SOME REMARKS ON THE GROMOV WIDTH OF HOMOGENEOUS HODGE MANIFOLDS

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ABSTRACT. We provide an upper bound for the Gromov width of compact homogeneous Hodge manifolds (M, ω) with $b_2(M) = 1$. As an application we obtain an upper bound on the Seshadri constant $\epsilon(L)$ where L is the ample line bundle on M such that $c_1(L) = [\frac{\omega}{\pi}]$.

1. INTRODUCTION

Consider the open ball of radius r,

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} x_j^2 + y_j^2 < r^2\} \tag{1}$$

in the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$. The Gromov width of a $2n$-dimensional symplectic manifold (M, ω), introduced in [4], is defined as

$$c_G(M, \omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega)\}. \tag{2}$$

By Darboux's theorem $c_G(M, \omega)$ is a positive number. In the last twenty years computations and estimates of the Gromov width for various examples have been obtained by several authors (see, e.g. [9] and reference therein).

Gromov's width is an example of *symplectic capacity* introduced in [6] (see also [7]). A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* if it satisfies the following conditions:

(**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;

(**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;

(**nontriviality**) $c(B^{2n}(r), \omega_0) = \pi r^2 = c(Z^{2n}(r), \omega_0)$.

Here $Z^{2n}(r)$ is the unitary open cylinder in the standard $(\mathbb{R}^{2n}, \omega_0)$, i.e.

$$Z^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}. \tag{3}$$

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Note that the monotonicity property implies that c is a symplectic invariant. The existence of a capacity is not a trivial matter. It is easily seen that the Gromov width is the smallest symplectic capacity, i.e. $c_G(M,\omega) \leq c(M,\omega)$ for any capacity c. Note that the nontriviality property for c_G comes from the celebrated *Gromov's nonsqueezing theorem* according to which the existence of a symplectic embedding of $B^{2n}(r)$ into $Z^{2n}(R)$ implies $r \leq R$. Actually it is easily seen that the existence of any capacity implies Gromov's nonsqueezing theorem.

Recently (see [9]) the authors of the present paper have computed the Gromov width of all Hermitian symmetric spaces of compact and noncompact type and their products extending the previous results of G. Lu [10] (see also [8]) for the case of complex Grassmanians.

The aim of this paper is to provide un upper bound of the Gromov width of homogeneous Hodge manifolds with second Betti number equal to one. In this paper a homogeneous Hodge manifold is a compact Kähler manifold (M,ω) such that $\frac{\omega}{\pi}$ is integral and such that the group of holomorphic isometries of M acts transitively on M.

Our main results are the following three theorems.

Theorem 1. *Let (M,ω) be a compact homogeneous Hodge manifold such that $b_2(M) = 1$ and ω is normalized so that $\omega(A) = \int_A \omega = \pi$ for the generator $A \in H_2(M,\mathbb{Z})$. Then*

$$c_G(M,\omega) \leq \pi. \tag{4}$$

Theorem 2. *Let (M_i,ω^i), $i = 1,\dots,r$, be homogeneous compact Hodge manifolds as in Theorem 1. Then*

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1 \oplus \cdots \oplus \omega^r\right) \leq \pi. \tag{5}$$

Moreover, if $a_1,\dots,a_r$ are nonzero constants, then

$$c_G\left(M_1 \times \cdots \times M_r, a_1\omega^1 \oplus \cdots \oplus a_r\omega^r\right) \leq \min\{|a_1|,\dots,|a_r|\}\pi. \tag{6}$$

Theorem 3. *Let (M,ω) be as in Theorem 1 and (N,Ω) be any closed symplectic manifold. Then, for any nonzero real number a,*

$$c_G(N \times M, \Omega \oplus a\omega) \leq |a|\pi. \tag{7}$$

Note that an Hermitian symmetric space of compact type is an example of compact Hodge manifold with $b_2(M) = 1$. It is worth pointing out that there exist many examples of manifolds satisfying the assumption of Theorem 1 which are not symmetric.

In the symmetric case inequalities (4) and (5) are equalities (see [9] for a proof) and we believe that this holds true also in the non symmetric cases. To this respect recall a conjecture due to P. Biran which asserts that π is a lower bound for the Gromov width of any closed integral symplectic manifold.

