ZERO FORCING SETS FOR GRAPHS

A Thesis

Submitted to the Faculty of Graduate Studies and Research

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

In

Mathematics

University of Regina

By

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Regina, Saskatchewan

August 2013

Abstract

For any simple *graph G* on n vertices, the (*positive semi-definite*) *minimum rank* of G is defined to be the smallest possible rank among all (positive semi-definite) real symmetric $n \times n$ matrices whose entry in position (i, j), for $i \neq j$, is non-zero if ij is an edge in G and zero otherwise. Also, the (*positive semi-definite*) *maximum nullity* of G is defined to be the largest possible nullity of a (positive semi-definite) matrix in the above set of matrices. In this thesis we study two graph parameters, namely the *zero forcing number* of G, $Z(G)$, and the *positive zero forcing number* of G, $Z_+(G)$, which bound the maximum nullity and the positive semi-definite maximum nullity from above, respectively. With regard to the zero forcing number, we introduce some new families of graphs for which the zero forcing number and the maximum nullity are the same. Also we establish an equality between the zero forcing number and the *path cover* number for a new family of graphs. In addition, we establish a connection between the zero forcing number and the *chromatic number* of graphs. With regard to the positive zero forcing number, we introduce the concept of *forcing*

trees in a graph and we establish a connection between the positive zero forcing number and the *tree cover number*. Also we study families of graphs for which these parameters coincide. In addition, we provide some new results on the connections of this parameter with other graph parameters, including the *independence* number and the chromatic number of G.

Acknowledgments

Foremost, I would like to express my sincere gratitude to my advisors professor Shaun Fallat and Dr. Karen Meagher for their continuous support of my PhD studies and research, for their patience, motivation, enthusiasm, and immense knowledge. I am so lucky to pursue my PhD under the supervision of such brilliant mathematicians.

I would also like to thank the other members of my PhD committee, Dr. Sandra Zilles and Dr. Shonda Gosselin and appreciate their valuable time spent on reading my thesis.

I also appreciate the financial supports of Dr. Fallat, Dr. Meagher and the department of Mathematics and Statistics at the University of Regina, during my PhD studies.

Last but not the least, I would like to thank my husband, Bahman, for all his helps, especially for drawing many of the graphs in my thesis. I really appreciate it.

Fatemeh Alinaghipour Taklimi

Regina, Canada, July 2013

Dedication

To my dear husband and best friend

Bahman

for his sincere love, endless support and encouragement, and to my lovely son

Ilia

for all the happiness he brought to us...

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Chapter 1

Introduction

Pattern classes of matrices represent an important part of the field of combinatorial matrix theory. Examples of related applications are models of problems in economics, mathematical biology (especially ecological food webs), statistical mechanics, and communication complexity.

1.1 The minimum rank problem

The minimum rank problem for a simple graph, or the minimum rank problem for short, determines the minimum rank among all real symmetric matrices whose zero-nonzero pattern of off-diagonal entries is described by a given simple graph G.

A graph or digraph "describes" the zero-nonzero pattern of a family of matrices; namely, the (i, j)-entry of the matrix is either non-zero or zero, depending on whether vertices v_i and v_j are adjacent or non-adjacent in the graph. Note that there is no

restriction on the diagonal entries of the matrix. In particular, the "adjacency matrix" of a graph G is described by G; this is the (0,1)-matrix whose rows and columns are labeled by the vertices of G and whose (i, j)-entry is 1 if and only if the vertices v_i and v_j are adjacent in G. The set of all real symmetric matrices which are described by a graph G, is denoted by $\mathcal{S}(G)$. For any graph G, we denote the minimum rank of G by $\mathrm{mr}(G)$.

The *minimum rank problem* is to determine the minimum among the ranks of the matrices in such a family; the determination of maximum nullity is then equivalent. Considerable progress has been made on the minimum rank problem for the family of real symmetric matrices described by a simple graph, although the problem is far from being solved in general. For example, the minimum rank problem has been completely solved for all types of tree patterns. The reader may refer to [14] to see a more detailed discussion on the recent results in the minimum rank problem.

We can also view the minimum rank problem as the problem of determining the maximum nullity. Since the nullity is, by definition, the number of vectors in a basis for the null space, the core idea becomes the study of subspaces with structured bases. Let matrix A be described by a graph G. Essentially, if "enough" of the entries of a vector in the null space of A are known to be zero, then, because of the pattern of A, we may deduce that the entire vector is zero. From this line of research, a new parameter was devised, and has become known as the *zero forcing*

number. This parameter is the minimum size of a "zero forcing set", which is a set of initially black-colored vertices of G which, through a coloring process, can "force" the rest of the vertices to be black. The zero forcing number gives an upper bound on the maximum nullity of A. However, a zero forcing parameter is a completely combinatorial parameter, and thus has the advantage of being intimately related to the structures within a pattern, or more precisely, a graph.

It turns out, perhaps surprisingly, that the zero-nonzero pattern described by the graph has significant influence on the minimum rank. For example, the minimum rank of a path graph is always equal to the number of edges in the path.

If in the definition of the minimum rank and maximum nullity, one restricts the matrices considered to the family of positive semi-definite matrices, then the resulting quantities will be called *positive semi-definite minimum rank* and *positive semi-definite maximum nullity*, respectively. In addition to studying the zero forcing number, in this thesis we investigate a graph parameter called *positive zero forcing number* which has the "same nature" as the zero forcing number. This is the minimum size of a *positive zero forcing set* for the graph which has, in turn, a similar definition as a zero forcing set with a different colour change rule. The goal for studying this parameter, similar to the conventional zero forcing number, is to bound the positive semi-definite maximum nullity. Although these two zero forcing numbers are analogous (as the two maximum nullities are), there has been more research conducted on

the conventional zero forcing number.

1.2 Relationship to other problems

The minimum rank of a family of matrices associated with a graph has also played a role in various other problems, such as the inverse eigenvalue problem, singular graphs, biclique decompositions and the biclique partition number, and orthonormal labellings of graphs (see the survey paper [14] for details on these problems).

It is well-known that the sum of the rank and the nullity of an $n \times n$ matrix is n. This implies that, for any graph G, the sum of the minimum rank and the maximum nullity of a graph on n vertices is n. Hence the solution of the minimum rank problem is equivalent to the determination of the maximum multiplicity of an eigenvalue among the same family of matrices in the symmetric free-diagonal case. An important related problem is then, the so-called *inverse eigenvalue problem* of a graph G, which can be stated as: what eigenvalues are possible for a real symmetric matrix A having nonzero off-diagonal entries determined by the edges of G? For example, if the minimum rank is m, then the multiplicity of an eigenvalue can be no more than $n - m$.

In spectral graph theory, an important tool is the $(0, 1)$-adjacency matrix $A(G)$ of the graph G, and a well-known open problem from 1957, is to characterize the graphs G whose adjacency matrix is singular (see [10] for a survey of this problem). Further,

many researchers have studied the dimension of the null space of the adjacency matrix; the nullity of $A(G)$ is an upper bound on the difference of n and the minimum rank of G.

Partitioning the edges of a graph is an important research area in graph theory (see [17], [28], [30], [33]). A *biclique partition* is a partition of the edges of a graph into subgraphs in such a way that each subgraph is a complete bipartite graph (or a biclique). The size of the smallest such partition is called the biclique partition number of G and is denoted by $\mathrm{bp}(G)$ (see also [20]). In [17] it is shown that

$$\mathrm{bp}(G) \geq \max\{i_+(A(G)), i_-(A(G))\},$$

also

$$n - \alpha(G) \geq \max\{i_+(A), i_-(A)\}, \quad A \in \mathcal{S}(G),$$

where $A(G)$ is the $(0,1)$-adjacency matrix of G and $i_+(X)$ and $i_-(X)$ are the number of positive and negative eigenvalues of the symmetric matrix X, respectively, and $\alpha(G)$ is the *independence number* of G; that is the maximum size of an independent set in G. The second bound is often called the *inertia bound* on the independence number. In the special case when G is a bipartite graph it follows easily that $\mathrm{bp}(G) \geq \frac{1}{2}\mathrm{rank}(A(G)) \geq \frac{1}{2}\mathrm{mr}(G)$.

The orthogonal labeling of the vertices is another prominent subject in graph theory that also has ties to minimum rank. If $G = (V, E)$ is a graph, then an *orthonormal labeling* of G of *dimension d* is a function $f : V \longrightarrow \mathbb{R}^d$ such that $f(u) \cdot f(v) = 0$

whenever vertices u and v are not adjacent, and $|f(u)| = 1$ for all $u \in V$ (see [12] for more details about orthonormal labeling). If $d(G)$ denotes the smallest dimension d over all orthonormal labellings of G, then $d(G)$ is equal to the minimum rank of a positive semidefinite matrix whose graph is given by G (see [12]). Lovász [29] has also shown that

$$\alpha(G) \leq \vartheta(G) \leq d(G) \leq \chi(\bar{G}), \tag{1.1}$$

where, $\vartheta(G)$ is the Lovász ϑ function and $\chi(\bar{G})$ is the chromatic number of the complement of G (i.e., the clique cover number of G). Note that (1.1) provides lower and upper bounds on $d(G)$; hence it approximates the minimum rank of a positive semidefinite matrix described by G.

1.3 Thesis objectives

One of the main objectives in this thesis is to determine the families of graphs for which the maximum nullity and the zero forcing number coincide. This is, in particular, important since the zero forcing number is a graph theoretical parameter which is more "tangible" than the maximum nullity. The latter is an algebraic parameter, and in the majority of the cases more difficult to deal with. This will, therefore, ease the study of this algebraic concept.

It has been shown [1, Proposition 2.10] that the path cover number of a graph is a lower bound for the zero forcing number of the graph. As another objective, we will

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investigate families of graphs for which these two parameters are equal.

One of the other objectives of this thesis is to develop new approaches to studying the (positive) zero forcing number and to understand its possible connections with other graph parameters. As the zero forcing number and the positive zero forcing number have a "graph colouring characteristic", one of the natural directions to this goal is studying the possible connections of these two parameters and other colouring parameters. As a result, we establish some relationships with the traditional graph colouring parameter, namely the chromatic number. Further, some results on the connection between positive zero forcing number and the chromatic number will be presented.

The other connection which will be established is the relationship between the positive zero forcing number and the *tree cover number*, which is the minimum number of induced trees of a graph which cover the vertices of the graph. We show, in particular, that the positive zero forcing number is an upper bound for the tree cover number of the graph. Also, we will investigate families of graphs in which the positive zero forcing number and the tree cover number agree.

1.4   Outline of the thesis

This thesis includes seven chapters. Chapter 2 will cover some necessary background material on graph theory and matrix theory and classical facts which we will

use in later chapters.

In Chapter 3 we will introduce the algebraic parameters (positive semi-definite) minimum rank and maximum nullity, defined for a given simple graph. We will then provide some known facts about these concepts.

Chapter 4 will provide all the information we need to begin the study of zero forcing sets and zero forcing numbers. This will illustrate how the zero forcing number bounds the maximum nullity of a graph. Then we study the graphs for which this bound is tight and provide some new results. Next we investigate graphs for which the zero forcing number meets the path cover number. In addition, the Colin de Verdière parameters for a graph and their connection to the zero forcing number will be discussed in Section 4.7. We conclude this chapter with a section on our new results on bounding the chromatic number with the zero forcing number.

