_i'; p_i)$, $B_i = B_{r_i}(p_i)$, and $B_i' = B^c_{r_i'}(p_i)$ for $i = 1, 2$. First assume $B_1' \cap B_2' = \emptyset$. Note that the assumption on $I_1 \cup I_2$ implies

$$B_i' \not\subset B_{i \bmod 2+1}. \tag{33}$$

[8] The possibility of more than one component appears when $genus(\tilde{\Sigma}) = 1$.

Indeed, suppose for example that $B_1' \subset B_2 \setminus B_2'$ then

$$I_1 \cup I_2 = I_2 \cup B_1.$$

By Lemma 4.19 this would imply $b_1(I_1 \cup I_2) \leq 1$ contradicting the assumption.

It follows that $r_1 < d(p_1, p_2) + r_2'$ and $r_2 < d(p_1, p_2) + r_1'$. The combination of these inequalities with the condition $r_i' \leq \frac{1}{5} r_i$ in the definition of admissibility implies that

$$d(p_1, p_2) > \frac{2}{5}(r_1 + r_2). \tag{34}$$

Let now $s_i = \frac{2r_i d(p_1,p_2)}{3(r_1+r_2)}$ for $i = 1, 2$. Write $J_i = A(s_i, r_i'; p_i)$. By equation (34),

$$s_i > \frac{1}{5} r_i \geq r_i'.$$

In particular, $J_i \neq \emptyset$. We have $s_1 + s_2 < d(p_1, p_2)$, so

$$J_1 \cap J_2 = \emptyset. \tag{35}$$

It thus suffices to show that there is a universal constant K such that $C(K, K; I_i) \subset J_i$ for $i = 1, 2$. Write $L_i := A(r_i, s_i; p_i)$. L_i is a subcylinder of I_i, so $I_i \backslash L_i = C(0, Mod(L_i); I_i)$. Note that $J_i = I_i \setminus L_i$, so $C(Mod(L_i), Mod(L_i); I) \subset J_i$. It therefore suffices to uniformly bound $Mod(L_i)$. We have

$$Mod(L_i) = \int_{s_i}^{r_i} \frac{dr}{h_\theta(r)} \tag{36}$$

We have either $h_\theta(r) = \sin(r)$ and $r_i \leq \pi/2$, or $h_\theta(r) = r$, or $h_\theta(r) = \sinh(r)$, so we need only verify the boundedness of expression (36) when $s_i \to 0$. But $s_i/r_i \geq \frac{4}{15}$, so this is obvious.

Now assume $B_1' \cap B_2' \neq \emptyset$. First note that the assumption on $I_1 \cup I_2$ implies that either $B_1' \not\subset B_2$ or $B_2' \not\subset B_1$. Indeed, otherwise

$$I_1 \cup I_2 = B_1 \cup B_2 \setminus B_1' \cap B_2'.$$

By Lemma 4.11 we would then have that $b_1(I_1 \cup I_2) = 1$ in contradiction to the assumption of the Lemma. Thus we may, without loss of generality, assume $B_1' \not\subset B_2$. Since $B_1' \cap B_2' \neq \emptyset$, $d(p_1, p_2) \leq r_1' + r_2'$. On the other hand , since $B_1' \not\subset B_2$, $r_2 \leq d(p_1, p_2) + r_1'$. The combination of these two inequalities implies that $r_2 \leq \frac{5}{2} r_1'$. We thus have

$$d(p_1, p_2) + r_2 \leq r_1' + r_2' + \frac{5}{2} r_1' \leq 4r_1'.$$

Therefore, letting $J = A(4r_1', r_1'; p_1)$, we have $(I_1 \setminus J) \cap I_2 = \emptyset$. On the other hand

$$C(Mod(J), Mod(J); I_1) \subset I_1 \setminus J.$$

We have that $Mod(J)$ is bounded from above by some constant K which is independent of r_1'. The claim follows. □

Lemma 4.21. *There is a constant K with the following significance. Let $I_1 = A(r_1, r_1'; p) \in \mathcal{A}_h$ be trivial and $I_2 \in \hat{\mathcal{A}}_h$ be nontrivial. Suppose $I_1 \not\subset I_2$. Then $I_1 \cap C(K, K; I_2) = \emptyset$.*

Proof. First assume $genus(\tilde{\Sigma}) \geq 1$. By definition, there is a simple closed geodesic γ such that I_2 is a sub-cylinder in $\mathcal{C}(\gamma)$. Recall the definition of (ρ, θ) coordinates on $\mathcal{C}(\gamma)$. Write $\rho_0 = \inf\{\rho(z)|z \in I_1 \cap I_2\}$ and $\rho_1 = \sup\{\rho(z)|z \in I_1 \cap I_2\}$. Let $I \subset \mathcal{C}(\gamma)$ be given in (ρ, θ) coordinates by

$$\{(\rho, \theta)|\rho_0 \leq \rho \leq \rho_1\}.$$

Denote by β_i the components $\{\rho = \rho_i\}$ of ∂I for $i = 0, 1$. Now note that since $B_{r_1}(p_1)$ is a disc of radius $r < \frac{1}{3}\operatorname{inj}(\tilde{\Sigma}; p_1)$, we have

$$\rho_1 - \rho_0 < \sup_{z \in I} \operatorname{inj}(\tilde{\Sigma}; z). \tag{37}$$

Indeed, this is obvious if $p_1 \in I$. Otherwise, suppose without loss of generality that β_0 lies between p_1 and β_1, and let p' be the intersection of the perpendicular from p_1 with β_0. Then $B_{\rho_1 - \rho_0}(p') \subset B_r(p_1)$. In particular, $\rho_1 - \rho_0 < \operatorname{inj}(\tilde{\Sigma}; p')$. This establishes inequality (37).

Now, $\operatorname{inj}(\tilde{\Sigma}, \cdot)$ is either constant or has no local maximum in I. When $genus(\tilde{\Sigma}) > 1$ this can be seen from relation (13). Else inj is constant. We may therefore assume without loss of generality that $\operatorname{inj}(\tilde{\Sigma}, \cdot)$ attains its supremum at ρ_0 (we no longer make the assumption from the previous paragraph about β_0). By the assumption on the genus, β_0 is not contractible. Therefore,

$$\operatorname{inj}(\Sigma; (\rho_0, \theta(\cdot))) \leq \frac{1}{2}\ell(\beta_0) = \pi h_\theta(\rho_0).$$

Thus, we have the estimate

$$Mod(I) \leq \int_{\rho_0}^{\rho_0 + \pi h_\theta(\rho_0)} \frac{d\rho}{h_\theta(\rho)}.$$

The last expression is bounded by a universal constant K. Indeed, in case $genus(\tilde{\Sigma}) = 1$, h_θ is constant and the bound is obvious. Otherwise, using Theorem 2.4(d), the last expression is estimated by $Ce^{\ell(\gamma)\pi\cosh(\rho_0)}$ for an a priori constant C. Using the definition of ρ_0 and $\mathcal{C}(\gamma)$ we have

$$\ell(\gamma)\pi\cosh(\rho_0) \leq \ell(\gamma)\pi\cosh(w(\gamma)) \leq C'.$$

Here $w(\gamma)$ is as defined in Theorem 2.4(c) and C' is an a priori constant.

