A simplified discharging proof of Grötzsch theorem

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Abstract

In this note, we combine ideas of several previous proofs in order to obtain a quite short proof of Grötzsch theorem.

Grötzsch [2] proved that every planar triangle-free graph is 3-colorable, using the discharging method. This proof was simplified by Thomassen [3] (who also gave a principally different proof [4]). Dvořák et al. [1] give another variation of the discharging proof. Both of the later arguments were developed in order to obtain more general results (the Thomassen's proof gives extensions to girth 5 graphs in the torus and the projective plane, while the proof of Dvořák et al. aims at algorithmic applications), and thus their presentation of the proof of Grötzsch theorem is not the simplest possible. In this note, we provide a streamlined version of the proof, suitable for teaching purposes.

We use the discharging method. Thus, we consider a hypothetical minimal counterexample to Grötzsch theorem (or more precisely, its generalization chosen so that we are able to deal with short separating cycles) and show that it does not contain any of several "reducible" configurations. Then, we assign charge to vertices and edges so that the total sum of charges is negative, and redistribute the charge (under the assumption that no reducible configuration appears in the graph) so that the final charge of each vertex and face is non-negative. This gives a contradiction, showing that there exists no counterexample to Grötzsch theorem.

A 3-coloring φ of a cycle C of length at most 6 is *valid* if either $|C| \leq 5$, or $|C| = 6$ and there exist two opposite vertices $u, v \in V(C)$ (i.e., both paths in C between u and v have length three) such that $\varphi(u) \neq \varphi(v)$. If G is a plane triangle-free graph whose outer face is bounded by an induced

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cycle C of length at most 6 and φ is a valid coloring of C, then we say that the pair (G, φ) is *valid*. We define a partial ordering $<$ on valid pairs as follows. We have $(G_1, \varphi_1) < (G_2, \varphi_2)$ if either $|V(G_1)| < |V(G_2)|$, or $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| > |E(G_2)|$. A valid pair (G, φ) is a *minimal counterexample* if φ does not extend to a 3-coloring of G, but for every valid pair $(G', \varphi') < (G, \varphi)$, the coloring φ' extends to a 3-coloring of G'.

Let us start with several basic reductions (eliminating short separating cycles, 4- and 6-faces), which are mostly standard. Usually, 6-faces are eliminated by collapsing similarly to 4-faces, which is necessary in the proofs that first eliminate the 4-cycles and then maintain girth five; in our setting, adding edges to transform them to 4-faces is more convenient.

Lemma 1. *If (G, φ) is a minimal counterexample, then G is 2-connected, $\delta(G) \geq 2$, all vertices of degree two are incident with the outer face, and every (≤ 5)-cycle in G bounds a face.*

Proof. If G contained a vertex v of degree at most two not incident with the outer face, then since (G, φ) is a minimal counterexample, the coloring φ extends to a 3-coloring of $G - v$. However, we can then color v differently from its (at most two) neighbors, obtaining a 3-coloring of G extending φ. This is a contradiction, and thus G contains no such vertex. Note that all vertices of G incident with the outer face have degree at least two, since the outer face is bounded by a cycle.

Suppose that a (≤ 5)-cycle K of G does not bound a face. Since G is triangle-free, the cycle K is induced. Let G_1 be the subgraph of G drawn outside (and including) K, and let G_2 be the subgraph of G drawn inside (and including) K. We have $(G_1, \varphi) < (G, \varphi)$, and thus there exists a 3-coloring ψ_1 of G_1 extending φ. Furthermore, $(G_2, \psi_1 \restriction V(K)) < (G, \varphi)$, and thus there exists a 3-coloring ψ_2 of G_2 that matches ψ_1 on K. The union of ψ_1 and ψ_2 is a 3-coloring of G extending φ, which is a contradiction. Hence, every (≤ 5)-cycle of G bounds a face.