The paper contains two other sections. In Section 2 we summarize Lu's work on pseudo symplectic capacities and their links with Gromov–Witten invariants needed in the proof of our main results. Section 3 is dedicated to the proofs of Theorem 1, 2 and 3. The paper ends with a remark on the Seshadri constant of an ample line bundle over a homogeneous Hodge manifold.

2. Pseudo symplectic capacities

G. Lu [10] defines the concept of *pseudo symplectic capacity* by weakening the requirements for a symplectic capacity (see the Introduction) in such a way that this new concept depends on the homology classes of the symplectic manifold in question (for more details the reader is referred to [10]). More precisely, if one denotes by $\mathcal{C}(2n,k)$ the set of all tuples $(M,\omega;\alpha_1,\dots,\alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M,ω) and k nonzero homology classes $\alpha_i \in H_*(M;\mathbb{Q})$, $i=1,\dots,k$, a map $c^{(k)}$ from $\mathcal{C}(2n,k)$ to $[0,+\infty]$ is called a *k-pseudo symplectic capacity* if it satisfies the following properties:

(**pseudo monotonicity**) if there exists a symplectic embedding $\varphi:(M_1,\omega_1)\to(M_2,\omega_2)$ then, for any $\alpha_i \in H_*(M_1;\mathbb{Q})$, $i=1,\dots,k$,

$$c^{(k)}(M_1,\omega_1;\alpha_1,\dots,\alpha_k) \le c^{(k)}(M_2,\omega_2;\varphi_*(\alpha_1),\dots,\varphi_*(\alpha_k));$$

(**conformality**) $c^{(k)}(M,\lambda\omega;\alpha_1,\dots,\alpha_k) = |\lambda| c^{(k)}(M,\omega;\alpha_1,\dots,\alpha_k)$, for every $\lambda\in\mathbb{R}\setminus\{0\}$ and all homology classes $\alpha_i\in H_*(M;\mathbb{Q})\setminus\{0\}$, $i=1,\dots,k$;

(**nontriviality**) $c(B^{2n}(1),\omega_0;pt,\dots,pt) = \pi = c(Z^{2n}(1),\omega_0;pt,\dots,pt)$, where pt denotes the homology class of a point.

Note that if $k>1$ a $(k-1)$-pseudo symplectic capacity is defined by

$$c^{(k-1)}(M,\omega;\alpha_1,\dots,\alpha_{k-1}) := c^{(k)}(M,\omega;pt,\alpha_1,\dots,\alpha_{k-1})$$

and any $c^{(k)}$ induces a true symplectic capacity

$$c^{(0)}(M,\omega) := c^{(k)}(M,\omega;pt,\dots,pt).$$

Observe also that (unlike symplectic capacities) pseudo symplectic capacities do not define symplectic invariants.

In [10] G. Lu was able to construct two 2-pseudo symplectic capacities denoted by $C^{(2)}_{HZ}(M,\omega;\alpha_1,\alpha_2)$ and $C^{(2o)}_{HZ}(M,\omega;\alpha_1,\alpha_2)$ respectively (see Definition 1.3 and Theorem 1.5 in [10]), where α_1 and α_2 are homology classes[1] in $H_*(M;\mathbb{Q})$. The

[1]In the notations of [10] the generic classes α_1 (resp. α_2) are called α_0 (resp. α_∞).

$C^{(2)}_{HZ}$ and $C^{(2o)}_{HZ}$ are called by Lu *pseudo symplectic capacities of Hofer–Zehnder type*. Denote by

$$C_{HZ}(M,\omega) := C^{(2)}_{HZ}(M,\omega;pt,pt)$$

(resp. $C^0_{HZ}(M,\omega) := C^{(2o)}_{HZ}(M,\omega;pt,pt)$) the corresponding true symplectic capacities associated to Lu's pseudo symplectic capacities. The next lemma summarizes some properties of the concepts involved so far.