When $genus(\tilde{\Sigma}) = 0$, $\mathcal{C}(\gamma)$ is an annulus in $\tilde{\Sigma}$ and the claim follows with slight modification in the same way as Lemma 4.20. □

Proof of Theorem 4.16. Let K_1 be a constant as in Lemmas 4.20 and 4.21. Suppose $C(K_1, K_1; I_1) \cap C(K_1, K_1; I_2) \neq \emptyset$. If I_1 and I_2 are both trivial annuli, the theorem is just a restatement of Lemma 4.20. If both I_1 and I_2 are nontrivial and $\Sigma_{\mathbb{C}}$ is not a torus covered by I_1 and I_2, Lemma 4.18 implies I_1 and I_2 intersect in a sub-cylinder I. So, $I_1 \cup I_2$ is a sub-cylinder of

$\mathcal{C}(\gamma)$ for some simple closed geodesic γ. In particular $b_1(I_1 \cup I_2) = 1$. Finally, if I_1 is trivial and I_2 is nontrivial. Then by Lemma 4.21, $I_1 \cup I_2 = I_2$. □

4.5. **Long annuli.**

Lemma 4.22. *There is a constant K_2 with the following significance. Let $L \geq K_2$ and let $I_1, I_2 \in \mathcal{A}_h$ be essentially disjoint. Suppose $Mod(I_i) > 4L$ for $i = 1, 2$. Suppose further that I_1 and I_2 are topologically related. Then*

$$Mod(m(I_1, I_2)) > \max\{Mod(I_1), Mod(I_2)\} + 2L. \tag{38}$$

Proof. When I_1 and I_2 are cylinders, it is straightforward to verify that the claim holds whenever $K_2 \geq K_1$ where K_1 is the constant from Definition 4.17. Otherwise, for $i = 1, 2$, write $I_i = A(r_i, r_i', p_i)$ and $B_i' = B^c_{r_i'}(p_i)$. Assume without loss of generality that $r_2' \leq r_1'$. Then Theorem 4.8(b) implies

$$d(p_2, \partial B_1) \geq r_1 - 2r_1',$$

and admissibility implies $p_2 \in B_1$. Let $r = d(p_2, \partial B_1)$ and write

$$I := m(I_1, I_2) = B_r(p_2) \backslash B_2'.$$

Let

$$\Delta = \int_{r_1 - 2r_1'}^{r_1} \frac{dr}{h_\theta(r)}.$$

Then, applying equation (7),

$$\begin{aligned} Mod(I) &= \int_{r_2'}^{r} \frac{dr}{h_\theta(r)} \\ &\geq \int_{r_2'}^{r_1 - 2r_1'} \frac{dr}{h_\theta(r)} \\ &= \int_{r_2'}^{r_1} \frac{dr}{h_\theta(r)} - \Delta \\ &\geq Mod(I_1) + Mod(I_2) - 2K_1 - \Delta. \end{aligned}$$

The claim of the lemma will follow if we find a K_2 such that

$$\Delta + 2K_1 \leq 2K_2.$$

In other words it suffices to bound Δ uniformly from above. By the restrictions on the range of r_1 in the definition of admissibility, h_θ is monotone increasing. See equation (8). Therefore,

$$\Delta \leq \frac{2r_1'}{h_\theta(r_1 - 2r_1')} \leq \frac{r_1'}{h_\theta(3r_1')}.$$

But whether the curvature of h is positive, negative or vanishing,

$$\lim_{r \to 0} \frac{r}{h_\theta(r)} = 1.$$

Relying again on equation (8) we get that Δ is uniformly bounded from above whenever the curvature is non-positive. When the curvature is positive, we still have that Δ is uniformly bounded in the range of admissibility

$$r_1' \in (0, \frac{\pi}{15}].$$

□

Lemma 4.23. *Let $I_i = A(r_i, r_i'; p_i) \in \mathcal{A}_h$ for $i = 1, 2$. Suppose $r_2 \leq r_1$, I_1 is topologically related to I_2, and $M(I_1, I_2) \neq I_1 \cup I_2$. Then*

(a)
$$d(p_1, p_2) \leq r_1' + r_2',$$

(b)
$$r_2 < \frac{5}{2} r_1',$$

(c)
$$M(I_1, I_2) = I_1 \cup m(I_1, I_2).$$

Proof. Write $B_i = B_{r_i}(p_i)$, $B_i' = B^c_{r_i'}(p_i)$ and $J = M(I_1, I_2)$. By Lemma 4.15,

$$B_1' \cap B_2' \neq \emptyset. \tag{39}$$

Part (a) is an immediate consequence. To prove part (b), verify using $J = B_1 \cup B_2 \setminus (B_1' \cap B_2')$ that

$$J \setminus (I_1 \cup I_2) \subset \left(B_1' \setminus B_2\right) \cup \left(B_2' \setminus B_1\right). \tag{40}$$

On the other hand, relation (39), admissibility, and the assumption $r_2 \leq r_1$, imply $B_2' \subset B_1$. Indeed, we have for any $q \in B_2'$

$$d(q, p_1) \leq 2r_1' + r_2' \leq \frac{2}{5} r_1 + \frac{1}{5} r_2 \leq \frac{3}{5} r_1 < r_1.$$

So, by relation (40), $B_1' \setminus B_2 \neq \emptyset$. Let $q \in B_1' \setminus B_2$. Then $d(q, p_1) < r_1'$ and $r_2 < d(q, p_2)$. Combining these inequalities we get

$$r_2 < d(q, p_2) \leq d(q, p_1) + d(p_1, p_2) < 2r_1' + r_2'. \tag{41}$$

Since I_2 is admissible, $r_2' \leq \frac{1}{5} r_2$. Thus, inequality (41) implies part (b).

We prove part (c). By definition, I_1 and $m(I_1, I_2)$ are subsets of $M(I_1, I_2)$. We prove the reverse inclusion. Using part (b) and admissibility, one verifies that $B_2 \subset B_1$. Therefore,

$$M(I_1, I_2) \setminus I_1 = B_1 \setminus \left((B_1' \cap B_2') \cup I_1\right) = B_1' \setminus B_2'.$$

Write $r = d(p_2, \partial B_1)$. We need to show that

$$B_1' \setminus B_2' \subset m(I_1, I_2) = B_r(p_2) \setminus B_2'.$$

For this it suffices to show that $B_1' \subset B_r(p_2)$. That is,

$$d(p_1, p_2) + r_1' < r.$$

But by parts (a) and (b) we get

$$\begin{aligned} r &= d(p_2, \partial B_1) \\ &\geq d(p_1, \partial B_1) - d(p_1, p_2) \\ &\geq r_1 - r_1' - r_2' \\ &\geq 3r_1' \\ &> d(p_1, p_2) + r_1'. \end{aligned}$$

□

Lemma 4.24. *There is a constant K_3 with the following significance. Let $I_i = A(r_i, r_i'; p_i) \subset \mathcal{A}_h$, for $i = 1, 2$. Let $L \geq K_3$ and suppose*

$$ModI_i \geq 2L.$$

Suppose I_1 is topologically related to I_2 and let $I = m(I_1, I_2)$. Then

(a) $M(I_1, I_2) = I_1 \cup I_2 \cup C(L, L; I)$.
(b) $I \setminus C(L, L; I) \subset I_1 \cup I_2$.

Proof. Write $B_i = B_{r_i}(p_i)$, $B_i' = B^c_{r_i'}(p_i)$, and $J = M(I_1, I_2)$. If

$$J = I_1 \cup I_2$$

there is nothing to prove. We thus assume $J \neq I_1 \cup I_2$. We first show that for some fixed K_3 chosen large enough, part (b) holds. Let $J_1 = I \setminus C(0, L; I)$ and $J_2 = I \setminus C(L, 0; I)$. Assume without loss of generality that $r_2 \leq r_1$. Then I is centered at p_2. Since, furthermore, $Mod(I_2) > L$, it follows that $J_2 \subset I_2$[9]. It remains to show that $J_1 \subset I_1$. Let $r = d(p_2, \partial B_1)$. There is a real number r' such that $J_1 = A(r, r', p_2)$. Clearly, $B_r(p_2) \subset B_1$. Therefore, to show the inclusion $J_1 \subset I_1$ it suffices to show that $B^c_{r_1'}(p_1) \subset B^c_{r'}(p_2)$. That is, it suffices to show that

$$r' - r_1' > d(p_1, p_2).$$

By parts (a) and (b) of Lemma 4.23, it suffices that $r' > 3r_1'$. Considering the definition of r', this is equivalent to the claim

$$Mod(A(r, 3r_1'; p_2)) > L.$$

Let $L' := Mod(A(r, 3r_1'; p_2))$. Since $B_1' \cap B_2' \neq \emptyset$, it is clear that

$$r \geq r_1 - 2r_1'.$$

[9]See remark 4.6.