In Chapter 5, we study the effect of graph operations on the zero forcing number. In particular, we calculate the zero forcing number of some graph products in terms of those of the initial graphs.

In Chapter 6, which also includes some of the main new results of this thesis, we will first introduce the parameter *positive zero forcing number* and then show how it bounds the positive semi-definite maximum nullity (and thus the positive semidefinite minimum rank). Furthermore, we will define the concept of *forcing trees* which provides connections between the positive zero forcing number and the *tree*

cover number. Next we introduce one of the most important families of graphs for which the positive zero forcing number equals the tree cover number. In addition, we study these two graph parameters in more families of graphs. Finally, we will compare the positive zero forcing number with the chromatic number and the independence number of a graph.

The thesis will be concluded with Chapter 7 in which we show some of the potential directions for future research work on this topic. We will list some open questions and conjectures which are closely related to our results.

Chapter 2

Graph Theory

This chapter presents some basic definitions, concepts and facts from graph theory and matrix theory needed throughout the thesis. The reader may also refer to [6] and [35] for additional details.

2.1 Basic concepts

Throughout this thesis all graphs are assumed to be finite, simple and undirected. We will often use the notation $G = (V, E)$ to denote the graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. An edge of G with end-points i and j is denoted by ij. The *order* of the graph G is defined to be $|V(G)|$, this may also be written as $|G|$. Vertex v is a *neighbour* of vertex u if $uv \in E(G)$. The set of all neighbours of v is denoted by $N(v)$. The *degree* of a vertex v is $|N(v)|$ and is denoted by $d(v)$. The minimum degree over all the vertices of a graph G is the *minimum degree* of the

graph G and is denoted by $\delta(G)$. A graph H is a *subgraph* of G (denoted $H \leq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is said to be *induced* if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. Let $S \subseteq V(G)$; then the subgraph H of G induced by S is denoted by $H = G[S]$. Also the subgraph induced by $V(G) \backslash S$ is denoted by $G \backslash H$. The *degree* of a vertex $v \in V(G)$ in an induced subgraph of the graph G, the subgraph H, is denoted by $d_H(v)$. A *supergraph* of a graph G is a graph of which G is a subgraph. A graph G is called *complete* if G has all possible edges between its vertices. The complete graph on n vertices is denoted by K_n. An *empty* graph is a graph with empty edge set. The complement of the graph G which is denoted by $\overline{G}$ is a graph on the same set of vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. In particular, the empty graph is denoted by $\overline{K_n}$.

A *path* on n vertices is a graph P_n with vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ in which v_i is adjacent only to v_{i+1}, for all $i \in \{1, \ldots, n-1\}$. We may also write P_n as $v_1 v_2 \cdots v_n$. The graph formed by adding the edge $v_1 v_n$ to P_n is called a *cycle* on n vertices and is denoted by C_n. The cycle C_n is said to be an *odd cycle* (*even cycle*) if n is odd (even). A graph G is said to be *connected* if for any pair x, y of vertices in G, there is a path between x and y. A *component* of a graph is a maximal connected subgraph of the graph. It is clear that any connected graph has only one component. Any graph G with more than one component is said to be *disconnected*. If a graph contains no

cycle as a subgraph, then it is called a *forest*. A connected forest is a *tree*.

The length of a path is the number of edges of the path, i.e. the length of P_n is $n - 1$. Let u and v be two vertices in graph G. Then the *distance* between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path in G between u and v. The *diameter* of a graph G is, then, defined to be the maximum distance over all vertices:

$$\text{diam}(G) = \max\{d_G(u, v) \mid u, v \in V(G)\}.$$

A subset S of the vertex set of a graph is said to be an *independent* set if no pair of elements of S are adjacent. A *multipartite* graph is a graph whose vertex set can be partitioned into subsets $X_1, \ldots, X_k$, with $k \geq 2$, such that each X_i is an independent set. The subsets X_i are called the *parts* of the graph. A multipartite graph with two parts is called a *bipartite* graph.

Theorem 2.1.1 (see [35]). *A graph is bipartite if and only if it does not contain an odd cycle.* □

It is obvious from Theorem 2.1.1 that all trees are bipartite. A *complete bipartite graph* is a bipartite graph with bipartition (X, Y) in which each vertex of X is adjacent to every vertex of Y; if $|X| = m$ and $|Y| = n$, such a graph is denoted by $K_{m,n}$. A multipartite graph which has all possible edges in its edge set is called a *complete multipartite graph*.

A *clique* in a graph is a subset of its vertices such that every pair of vertices in the subset are adjacent. A clique covering of a graph is a family of cliques in the graph

that cover all edges of the graph. The *clique cover number* of a graph G, denoted by $\text{cc}(G)$, is the minimum number of cliques in the graph needed to "cover" all edges of G; i.e. each edge of G lies in at least one of those cliques. Figure 5.3 has an example of a graph with $\text{cc}(G) = 4$. The reader may refer to [6] for more facts about the clique cover number.

There are similar concepts for covering the vertices of a graph. For instance, a *path covering* of a graph is a family of induced vertex-disjoint paths in the graph that cover all vertices of the graph. The minimum number of such paths that cover the vertices of a graph G is the *path cover number* of G and is denoted by $P(G)$. Also a *tree covering* of a graph is a family of induced vertex disjoint trees in the graph that cover all vertices of the graph. The minimum number of such trees that cover the vertices of a graph G is the *tree cover number* of G and is denoted by $T(G)$.

Next, we define the *chromatic number* of a graph. To this end, recall that a k-vertex colouring of G assigns to each vertex one colour from a set of k colours; the colouring is proper if no two adjacent vertices have the same colour. A graph G is *k-colourable* if G has a proper k-vertex colouring. The chromatic number, $\chi(G)$, of G is the minimum k for which G is k-colourable. A k-chromatic graph G is a graph with $\chi(G) = k$.

Identifying two vertices is a way to form new graphs. Let $v_1, v_2 \in V(G)$. The simple graph formed by identifying v_1 and v_2 has vertex set $(V(G)\backslash\{v_1, v_2\})\cup\{v\}$ and $N(v) =$

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$N(v_1) \cup N(v_2)$ with all possible multiple edges changed to single edges. Similarly we can identify vertices from different graphs. If $v_1 \in G$ and $v_2 \in H$, then identifying the vertices v_1 and v_2 produces the *vertex-sum* of the graphs G and H at vertex v, which is denoted by $G \underset{v}{+} H$. This simple graph has $V(G) \cup (V(H) \backslash \{v_1, v_2\}) \cup \{v\}$ as its vertex set and $E(G) \cup E(H)$ as its edge set and with $N(v) = N(v_1) \cup N(v_2)$. The graph formed by connecting G and H by adding the edge $v_1 v_2$ is the *edge-sum* of the graphs G and H and is denoted by $G = G \underset{v_1 v_2}{+} H$. This graph has $V(G) \cup V(H)$ as its vertex set and $E(G) \cup E(H) \cup \{v_1 v_2\}$ as its edge set.

The graph resulting from the removal of a vertex v and all edges incident with v from the graph G is denoted by $G - v$. Also the graph resulting from removing an edge e from the graph G is denoted by $G - e$. The *subdivision* of an edge $e = uv$ in a graph G yields a graph with vertex set $V(G) \cup \{w\}$ and edge set $E(G) \backslash \{uv\} \cup \{uw\} \cup \{wv\}$.

A *cut vertex* of a graph G is a vertex whose deletion disconnects G. A *cut set* of the graph G is a set $V' \subseteq V(G)$ whose deletion disconnects G. The *vertex connectivity* of the graph G, which is denoted by $\kappa(G)$, is the minimum size over all cut sets of G.

Recall that a subgraph of a graph G is a graph "smaller" than G that is obtained from G by removing some vertices and/or edges of G. There are other "smaller" graphs, which are not necessarily subgraphs of G, yet are obtained from G by a series of edge and vertex removing and identifying operations. The next definition introduces an example of these "smaller" graphs. If $e = xy$ is an edge of G, then the

contraction of e, which is denoted by G/e, is the operation that deletes the edge e and identifies x and y and removing all multiples edges. The resulting graph is a simple graph with fewer number of vertices and edges. A *minor* of a graph G arises by performing a sequence of the following operations:

(a) deletion of edges;

(b) deletion of isolated vertices;

(c) contraction of edges.

Note that a graph obtained from a graph G by deleting some of its vertices is also a minor of G; therefore any subgraph is a minor. If a graph G has a graph H as a minor, we say that "G has an H minor".

A graph parameter λ is said to be *minor-monotone*, if for any graph G and any minor H of G, we have $\lambda(H) \leq \lambda(G)$, or if for any graph G and any minor H of G, we have $\lambda(H) \geq \lambda(G)$. For example, $\kappa(G)$ is a minor-monotone graph parameter. Based on the definition, a graph has lots of different minors but the existence of certain minors in the graph can give information about its minor monotone parameters.

A graph is said to be *embeddable* in the plane, or *planar*, if it can be drawn in the plane so that its edges intersect only at their incident vertices. Such a drawing of a planar graph G is called a *planar embedding* or *crossing-free embedding* of G. We refer the reader to the books [11] and [35] for detailed discussions about planar

graphs. A graph is planar if and only if it has neither a K_5 minor nor a $K_{3,3}$ minor (see [35, Theorem 6.2.2]). In a planar graph a *face* is a region bounded by edges that does not have any edges going through them. The area outside the planar graph is also a face, called the *outer face*, and the other faces are called the *inner faces*. Similarly a graph is *outerplanar* if it has a planar embedding in which the outer face contains all the vertices. The edges touching the outer face are called *outer edges* and other edges are called *inner edges*. In the graph depicted in Figure 6.7, wu is an outer edge while wv is an inner edge and the region bounded by edges wu, wv and uv is an inner face. A graph is outerplanar if and only if it has neither a K_4 minor nor a $K_{2,3}$ minor [11].

The *corona* of G with H, denoted by $G \circ H$, is the graph of order $|V(G)||V(H)| + |V(G)|$ obtained by taking one copy of G and $|V(G)|$ copies of H, and joining all vertices in the i-th copy of H to the i-th vertex of G. See Figure 2.1 for an illustration of $C_5 \circ K_2$. In Section 4.6 we give a generalization of the corona, in keeping with the notation of this generalization, the corona of G with H will be denoted by $G \prec H, \ldots, H \succ$.

2.2 Spectral graph theory

This section represents a brief introduction to an important part of algebraic graph theory, namely spectral graph theory. Here we introduce some linear algebraic tools
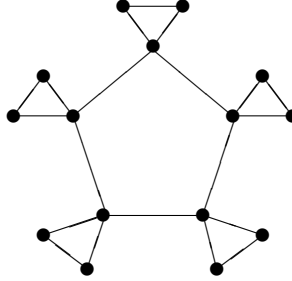
Corona of $C_5 \circ K_2$ Figure 2.1

which are used to establish some properties of graphs. For more details on spectral graph theory, the reader may refer to [5] or [16] and for more details on matrix theory to [21] or [22].

First we define the concept of an *adjacency matrix*. Assume that X is a graph with vertex set $V(X) = \{v_1, \ldots, v_n\}$. Then the *adjacency matrix* of X, which is denoted by $A(X)$, is a square (0,1)-matrix of size n, whose (i, j)-th entry is 1 if and only if v_i is adjacent to v_j. Since there are no loops in the graph, the diagonal entries of the adjacency matrix are zero. Also, changing the order of the vertices produces a distinct, but similar adjacency matrix.