Lemma 4. *Let (M,ω) be any symplectic manifold. Then, for arbitrary homology classes $\alpha_1,\alpha_2 \in H_*(M;\mathbb{Q})$ and for any nonzero homology class α, with $\dim\alpha \leq \dim M - 1$, the following inequalities hold true:*

$$C^{(2)}_{HZ}(M,\omega;\alpha_1,\alpha_2) \leq C^{(2o)}_{HZ}(M,\omega;\alpha_1,\alpha_2) \tag{8}$$

$$c_G(M,\omega) \leq C^{(2)}_{HZ}(M,\omega;pt,\alpha), \tag{9}$$

Proof. See Lemma 1.4 and (12) in [10]. □

When the symplectic manifold M is closed the pseudo symplectic capacities $C^{(2)}_{HZ}(M,\omega;\alpha_1,\alpha_2)$ and $C^{(2o)}_{HZ}(M,\omega;\alpha_1,\alpha_2)$ can be estimated by other two pseudo symplectic capacities $GW(M,\omega;\alpha_1,\alpha_2)$ and $GW_0(M,\omega;\alpha_1,\alpha_2)$. These GW and GW_0 are defined in terms of *Liu–Tian type Gromov-Witten invariants* as follows. Let $A \in H_2(M,\mathbb{Z})$: the Liu–Tian type Gromov–Witten invariant of genus g and with k marked points is a homomorphism

$$\Psi^M_{A,g,k} : H_*(\overline{\mathcal{M}}_{g,k};\mathbb{Q}) \times H_*(M;\mathbb{Q})^k \to \mathbb{Q},\ 2g+k \geq 3$$

where $\overline{\mathcal{M}}_{g,k}$ is the space of isomorphism classes of genus g stable curves with k marked points. When there is no risk of confusion, we will omit the superscript M in $\Psi^M_{A,g,k}$. Roughly speaking, one can think of $\Psi^M_{A,g,k}(\mathcal{C};\alpha_1,\ldots,\alpha_k)$ as counting, for suitable generic ω-tame almost complex structure J on M, the number of J-holomorphic curves of genus g representing A, with k marked points p_i which pass through cycles X_i representing α_i, and such that the image of the curve belongs to a cycle representing $\mathcal{C}$ (for details the reader is referred to the Appendix in [10] and references therein for details).

In fact, several different constructions of Gromov-Witten invariants appear in the literature and the question whether they agree is not trivial (see [10] and also Chapter 7 in [11]). The Gromov–Witten invariants described in the book of D. McDuff and D. Salamon [11] are the most commonly used: these are homomorphisms

$$\Psi_{A,g,m+2} : H_*(M;\mathbb{Q})^{m+2} \to \mathbb{Q},\ m \geq 1$$

The reason for this notation comes from the concept of hypersurface $S \subset M$ separating the homology classes α_0 and α_∞ (see Definition 1.3 and the (α_0,α_∞)-Weinstein conjecture at p.6 of [10]).

which play an important role in the proofs of this paper. The conditions under which these invariants agree with the ones considered by Lu are given in Lemma 6 below.

Let $\alpha_1, \alpha_2 \in H_*(M, \mathbb{Q})$. Following [10], one defines

$$GW_g(M, \omega; \alpha_1, \alpha_2) \in (0, +\infty]$$

as the infimum of the ω-areas $\omega(A)$ of the homology classes $A \in H_2(M, \mathbb{Z})$ for which the Liu–Tian Gromov–Witten invariant $\Psi_{A,g,m+2}(C; \alpha_1, \alpha_2, \beta_1, \dots, \beta_m) \neq 0$ for some homology classes $\beta_1, \dots, \beta_m \in H_*(M, \mathbb{Q})$ and $C \in H_*(\overline{\mathcal{M}}_{g,m+2}; \mathbb{Q})$ and integer $m \geq 1$ (we use the convention $\inf \emptyset = +\infty$). The positivity of GW_g reflects the fact that $\Psi_{A,g,m+2} = 0$ if $\omega(A) < 0$ (see, for example, Section 7.5 in [11]). Set

$$GW(M, \omega; \alpha_1, \alpha_2) := \inf\{GW_g(M, \omega; \alpha_1, \alpha_2) \mid g \geq 0\} \in [0, +\infty]. \tag{10}$$