So,

$$\begin{aligned} L' &\geq Mod\left(A(r_1 - 2r_1', 3r_1'; p_2)\right) \\ &= Mod\left(A(r_1 - 2r_1', 3r_1'; p_1)\right) \\ &> Mod\left(A\left(\frac{1}{2}r_1, 3r_1'; p_1\right)\right) \\ &= Mod(I_1) - Mod\left(A\left(r_1, \frac{1}{2}r_1; p_1\right)\right) - Mod\left(A(3r_1', r_1'; p_1)\right) \\ &\geq 2L - \int_{\pi/6}^{\pi/3} \frac{dr}{\sin r} - \int_{\pi/9}^{\pi/3} \frac{dr}{\sin r} \end{aligned}$$

For the last line, see equations (7), (8) and the definition of admissibility. Choosing

$$K_3 = \int_{\pi/6}^{\pi/3} \frac{dr}{\sin r} + \int_{\pi/9}^{\pi/3} \frac{dr}{\sin r},$$

we thus get $J_1 \subset I_1$. This establishes part (b). We prove part (a). The inclusion

$$M(I_1, I_2) \supset I_1 \cup I_2 \cup C(L, L; I),$$

follows from definitions. The reverse inclusion is an immediate consequence of Lemma 4.23(c) and part (b).

□

5. Construction of bubble decomposition

Let $L_0 > \max\{c_2, \log 3/c_3\}$. It follows from the cylinder inequality that for any $(\Sigma, \mu) \in \mathcal{M}$ and any clean $I \subset \Sigma_{\mathbb{C}}$ with $\mu(I) \leq \delta_2$ and $Mod(I) > 2L_0$, we have

$$\mu\left(C(L_0, L_0; I)\right) \leq \frac{\mu(I)}{3}. \tag{42}$$

Let L_0 satisfy, further, $L_0 > \max\{K_1, K_2, K_3\}$. Here, K_1, K_2 and K_3 are the constants from Definition 4.17, Lemma 4.22 and Lemma 4.24, respectively.

Definition 5.1. A **neck** is an $I \in \mathcal{A}_h$ with the property that $\tilde{\Sigma} \setminus I$ is μ-stable. Write $L_1 = 4L_0$. A **long neck** is a neck I which satisfies $\mu(I) \leq \delta_2/6$ and $Mod(I) \geq L_1$.

When $genus(\tilde{\Sigma}) \neq 1$, let LN denote the set of long necks. Otherwise, recall the definition of I_0 appearing in the definition of α_0. Define LN to be the set of long necks contained in $\Sigma \setminus I_0$. If $Mod(I_0) \geq L_1$, let $\tilde{LN} := LN \cup \{I_0\}$. In all the other cases, whatever the genus, define $\tilde{LN} := LN$.

Definition 5.2. A **maximal μ-decomposition** is a bubble decomposition $\mathcal{B}$ with the following properties.

(a) There exists a set $\tilde{\mathcal{B}}$ of pairwise essentially disjoint elements of $\tilde{LN}$ such that
$$\mathcal{B} := \{C(2K_1, 2K_1; I) | I \in \tilde{\mathcal{B}}\}.$$

(b) For any $v \in V_{\mathcal{B}}$, Σ_v is μ-stable.
(c) For any $v \in V_{\mathcal{B}}$, Σ_v contains no long necks.

Theorem 5.3. *For any $(\Sigma, \mu) \in \mathcal{M}$ satisfying Assumption 4.3.1, there exists a maximal μ-decomposition.*

To prove Theorem 5.3, we define a relation $\sim$ on LN as follows. For $I_1, I_2 \in LN$, $I_1 \sim I_2$ if and only if I_1 and I_2 are topologically related and $\mu(M(I_1, I_2)) \leq \delta_2/2$.

Lemma 5.4. *Let $I_1 \in LN$ be trivial. Write $I_1 = B_1 \setminus B_1$ for appropriate concentric discs in $\tilde{\Sigma}$. Let $I_2 \in LN$. Then $B_1' \not\subset I_2$. Furthermore, if $I_3 \in LN$ and $I_2 \sim I_3$ then $B_1' \not\subset M(I_2, I_3)$.*

Proof. The component B_1' of $\tilde{\Sigma} \setminus I_1$ is μ-stable. That is,

$$\mu(B_1') \geq \delta_1/2 > \delta_2/6.$$

On the other hand, $\mu(I_2) \leq \delta_2/6$, and, by $\sim$-equivalence,

$$\mu(M(I_2, I_3)) \leq \delta_2/2 < \delta_1/2.$$

Both parts of the claim follow. □

Lemma 5.5. *The relation $\sim$ is an equivalence relation.*

Proof. Symmetry and reflexivity are obvious, so we need only establish transitivity. Let $I_i \in LN$ for $i = 1, 2, 3$, and suppose $I_1 \sim I_2$ and $I_2 \sim I_3$. It follows from Theorem 4.8(a) that either the three annuli are all trivial or all nontrivial. Suppose all are non trivial. Observe, using 4.8(a), that I_1 and I_3 are topologically related.

Let now $J = M(I_1, I_3)$. We show first that

$$\mu(J) \leq \delta_2. \tag{43}$$

Let $I' = M(I_1, I_2)$ and $I'' = M(I_2, I_3)$. If $J \subset I' \cup I''$ we have

$$\mu(J) \leq \mu(I') + \mu(I'') \leq \delta_2/2 + \delta_2/2 = \delta_2.$$

Otherwise, by Lemma 4.13 we may assume without loss of generality that $J \subset M(I_1, \overline{I}_1)$ and $I'' = \overline{I''}$. Let $J' = J \cap \Sigma$ and $J'' = \overline{J'}$. It is easy to verify that $J' \subset I' \cup I''$. So, $\mu(J') \leq \delta_2$. Similarly, one verifies that

$$J' \setminus C(L_0, L_0; J') \subset I_1 \cup C(L_0, L_0; I'').$$

Applying inequality (42) we get

$$\mu(J') \leq \frac{3}{2}(\delta_2/6 + \delta_2/6) = \delta_2/2.$$

Similarly, $\mu(J'') \leq \delta_2/2$. Inequality (43) follows.

Applying inequality (42) again,

$$\mu(J) \leq \frac{3}{2}\mu(J \setminus C(L_0, L_0; J)). \tag{44}$$

Now note that by definition of J, $J \setminus C(L_0, L_0; J)$ is contained within one the following sets: $I_1 \cup I_3$, $I_1 \cup \overline{I}_1$, or $I_3 \cup \overline{I_3}$. But

$$\mu(I_i) = \mu(\overline{I}_i) \leq \frac{\delta_2}{6}$$

for $i = 1, 3$. Thus in any case we get that

$$\mu(J \setminus C(L_0, L_0; J)) \leq \frac{\delta_2}{3}.$$

Therefore by inequality (44)

$$\mu(J) \leq \frac{\delta_2}{2}$$

as was to be proven.

Let now I_i all be trivial. Write $I_i = B_i \setminus B_i'$ where $B_i = B_{r_i}(p_i)$, $B_i' = B^c_{r_i'}(p_i)$ for some $p_i \in \tilde{\Sigma}$, $r_i' < r_i \in (0, \infty)$, and $i = 1, 2, 3$. By Lemmas 4.15 and 5.4, I_1 and I_3 are topologically related. Let $J = M(I_1, I_3)$. We need to show that $\mu(J) \leq \delta_2/2$. By Lemma 4.24,

$$J = I_1 \cup I_3 \cup C(L_0, L_0; I)$$

where $I = m(I_1, I_3)$. Thus, $\mu(J) \leq \delta_2/3 + \mu(C(L_0, L_0; I))$. To finish the proof we need to show that $\mu(C(L_0, L_0; I)) \leq \delta_2/6$. Then

$$I \subset J \subset M(I_1, I_2) \cup M(I_2, I_3).$$

So, $\mu(I) \leq \delta_2$. On the other hand, by Lemma 4.24,

$$I \setminus C(L_0, L_0; I) \subset I_1 \cup I_3.$$

Therefore, by definition of L_0,

$$\mu(C(L_0, L_0; I)) \leq \frac{1}{3}(\mu(I_1 \cup I_3) + \mu(C(L_0, L_0; I))).$$

So,

$$\mu(C(L_0, L_0; I)) \leq \frac{1}{2}\mu(I_1 \cup I_3) \leq \frac{\delta_2}{6}$$

as required. □

Lemma 5.6. *For every $\sim$-equivalence class c there is an $I \in c$ such that $Mod(I)$ is maximal in c .*