Suppose that G is not 2-connected, and thus there exist graphs G_1, G_2 intersecting in at most one vertex such that $G = G_1 \cup G_2$, $C \subseteq G_1$ and $|V(G_1)|, |V(G_2)| \geq 4$. Observe that for $i \in \{1, 2\}$, there exists a vertex $v_i \in V(G_i)$ incident with the common face of G_1 and G_2 such that if G_1 and G_2 intersect, then the distance between v_i and the vertex in $G_1 \cap G_2$ is at least two. Then $G + v_1 v_2$ is triangle-free and $(G + v_1 v_2, \varphi) < (G, \varphi)$. However, this implies that there exists a 3-coloring of $G + v_1 v_2$ extending φ, which also gives such a 3-coloring of G. This is a contradiction. $\qquad \square$

Lemma 2. *If (G, φ) is a minimal counterexample with the outer face bounded by a cycle C, then G contains no induced 6-cycle other than C.*

Proof. Suppose that G contains an induced 6-cycle $K \neq C$. Let G_1 be the subgraph of G drawn outside (and including) K, and let G_2 be the subgraph of G drawn inside (and including) K. Since $K \neq C$ and C is an induced cycle, we have $V(K) \nsubseteq V(C)$. Let us label the vertices of K by $v_1, v_2, \ldots v_6$ in order so that $v_1 \notin V(C)$ and subject to that, the degree of v_1 in G_1 is as small as possible. Let $G_1' = G_1 + v_1 v_4$.

Note that C is an induced cycle bounding the outer face of G_1'. If G_1' contains a triangle, then G contains a 5-cycle $Q = v_1 v_2 v_3 v_4 x$ with $x \in V(G_1) \setminus V(K)$, which bounds a face by Lemma 1. Hence, the path $v_1 v_2 v_3$ is contained in boundaries of two distinct faces (K and Q) in G_1, and thus v_2 has degree two in G_1. However, v_1 has at least three neighbors v_2, v_3 and x in G_1, which contradicts the choice of the labels of the vertices of K. Therefore, G_1' is triangle-free. Note also that either $|V(G_1')| < |V(G)|$ (if K does not bound a face), or $|V(G_1')| = |V(G)|$ and $|E(G_1')| > |E(G)|$ (if K bounds a face). Hence, $(G_1', \varphi) < (G, \varphi)$, and thus there exists a 3-coloring ψ_1 of G_1' extending φ. Because of the edge $v_1 v_4$, $\psi_1 \restriction V(K)$ is a valid coloring of K. Since K is an induced cycle, we have $V(C) \nsubseteq V(K)$, and thus $|V(G_2)| < |V(G)|$ and $(G_2, \psi_1 \restriction V(K)) < (G, \varphi)$. Therefore, there exists a 3-coloring ψ_2 of G_2 that matches ψ_1 on K. The union of ψ_1 and ψ_2 is a 3-coloring of G extending φ, which is a contradiction. $\square$

Lemma 3. *If (G, φ) is a minimal counterexample with the outer face bounded by a cycle C, then G contains no 4-cycle other than C.*

Proof. Suppose that G contains a 4-cycle $K \neq C$. By Lemma 1, K bounds a face. Let $v_1, \ldots, v_4$ be the vertices of K in order. Since $K \neq C$ and C is an induced cycle, we can assume that $v_3 \notin V(C)$. Let G_1 be the graph obtained from G by identifying v_1 with v_3. Note that each 3-coloring of G_1 corresponds to a 3-coloring of G, and thus φ does not extend to a 3-coloring of G_1. Since $|V(G_1)| < |V(G)|$, it follows that the pair (G_1, φ) is not valid. There are two possibilities: either G_1 contains a triangle or its outer face is not an induced cycle.

If G_1 contains a triangle, then G contains a 5-cycle $Q = v_1 v_2 v_3 xy$. By Lemma 1, Q bounds a face, hence the path $v_1 v_2 v_3$ is contained in boundaries of two distinct faces (K and Q). It follows that v_2 has degree two, and by Lemma 1, v_2 is incident with the outer face. However, this implies that v_3 is incident with the outer face as well, contrary to its choice.