Recall from linear algebra that for a matrix $A = [a_{i,j}]$, the *transpose of a matrix* of A, denoted by $A^\top$, is defined to be the matrix $A^\top = [a'_{i,j}]$, where $a'_{i,j} = a_{j,i}$, for all i and j. An $n \times n$ matrix A is said to be a *symmetric matrix* if $A^\top = A$. The set

of all square $n \times n$ matrices with entries in $\mathbb{C}$ is denoted by M_n. The *conjugate* of a matrix $A = [a_{ij}] \in M_n$ is denoted by $\bar{A} = [\bar{a}_{ij}]$, where $\bar{a}_{ij}$ is the conjugate of a_{ij}. The *conjugate transpose* of a matrix A, denoted by A^*, is defined to be

$$A^* \equiv \left(\bar{A}\right)^\top = [\bar{a}_{ji}].$$

A matrix $A = [a_{ij}] \in M_n$ is said to be *Hermitian* if $A = A^*$. The set of real symmetric $n \times n$ matrices is denoted by $\mathcal{S}_n$ and the set of Hermitian $n \times n$ matrices is denoted by H_n.

The *null space* (or the *kernel*) of a matrix A is the vector space consisting of all vectors v for which $Av = 0$; we denote the null space of A by $\ker(A)$. The matrix A is said to be *nonsingular* (or *invertible*) if there is no nonzero vector in its null space, otherwise, it is *singular*. The dimension of the null space of a matrix A is called the *nullity* of A and is denoted by $\operatorname{null}(A)$. The *rank* of a matrix is defined to be the dimension of the row or column space of the matrix; that is, the dimension of the vector space spanned by the rows or columns of the matrix. The rank of A is denoted by $\operatorname{rank}(A)$. The following fact is well-known.

Proposition 2.2.1 (see [21]). *For an $n \times n$ matrix A,*

$$\operatorname{rank}(A) + \operatorname{null}(A) = n. \quad \square$$

For a square matrix A, the *eigenvalues* are defined to be the roots λ of the *characteristic polynomial* $\phi(A; x) = \det(xI - A)$. Equivalently, a complex number λ is an

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eigenvalue of A, if the determinant of the matrix $\lambda I - A$ is zero. This is, in turn, equivalent to the fact that there is a non-zero vector in the null space of $\lambda I - A$. If λ is an eigenvalue of A, then the null space of $\lambda I - A$ is called the *eigenspace* of A corresponding to the eigenvalue λ, and its non-zero elements are called the *eigenvectors* of A corresponding to λ. The dimension of the eigenspace corresponding to the eigenvalue λ is called the *geometric multiplicity* of the eigenvalue λ. The *algebraic multiplicity* of the eigenvalue λ of a matrix A, is the maximum power of the factor $x - \lambda$ in the characteristic polynomial $\phi(A; x)$ and is denoted by $m(\lambda)$.

Proposition 2.2.2. *If a square matrix A is symmetric, then all the eigenvalues of A are real and the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity.* □

Note that in this thesis, we call the the eigenvalues of the adjacency matrix of the graph the eigenvalues of the graph. So we have the following simple corollary of the previous results.

Corollary 2.2.3. *All the eigenvalues of a graph are real.* □

Therefore, we usually order the eigenvalues of a graph on n vertices as $\lambda_n \geq \lambda_{n-1} \geq \cdots \geq \lambda_1$. The *spectrum* of a graph X is the set of its eigenvalues with their multiplicities. It is usually written as the array:

$$
\mathrm{Spec}(X) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_s) \end{pmatrix},
$$

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where $\lambda_s > \cdots > \lambda_1$ are the distinct eigenvalues of X. For example, the spectra of the complete graph K_n and the complete bipartite graph $K_{m,n}$ are as follows:

$$\text{Spec}(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix},$$

$$\text{Spec}(K_{m,n}) = \begin{pmatrix} -\sqrt{mn} & 0 & \sqrt{mn} \\ 1 & m+n-2 & 1 \end{pmatrix}.$$

A symmetric matrix A is *positive semi-definite* if all the eigenvalues of A are non-negative. For example $A(K_n)$ is not a positive semi-definite matrix while $A(K_n) + I$ is. The matrix A is said to be a *tridiagonal matrix* if it has non-zero entries only on the diagonal, the super-diagonal and the sub-diagonal. For example $A(P_n)$ is a tridiagonal matrix. The following is another example of a tridiagonal matrix :

$$A = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -5 \end{pmatrix}.$$

Lemma 2.2.4. *Let A be an $n \times n$ tridiagonal matrix in which all the entries of the super-diagonal and sub-diagonal are non-zero. Then the $(1,n)$-entry of A^r is zero, for any integer $0 \le r \le n-2$, and non-zero for $r = n-1$.*

Proof. We assume $n \ge 3$ as the lemma is trivial for $n = 1, 2$. First we claim that for any $0 \le r \le n-1$ and for any $1 \le i < n-r$, the (i,n)-entry of A^r is zero. The proof is by

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induction on r. If $r = 0$ or 1, the claim is, obviously, true. Thus, assume $2 \leq r \leq n - 1$.

Then for any $1 \leq i < n - r$, since A is tridiagonal, we have

$$(A^r)_{i,n} = (A\,A^{r-1})_{i,n} = \sum_{k=1}^{n} A_{i,k}(A^{r-1})_{k,n}$$

$$= A_{i,i-1}(A^{r-1})_{i-1,n} + A_{i,i}(A^{r-1})_{i,n} + A_{i,i+1}(A^{r-1})_{i+1,n},$$

where $A_{i,i-1}$ is considered to be zero if $i = 1$. Then, using the induction hypothesis, we

have $(A^{r-1})_{i-1,n} = (A^{r-1})_{i,n} = (A^{r-1})_{i+1,n} = 0$; thus the claim is proved. Now it follows

from the claim that, for any $0 \leq r \leq n - 2$, the $(1, n)$-entry of A^r is zero. To complete

the proof of lemma, it is enough to note that the $(1, n)$-entry of A^{n-1} is

$$A_{1,2}\,A_{2,3}\cdots A_{n-1,n} \neq 0. \qquad \square$$

Chapter 3

The Minimum Rank of Graphs

In this chapter we introduce the "minimum rank" and the "maximum nullity" of a graph. We will also consider the positive semi-definite minimum rank and the positive semi-definite maximum nullity of graphs. The problem of evaluating or approximating these parameters is of great importance in this thesis. In the future chapters we will approach these problems using some combinatorial tools.

3.1 Definition

In this chapter we introduce the concept of minimum rank of graphs and related concepts, and provide some examples. To a given graph G with vertex set $\{1, \ldots, n\}$, we associate a class of real, symmetric matrices as follows.

$$\mathcal{S}(G) = \{A = [a_{ij}] \mid A \in \mathcal{S}_n, \ \text{for} \ \ i \neq j, \ a_{ij} \neq 0 \iff ij \in E(G)\},$$

Note that there is no restriction on the value of a_{ii}, with $i = 1, 2, \ldots, n$ and the adjacency matrix $A(G)$ belongs to $\mathcal{S}(G)$.

On the other hand, the *graph* of an $n \times n$ Hermitian matrix A, denoted by $\mathcal{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and the edge set

$$\{ij \mid a_{ij} \neq 0, \ 1 \leq i \neq j \leq n\}.$$

The *minimum rank* of G is defined to be

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) \mid A \in \mathcal{S}(G)\},$$

while the *maximum nullity* of G is defined as

$$M(G) = \max\{\mathrm{null}(A) \mid A \in \mathcal{S}(G)\}.$$

By Proposition 2.2.1 we have

$$\mathrm{mr}(G) + M(G) = |V(G)|. \tag{3.2}$$

Note that for any non-empty graph each matrix in the class of all symmetric matrices associated with a non-empty graph is non-zero, so the minimum rank of any graph is bounded below by 1. Also, as the diagonal entries of the matrices are free to be chosen, they can always be adjusted in such a way that the matrix admits a zero eigenvalue; this implies that $\mathrm{mr}(G)$ is bounded above by $|V(G)| - 1$. Thus we have the following lemma.

Lemma 3.1.1. *For any non-empty graph G*

$$1 \le \mathrm{mr}(G) \le |V(G)| - 1.$$

Example 3.1.2. Consider the complete graph K_3. Then, for example

$$\begin{pmatrix} 0 & -3 & 1 \\ -3 & 0 & 7 \\ 1 & 7 & -4 \end{pmatrix} \in \mathcal{S}(K_3);$$

and also

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in \mathcal{S}(K_3);$$

Thus since there is a matrix of rank 1 in $\mathcal{S}(K_3)$, $\mathrm{mr}(K_3) = 1$.

Example 3.1.3. As another illustrative example, consider any matrix A associated with a path $v_1 v_2 \cdots v_n$. Up to labelling of the vertices, A is a symmetric tridiagonal matrix with nonzero sub-diagonal and super-diagonal. From Lemma 2.2.4 the $(1, n)$-entry of A^i is zero, for any $0 \le i \le n-2$, and non-zero for $i = n-1$. We deduce that the set $\{I = A^0, A^1, A^2, \ldots, A^{n-1}\}$ is linearly independent. Thus the minimal polynomial of A must be of degree at least n. This implies that A must have n distinct eigenvalues. This, in turn, implies that the number of non-zero eigenvalues of A is at least $n - 1$; that is $\mathrm{mr}(G) \ge n - 1$ and, therefore, $\mathrm{mr}(G) = n - 1$. In fact a path on n vertices is the only graph on n vertices whose minimum rank is equal to $n-1$ (see [14, Theorem 1.4] and [15]).

3.2 Basic facts about the minimum rank

In this section we provide some basic results about the minimum rank of graphs. The set of real positive semidefinite matrices described by a graph G and the set of Hermitian positive semidefinite matrices described by G are

$$\mathcal{S}_+(G) = \{A \in \mathcal{S}_n \; : \; \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite}\}$$

and

$$\mathcal{H}_+(G) = \{A \in H_n \; : \; \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite}\},$$

respectively. The *minimum positive semidefinite rank* of G and *minimum Hermitian positive semidefinite rank* of G are

$$\mathrm{mr}_+^{\mathbb{R}}(G) = \min\{\mathrm{rank}(A) \mid A \in \mathcal{S}_+(G)\}$$

and

$$\mathrm{mr}_+^{\mathbb{C}}(G) = \min\{\mathrm{rank}(A) \mid A \in \mathcal{H}_+(G)\},$$

respectively. The *maximum positive semidefinite nullity* of G and *maximum Hermitian positive semidefinite nullity* of G are

$$M_+^{\mathbb{R}}(G) = \max\{\mathrm{null}(A) \mid A \in \mathcal{S}_+(G)\}$$

and

$$M_+^{\mathbb{C}}(G) = \max\{\mathrm{null}(A) \mid A \in \mathcal{H}_+(G)\},$$

respectively.

Clearly $\mathrm{mr}_+^{\mathbb{R}}(G) + M_+^{\mathbb{R}}(G) = |V(G)|$ and $\mathrm{mr}_+^{\mathbb{C}}(G) + M_+^{\mathbb{C}}(G) = |V(G)|$.

The following two observations are well known and straightforward to prove.