Proof. Write LN_0 for the set of trivial long necks, and LN_1 for the set of nontrivial long necks. Let $R := \max_{z \in \Sigma} \operatorname{inj}(\Sigma, z; h)$. To give a trivial element of $\mathcal{A}_h$ is to give a point and two real numbers r_1, r_2 subject to some restrictions. This induces on LN_0 the topology of a subset of the compact bordered manifold

$$X_0 = \Sigma \times \left[0, \frac{1}{3}R\right] \times \left[0, \frac{1}{10}R\right].$$

To give a nontrivial element of $\mathcal{A}_h$ is to give a simple closed geodesic and two real numbers subject to some restrictions. Thus, when

$$genus(\tilde{\Sigma}) > 1,$$

LN_1 can be assigned the topology of a subset of the compact bordered manifold

$$X_1 = \bigcup_{\{\gamma|\ell(\gamma)<\sinh^{-1}(1)\}} \left[-\frac{1}{2}Mod(\mathcal{C}(\gamma)), \frac{1}{2}Mod(\mathcal{C}(\gamma))\right]^2.$$

X_1 is indeed compact since the number of simple closed geodesics γ for which $\ell(\gamma) < \sinh^{-1}(1)$ is finite. See [2]. When $genus(\tilde{\Sigma}) = 0$ we have that LN_1 can be thought of as a subset of

$$[-\pi, \pi]^2.$$

Finally, when $genus(\tilde{\Sigma}) = 1$, LN_1 is a subset of

$$X_1 = \left[-\frac{1}{2}Mod(\mathcal{C}(\alpha_0)), \frac{1}{2}Mod(\mathcal{C}(\alpha_0))\right].$$

We show that LN_i is closed in X_i. The conditions of stability, length and cleanness are closed conditions. However, admissibility alone is not a closed condition for trivial annuli because the inner radius of a trivial annulus must be positive. For nontrivial annuli it is not closed when $genus(\tilde{\Sigma}) = 0$, since $\mathcal{C}(\partial\Sigma)$ is not closed in this case. We show that the intersection of the set of admissible annuli with those having stable complement is closed.

First we show this for trivial annuli. The non-admissible points of X_0 in the closure of the trivial admissible annuli are points of the form $(p, r, 0)$. That is, annuli with internal radius 0. Let

$$r = \inf\left\{r' \in \left[0, \frac{1}{10}R\right] \middle| p \in \Sigma, \mu(B_{r'}(p)) \geq \delta_1/2\right\}.$$

Since Σ is compact, $\frac{d\mu}{d\nu_h}$ is bounded. So, $r > 0$. Thus, any trivial element of LN_0 has internal radius no less than r. The claim follows. For nontrivial annuli the claim follows in a similar manner.

Now we need to show that on LN, the condition of equivalence is closed. The only non trivial point is to show that topological relatedness is a closed condition. By Theorem 4.8 it suffices to show that there is an $a > 0$ such that for any two equivalent trivial long necks of the form $I_i = B_{r_i}(p_i) \setminus B^c_{r'_i}(p_i)$, $i = 1, 2$, we have

$$A := Area(B^c_{r'_1}(p_1) \cap B^c_{r'_2}(p_2); h_{can}) \geq a. \tag{45}$$

Write $B_i = B_{r_i}(p_i)$ and $B'_i = B^c_{r'_i}(p_i)$. We have $\mu(B'_i) \geq \delta_1/2$ and

$$\mu(B'_1 \cup B'_2 \setminus B'_1 \cap B'_2) = \mu(M(I_1, I_2)) \leq \delta_2/2.$$

Therefore,

$$\mu(B^c_{r'_1}(p_1) \cap B^c_{r'_2}(p_2)) \geq \delta_1/2 - \delta_2/2 \geq \delta_2/2.$$

Since Σ is compact, $\frac{d\mu}{d\nu_h}$ is bounded on Σ by some constant d. Clearly,

$$\delta_2/2 \leq \mu(B^c_{r'_1}(p_1) \cap B^c_{r'_2}(p_2)) \leq Ad.$$

Inequality (45) follows.

Finally, equation (7) shows that $Mod : LN \to \mathbb{R}$ is continuous with respect to the topology on LN. □

Lemma 5.7. *Let $I_i \in LN$ for $i = 1, 2$. Suppose $I_1 \sim I_2$ and I_1 is essentially disjoint from I_2. Write $I = m(I_1, I_2)$. Then $C(L_0, L_0; I)$ is a long neck which is $\sim$-equivalent to each of the I_i.*

Proof. Relying on Lemma 4.22 one verifies that $I \in \mathcal{A}_h$ and, furthermore, that $Mod(I) > L_1$. We have

$$I \subset M(I_1, I_2),$$

so

$$\mu(I) \leq \mu(M(I_1, I_2)) \leq \delta_2/2.$$

Therefore, $\mu(C(L_0, L_0; I)) \leq \delta_2/6$. We show that each component of

$$\tilde{\Sigma} \setminus C(L_0, L_0; I)$$

is μ-stable. Let A be one such connected component. If I is an admissible annulus, then, by construction, A contains a component of $\tilde{\Sigma} \setminus I_i$ for either $i = 1$ or $i = 2$. If I is an admissible cylinder then if

$$genus(\tilde{\Sigma}) > 1,$$

stability is automatic. It is left to treat the exceptional cases. When $genus(\tilde{\Sigma}) = 0$ the claim follows as in the case of trivial annuli. When $genus(\tilde{\Sigma}) = 1$, the complement of $C(L_0, L_0; I)$ consists of a single component and so the claim follows by Assumption 4.3.1. We thus showed that I is a long neck. For the remaining part of the claim, I and each of the I_i are nontrivially embedded in $M(I_1, I_2)$ and so are topologically related. Furthermore, we have $M(I_i, I)) = M(I_1, I_2)$, so $\mu(M(I_i, I)) \leq \delta_2$. The claim follows. □

Lemma 5.8. *Let $I_1, I_2 \in LN$ and suppose $b_1(I_1 \cup I_2) = 1$. Then $I_1 \sim I_2$.*

Proof. Suppose first that I_1 and I_2 are both trivial. Let $I_i = A(r_i, r'_i; p_i)$. By the Mayer Vietoris sequence, the assumption implies that

$$b_1(I_1 \cap I_2) = 1.$$

From this it follows that $B^c_{r'_1}(p_1) \subset B_{r_2}(p_2)$ and $B^c_{r'_2}(p_2) \subset B_{r_1}(p_1)$. It easily follows that $I_1 \cup I_2$ is clean. Since I_1 and I_2 are nontrivially embedded in $I_1 \cup I_2$, they are topologically related. Furthermore, $M(I_1, I_2) = I_1 \cup I_2$. In particular

$$\mu\left(M(I_1, I_2)\right) \leq \delta_2/3 \leq \delta_2/2.$$

Suppose now that I_1 and I_2 are both non trivial. If $I_1 \cup I_2$ is clean then it is straightforward that $M(I_1, I_2) = I_1 \cup I_2$. Otherwise, I_1 or I_2 is conjugation invariant. Without loss of generality assume I_1 is conjugation invariant. Then $M(I_1, I_2) = I_1 \cup I_2 \cup \overline{I}_2$. In any case, $\mu(M(I_1, I_2)) \leq \delta_2/2$. □

Lemma 5.9. *For any $I_1, I_2 \in LN$, $b_1(I_1 \cup I_2) > 0$.*

Proof. If I_1 is nontrivial, this is immediate. Otherwise, the claim is a consequence of Lemma 5.4. □

Lemma 5.10. *Let c be a $\sim$-equivalence class. Let $I_1 \in c$ have maximal modulus. Let $I_2 \in LN$. I_1 and I_2 are essentially disjoint if and only if $I_2 \notin c$.*

Proof. Suppose I_1 and I_2 are essentially disjoint and suppose by contradiction $I_2 \in c$. Write $I := C(L_0, L_0; m(I_1, I_2))$. By Lemma 5.7, $I \in LN \cap c$. By Lemma 4.22, $Mod(I) > Mod(I_1)$. This is a contradiction. Conversely, suppose I_1 is not essentially disjoint from I_2. Recall that we excluded the trivial case $\tilde{\Sigma} \neq I_1 \cup I_2$. Therefore, combining Theorem 4.16 and Lemma 5.9, we have $b_1(I_1 \cup I_2) = 1$ as long as $\tilde{\Sigma} \neq I_1 \cup I_2$, we conclude $I_1 \sim I_2$. □

Lemma 5.11. *Let c be a $\sim$-equivalence class.*

(a) The elements of c are either all trivial or all nontrivial. In the first case we say that c is trivial, in the second case we say that it is nontrivial.
(b) If c is trivial and then either all elements of c are conjugation invariant or there is a component A of $\Sigma_{\mathbb{C}} \setminus \partial\Sigma$ such that they are all contained in A.
(c) If c is nontrivial then either the maximal elements of c are conjugation invariant or there is a component A of $\Sigma_{\mathbb{C}} \setminus \partial\Sigma$ such that they are all contained in A.