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It remains to consider the case that the outer face of G_1 is not an induced cycle. Since G_1 contains no triangle, it follows that the outer face of G_1 has length 6. Hence, $C = v_1 w_2 w_3 w_4 w_5 w_6$ and v_3 is adjacent to w_4. We choose the labels so that either $v_2 = w_2$ or v_2 is contained inside the 6-cycle $Q = v_1 v_4 v_3 w_4 w_3 w_2$. By Lemma 2, Q is not an induced cycle, and since C is an induced cycle and G is triangle-free, we conclude that $v_3 w_2 \in E(G)$. The symmetric argument for the 6-cycle $v_1 v_2 v_3 w_4 w_5 w_6$ implies that $v_3 w_6 \in E(G)$. By Lemma 1, $w_2 v_1 w_6 v_3$, $w_2 v_3 w_4 w_3$ and $w_6 v_3 w_4 w_5$ bound faces, hence $V(G) = V(C) \cup \{v_3\}$. Since φ is a valid coloring of C, two opposite vertices of C have different colors; say $\varphi(v_1) \neq \varphi(w_4)$. Then, we can properly color v_3 by $\varphi(v_1)$. This is a contradiction. $\square$

Corollary 4. *If (G, φ) is a minimal counterexample with the outer face bounded by a cycle C, then G contains no 6-cycle other than C.*

Proof. No 6-cycle in G other than C is induced by Lemma 2. However, a non-induced 6-cycle would imply the presence of at least two 4-cycles, contradicting Lemma 3. $\square$

The following is the main reduction enabling us to eliminate 5-faces incident with too many vertices of degree three. Thomassen [3] uses a different reduction in this case, which however is slightly more difficult to argue about.

Lemma 5. *Let (G, φ) be a minimal counterexample whose outer face is bounded by a cycle C. Let $K = v_1 v_2 v_3 v_4 v_5$ be a cycle bounding a 5-face in G such that v_1, v_2, v_3 and v_4 have degree three and do not belong to $V(C)$. Then at least one of the neighbors of v_1, ..., v_4 outside K belongs to $V(C)$.*

Proof. Let x_1, ..., x_4 be the neighbors of v_1, ..., v_4, respectively, outside of K. Suppose that none of these vertices belongs to $V(C)$. Let G' be the graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by adding the edge $x_1 x_4$ and by identifying x_2 with x_3. Note that C is an induced cycle bounding the outer face of G'.

If G' contained a triangle, then G would contain a 6-cycle $x_2 v_2 v_3 x_4 ab$ or $x_1 v_1 v_5 v_4 x_4 a$ (contrary to Corollary 4) or a matching between $\{x_1, x_4\}$ and $\{x_2, x_3\}$ (contrary to either planarity or Lemma 3). Hence, $(G', \varphi) < (G, \varphi)$ is valid and there exists a 3-coloring ψ of G' extending φ. Note that $\psi(x_1) \neq \psi(x_4)$; hence, we can choose colors $\psi(v_1) \notin \{\psi(x_1), \psi(v_5)\}$ and $\psi(v_4) \notin \{\psi(x_4), \psi(v_5)\}$ so that $\psi(v_1) \neq \psi(v_4)$. Since $\psi(x_2) = \psi(x_3)$, observe that we can extend this coloring to v_2 and v_3. This gives a 3-coloring of G extending φ, which is a contradiction. $\square$

We can now proceed with the discharging phase of the proof.

Lemma 6. *If (G, φ) is a valid pair, then φ extends to a 3-coloring of G.*

Proof. Suppose that φ does not extend to a 3-coloring of G; choose a valid pair (G, φ) with this property that is minimal with respect to $<$. Thus, (G, φ) is a minimal counterexample. Clearly, G has a vertex not incident with its outer face. Let the *initial charge* $c_0(v)$ of a vertex v of G be defined as $\deg(v) - 4$ and the initial charge $c_0(f)$ of a face f of G as $|f| - 4$.