Observation 3.2.1. *If the connected components of the graph G are $G_1, \ldots, G_t$, then*

$$\mathrm{mr}(G) = \sum_{i=1}^{t} \mathrm{mr}(G_i) \quad and \quad M(G) = \sum_{i=1}^{t} M(G_i). \quad \square$$

The following fact has been shown in [31].

Proposition 3.2.2. *For any vertex v of G, $0 \le \mathrm{mr}(G) - \mathrm{mr}(G - v) \le 2$.* $\quad\square$

Clearly if A' is a submatrix of A then $\mathrm{rank}(A') \le \mathrm{rank}(A)$. Based on this fact we have the following corollary.

Corollary 3.2.3. *If G' is an induced subgraph of G then*

$$\mathrm{mr}(G') \le \mathrm{mr}(G). \quad \square$$

Note that a similar result does not hold for maximum nullity. To show this note that $M(C_n) = 2$ and P_{n-1} is an induced subgraph of it with $M(P_{n-1}) = 1$. Also the graph shown in Figure 5.3 has $M(G) = 2$ while $M(G - v) = 3$.

Since a connected graph G contains an induced path on $\mathrm{diam}(G) + 1$ vertices we observe the following.

Observation 3.2.4. *For a connected graph G we have $\mathrm{diam}(G) \le \mathrm{mr}(G)$.* $\quad\square$

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The *union* of the graphs $G_i = (V_i, E_i)$, $i = 1, \ldots, h$, is defined to be

$$\bigcup_{i=1}^{h} G_i = \left(\bigcup_{i=1}^{h} V_i, \bigcup_{i=1}^{h} E_i \right).$$

If $G = \bigcup_{i=1}^{h} G_i$, a matrix A of rank at most $\sum_{i=1}^{h} \mathrm{mr}(G_i)$ having $\mathcal{G}(A) = G$ can be obtained by choosing (for each $i = 1, \ldots, h$) a matrix A_i that realizes $\mathrm{mr}(G_i)$, embedding A_i in a matrix $\tilde{A}_i$ of size $|V(G)|$, choosing $a_i \in \mathbb{R}$ such that no cancelation of nonzero entries occurs, and letting $A = \sum_{i=1}^{h} a_i \tilde{A}_i$. Thus we have the following.

Observation 3.2.5. *If $G = \bigcup_{i=1}^{h} G_i$, then $\mathrm{mr}(G) \leq \sum_{i=1}^{h} \mathrm{mr}(G_i)$.* ☐

The inequality in Observation 3.2.5 can be strict. For example $\mathrm{mr}(K_3) = 1$ and K_3 can be considered as the union of 3 copies of K_2 with $\mathrm{mr}(K_2) = 1$. Since $1 < 1 + 1 + 1$, the inequality in this example is strict.

If in the definition of the union of graphs we have $V_i \cup V_j = \varnothing$ for $i \neq j$, then we have the *disjoint union* of graphs. Under these assumptions the matrices $\tilde{A}_i$, for $1 \leq i \leq h$, don't overlap on the position of nonzero entries. Thus in this case we have the same result for the minimum rank of disjoint union of graphs as Observation 3.2.1, i.e. $\mathrm{mr}(\bigcup_{i=1}^{h} G_i) = \sum_{i=1}^{h} \mathrm{mr}(G_i)$.

The following well-known fact is a special case of Observation 3.2.5.

Corollary 3.2.6. *If G is a graph, $\mathrm{mr}(G) \leq \mathrm{cc}(G)$.* ☐

The problem of the minimum rank of trees has been completely solved (see [25]).

Theorem 3.2.7. $M(T) = P(T) = |V(T)| - \mathrm{mr}(T)$. ☐

The *rank-spread* of a graph G at a vertex v, denoted by $r_v(G)$, is the amount of change in the minimum rank of G when v is deleted. More precisely

$$r_v(G) = \text{mr}(G) - \text{mr}(G - v).$$

From Proposition 3.2.2, for any vertex v of G, we have $0 \le r_v(G) \le 2$. It has been shown in $[2,23]$ that if a graph has a cut-vertex, then the problem of finding the minimum rank of the graph can be reduced to the problem of determining the minimum rank of several graphs of smaller orders.

Theorem 3.2.8 (Cut-vertex reduction). *Let v be a cut-vertex of a graph G. For $i = 1, \ldots, h$, let $W_i \subseteq V(G)$ be the vertices of the i-th component of $G \backslash v$ and let G_i be the subgraph of G induced by $\{v\} \cup W_i$. Then*

$$r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}$$

and hence

$$\text{mr}(G) = \sum_{i=1}^{h} \text{mr}(G_i - v) + \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\}. \quad \square$$

Example 3.2.9. Consider the graph K_3 with a vertex labeled v. The vertex v is a cut-vertex of the graph $G = K_3 \underset{v}{+} K_3$. Using Theorem 3.2.8, we calculate the minimum rank of the graph G. The removal of the vertex v from G results in two components $G_1 - v = G_2 - v = K_2$ with $G_1 = G_2 = K_3$. We have $r_v(G_1) = r_v(G_2) = 0$ therefore, $r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i),\ 2\right\} = 0$. Also $\text{mr}(G_1 - v) = \text{mr}(G_2 - v) = 1$ thus $\text{mr}(G) = 1 + 1 = 2$.

Chapter 4

Zero Forcing Sets

In this chapter we will introduce a new type of graph colouring which defines a graph parameter called the zero forcing number, denoted by $Z(G)$, which is the minimum size of a zero forcing set. This parameter was first introduced and defined at the workshop "Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns", which was held at the American Institute of Mathematics on October, 2006 (see [18]). Also, in that workshop it was shown that $Z(G)$ is an upper bound for $M(G)$. Somewhat surprisingly, $M(G) = Z(G)$, for most of the graphs for which $M(G)$ is known. For instance, these two parameters are equal for all graphs with fewer than seven vertices and for some families of chordal graphs. We will establish this equality for more families of graphs and we show the equality $Z(G) = P(G)$ in some families of graphs as well. In addition, we will establish a relationship between $Z(G)$ and the chromatic number $\chi(G)$ of a graph G.

4.1 Definition

This section includes basic definitions and facts about the zero forcing number of a graph G, which will be used throughout this thesis. The zero forcing number is a graph parameter that arises from a type of graph colouring, therefore, first we turn our attention to the rules of this new graph colouring. This graph colouring is based on a colour-change rule that describes how to colour the vertices of the graph.

Let G be a graph with all vertices initially coloured either black or white. If u is a black vertex of G and u has exactly one neighbour that is white, say v, then we change the colour of v to black; this rule is called the *colour change rule*. In this case we say "u forces v" which is denoted by $u \rightarrow v$. The procedure of colouring a graph using the colour rule is called the zero forcing process (shortly the forcing process). Note that each vertex will force at most one other vertex. Given an initial colouring of G, the *derived set* is the set of all black vertices resulting from repeatedly applying the colour-change rule until no more changes are possible. A *zero forcing set Z*, is a subset of vertices of G such that if initially the vertices in Z are coloured black and the remaining vertices are coloured white, then the derived set of G is $V(G)$. The *zero forcing number* of a graph G, denoted by $Z(G)$, is the smallest size of a zero forcing set of G. We abbreviate the term zero forcing set as ZFS. A zero forcing process is called *minimal* if the initial set of black vertices is a minimal ZFS. Note

that for any nonempty graph G

$$1 \le Z(G) \le |V(G)| - 1.$$

4.2 Examples and basic results

This section presents examples of zero forcing sets of several graphs in order to illustrate this concept. As an example, each endpoint of a path is a zero forcing set for the path, thus $Z(P_n) = 1$ (note that no other vertex of the path is a zero forcing set). In a cycle, any set of two adjacent vertices is a zero forcing set and there is no ZFS of size one for the cycle, thus $Z(C_n) = 2$. It is, also, easy to see that $Z(K_n) = n - 1$.

Example 4.2.1. Let G be the graph in Figure 4.1. Then, Figure 4.2 illustrates why the set $Z = \{v_1, v_2\}$ is a ZFS for G.
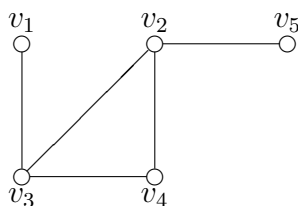
The   graph G for   which   we   want   to   find   a   ZFS

Figure 4.1

Any three pairwise adjacent vertices of a wheel form a ZFS for it. Any set of

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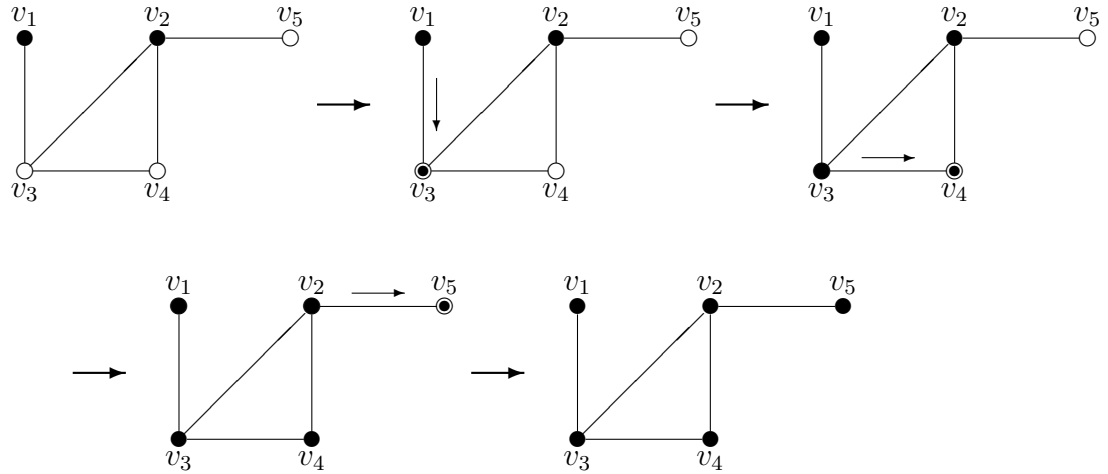
Finding a zero forcing set   Figure 4.2

pendant vertices of size $n - 1$ of a star $(K_{1,n-1})$ is a ZFS for it. As another example we have the following proposition.

Proposition 4.2.2. *Let* $K_{n_1,\ldots,n_m}$ *be a complete multipartite graph with at least one* $n_i > 1$ *for* $1 \le i \le m$, *then*

$$Z\left(K_{n_1,\ldots,n_m}\right) = \left(n_1 + n_2 + \cdots + n_m\right) - 2.$$

Proof. Let $G = K_{n_1,\ldots,n_m}$ and Z be the set of all vertices of the graph except for two vertices say, u and v, which lie in different parts, A and B. Since the graph is a complete multipartite graph, any vertex in A can force v and any vertex in B can force u. Thus Z is a ZFS for G and

$$Z(G) \le |G| - 2.$$

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Next we show that $Z(G) \geq |G| - 2$. Suppose that $Z(G) < |G| - 2$, therefore we have a set of size at most $|G| - 3$ as the set of initial black vertices and this means we initially have at least 3 white vertices, which leads to the following three different cases:

(a) All of them are in the same part.

(b) They are all in different parts.

(c) Exactly two of them are in the same part.