Proof. (a) This follows from Theorem 4.8 since $\sim$-equivalence entails topological relatedness.
(b) For any $I_1, I_2 \in c$, I_1 and I_2 embed nontrivially in $M(I_1, I_2)$ which is clean and doubly connected. Since these annuli are all trivial, none of them contains a component of $\partial\Sigma$. It thus follows from Lemma 2.9 that I_1 and I_2 are either both conjugation invariant or both contained in the same component of $\Sigma_{\mathbb{C}} \setminus \partial\Sigma$.
(c) Suppose there is a maximal element I_1 of c that is not conjugation invariant. Since I_1 is clean, there is a component A of $\Sigma_{\mathbb{C}} \setminus \partial\Sigma$ such that $I_1 \subset A$. Suppose by contradiction that there is an $I_2 \in c$ such that $I_2 \not\subset A$. Write $J = C(L_0, L_0; M(I_1, I_2))$. Then maximality of I_1 easily implies
$$J = M(I_1, \overline{I}_1).$$
But then we get the contradiction $J \in LN \cap c$ and
$$Mod(J) > Mod(I_1).$$
□

Lemma 5.12. *Let $\mathcal{B}$ be a bubble decomposition consisting of elements of LN. Let $I = \Sigma_v$ for some $v \in V_{\mathcal{B}}$ and suppose $b_1(I) = 1$. Suppose I is bordered by $\sim$-inequivalent elements $I_1, I_2 \in \mathcal{B} \subset LN$. Then I is μ-stable.*

Proof. First we claim that I_1 and I_2 are topologically related. To see this note first that I_1 and I_2 freely homotopic, so they are either both trivial or both non-trivial. If both are nontrivial, the claim follows from Lemma 2.8. Suppose both are trivial. Write $I_i = B_i \setminus B_i'$ where B_i and B_i' are concentric discs. Clearly we may assume with no loss of generality that $B_2 \subset B_1'$ and so, $I = B_1' \setminus B_2$. In particular $B_1' \cap B_2' \neq \emptyset$. Further, $I_1 \cap I_2$ is clean. Indeed, the only alternative is that B_1 is conjugation invariant while B_2 is not, but in that case I is not conjugation invariant. Since by definition $\mathcal{B}$ is conjugation invariant, this is a contradiction. By Theorem 4.8(b), I_1 is topologically related to I_2.

We now show that I is μ-stable. In case the I_i are trivial,

$$I = M(I_1, I_2) \setminus (I_1 \cup I_2).$$

So, by $\sim$-inequivalence, $\mu(I) > \delta_2/6$. Suppose the I_i are nontrivial. If $I_1 \cup I_2$ is clean, we have $I = M(I_1, I_2) \setminus (I_1 \cup I_2)$ and the claim follows as before. Otherwise, without loss of generality I_1 is conjugation invariant while I_2 is not. Then

$$M(I_1, I_2) = I_1 \cup I \cup I_2 \cup \overline{I \cup I_2}.$$

Suppose by contradiction that $\mu(I) < \delta_2/6$. Write $M = M(I_1, I_2)$. Then $\mu(M) < \delta_2$. Let $N = M \setminus C(L_0, L_0; M)$. We have $N \subset I_2 \cup \overline{I}_2$. In particular $\mu(N) \leq \delta_2/3$. By definition of L_0 it follows that

$$\mu(M) < \delta_2/2.$$

That is, $I_1 \sim I_2$. A contradiction. □

Proof of Theorem 5.3. Denote by S the set of $\sim$ equivalence classes. Pick a component $A \subset \tilde{\Sigma} \setminus \partial\Sigma$. For each $c \in S$ whose elements lie in A or which is conjugation invariant assign an element $I_c \in c$ of maximal modulus. For any other c define $I_c := \overline{I}_{\overline{c}}$. Let now

$$\tilde{\mathcal{B}} = \{I_c | c \in S\}.$$

In the exceptional case where $genus(\tilde{\Sigma}) = 1$ and $Mod(I_0) \geq L_1$, we add I_0 to $\tilde{\mathcal{B}}$. It is follows from Lemma 5.11 that $\tilde{\mathcal{B}}$ is conjugation invariant. By Lemma 5.10 the elements of $\tilde{\mathcal{B}}$ are pairwise essentially disjoint. Now let

$$\mathcal{B} = \{C(2K_1, 2K_1; I) | I \in \tilde{\mathcal{B}}\}.$$

We show that $\mathcal{B}$ is a maximal μ-decomposition. We check the stability condition. Let $v \in V_{\mathcal{B}}$. We distinguish between the following cases.

(a) $2genus(\Sigma_v) + |\pi_0(\partial\Sigma_v)| \geq 3$. In this case stability is automatic.
(b) $genus(\Sigma_v) = 1$ and $\partial\Sigma_v = \emptyset$. Stability is a consequence of Assumption 4.3.1.

(c) $genus(\Sigma_v) = 0$ and $|\pi_0(\partial\Sigma_v)| = 2$. Then if Σ_v is bordered by two inequivalent elements of LN, this case is covered by Lemma 5.12. Otherwise, we must have that $genus(\tilde{\Sigma}) = 1$, and either Σ_v is the complement of a long neck, or Σ_v is bordered by I_0 and some element I_1 of LN. In the first case, stability follows by definition of long necks. We treat the second case. Write $I = \Sigma_v$ and suppose by contradiction that $\mu(I) < \delta_2/6$. Let

$$M = I_0 \cup I_1 \cup I \cup \overline{I_1 \cup I},$$

and let $N = C(L_0, L_0; M)$. Suppose first $\partial\Sigma = \emptyset$. Then $Mod(N) > Mod(I_0)$ and $\mu(N) \leq \delta_2/6$. This contradicts the choice of I_0. Suppose now $\partial\Sigma \neq \emptyset$. Then $I_0 \subset N$ and $M \setminus N \subset I_1 \cup \overline{I}_1$. By assumption $\mu(M) < \delta_1$, so

$$\mu(I_0) \leq \mu(N) \leq \delta_2/9.$$

This again contradicts the choice of I_0.

(d) $|\pi_0(\partial\Sigma_v)| = 1$. In this case Σ_v is a component of the complement of a long neck and so the claim follows by definition.

It remains to check the maximality condition. Suppose by contradiction that Σ_v contains a long neck I'. Then there is a $c \in S$ such that $I' \in c$. Clearly,

$$C(K_1, K_1; I') \cap C(K_1, K_1; I_c) = \emptyset.$$

That is, I' and I_c are essentially disjoint. Since I_c has maximal modulus in c, this contradicts Lemma 5.10.

□

6. (μ, h)-ADAPTEDNESS

Theorem 6.1. *Let $\mathcal{F}$ be a uniformly thick thin family. Let $(\Sigma, \mu) \in \mathcal{F}$ satisfy Assumption 4.3.1.*

(a) If $\partial\Sigma = \emptyset$, there is a conformal constant curvature metric h on Σ and a (μ, h)-adapted bubble decomposition $\mathcal{B}$ of Σ with constants independent of (Σ, μ).

(b) If $\partial\Sigma \neq \emptyset$, there is a conjugation invariant conformal constant curvature metric h on $\Sigma_{\mathbb{C}}$ and a conjugation invariant (μ, h)- adapted bubble decomposition $\mathcal{B}$ of $\Sigma_{\mathbb{C}}$ with constants independent of (Σ, μ).

For the rest of this section fix a $(\Sigma, \mu) \in \mathcal{F}$ satisfying Assumption 4.3.1 with the understanding that all constants depend only on $\mathcal{F}$ and not on the particular (Σ, μ) we chose. Let $\mathcal{B}$ be a maximal μ-decomposition as in Theorem 5.3. To prove that $\mathcal{B}$ satisfies the estimates in part (b) of Definition 6.1, we need to introduce some notation.