Lemma 5. *Let (M, ω) be a closed symplectic manifold. Then*

$$0 \leq GW(M, \omega; \alpha_1, \alpha_2) \leq GW_0(M, \omega; \alpha_1, \alpha_2).$$

Moreover $GW(M, \omega; \alpha_1, \alpha_2)$ and $GW_0(M, \omega; \alpha_1, \alpha_2)$ are pseudo symplectic capacities and, if $\dim M \geq 4$ then, for nonzero homology classes α_1, α_2, we have

$$C_{HZ}^{(2)}(M, \omega; \alpha_1, \alpha_2) \leq GW(M, \omega; \alpha_1, \alpha_2),$$

$$C_{HZ}^{(2o)}(M, \omega; \alpha_1, \alpha_2) \leq GW_0(M, \omega; \alpha_1, \alpha_2).$$

In particular, for every nonzero homology class $\alpha \in H_*(M, \mathbb{Q})$,

$$C_{HZ}^{(2)}(M, \omega; pt, \alpha) \leq GW(M, \omega; pt, \alpha), \tag{11}$$

$$C_{HZ}^{(2o)}(M, \omega; pt, \alpha) \leq GW_0(M, \omega; pt, \alpha). \tag{12}$$

Proof. See Theorems 1.10 and 1.13 in [10]. □

We end this section with the following lemmata fundamental for the proof of our results. Recall that a closed symplectic manifold is *monotone* if there exists a number $\lambda > 0$ such that $\omega(A) = \lambda c_1(A)$ for A spherical (a homology class is called spherical if it is in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M, \mathbb{Z})$). Further, a homology class $A \in H_2(M, \mathbb{Z})$ is *indecomposable* if it cannot be decomposed as a sum $A = A_1 + \cdots + A_k$, $k \geq 2$, of classes which are spherical and satisfy $\omega(A_i) > 0$ for $i = 1, \dots, k$.

Lemma 6. *Let (M, ω) be a closed monotone symplectic manifold. Let $A \in H_2(M, \mathbb{Z})$ be an indecomposable spherical class, let pt denote the class of a point in $H_*(\overline{\mathcal{M}}_{g,m+2}; \mathbb{Q})$ and let $\alpha_i \in H_*(M, \mathbb{Z})$, $i = 1, 2, 3$. Then the Liu–Tian Gromov–Witten invariant $\Psi_{A,0,3}(pt; \alpha_1, \alpha_2, \alpha_3)$ agrees with the Gromov–Witten invariant $\Psi_{A,0,3}(\alpha_1, \alpha_2, \alpha_3)$.*

Proof. See [10, Proposition 7.6]. □

Lemma 7. *Let (N_1, ω_1) and (N_2, ω_2) be two closed symplectic manifolds. Then for every integer $k \geq 3$ and homology classes $A_2 \in H_2(N_2; \mathbb{Z})$ and $\beta_i \in H_*(N_2; \mathbb{Z})$, $i = 1, \ldots, k$,*

$$\Psi^{N_1 \times N_2}_{0 \oplus A_2, 0, k}(pt; [N_1] \otimes \beta_1, \ldots, [N_1] \otimes \beta_{k-1}, pt \otimes \beta_k) = \Psi^{N_2}_{A_2, 0, k}(pt; \beta_1, \ldots, \beta_k).$$

Proof. See [10, Proposition 7.4]. □

3. The proofs of Theorems 1, 2, 3

The following lemma is the key tool for the proofs of our main results.

Lemma 8. *Let (M, ω) be a compact homogeneous Hodge manifold of complex dimension n such that $b_2(M) = 1$ and ω is normalized so that $\omega(A) = \int_A \omega = \pi$ for the generator $A \in H_2(M, \mathbb{Z})$. Then there exist $\alpha(M, \omega)$ and $\beta(M, \omega)$ in $H_*(M, \mathbb{Z})$ such that*