In the first and second cases there is no black vertex with a single white neighbour in the graph thus no vertex can perform a force. In the third case neither of the two white vertices that are in the same part can be turned black since any existing black vertex is either adjacent to both of them or adjacent to neither of them. Therefore the initial set of black vertices can not be a zero forcing set for the graph. Thus $Z(G) \geq |G| - 2$. □

Observation 4.2.3. *For every graph G we have*

$$Z(G) \geq \delta(G). \quad \square$$

This bound is tight when G is a complete graph. But for trees with a large path cover number, such as stars, this can be a very bad bound (see Proposition 4.4.5).

Since the neighbours of the first black vertex which is performing a force in a zero forcing process are all black, it is easy to observe the following. The i-th *level* of a

forcing process is the colouring of G after applying the colour-change rule in a zero forcing process i times. As noted in [18], since any vertex that turns black under one sequence of application of the colour-change rule can always be turned black, regardless of the order of colour changes, we have the following.

Proposition 4.2.4. *Let G be a graph. The derived set of any forcing process in the graph G, starting with a specific initial set of black vertices, is unique.*

4.3 Connection to the maximum nullity

In this section we show how the zero forcing number of a graph bounds the maximum nullity of the graph. To see this we need some additional definitions and theorems from linear algebra. The *support* of a vector $\mathbf{x} = (x_i)$, denoted by $\mathrm{Supp}(\mathbf{x})$, is the set of indices i such that $x_i \neq 0$. The following proposition relates the support of the vectors in the null space of a matrix to its nullity, (from [18, Proposition 2.2]).

Proposition 4.3.1. *Let A be an $n \times n$ matrix and suppose $\mathrm{null}(A) > k$. Then there is a nonzero vector $\mathbf{x} \in \mathrm{null}(A)$ vanishing at any k specified positions.* □

In other words, if W is a set of k indices, then there is a nonzero vector $\mathbf{x} \in \mathrm{null}(A)$ such that $\mathrm{Supp}(\mathbf{x}) \cap W = \varnothing$.

Proposition 4.3.2 (see [18, Proposition 2.3]). *Let Z be a zero forcing set of $G = (V, E)$ and $A \in \mathcal{S}(G)$. If $\mathbf{x} \in \mathrm{null}(A)$ and $\mathrm{Supp}(\mathbf{x}) \cap Z = \varnothing$, then $\mathbf{x} = 0$.*

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Proof. Assume $\mathbf{x} \in \mathrm{null}(\mathbf{A})$ and $\mathrm{Supp}(\mathbf{x}) \cap Z = \varnothing$. If $Z = V$, the statement clearly holds, so suppose $Z \neq V$. If $v \in Z$ then $x_v = 0$. Since Z is a zero forcing set we must be able to perform a colour change. That is, there exists a vertex u coloured black (x_u is required to be 0) with exactly one neighbour v coloured white (so x_v is not yet required to be 0). Upon examination, the equation $(A\mathbf{x})_u = 0$ reduces to $a_{uv}x_v = 0$, which implies that $x_v = 0$. Similarly each colour change corresponds to requiring another entry in $\mathbf{x}$ to be zero. Thus $\mathbf{x} = 0$. $\qquad\square$

Theorem 4.3.3 (see [18, Proposition 2.4]). *If G is a graph, then $M(G) \leq Z(G)$.*

Proof. Let Z be a ZFS for G. Assume $M(G) > |Z|$, and let $A \in \mathcal{S}(G)$ with $\mathrm{null}(A) > |Z|$. By Proposition 4.3.1, there is a nonzero vector $\mathbf{x} \in \mathrm{ker}(A)$ that vanishes on all vertices in Z. By Proposition 4.3.2, $\mathbf{x} = 0$, which is a contradiction. $\qquad\square$

Note that Theorem 4.3.3 displays a nice relationship between a linear algebraic quantity, $M(G)$, and a purely graph theoretical parameter, $Z(G)$. Note also that the inequality in this theorem can be tight or strict. For example, for paths, it holds with equality while the corona $C_5 < K_1, \ldots, K_1 >$ (also called the penta-sun) has a zero forcing number equal to three (this follows from the fact that is proven in Corollary 4.5.4) but the maximum nullity of the penta-sun is equal to two, see [18, Example 4.1].

4.4 Zero forcing chains

In this section we study the *zero forcing chains* produced by a zero forcing process in a graph and how this concept relates the zero forcing number of a graph to the path cover number of the graph.

Let Z be a zero forcing set of a graph G. Construct the derived set, making a list of the forces in the order in which they are performed. This list is called the *chronological list of forces*. A *forcing chain* (for a particular chronological list of forces) is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $v_i \to v_{i+1}$, for $i = 1, \ldots, k - 1$. Not that a minimal zero forcing process produces a minimal collection of forcing chains. For the graph G in Example 4.2.1, we have the forcing chains (v_1, v_3, v_4) and (v_2, v_5). A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another zero forcing chain (the previous example has two maximal forcing chains). Note that a zero forcing chain can consist of a single vertex (v_1) and such a chain is maximal if $v_1 \in Z$ and v_1 does not perform a force. In each step of a forcing process, each vertex can force at most one other vertex and can be forced by at most one other vertex, therefore the maximal forcing chains are disjoint. Thus the vertices of the zero forcing set partition the vertices of the graph into disjoint paths (this will be discussed in Proposition 4.4.4). As we showed in Theorem 4.2.4, the derived set of a given set of black vertices is unique; however, a chronological list of forces and the forcing chains of a particular zero forcing set usually is not.

The number of chains in a zero forcing process starting with a zero forcing set Z is equal to the size of Z and the elements of Z are the initial vertices of the forcing chains. Let Z be a zero forcing set of a graph G. A *reversal* of Z is the set of last vertices of the maximal zero forcing chains of a chronological list of forces. Thus the cardinality of a reversal of Z is the same as the cardinality of Z.

Theorem 4.4.1 (see [1, Theorem 2.6]). *If Z is a zero forcing set of G, then so is any reversal of Z.*

Proof. We prove this by induction on the number of vertices. For the base case consider the graph K_2. Either of the vertices of the graph is a reversal for the other one and also is a ZFS for the graph. Let G be a graph and assume that this is true for all graphs G' with $|V(G')| < |V(G)|$. Now in graph G write the chronological list of forces in reverse order. Reversing each force, this is the reverse chronological list of forces. Let the reversal of Z for this list be denoted by W. We show the reverse chronological list of forces is a valid list of forces for W. Consider the first force, $u \to v$ on the reverse chronological list. We need to show that all neighbours of u except v must be in W, so that u can force v. Since then the last force in the original chronological list of forces was $v \to u$, each of the neighbours of u had u as a white neighbour and thus could not have forced any vertex previously (in the original chronological list of forces). Then $(Z \backslash \{u\}) \cup \{v\}$ is a ZFS for the graph $G - u$. Then according to the induction hypothesis, the rest of forces in the reverse chronological

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list of forces is a valid list of forces for the graph $G - u$. Thus W is a zero forcing set

of G. $\square$

Since we can always reverse a ZFS, every connected graph (except K_1) has multiple

ZFS. Therefore, we have the following.

Corollary 4.4.2 (see [1, Corollary 2.7]). *No connected graph of order greater*

than one has a unique minimum zero forcing set. $\square$

The following theorem shows that there is no connected graph G with a vertex

$v \in V(G)$ such that v is in every minimum zero forcing set. To see a proof of it refer

to [1, Theorem 2.9]. Let ZFS(G) be the set of all minimum zero forcing sets of G.

Theorem 4.4.3. *If G is a connected graph of order greater than one, then*

$$\bigcap_{Z \in ZFS(G)} Z = \varnothing. \quad \square$$

The next proposition shows that the path cover number is a lower bound for the

zero forcing number (see [1, Proposition 2.10]).

Proposition 4.4.4. *For any graph G, $P(G) \leq Z(G)$.* $\square$

The most famous family of graphs for which the path cover number agrees with

the zero forcing number is trees (see [18, Proposition 4.2]). Two forcing chains P_1 and

P_2 are called *adjacent* if there are two vertices $v \in P_1$ and $u \in P_2$ such that $uv \in E(G)$.

Proposition 4.4.5. *For any tree T, $P(T) = Z(T)$. Moreover, any minimal path covering of a tree $\mathcal{P}(T)$ coincides with a collection of forcing chains with $|\mathcal{P}(T)| = Z(G)$ and the set consisting of one end-point from each path in $\mathcal{P}(T)$ is a ZFS for T.*

Proof. We prove this by induction on the path cover number. For any tree with $\mathcal{P}(T) = 1$ (a path), the theorem applies. To perform induction step we need to show the following claim.

Claim. In any minimal path covering of a tree there always is a path that is connected (through an edge) to only one other path in the path covering. We call such path a *pendant path*. To observe this, suppose there is no such a path in a minimal path covering of a tree T. Thus any path is connected to at least two other paths in the path covering. This means the graph has a cycle as a subgraph which contradicts T being a tree.

Assume the theorem holds for all trees T' with $P(T') < P(T)$. Let $\mathcal{P}(T)$ be a path covering of T with $|\mathcal{P}(T)| = P(T)$. Let Z be the set consisting of one end-point of each path in $\mathcal{P}(T)$ and P_1 be a pendant path in $\mathcal{P}(T)$ that is joined to the rest of T by only one edge uv with $v \in V(P_1)$ and $u \notin V(P_1)$. Then by repeatedly applying the colour-change rule starting at the black end-point of P_1, all vertices from the black end-point through to v are coloured black. Now the path P_1 is irrelevant to the analysis of the tree $T - V(P_1)$, thus by the induction hypothesis, the black end-points of the remaining paths are a zero forcing set for $T - V(P_1)$, and all vertices not in

P_1, including u, can be coloured black. Hence the remainder of path P_1 can also be coloured black and Z is a zero forcing set for T. Moreover all the forces are performed along the paths in $\mathcal{P}(T)$ which completes the proof. □

We will basically follow a similar idea as in the proof of Proposition 4.4.5 in order to prove Theorem 4.5.3 and Theorem 6.4.9 in the following sections. In [25] it has been shown that:

Theorem 4.4.6. *For any tree T, $M(T) = P(T)$.*

Combining Proposition 4.4.5 and Theorem 4.4.6 we obtain the following.

Corollary 4.4.7. *For any tree T, $M(T) = Z(T)$.*

4.5 Graphs with $Z(G) = P(G)$

The most famous family of graphs satisfying $Z(G) = P(G)$ are trees (see Proposition 4.4.5). In this section we establish this equality for another family of graphs namely the block-cycle graphs. We will, also, try to give some evidence to show that not so many families of graphs satisfy this equality.

4.5.1 Block-cycle graphs

A graph is called *non-separable* if it is connected and has no cut-vertices. A *block* of a graph is a maximal non-separable induced subgraph. A *block-cycle* graph is a

graph in which every block is either an edge or a cycle (see Figure 4.3). A block-cycle graph with only one cycle is a *unicycle* graph.
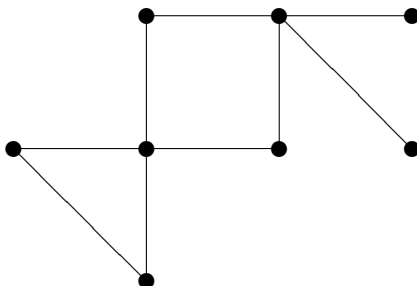
A block-cycle graph   Figure 4.3

Let $b(G)$ be the number of blocks in a block-cycle graph G. According to the definition, the only block-cycle graph with no cut vertex is either a cycle or an edge. In a block-cycle graph each pair of cycles can intersect in at most one vertex, otherwise there will exist a block in the graph which is neither a cycle nor an edge. Two blocks are called *adjacent* if they have one vertex in common. A block in a block-cycle graph is *pendant* if it shares only one of its vertices with the other blocks.