Associate to $\mathcal{B}$ a graph $G_{\mathcal{B}}$ as follows. As the vertex set of $G_{\mathcal{B}}$ take $V_{\mathcal{B}}$. Add an outgoing half edge l from v for each element of $\pi_0(\partial\Sigma_v)$. For any $v \in V_{\mathcal{B}}$, denote by $\mathcal{H}_v$ the set of half edges going out of v. For $l \in \mathcal{H}_v$ denote by γ_l the boundary component corresponding to l. Half edges l_1 and l_2 are connected to one another in $G_{\mathcal{B}}$ if and only if there is an element of $I \in \mathcal{B}$

such that $\partial I = \gamma_{l_1} \cup \gamma_{l_2}$. There is thus a two to one correspondence between half edges and elements of $\mathcal{B}$. For $l \in \mathcal{H}_v$, write I_l for the corresponding element of $\mathcal{B}$.

Let $v \in V_{\mathcal{B}}$. An external boundary component of Σ_v is an element $\gamma \in \pi_0(\partial \Sigma_v)$ such that γ is either not contractible in $\tilde{\Sigma}$ or satisfies

$$Diam(\gamma; h) = Diam(\Sigma_v; h)^{10}.$$

Let $E_v \subset \mathcal{H}_v$ denote the half edges corresponding to the external boundary components of Σ_v, and let

$$F_v := \mathcal{H}_v \setminus E_v.$$

For each $l \in F_v$, γ_l is the boundary of a disc $B_l \subset \tilde{\Sigma}$. Write

$$Cl(\Sigma_v) := \Sigma_v \bigcup_{\{l \in F_v\}} B_l \subset \Sigma_{\mathbb{C}}.$$

Lemma 6.2. *There is a constant f_1 with the following significance. Let $v \in V_{\mathcal{B}}$ and let $I \subset Cl(\Sigma_v)$ be a neck. In case $I = B \setminus B'$ for discs $B' \subset B \subset \Sigma$, suppose that $\partial B' \subset \Sigma_v$. Let*

$$n(I) := |\{l \in F_v : \gamma_l \cap I \neq \emptyset\}|\,.$$

Then $Mod(I) \leq f_1(\mu(I \cap \Sigma_v) + n(I) + 1)$.

Proof. Let $L = Mod(I)$. For any integer $0 \leq i < \lfloor L/L_1 \rfloor$ let

$$I_i := S(iL_1, (i+1)L_1; I).$$

Since Σ_v contains no long necks, we must have

$$\mu(I_i) > \delta_2/6.$$

Let S_1 denote the set of those $0 \leq i < \lfloor L/L_1 \rfloor$ that satisfy $I_i \subset \Sigma_v$ and let S_2 be the rest. Clearly,

$$\frac{L}{L_1} \leq |S_1| + |S_2| + 1,$$

and

$$|S_1| \leq \frac{6\mu(I \cap \Sigma_v)}{\delta_2}.$$

To complete the proof we need to bound $|S_2|$.

If $i \in S_2$, there is an $l \in F_v$ such that $I_i \cap B_l \neq \emptyset$. We show that there as at most one $j \neq i$ such that B_l meets I_j. For this, let $J_0 \in \mathcal{B}$ be the unique element such that

$$\gamma_l \subset \partial J_0.$$

There is a $J_1 \in LN$ such that $J_0 = C(2K_1, 2K_1; J_1)$. Let

$$J = S(Mod(J_1) - 3K_1, Mod(J_1) - K_1; J_1).$$

[10]Note that if Σ_v is formed by removing any number of small discs from sphere, then $E_v = \emptyset$.

Suppose now by contradiction that B_l meets three successive sub-cylinders I_{i-1}, I_i and I_{i+1}. By the assumption of the lemma, B_l does not contain any of the I_i. Therefore, $\gamma_l \cap C(K_1, K_1; I_i) \neq \emptyset$. But $\gamma_l \subset C(K_1, K_1; J)$. So, J and I_i are not essentially disjoint.

On the other hand, we show that the fact that $B_l \not\subset I_i$ implies that J and I_i are essentially disjoint. Let $k := b_1(I_i \cup J)$. By Theorem 4.16 it suffices to show that $k > 1$. Suppose by contradiction that $k \leq 1$. If $k = 0$ then I is trivial and its interior disc B is contained in J. But since I is a neck, $\mu(B) \geq \delta_1/2$ whereas $\mu(J) \leq \mu(J_1) \leq \delta_2/6$. Suppose now that $k = 1$. Since we are assuming $B_l \not\subset I_i$, this is only possible if I is trivial and $B_l \cap B \neq \emptyset$. But then by the assumption of the Lemma we have that $B_l \subset B$, in contradiction to $I_i \cap B \neq \emptyset$. We conclude that I_i and J are essentially disjoint. The contradiction shows that B_l meets at most two sub-cylinders. We thus conclude that

$$|S_2| \leq 2n(I).$$

□

For any $v \in V_{\mathcal{B}}$ let $n_v = |F_v|$ and $\mu_v = \mu(\Sigma_v)$.

Lemma 6.3. *There are constants f_i, for $i = 2, ..., 9$, with the following significance. Let $v \in V_{\mathcal{B}}$ and let $l \in E_v$.*

(a)

$$\ell(\gamma_l; h_v) \geq f_2 e^{-f_3(\mu_v + n_v)}.$$

(b) For all $x \in \Sigma_v$

$$\mathrm{inj}(\Sigma_v, x; h_v) \geq f_4 e^{-f_5(\mu_v + n_v)}.$$

(c) Let $l' \neq l \in \mathcal{H}_v$. Then

$$d(\gamma_l, \gamma_{l'}; h_v) \geq f_6 e^{-f_7(\mu_v + n_v)}.$$

Remark 6.4. $\mathrm{inj}(\Sigma_v, x; h_v)$ is defined as the supremum of all r such that any unit speed geodesic ray

$$\alpha : [0, \min\{r, d_{h_v}(p, \partial\Sigma_v)\}] \to \Sigma_v$$

emanating from p minimizes length.

Proof. Let let $g = genus(\tilde{\Sigma})$. Let

$$m(v) := 2genus(Cl(\Sigma_v)) + |E_v|.$$

We distinguish between various possibilities for $m(v)$ and g.

(a) $m(v) = 0$. In this case $E_v = \emptyset$, so only part (b) is not vacuous. But part (b) is obvious.

(b) $m(v) = 1$ and $g = 0$. By carefully inspecting the definition of external boundary parts (a) and (b)are seen to hold. We show part (c). By construction, there are $J, J' \in LN$ such that

$$I_l = C(2K_1, 2K_1; J),$$

and

$$I_{l'} = C(2K_1, 2K_1; J').$$

Let N be the component of

$$C(K_1, K_1; J) \setminus I_l,$$

for which $\gamma_l \subset \partial N$. N is a tubular neighborhood of γ_l. By essential disjointness of J_l and $J_{l'}$ we have that $N \cap \gamma_{l'} = \emptyset$. We have

$$Mod(N) = K_1.$$

Denote by r the metric width of N. That is, the distance between the two boundary components. Then

$$K_1 = \frac{1}{s_v}\int_0^r \frac{dx}{h_{\theta,FS}} \leq \int_0^r \frac{dx}{h_{\theta,FS}},$$

where $h_{\theta,FS}$ is Fubini Study metric in appropriate coordinates. Take f_6 to be the solution of

$$K_1 = \int_0^{f_6} \frac{dx}{h_{\theta,FS}}.$$

f_7 may be taken to vanish.

(c) $m(v) = 1$ and $g > 0$. This case is similar to the previous case.

(d) $m(v) = 2$ and $g = 0$. In this case it can be verified that $Cl(\Sigma_v) = B \setminus B'$ for two concentric discs in $\Sigma_{\mathbb{C}}$. Suppose first that $Cl(\Sigma_v)$ is contained in a hemisphere. Then the only additional thing to address after the case $m(v) = 1$ is to estimate $\ell(\partial B'; h_v)$. Applying Lemma 6.2 to $Cl(\Sigma_v)$ we have

$$Mod(cl(\Sigma_v)) \leq f_1(\mu(I \cap \Sigma_v) + n(I) + 1).$$

On the other hand we denote by r and r' the radii of B and B' with respect to h_v, then

$$\log(r/r') \leq cMod(I),$$

for an appropriate constant. Now note that $r = 1$, so the claim follows. If $Cl(\Sigma_v)$ is not contained in a hemisphere, cut $Cl(\Sigma_v)$ in two along a concentric equator and repeat the same argument.