$$\dim \alpha(M, \omega) + \dim \beta(M, \omega) = 4n - 2c_1(A)$$

and

$$\Psi_{A,0,3}(pt; \alpha(M, \omega), \beta(M, \omega), pt) \neq 0. \tag{13}$$

Proof. Since the symplectic form ω is Kähler-Einstein (being $b_2(M) = 1$), it follows that (M, ω) is monotone, so that Lemma 6 applies under our assumptions. We need then to show the existence of a non-vanishing Gromov-Witten invariant $\Psi_{A,0,3}(\alpha(M, \omega), \beta(M, \omega), pt)$, which follows by Fulton's results on the *quantum cohomology* of homogeneous spaces proved in [3]. In order to explain this, let us recall that a compact homogeneous space M writes as $M = G/P$, where G is a semisimple complex group and P is parabolic (i.e. contains a maximal solvable subgroup of G) in G. Let R be the *root system* associated to G. As it is known from the theory of semisimple complex Lie algebras (see, for example, [5] for the details), one chooses in R a finite set $\Delta \subset R$, the set of *simple roots* (with the property that every root can be written as a linear combination of the elements of Δ with the coefficients either all non-negative or all non-positive) and associates to every $\alpha \in R$ a reflection s_α in a suitable euclidean vector space: the group W generated by these reflections is called the *Weyl group* of G. For every $w \in W$ the *length* $l(w)$ of w is defined as the minimum number of reflections associated to simple roots whose product is w. Then, as recalled in Section 3 of [3], the homology classes of M correspond to the classes of the quotient W/W_P, where W_P is the subgroup of W generated by the reflections s_α for which α belongs to the root system of the reductive part of P. More precisely, for every $u \in W/W_P$ we denote by $\sigma(u)$ (resp. σ_u) the class of the corresponding so called *Schubert variety* (resp. *opposite*

Schubert variety) of complex dimension (resp. codimension) $l(u) := \inf_{[w]=u} l(w)$. The classes σ_u and $\sigma(u)$ are dual under the intersection pairing, and moreover one has $\sigma(u) = \sigma_{u^\vee}$, where $u^\vee := w_0 u$, being w_0 the element of longest length in W. In particular, $\sigma_{[w_0]} = \sigma_{[1]^\vee} = \sigma([1])$ has zero dimension, i.e. is the class of a point.

Under the assumption $b_2(G/P) = 1$, there exists a simple root β such that $H_2(G/P)$ is generated by the class $\sigma([s_\beta])$. Now, for every $u = [\tilde{u}] \in W/W_P$, let $v = [\tilde{u}s_\beta]^\vee$. Then, in the terminology of Section 4 in [3], u and $v^\vee$ are *adjacent* and $u_0 = u, u_1 = v^\vee$ is a *chain of degree* $\sigma([s_\beta])$ between u and v. By Theorem 9.1 in [3], it follows that there exists $w \in W/W_P$ such that the Gromov-Witten invariant $\Psi_{\sigma([s_\beta]),0,3}(\sigma_u, \sigma_v, \sigma_{w^\vee})$ does not vanish. In particular, when $u = [w_0]$, we get $\Psi_{\sigma([s_\beta]),0,3}(pt, \sigma_v, \sigma_{w^\vee}) \neq 0$, as required. The relation between the (real) dimensions of the classes in the statement easily follows from the general condition necessary for a Gromov-Witten invariant to be well-defined and non-vanishing (see, for example, Section 6 in [3]). □

Proof of Theorem 1. In order to use Lemma 5 we can assume, without loss of generality, that $\dim M \geq 4$. Indeed the only compact homogeneous Hodge manifold of (real) dimension < 4 (and hence of dimension 1) is $(\mathbb{C}\mathrm{P}^1, \omega_{FS})$ whose Gromov width is well-known to be equal to π. Let $A = [\mathbb{C}P^1]$ be the generator of $H_2(M, \mathbb{Z})$ as in the statement of Theorem 1. Then the value $\omega(A) = \pi$ is clearly the infimum of the ω-areas $\omega(B)$ of the homology classes $B \in H_2(M, \mathbb{Z})$ for which $\omega(B) > 0$. By Lemma 8 we have $\Psi_{A,0,3}(pt; pt, \alpha, \beta) \neq 0$, with $\alpha = \alpha(M, \omega)$ and $\beta = \beta(M, \omega)$, and hence, by definition of GW_g,

$$GW(M, \omega; pt, \gamma) = GW_0(M, \omega; pt, \gamma) = \pi \tag{14}$$

with $\gamma = \alpha(M, \omega)$ or $\gamma = \beta(M, \omega)$. It follows by the inequalities (8), (9), (11) and (12) that

$$c_G(M, \omega) \leq C^{(2)}_{HZ}(M, \omega; pt, \gamma) \leq C^{(2o)}_{HZ}(M, \omega; pt, \gamma) \leq \pi,$$

i.e. the desired inequality. □

Proof of Theorem 2. The proof of Theorem 2 is an immediate consequence of the following result combined with (8) and (9) in Lemma 4.