Lemma 4.5.1. *Any block-cycle graph has at least two pendant blocks.*

Proof. Assume G is a given block-cycle graph. To prove this lemma, we will construct a minor G' of the graph G and we will show that the end-points of the longest induced path in G' are associated to the pendant blocks in the original graph. Let $B_1, \ldots, B_N$ be the blocks in G. Note that B_i is either an edge or a cycle, for any $1 \le i \le N$. If B_i

is a cycle, contract all edges in B_i until all that remains is a single vertex, call this v_i. In this case we say the vertex v_i in G' is associated to the cycle B_i in G. Also we say $v_i \in G'$ is associated to the edge B_i in G, if v_i is an end-point of the edge B_i in G. Note that all the edges adjacent to B_i, are adjacent to v_i after this operation. If two cycles B_i and B_j share a vertex in G, draw an edge between the associated vertices v_i and v_j in G'.

Let $P = \{u_1, u_2, \cdots, u_k\}$ be the longest induced path in G'. Let $u_1 = v_i$ and $u_2 = v_j$ for some i and j and v_i and v_j correspond to blocks B_i and B_j. We first consider the case where B_i is a cycle while B_j is either an edge or a cycle. We claim that the cycle B_i is a pendant block in G. Otherwise there is another block, B, which can be an edge or a cycle that shares a vertex with B_i which is different from the vertex that B_i shares with B_j. First, assume that B is a cycle in G. Let z be the vertex associated to B. The only reason z is not in P is that there is an edge between z and some other vertices, say u_i, in P. The translation of this in the graph G is that there is either an edge or a cycle associated with u_i, with $i \neq 1$, which B shares a vertex with. Thus, there exists a cycle in G which shares more than one vertex with at least one other cycle in G. This means that there is a block in G that is neither a cycle nor an edge which contradicts with the fact that G is a block-cycle graph. A similar argument applies when B is an edge in G.

Using a similar reasoning when B_i is an edge in G, we get the same contradiction

which proves the theorem. □

The following lemma is straightforward to prove.

Lemma 4.5.2. *If B is a pendant block in a block-cycle graph, then*

$$P(G \backslash B) \le P(G). \quad \square$$

Theorem 4.5.3. *Let G be a block-cycle graph. Then*

$$Z(G) = P(G).$$

Furthermore, the paths in any minimal path covering of G are precisely the forcing chains in a minimal zero forcing process initiated by a proper selection of the end-points of the paths in this collection.

Proof. We prove the equation by induction on the number of blocks in G and applying Lemma 4.5.1. The only block-cycle graph with $b(G) = 1$ is either an edge or a cycle and the theorem is clearly true for these graphs. Assume that it is true for all graphs G' with $b(G') < n$. Let $b(G) = n$. According to Lemma 4.5.1, there is a pendant block, B, in G which is connected to the other blocks through a single vertex, u. Let $G = G' +_{u} B$. The induction hypothesis holds for G', that is $Z(G') = P(G')$ and a proper selection of the end-points of the paths in a minimal path covering $\mathcal{P}$ of G' constructs a zero forcing set for it. By Lemma 4.5.2, we have $P(G') \le P(G)$. Two cases are then possible:

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(a) There is a path-cover $\mathcal{P}$ for the graph $G' = G \backslash B$ in which there is a path P of length more than one such that u is an end-point of P.

First assume that B is the edge uv. Then $G \backslash B$ is the graph obtained from G by removing the pendant vertex v of G. Since u is an end-point of P and v is only connected to u, returning B to G' doesn't change the path cover number of the graph. By the induction hypothesis, the paths in the path-cover $\mathcal{P}$ are the forcing chains of the forcing process initiated by the end-points of the paths in $\mathcal{P}$. Also since u is an end-point of P, we can assume that it doesn't perform any force. Therefore, the zero forcing process will be continued by using u to force v. Thus,

$$P(G') = P(G) \leq Z(G) \leq Z(G') = P(G'),$$

which implies $Z(G) = P(G)$.

If B is a cycle, then since it is a pendant cycle, returning it to G' increases the path cover number of G' at least by one, since we will need at least two paths to cover the vertices of a cycle. Assume that v and w are two neighbours of u in B. Then, since u is an end-point of P, $P \cup (V(B) \backslash \{v\})$ can cover all vertices in the cycle B except v. Thus the path cover for G' along with $\{v\}$ is a path cover for G and $P(G) = P(G') + 1$. Also we need at least two initial black vertices in the cycle to colour its vertices. By assigning colour black to the vertex v, all vertices

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in B will be coloured by continuing the forcing process in P through u. Thus,

$$P(G') + 1 = P(G) \leq Z(G) \leq Z(G') + 1 = P(G') + 1,$$

which implies $Z(G) = P(G)$.

(b) In every minimal path covering $\mathcal{P}$ of G', u is an inner vertex of a path in P.

If B is an edge uv, then $P(G) \neq P(G')$. Otherwise v is covered in the same path as u is, which contradicts the fact that u is not an end-point of any path in any path covering of G. Thus $P(G) = P(G') + 1$ since $\mathcal{P} \cup \{v\}$ covers all vertices of G. Also by assigning a black colour to the vertex v we are able to colour the graph G following the same forcing process which we followed to colour the graph G'. Thus,

$$P(G') + 1 = P(G) \leq Z(G) \leq Z(G') + 1 = P(G') + 1,$$

which implies $Z(G) = P(G)$.

If B is a cycle, then $P(G) = P(G') + 1$, since $\mathcal{P}$ along with a path covering all vertices of B, except u, covers all vertices of G. Also, by assigning a black colour to one of the neighbours of u in B, w, we can colour G using the initial set of black vertices in the zero forcing set of G' along with w. Thus,

$$P(G') + 1 = P(G) \leq Z(G) \leq Z(G') + 1 = P(G') + 1,$$

which implies $Z(G) = P(G)$. □

The following corollary is obtained from the fact that any unicycle graph is a block-cycle graph.

Corollary 4.5.4. *If G is a unicycle graph, then $Z(G) = P(G)$.* $\qquad\qquad\square$

4.5.2 Graphs for which the path cover number is two

It seems that for a general graph it is rare to satisfy the equality $Z(G) = P(G)$. To show this along with the fact that the discrepancy between $Z(G)$ and $P(G)$ can be arbitrarily large we focus on the family of graphs with $P(G) = 2$.

Proposition 4.5.5. *Let G be a graph with $P(G) = 2$ and two covering paths P_1 and P_2 with $|P_1| = m$ and $|P_2| = n$. Then*

$$2 \le Z(G) \le min\{n, m\} + 1.$$

Moreover for any number k in this interval, there is a graph G satisfying $P(G) = 2$ with $Z(G) = k$.

Proof. The claim that $Z(G) \ge 2$ is trivial. Suppose $m \le n$ and let B be the set consisting of $V(P_1)$ and an end-point of P_2. Obviously B is a ZFS for G. Thus $Z(G) \le |B| = m + 1 = min\{n, m\} + 1$. For the given number k in the interval $[2, min\{m, n\} + 1]$ let G be the following graph. Set two paths P_1 and P_2 with $|P_1| = m$ and $|P_2| = n$ and assume that $m \le n$. Starting with an end-point of P_1 make each of k consecutive

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vertices of P_1 adjacent to all of the vertices of P_2. Then, it is easy to observe that

$Z(G) = k$. □

Among all the graphs with $P(G) = 2$, only those that are also outerplanar satisfy

$Z(G) = 2$ (see [27, Theorem 5.1]).

4.6   Graphs with $Z(G) = M(G)$

One of the most important questions here, which is motivated by Theorem 4.3.3, is to characterize the graphs for which $M(G) = Z(G)$ holds. Although, all families of graphs with $M(G) = Z(G)$ have not yet been fully characterized, there are some families of graphs for which this equality has been established. For instance, all the graphs with less than seven vertices (see [18, Proposition 4.4]), and the families of graphs are listed in [18, Table 1]. In this chapter we expand this list.

A graph G is said to be *chordal*, if it has no induced cycle of length more than three. In [18, Proposition 3.23] as well as in [24, Theorem 7], it has been shown that the equality $M(G) = Z(G)$ also holds for some families of chordal graphs. In this section, also, we prove that this equality holds for some new families of chordal graphs.

For any graph G, assume that Z_G is a minimum zero forcing set of G. Given graph G with $|G| = n$ and the graphs $H_1, \ldots, H_n$, let

$$G < H_1, \ldots, H_n >$$

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be the graph obtained by joining all the vertices of the graph H_i to the i-th vertex of

G, where $i = 1, \ldots, n$. Note that $|H_i|$ can be zero, in which case no extra vertices will

be joined to the vertex i. We call the graph

$$G < H_1, \ldots, H_n >$$

a *generalized corona* of G with $H_1, \ldots, H_n$. In the next theorem, we use the convention

that $Z(H_i) = 0$ if $|H_i| = 0$.

Theorem 4.6.1. *For graph G with $|G| = n$ and graphs $H_1, \ldots, H_n$ we have*

$$Z(G < H_1, \ldots, H_n >) \leq Z(G) + \sum_{i=1}^{n} Z(H_i).$$

Proof. Let $G = \{v_1, \ldots, v_n\}$. Without loss of generality, we may assume that $Z_G = \{v_1, \ldots, v_k\}$ is a zero forcing set for G, and that one can colour all the other vertices

of G by a sequence of forces in the following order:

$$v_{k+1}, v_{k+2}, \ldots, v_n. \tag{4.3}$$

Define

$$\Delta = Z_G \cup Z_{H_1} \cup \ldots \cup Z_{H_n}.$$

We claim that Δ is a zero forcing set for $G < H_1, \ldots, H_n >$, which is of size

$$|\Delta| = Z(G) + \sum_{i=1}^{n} Z(H_i).$$

To prove the claim, first assume that Δ is initially coloured black and we colour all

the vertices of the graphs H_i, with $i = 1, \ldots, k$, which are associated to the vertices of

Z_G using the corresponding set Z_{H_i}'s. Now note that there is a vertex $v_i \in Z_G$ whose only white neighbour in G is v_{k+1}; because we were able to colour the vertices of G using Z_G in the order (4.3). Now, since all the vertices in the graph H_i associated with v_i are black, v_{k+1} is, still, the only white neighbour of v_i in $G \prec H_1, \ldots, H_n \succ$. Thus v_{k+1} can be forced by v_i. Then we colour all the vertices in the H_{k+1} using the black vertices in $Z_{H_{k+1}}$. Continuing this process, therefore, we can colour all the vertices of $G \prec H_1, \ldots, H_n \succ$ which proves the claim. □

The following is an immediate consequence of Theorem 4.6.1.

Corollary 4.6.2. *For graphs G and H we have*

$$Z(G \prec H, \ldots, H \succ) \leq Z(G) + |G|Z(H). \quad \square$$

The bound in Corollary 4.6.2 is, in fact, an improvement of the bound in [18, Proposition 2.12]. Using Theorem 4.6.1, Theorem 4.3.3 and Corollary 3.2.6, we can prove that equality in Theorem 4.3.3 holds for the following families of chordal graphs.