(e) $m(v) = 2$ and $g = 1$. Only part (b) is not vacuous. But d_v in this case is proportional to the modulus of Σ_v which is appropriately bounded by Lemma 6.2.

(f) $m(v) = 2$ and $g > 1$. Let $e \in E_v$. Then there is a simple closed geodesic γ such that $\gamma_e \subset \mathcal{C}(\gamma)$. Write $I = Cl(\Sigma_v)$ and let γ_1 and γ_0 be the components of ∂I. It is easy to see that I is a sub-cylinder of $\mathcal{C}(\gamma)$. Therefore, γ_0 and γ_1 have constant ρ coordinates x_0 and x_1, respectively. For $r \in [x_0, x_1]$ let

$$\gamma_r := \{z \in I | \rho(z) = r\},$$

and let $r_{\min} \in [x_0, x_1]$ be the point where $\ell(\gamma_r)$ obtains its minimum, $\ell_{\min}$. Without loss of generality, assume $|x_0| \leq |x_1|$. We have

$$\begin{aligned} \ln \frac{\ell_{\min}}{\ell_1} &= \ln \frac{h_\theta(r_{\min})}{h_\theta(x_1)} \\ &\geq -\int_{x_0}^{x_1} \frac{h'_\theta(x)}{h_\theta(x)} dx \\ &\geq -\int_{x_0}^{x_1} \frac{1}{\pi h_\theta(x)} dx \\ &= -\frac{1}{\pi} ModI \\ &\geq -\frac{1}{\pi} f_1(\mu_v + n_v). \end{aligned} \tag{46}$$

Here we rely on the inequality $h'_\theta(x) = \ell(\gamma) \sinh x/(2\pi) \leq 1/\pi$ for $x \in w(\gamma)$. On the other hand,

$$\frac{d_v}{\ell(\gamma_1; h)} \leq \frac{|x_1 - x_0| + \ell(\gamma_0; h)}{\ell(\gamma_1; h)}. \tag{47}$$

But

$$|x_1 - x_0| \leq \frac{\ell(\gamma_1; h)}{2\pi} \int_{x_0}^{x_1} \frac{dx}{h_\theta(x)} = \frac{\ell(\gamma_1; h)}{2\pi} Mod(I), \tag{48}$$

where for the inequality we relied on the equation

$$h_\theta(x) = \frac{1}{2\pi} \ell(\{\rho = x\}; h) \leq \ell_1.$$

Combining estimates (46), (47) and (48), we obtain

$$\frac{\ell_{min}}{d_v} \geq \frac{exp\left(-\frac{1}{\pi} f_1(\mu_v + n_v)\right)}{Mod(I) + 1}.$$

Together with Lemma 6.2, this implies part (a)

Part (b) is a consequence of Eq. (13) as follows. For any $p \in I$, let $x = \rho(p)$ and $d = w(\gamma) - |x|$. We have,

$$\begin{aligned} \text{inj}(p; \Sigma, h) &= \sinh^{-1}(\cosh \frac{1}{2}\ell(\gamma) \cosh d - \sinh d) \\ &= \sinh^{-1}(e^{-d} + (\cosh \frac{1}{2}\ell(\gamma) - 1) \cosh d) \\ &\geq \sinh^{-1}(e^{-d}) \\ &= \ln(e^{-d} + \sqrt{e^{-2d} + 1}) \\ &= e^{-d} + o(e^{-d}). \end{aligned} \tag{49}$$

Let $\xi = |x_1| - |x| \in [0, |x_1| - |x_0|]$. We have

$$\begin{aligned}\mathrm{inj}(p;\Sigma,h_v) &\geq c\frac{e^{-d}}{\frac{d_v}{\ell_1}\ell_1}\\ &\geq c\frac{e^{-d}}{\ell(\gamma)\cosh x_1(Mod(I)+1)}\\ &\geq c\frac{e^{-w(\gamma)}}{\ell(\gamma)(Mod(I)+1)}e^{-\xi}.\end{aligned}$$

It is straightforward to verify that there is lower bound on the expression $\frac{e^{-w(\gamma)}}{\ell(\gamma)}$ which is independent of γ. Since $\xi \leq |x_1| - |x_0| < diam(I;h)$, the claim follows.

Given the estimate on $\ell(\gamma_i)$ the proof of part (c) in the current case is similar to that of the case $m(v) = 1$ and $g = 0$. We omit the details.

(g) $m(v) >$ and $g = 0$. Considering the definition of external boundary components, there is no such case.

(h) $m(v) > 2$ and $g > 1$. Decompose

$$\Sigma_v = (Thick(\Sigma;h) \cap \Sigma_v) \cup (Thin(\Sigma;h) \cap \Sigma_v).$$

The components of $(Thin(\Sigma;h) \cap \Sigma_v)$ behave exactly as the case $m(v) = 2$ and $g = 1$ and contain all the external boundary components. It remains to estimate on inj and $(Thick(\Sigma;h) \cap \Sigma_v)$, but this is a tautology.

□

To establish the rest of the estimates in Definition 1.2, we introduce some further notation. For any $v \in V_{\mathcal{B}}$ let

$$r_v := \begin{cases} \frac{1}{3}\min_{z\in Cl(\Sigma_v)} \mathrm{inj}(\Sigma_{\mathbb{C}}, z; h), & genus(\Sigma_{\mathbb{C}}) > 0,\\ \min\left\{\sqrt{\frac{\delta_1}{2\pi K_0}}, \frac{\pi}{3}\right\}, & genus(\Sigma_{\mathbb{C}}) = 0.\end{cases}$$

Let $B = B_r(p;h_v) \subset Cl(\Sigma_v)$ be a clean geodesic disc. Define

$$r_B := \begin{cases} \min\{s_v r_v, d(p, \partial Cl(\Sigma_v); h_v)\}, & p \in \partial\Sigma,\\ \min\{s_v r_v, d(p, \partial Cl(\Sigma_v); h_v), \frac{1}{2}d(p,\overline{p};h_v)\}, & p \notin \partial\Sigma.\end{cases}$$

Lemma 6.5. *There is a constant f_2 with the following significance. Let $B = B_r(p;h_v) \subset Cl(\Sigma_v)$ be a clean disc of radius r satisfying*

$$\mu(B) \geq \delta_1/2.$$

Then

$$r \geq \frac{1}{5}r_B e^{-f_2(\mu_v+n_v)}.$$

Proof. Let
$$I = A(r_B, r, p).$$
If $r_B < 5r$ we are done, so suppose $r_B \geq 5r$. It follows that $I \in \mathcal{A}_h$. Also, $I \subset Cl(\Sigma_v)$. We claim that I is a neck. For this we need to verify that both B and $\Sigma' := \Sigma_{\mathbb{C}} \backslash B_{r_B}(p; h_v)$ are stable. But B is stable by assumption. In the case where
$$genus(\Sigma_{\mathbb{C}}) > 0,$$
Σ' is immediately seen to be stable. When $genus(\Sigma) = 0$, stability of Σ' follows from the fact that h satisfies the condition of Lemma 4.1. Furthermore, I satisfies the condition of Lemma 6.2. So,
$$f_1\{\mu(Cl(I)) + n(Cl(I)) + 1\} \geq Mod(Cl(I)) > c \log \frac{r_B}{2r},$$
for an appropriate constant c. This inequality gives the claim. □

Lemma 6.6. *There are constants f_i, $10 \leq i \leq 13$, with the following significance. Let $B = B_r(p; h_v) \subset Cl(\Sigma_v)$ be a clean disc such that $\mu(B) \geq \delta_1/2$. Suppose*
$$d(p, \partial Cl(\Sigma_v); h_v) > f_6 e^{-f_7(\mu_v + n_v)}. \tag{50}$$
Then
$$r \geq f_{10} e^{-f_{11}(\mu_v + n_v)}.$$

Proof. By Lemma 6.3 we have that
$$\text{inj}(x; h_v) \geq f_4 e^{-f_5(\mu_v + n_v)}$$
for all $x \in Cl(\Sigma_v)$. So, by assumption (50), when B is conjugation invariant we have
$$r_B \geq \min \left\{ f_4 e^{-f_5(\mu_v + n_v)}, \frac{1}{2} f_6 e^{-f_7(\mu_v + n_v)}, s_v \sqrt{\frac{\delta_1}{2\pi K_0}}, s_v \frac{\pi}{3} \right\}.$$
the claim now follows by Lemma 6.5.