Lemma 9. *Let (M, ω) be as in Theorem 1 and let (N, Ω) be any closed symplectic manifold. Then*

$$C^{(2o)}_{HZ}(N \times M, \Omega \oplus a\omega; pt, [N] \times \gamma) \leq |a|\pi \tag{15}$$

for any $a \in \mathbb{R} \setminus \{0\}$ and $\gamma = \alpha(M, \omega)$ or $\gamma = \beta(M, \omega)$, with $\alpha(M, \omega)$ and $\beta(M, \omega)$ given by Lemma 8.

Proof. Since by (13) we have $\Psi^M_{A,0,3}(pt; \alpha, \beta, pt) \neq 0$, with $\alpha = \alpha(M, \omega)$ and $\beta = \beta(M, \omega)$, it follows by Lemma 7 that

$$\Psi^{N\times M}_{B,0,3}(pt; [N] \times \alpha(M, \omega), [N] \times \beta(M, \omega), pt) \neq 0$$

for $B = 0 \times A$, where 0 denotes the zero class in $H_2(N, \mathbb{Z})$ and A the generator of $H_2(M, \mathbb{Z})$. Hence (15) easily follows from (12) in Lemma 5. □

Proof of Theorem 3. From (8) and (9) in Lemma 4 and by (15) it follows that

$$c_G(N \times M, \Omega \oplus a\omega) \leq C^{(2o)}_{HZ}(N \times M, \Omega \oplus a\omega; pt, [N] \times \gamma) \leq |a|\pi,$$

where $\gamma = \alpha(M, \omega)$ (or $\gamma = \beta(M, \omega)$), which yields the desired inequality (7). □

On Seshadri constants of homogeneous manifolds

Given a compact complex manifold (N, J) and a holomorphic line bundle $L \to N$ the *Seshadri constant* of L at a point $x \in N$ is defined as the nonnegative real number

$$\epsilon(L, x) = \inf_{C \ni x} \frac{\int_C c_1(L)}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible holomorphic curves C passing through the point x and $\mathrm{mult}_x C$ is the multiplicity of C at x (see [2] for details). The (global) Seshadri constant is defined by

$$\epsilon(L) = \inf_{x \in N} \epsilon(L, x).$$

Note that Seshadri's criterion for ampleness says that L is ample if and only if $\epsilon(L) > 0$. P. Biran and K. Cieliebak [1, Prop. 6.2.1] have shown that

$$\epsilon(L) \leq c_G(N, \omega_L),$$

where ω_L is any Kähler form which represents the first Chern class of L, i.e. $c_1(L) = [\omega_L]$. Let now (M, ω) be a compact homogeneous Hodge manifold and L be the (very) ample line bundle $L \to M$ such that $c_1(L) = [\frac{\omega}{\pi}]$ (L can be taken as the pull-back of a suitable normalized Kodaira embedding $M \to \mathbb{C}P^N$ of the universal bundle of $\mathbb{C}\mathrm{P}^N$). Then, by using the upper bound $c_G(M, \omega) \leq \pi$ and the conformality of c_G we get:

Corollary 10. *Let (M_i, ω^i), $i = 1, \dots, r$, be homogeneous compact Hodge manifolds as in Theorem 1. Let $(M, \omega) = (M_1 \times \cdots \times M_r, \omega^1 \oplus \cdots \oplus \omega^r)$ and $L \to M$ as above. Then $\epsilon(L) \leq 1$.*

Remark 11. *The previous inequality has been found in* [9] *when (M, ω) is a Hermitian symmetric space of compact type*

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