Theorem 4.6.3. *Let G be a graph satisfying the equalities $Z(G) = M(G)$ and $\mathrm{mr}(G) = \mathrm{cc}(G)$ and $H_1, \ldots, H_n$ be graphs satisfying the equalities $Z(H_i) = M(H_i)$ and $\mathrm{mr}(H_i) = \mathrm{cc}(H_i)$ with $i = 1, \ldots, n$. Then*

$$Z(G \prec H_1, \ldots, H_n \succ) = M(G \prec H_1, \ldots, H_n \succ),$$

and

$$\mathrm{mr}(G \prec H_1, \ldots, H_n \succ) = \mathrm{cc}(G \prec H_1, \ldots, H_n \succ).$$

Proof. By (3.2) and Corollary 3.2.6 we have

$$M(G < H_1, \ldots, H_n >) \geq |V(G < H_1, \ldots, H_n >)| - \mathrm{cc}(G < H_1, \ldots, H_n >). \qquad (4.4)$$

Note that

$$|V(G < H_1, \ldots, H_n >)| = |V(G)| + \sum_{i=1}^{n} |V(H_i)|.$$

On the other hand we have

$$\mathrm{cc}(G < H_1, \ldots, H_n >) \leq \mathrm{cc}(G) + \sum_{i=1}^{n} \mathrm{cc}(H_i). \qquad (4.5)$$

In order to show (4.5) note that each clique K_{j_i} in a minimal clique covering of H_j joined to the vertex j of G constructs a clique isomorphic to K_{j_i+1}. Thus by using the same number of cliques we use to cover the edges of G and H_i with $i = 1, \ldots, n$ we can cover the edges of $G < H_1, \ldots, H_n >$. Also by Theorem 4.6.1 and (3.2), and the assumptions of the theorem we have

$$
\begin{aligned}
&M(G < H_1, \ldots, H_n >) \\
&= |V(G < H_1, \ldots, H_n >)| - \mathrm{mr}(G < H_1, \ldots, H_n >) \\
&\leq Z(G < H_1, \ldots, H_n >) \\
&\leq Z(G) + \sum_{i=1}^{n} Z(H_i) \qquad\qquad\qquad\qquad (4.6) \\
&= M(G) + \sum_{i=1}^{n} M(H_i) \\
&= |V(G)| + \sum_{i=1}^{n} |V(H_i)| - \left[\mathrm{cc}(G) + \sum_{i=1}^{n} \mathrm{cc}(H_i) \right] \\
&\leq |V(G < H_1, \ldots, H_n >)| - \mathrm{cc}(G < H_1, \ldots, H_n >).
\end{aligned}
$$

The theorem follows from (4.4) and (4.6). $\qquad\qquad\qquad\qquad\square$

Theorem 4.6.3 provides a recursive construction to generate an infinite family of graphs that satisfy the equality $Z(G) = M(G)$. The following is a consequence of Theorem 4.6.3.

Corollary 4.6.4. *For the following graphs, $Z(G) = M(G)$:*

(a) $G_1 = K_t < P_{s_1}, \ldots, P_{s_r}, K_{q_1}, \ldots, K_{q_\ell} >$, *where* $s_i, q_j \geq 2$;

(b) $G_2 = P_t < P_{s_1}, \ldots, P_{s_r}, K_{q_1}, \ldots, K_{q_\ell} >$, *where* $s_i, q_j \geq 2$.

Moreover

(1) $M(G_1) = Z(G_1) = t - 1 - \ell + q_1 + \cdots + q_\ell + r$;

(2) $\mathrm{mr}(G_1) = \mathrm{cc}(G_1) = 1 + \ell + s_1 + \cdots + s_r - r$;

(3) $M(G_2) = Z(G_2) = 1 - \ell + q_1 + \cdots + q_\ell + r$;

(4) $\mathrm{mr}(G_2) = \mathrm{cc}(G_2) = t - 1 + \ell + s_1 + \cdots + s_r - r$.

Proof. Statements (1), (2), (3) and (4) follow from the facts $Z(K_t) = t - 1$, $Z(P_t) = 1$, $\mathrm{cc}(K_t) = 1$ and $\mathrm{cc}(P_t) = t - 1$, for any t. □

The next theorem demonstrates the existence of another family of chordal graphs for which the equality holds in Theorem 4.3.3.

Theorem 4.6.5. *Let G be a chordal graph consisting of $N \geq 1$ cliques, $K_{n_1}, \ldots, K_{n_N}$, such that the intersection of any two cliques is a complete graph and no vertex is*

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contained in more than two cliques. Also, let $k_{i,j}$ be the size of the clique in the intersection of K_{n_i} and K_{n_j} then

$$M(G) = Z(G) = \sum_{i=1}^{N} n_i - \sum_{i<j} k_{ij} - N,$$

and

$$\mathrm{mr}(G) = \mathrm{cc}(G) = N.$$

Proof. The proof is by induction on the number of cliques. It is clearly true for $N = 1$. Since no vertex of G is contained in more than two cliques, it has a pendant clique, that is, a clique which intersects only one other clique in the graph. Otherwise we have an induced cycle of order more than three in G which contradicts with G being chordal. Without loss of generality, assume that, this clique is K_{n_1} and that it only intersects with K_{n_2}. Let $H = K_{n_1} \setminus (K_{n_1} \cap K_{n_2})$ and $G' = G \setminus H$. By the induction hypothesis we have,

$$M(G') = Z(G') = \sum_{i=2}^{N} n_i - \sum_{1<i<j} k_{ij} - (N-1).$$

It is clear that

$$\mathrm{cc}(G') \leq N - 1.$$

Then it follows that a ZFS from G' with an additional $n_1 - k_{1,2} - 1$ black vertices from H forms a ZFS for G. Thus,

$$M(G) \leq Z(G) \leq \sum_{i=2}^{N} n_i - \sum_{1<i<j} k_{ij} - N + 1 + n_1 - k_{1,2} - 1 = \sum_{i=1}^{N} n_i - \sum_{i<j} k_{ij} - N,$$

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and

$$\mathrm{cc}(G) \le N.$$

It is clear that

$$|G| = \sum_{i=1}^{N} n_i - \sum_{i<j} k_{ij}.$$

Thus, using (3.2) and Corollary 3.2.6, we obtain the following lower bound for $M(G)$:

$$M(G) \ge \sum_{i=1}^{N} n_i - \sum_{i<j} k_{ij} - N,$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that the equality in Theorem 4.3.3 does not hold for all chordal graphs. See for instance [1, Example 2.11].

4.7 The Colin de Verdière graph parameter

In 1990, Colin de Verdière [8] introduced an interesting new parameter $\mu(G)$ for any undirected graph G. The parameter $\mu(G)$ can be fully described in terms of properties of matrices related to G. One of the interesting applications of μ is that certain topological properties of a graph G can be characterized by spectral properties of matrices associated with G including values of μ, see [8, 34]. Another interesting graph theoretical property of μ is that it is monotone on graph minors. Before defining this parameter, we need to introduce the Strong Arnold Property, which will be abbreviated by SAP throughout the thesis.

The *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size is just their element-wise product $A \circ B \equiv [a_{ij}b_{ij}]$. Let A and X be symmetric $n \times n$ matrices. We say that X *fully annihilates* A if

1. $AX = 0$,

2. $A \circ X = 0$, and

3. $I_n \circ X = 0$.

The matrix A has the Strong Arnold Property (SAP) if the zero matrix is the only symmetric matrix that fully annihilates A. The following is a basic, yet useful, observation concerning symmetric matrices with small nullity. In particular, it demonstrates that there are matrices in $\mathcal{S}(G)$ that satisfy SAP, for any graph G.

Lemma 4.7.1. *If* $\mathrm{null}(A) \leq 1$, *then* A *has SAP.*

Proof. If $\mathrm{null}(A) = 0$, then A is nonsingular and the only matrix X that fully annihilates A is the zero matrix. Suppose, now, that $\mathrm{null}(A) = 1$ and let X fully annihilate A. Thus by condition (3), the diagonal of X is 0. Since X is symmetric, this implies that X is not a rank 1 matrix. Hence, if $X \neq 0$, then $\mathrm{rank}(X) \geq 2$ and $AX = 0$ would imply $\mathrm{null}(A) \geq 2$. Therefore $X = 0$ and A has SAP. $\qquad\square$

Now we can define the Colin de Verdière parameter. For a given graph G, $\mu(G)$ is defined to be the maximum multiplicity of 0 as an eigenvalue of L, where $L = [l_{i,j}]$ satisfies all of the following conditions:

1. $L \in \mathcal{S}(G)$ and $l_{i,j} \le 0$, for $i \ne j$;

2. L has exactly one negative eigenvalue (with multiplicity one);

3. L has SAP.

In other words $\mu(G)$ is the maximum nullity among the matrices satisfying (1)-(3) above. Further, observe that

$$\mu(G) \le M(G) = n - \mathrm{mr}(G) \le Z(G). \tag{4.7}$$

Hence there is an obvious relationship between $\mu(G)$ and $\mathrm{mr}(G)$.

the *linking number* is a numerical invariant that describes the linking of two closed curves in three-dimensional space. Intuitively, the linking number represents the number of times that each curve winds around the other. A *linkless embedding* of an undirected graph is an embedding of the graph into Euclidean space in such a way that no two cycles of the graph have nonzero linking number.

Colin de Verdière et al [8, 34], through some sophisticated theorems, have shown that :

- $\mu(G) \le 1$ if and only if G is a disjoint union of paths;

- $\mu(G) \le 2$ if and only if G is outerplanar;

- $\mu(G) \le 3$ if and only if G is planar;

- $\mu(G) \le 4$ if and only if G is linklessly embeddable.

A related parameter, also introduced by Colin de Verdière (see [9]), is denoted by $\nu(G)$ and is defined to be the maximum nullity among matrices A that satisfy the following conditions:

1. $A \in \mathcal{S}(G)$;

2. A is positive semidefinite;

3. A has SAP.

Note, also, that

$$\nu(G) \leq M_+(G) = n - \mathrm{mr}_+(G). \tag{4.8}$$

Thus, there is an obvious relationship between $\nu(G)$ and $\mathrm{mr}_+(G)$.

Properties analogous to $\mu(G)$ have been established for $\nu(G)$. For example, $\nu(G) \leq 2$ if the dual of G is outerplanar, see [9]. Furthermore, $\nu(G)$, like $\mu(G)$ is a minor-monotone graph parameter.

Consequently, in order to learn more about the minimum rank of graphs, another related parameter which is denoted by $\xi(G)$ is been introduced by Fallat et al. [3] with the following definition.

Definition 4.7.2. *For a graph G, $\xi(G)$ is the maximum nullity among matrices $A \in \mathcal{S}(G)$ having SAP.*

For example, based on Lemma 4.7.1 we have :

$$\xi(K_n) = n - 1. \tag{4.9}$$

One of the most important properties of $\xi(G)$ analogous to $\mu(G)$ is its minor-monotonicity [3, Corollary 2.5] . Also, note that

$$\mu(G) \le \xi(G) \le M(G) \le Z(G), \tag{4.10}$$

and

$$\nu(G) \le \xi(G) \le M(G) \le Z(G). \tag{4.11}$$

Two of the most interesting open questions in this area that have been proposed by Colin de Verdière in 1998, are the following.