To prove the claim for any clean B we need to further bound
$$r' := \frac{1}{2} d(p, \overline{p}; h_v)$$
from below by an exponent in $\mu_v + n_v$. In fact, to prove the Lemma, it suffices to estimate $r + r'$ by such an exponent. We may suppose
$$r + r' < \frac{1}{2} f_6 e^{-f_7(\mu_v + n_v)}, \tag{51}$$
for otherwise we are done. Let p' be the midpoint of the shortest geodesic segment connecting p with $\overline{p}$. Combining inequality (51) with inequality (50) we get
$$d(p', \partial Cl(\Sigma_v)) > \frac{1}{2} f_6 e^{-f_7(\mu_v + n_v)}.$$

Let $B' := B_{r+r'}(p')$. Then $\subset Cl(\Sigma_v)$. Furthermore,

$$r_{B'} > \frac{1}{4} f_6 e^{-f_7(\mu_v+n_v)},$$

By Lemma 6.5 this implies

$$r' + r \geq \frac{1}{20} f_6 e^{-(f_7+f_2)(\mu_v+n_v)}.$$

□

Corollary 6.7. *(a) For any $l \in F_v$*

$$\ell(\Sigma_v; h_v) \geq f_{10} e^{-f_{11}(\mu_v+n_v)}.$$

(b) For any $l_1, l_2 \in F_v$ we have

$$d(\gamma_{l_1}, \gamma_{l_2}; h_v) \geq f_{12} e^{-f_{13}(\mu_v+n_v)}.$$

Proof. (a) By Lemma 6.3(c), the assumptions of Lemma 6.6 hold in particular for $B = B_l$ where $l \in F_v$.
(b) This follows by the same proof as that of Lemma 6.3(c).

□

In the following, for any $\gamma \in \pi_0(\partial\Sigma)$, let $N_\gamma := B_{f_{12}e^{-f_{13}(\mu_v+n_v)}}(\gamma; h_v)$. Without loss of generality we assume $f_{12} \leq f_6$ and $f_{13} \geq f_7$.

Corollary 6.8. *For any $\gamma \in \pi_0(\partial\Sigma_v)$*

$$\left.\frac{d\nu_{h_v}}{d\nu_{h_{st}}}\right|_{N_\gamma} \geq f_{10} e^{-f_{11}(\mu_v+n_v)}.$$

Proof. Using cylindrical coordinates on N_γ let

$$\gamma_r = \{z \in N_\gamma | \rho(z) = r\}.$$

We have

$$\frac{d\nu_{h_v}}{d\nu_{h_{st}}}(r,\theta) = \frac{1}{2\pi}\ell(\gamma_r).$$

If $\gamma \in F_v$, Lemmas 6.6 and 6.3(c) imply

$$\ell(\gamma_r) \geq f_{10} e^{-f_{11}(\mu_v+n_v)}.$$

Otherwise, this is just Lemma 6.3(a). □

Lemma 6.9. *There are constants f_{14}, f_{15}, such that for any $p \in \Sigma_v$,*

$$\frac{d\mu}{d\nu_{h_v}}(p) \leq f_{15} e^{f_{14}(\mu_v+n_v)}.$$

Proof. Let $p \in \Sigma_v$ be the point where the supremum of $\frac{d\mu}{d\nu_{h_v}}$ is obtained and let d be its value. Let $c = f_6 e^{-f_7(\mu_v+n_v)}$. If $d(p, \partial Cl(\Sigma_v); h_v) < c$ then, by construction of $\mathcal{B}$, there is a long neck I so that p is contained in $C(\pi + c_2, \pi + c_2; I) \subset C(K_1, K_1; I)$. Thus, by Lemma 3.4, $\frac{d\mu}{d\nu_{h_{st}}} \leq a$. The claim now follows from Corollary 6.8.

Otherwise, if $d < 1/c$ we are done. If $d > 1/c$, consider the disc $B = B_{\frac{1}{d}}(p; h_v)$. Then $\mu(B) > \delta_1$ by Remark 3.3 and $B \subset \Sigma_v$, so the bound follows immediately from Lemma 6.6. □

Proof of Theorem 6.1. Let $\mathcal{B}$ be a maximal μ-decomposition as in Theorem 5.3. Note that this $\mathcal{B}$ satisfies part (a) of Definition 1.2. Indeed, for any I in $\mathcal{B}$ there is an $I' \in LN$ such that $I = C(K_1, K_1; I')$. But we assumed in Definition 4.17 that $K_1 \geq c_2 + \pi$. By definition of LN, $\mu(I') < \delta_2$. The claim now follows from Lemma 3.4. That $\mathcal{B}$ satisfies part (c) of Definition 1.2 is just Definition 5.2(b). The estimates of part 1.2(b) are the content of Lemmas 6.3 and 6.6, Corollary 6.7, and Lemma 6.9. □

7. Proof of Theorems 1.6, 1.11, and 1.14

Proof of Theorems 1.6, 1.11, and 1.14. Let

$$\mathcal{M} = \{(\Sigma, \mu_u) | (\Sigma, u) \in \mathcal{F}).$$

According to Theorem 2.8 in [3], the hypotheses of Theorems 1.6, 1.11, and 1.14 imply that $\mathcal{M}$ is uniformly thick thin. If $(\Sigma, u) \in \mathcal{M}$ satisfies Assumption 4.3.1 the theorems follow from Theorem 6.1. Otherwise, let $\mathcal{B} = \emptyset$. If $genus(\Sigma_{\mathbb{C}}) = 0$ we must have a metric h satisfying condition (a) in Lemma 4.1. Stability follows from the fact that u is non-constant and the rest of the claims are obvious. Now assume $genus(\Sigma_{\mathbb{C}}) = 1$. All parts of the theorem hold vacantly except for stability, the derivative estimate and the injectivity radius estimate. Stability follows from the monotonicity inequality as follows. The injectivity radius of M is uniformly bounded away from zero by a constant r. Let $p \in (\Sigma)$. Since u represents a nontrivial homology class $u(\Sigma) \not\subset B_r(p; g_J)$. By the boundedness of the curvature and by the monotonicity inequality,

$$E(\Sigma_{\mathbb{C}}; u) > Area(u(\Sigma) \cap B_r(p; g_J)) \geq cr^2$$

for a constant $c > 0$.

To bound the injectivity radius and derivative we need to bound

$$Diam(\Sigma_{\mathbb{C}}; h).$$

For this, it suffices to bound the modulus of $\Sigma_{\mathbb{C}}$. For any $x > 0$, let

$$L := c_2 + \pi + \frac{\ln\{a(c_2 + \pi + x)\}}{c_3},$$

where the constants are as in Lemma 3.4. If $ModI > 2L$, any point $p \in \Sigma_{\mathbb{C}}$ is at the center of a cylinder of modulus $2L$. Lemma 3.4 then implies that

$$\frac{d\mu}{d\nu_{h_{st}}}(p) \leq \frac{1}{(c_2 + \pi + x)} \mu(\Sigma_{\mathbb{C}}).$$

Pick x large enough so that

$$\frac{4\pi L}{c_2 + \pi + x} < 1.$$

We then have the contradiction

$$\mu(\Sigma_{\mathbb{C}}) \leq 2L \sup_{p\in\Sigma_{\mathbb{C}}} \frac{d\mu}{d\nu_h}(p) \leq \frac{4\pi L}{c_2+\pi+x}\mu(\Sigma_{\mathbb{C}}) < \mu(\Sigma_{\mathbb{C}}).$$

The derivative estimate is an immediate consequence of Remark 3.3 and the global bound $\mu(\Sigma_{\mathbb{C}}) < \delta_2 < \delta_1$. The radius on injectivity of h_v is just the inverse of the diameter multiplied by a suitable constant. □

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