Let C be the cycle bounding the outer face of G. A 5-face Q is *tied* to a vertex $z \in V(C)$ if $z \notin V(Q)$ and z has a neighbor in $V(Q) \setminus V(C)$ of degree three. Let us redistribute the charge as follows: each face other than the outer one sends $1/3$ to each incident vertex that either has degree two, or has degree three and does not belong to $V(C)$. Each vertex of C sends $1/3$ to each 5-face tied to it. Let the charge obtained by these rules be called *final* and denoted by c.

First, let us argue that the final charge of each vertex $v \in V(G) \setminus V(C)$ is non-negative: by Lemma 1, v has degree at least three. If v has degree at least four, then $c(v) \geq c_0(v) = \deg(v) - 4 \geq 0$. If v has degree three, then it receives $1/3$ from each incident face, and $c(v) = c_0(v) + 1 = 0$.

Next, consider the charge of a face f distinct from the outer one. By Lemma 3, we have $|f| \geq 5$. The face f sends at most $1/3$ to each incident vertex, and thus its final charge is $c(f) \geq c_0(f) - |f|/3 = 2|f|/3 - 4$. Hence, $c(f) \geq 0$ unless $|f| = 5$. Suppose that $|f| = 5$ and let k be the number of vertices to that f sends charge. We have $c(f) = c_0(f) - k/3 = 1 - k/3$. If $k \leq 3$, then $c(f) \geq 0$, and thus we can assume that $k \geq 4$. If f is incident with a vertex v of degree two, then note that $v \in V(C)$ by Lemma 1. Furthermore, since G is 2-connected and $G \neq C$, we conclude that f is incident with at least two vertices of degree three belonging to $V(C)$, to which f does not send charge. This contradicts the assumption that $k \geq 4$. Hence, no vertex of degree two is incident with f, and thus k is the number of vertices of $V(f) \setminus V(C)$ of degree three. By Lemma 5, f is tied to at least $k - 3$ vertices of C, and thus $c(f) \geq c_0(f) - k/3 + (k - 3)/3 = 0$.

The final charge of the outer face is $|C| - 4$. Consider a vertex $v \in V(C)$. If $\deg(v) = 2$, then v receives $1/3$ from the incident non-outer face and $c(v) = -5/3$. If $\deg(v) \geq 3$, then v sends $1/3$ to at most $\deg(v) - 2$ faces tied to it, and thus $c(v) \geq c_0(v) - (\deg(v) - 2)/3 = 2\deg(v)/3 - 10/3 \geq -4/3$.

Note that since G is 2-connected and $G \neq C$, the outer face is incident with at least two vertices of degree greater than two. Therefore, the sum of the final charges is at least $(|C| - 4) - 5(|C| - 2)/3 - 2 \cdot 4/3 = -10/3 - 2|C|/3 > -8$, since $|C| \leq 6$. On the other hand, the sum of final charges is equal to

the sum of the initial charges, which (if G has n vertices, m edges and s faces) is

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\begin{aligned}
\sum_v c_0(v) + \sum_f c_0(f) &= \sum_v (\deg(v) - 4) + \sum_f (|f| - 4) \\
&= (2m - 4n) + (2m - 4s) = 4(m - n - s) \\
&= -8
\end{aligned}
$$

by Euler's formula. This is a contradiction. $\square$

The proof of Grötzsch theorem is now straightforward.

Theorem 7. *Every planar triangle-free graph is* 3*-colorable.*

Proof. Suppose for a contradiction that G is a planar triangle-free graph that is not 3-colorable, chosen with as few vertices as possible. Clearly, G has minimum degree at least three (as otherwise we can remove a vertex v of degree at most two, 3-color the rest of the graph by the minimality of G, and color v differently from its neighbors). Hence, Euler's formula implies that every drawing of G in the plane has a face of length at most 5. Fix a drawing of G such that the outer face is bounded by a cycle C of length at most 5. Since G is triangle-free, the cycle C is induced. Let φ be an arbitrary 3-coloring of C. By Lemma 6, φ extends to a 3-coloring of G, which is a contradiction. $\square$

References

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