Conjecture 4.7.3. *For any graph G, $\chi(G) \le \mu(G) + 1$.*

Conjecture 4.7.4. *For any graph G, $\chi(G) \le \nu(G) + 1$.*

According to Conjecture 4.7.3 and Equations (4.3.3) and (4.7), a weaker comparison can be stated as follows:

Question. Does the following inequality hold for any graph G;

$$\chi(G) \le Z(G) + 1?$$

The next section is devoted to answering this question.

4.8 The zero forcing number and the chromatic number

Since the zero forcing number of a graph is a graph parameter associated with a new type of graph colouring, it seems natural to ask if there is any relationship

between this parameter and the traditional graph colouring parameter called the chromatic number of the graph. This, along with the second last question of Section 4.7 motivates us to find a relationship between the zero forcing number and the chromatic number of a graph.

Theorem 4.8.1. *Let G be a graph. Then we have*

$$Z(G) \geq \max\{\delta(H') \mid H' \text{ is any induced subgraph of } G\}.$$

Proof. Let H be an induced subgraph of G such that

$$\delta(H) = \max\{\delta(H') \mid H' \text{ is any induced subgraph of } G\}.$$

Let Z be a zero forcing set of G with $|Z| = Z(G)$. If Z includes all vertices of H, then there is nothing to prove. If not, and no vertex of H performs a force in the zero forcing process with the initial black vertices in Z, then there is a zero forcing set Z' (the reversal of Z) with $|Z'| = |Z|$ that includes all vertices in H and again the result follows. Now assume that there is at least one vertex in H performing a force and let v be the first vertex of H which performs a force in the process (this vertex is either in Z or it is forced by a vertex not in H). Since v is the first one which performs a force, all the neighbours of v in H, except one, should already have been forced in some distinct forcing chains. The initial vertices of these chains are in Z. Thus,

$$Z(G) \geq d_H(v) - 1 + 1 \geq \delta(H),$$

which completes the proof. □

The following inequality shows the connection between the chromatic number of a graph and the minimum degree among all the induced subgraphs of the graph (see [6, Section 4.2]).

Lemma 4.8.2. *For any graph* G,

$$\chi(G) \le 1 + \max\{\delta(H') \mid H' \text{ is any induced subgraph of } G\}. \quad \square$$

Corollary 4.8.3. *For any graph* G *we have*

$$\chi(G) \le Z(G) + 1.$$

Proof. The inequality follows directly from Theorem 4.8.1 and Lemma 4.8.2. $\quad \square$

Note that the bound in the above inequality is tight for some graphs such as paths and complete graphs. The following corollary is a direct result of Theorem 4.8.1.

Corollary 4.8.4. *Let* G *be a graph.*

1. *If* K_n *is a subgraph of* G *then* $Z(G) \ge n - 1$.

2. *If* $K_{p,q}$ *is a subgraph of* G *then* $Z(G) \ge \min\{p, q\}$. $\quad \square$

In fact there is a stronger result than the first statement of Corollary 4.8.4 based on the minor monotonicity property of $\xi(G)$ and (4.11) and (4.9) which is as follows:

Proposition 4.8.5. *If* G *has a* K_n *minor then*

$$Z(G) \ge n - 1.$$

Also we can improve the bound in the second statement of Corollary 4.8.4 as follows.

Proposition 4.8.6. *If the complete bipartite graph $K_{p,q}$ is a subgraph of G then,*

$$Z(G) \geq \min\{p, q\} + 1,$$

provided that $(p, q) \neq (1, 1), (1, 2), (2, 1), (2, 2)$.

Proof. Let the graph $K_{p,q}$ be a subgraph of the graph G. Assume that (X, Y) are the partitions of $K_{p,q}$ with $|X| = p$ and $|Y| = q$ where $p \leq q$. Let Z be a zero forcing set of G with $|Z| = Z(G)$. If Z includes all vertices of $K_{p,q}$, there is nothing to prove. If not and no vertex of $K_{p,q}$ performs a force in the zero forcing process with the initial black vertices in Z, then there is a zero forcing set Z' (the reversal of Z) with $|Z'| = |Z|$ that includes all vertices in $K_{p,q}$ and the inequality follows.

Now assume that there is at least one vertex in $K_{p,q}$ performing a force. Let v be the first vertex of $K_{p,q}$ which is forcing a vertex u in a zero forcing process starting with the vertices in Z (v is either in Z or it is forced by a vertex not in $K_{p,q}$). First assume that $v \in X$. If u is not a vertex of $K_{p,q}$, since v is the first vertex of $K_{p,q}$ which performs a force, all q neighbours of v in $K_{p,q}$ should already have been forced in some distinct forcing chains and the result follows. If u is in $K_{p,q}$, then there are $q - 1$ distinct forcing chains having the neighbours of v in $K_{p,q}$ except u as their end-points. If u has no other neighbours in X, then $\min\{p, q\} = 1$ and the

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inequality follows from the fact that $Z(G) \geq q \geq 2 = \min\{p, q\} + 1$. If not and u is the second vertex of $K_{p,q}$ that performs a force, this requires at least all the neighbours of u in $K_{p,q}$, except one, are already coloured black in some distinct forcing chains. Therefore, $Z(G) \geq q + 1 \geq \min\{p, q\} + 1$. If not and the forcing chain containing v ends with u, then there is at least one more forcing chain to colour the rest of the vertices in X. Thus $Z(G) \geq q + 1 \geq \min\{p, q\} + 1$. A similar argument applies when $v \in Y$. $\square$

Chapter 5

Zero forcing number and graph operations

In this chapter we study effects of some graph operations, including operations on a single graph or operations on several graphs, on the zero forcing number of the resulting graph. The impact of the graph operations on zero forcing number shows this parameter, similar to maximum nullity, is not monotone on subgraphs.

5.1 Simple operations

Deleting a vertex from a graph can increase or decrease the zero forcing number, but by at most one in either direction. It is somehow predictable that vertex deletion could either decrease the zero forcing number or keep it the same, but surprisingly, there are examples in which vertex deletion increases the zero forcing number.

Proposition 5.1.1 (see [32, Theorem 6.4]). *Let v be a vertex of the graph G. Then $Z(G - v) - 1 \leq Z(G) \leq Z(G - v) + 1$.* $\qquad\qquad\square$

The following are examples of all three possible cases in Proposition 5.1.1.

- Decrease:

$$Z(G) = 2 \qquad\qquad Z(G - v) = 1$$

An example of a graph for which removing a vertex decreases the zero forcing number

Figure 5.1

- No change:

$$Z(G) = 1 \qquad\qquad Z(G - v) = 1$$
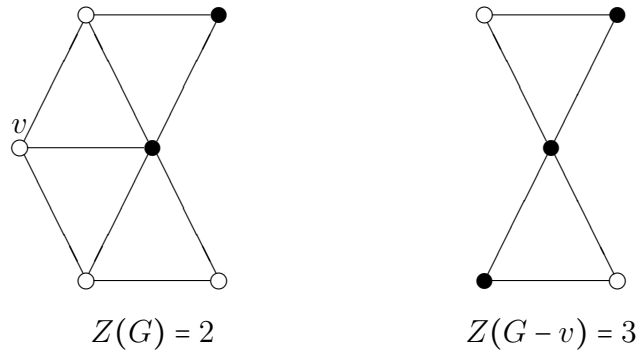
An example of a graph for which removing a vertex does not change the zero forcing number

Figure 5.2

- Increase:

$$Z(G) = 2 \qquad\qquad Z(G - v) = 3$$

An example of a graph for which removing a vertex increases the zero forcing number

Figure 5.3

It is not hard to observe the following.

Proposition 5.1.2. *Let G be a graph with a vertex labeled v.*

(a) If $v \in Z$ where Z is a minimal ZFS of G, then $Z(G) - 1 \le Z(G - v) \le Z(G)$.

(b) If $Z(G - v) = Z(G) + 1$, then there is no minimal ZFS of G including v. $\qquad\square$

Similarly edge deletion can change the zero forcing number by at most 1.

Proposition 5.1.3 (see [32, Theorem 5.2]). *Let G be a connected graph. If $e = uv$ is an edge of G, then*

$$Z(G) - 1 \le Z(G - e) \le Z(G) + 1. \quad \square$$

The following are examples of all three possible cases in Proposition 5.1.3.

- Decrease:
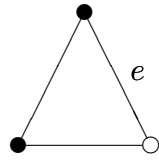
$$Z(G) = 2 \qquad Z(G - e) = 1$$
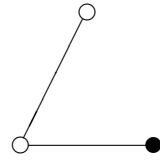
An example of a graph for which removing an edge decreases the zero forcing number

Figure 5.4

- No change:

$$Z(G) = 2 \qquad\qquad Z(G - e) = 2$$

An example of a graph for which removing an edge does not change the zero forcing number

Figure 5.5

- Increase:

$$Z(G) = 2 \qquad\qquad Z(G - e) = 3$$
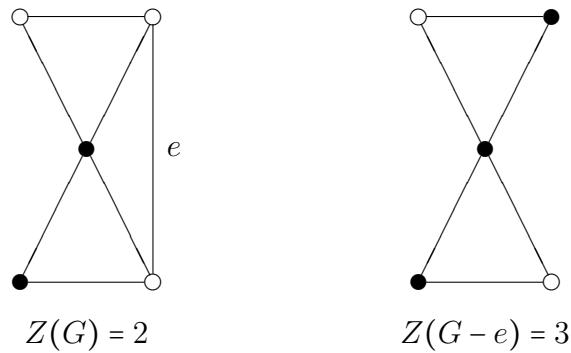
An example of a graph for which removing an edge increases the zero forcing number

Figure 5.6

It is not hard to observe the following

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Proposition 5.1.4. *Let* $e = uv$ *be an edge in a graph* G. *If there is a minimal zero forcing process for* G *in which neither* u *forces* v *nor* v *forces* u, *then* $Z(G) - 1 \leq Z(G - e) \leq Z(G)$. *Otherwise* $Z(G) \leq Z(G - e) \leq Z(G) + 1$.

Proof. In order to prove the case "otherwise", assume that $\mathcal{P}$ is a minimal collection of forcing chains of G in which u forces v in the chain P. Let $P \backslash \{e\} = P_1 \cup P_2$. Then by assigning the black colour to v, $(\mathcal{P} \backslash \{P\}) \cup \{P_1, P_2\}$ is a minimal collection of forcing chains for $G - e$. Thus $Z(G - e) \leq Z(G) + 1$. To show that $Z(G) \leq Z(G - e)$, suppose that $Z(G - e) \leq Z(G) - 1$. Then there is a collection of forcing chains $\mathcal{P}'$ of $G - e$ of size $Z(G) - 1$. First suppose that u and v are in the same chain $P' \in \mathcal{P}'$ and u is in a lower level than v. Assume u forces w and w' forces v in this chain. Let $P' \backslash \{uw, w'v\} = P_1' \cup P_2' \cup P_3'$ in which P_1' ends with u and P_2' starts with w and ends with w' and P_3' starts with v. Let P'' be the chain obtained by merging two chains P_1' and P_3' by adding the edge uv. Then by assigning the black colour to w, $(\mathcal{P} \backslash \{P\}) \cup \{P_3', P''\}$ is a minimal collection of forcing chains of G in which u forces v. A similar argument applies when u and v are in different chains. $\qquad \square$

Edge contraction affects the zero forcing number in the following ways.

**Proposition 5.1.5 (see [32, Theorem 5.1]).** *Let* $e = uv$ *be an edge in a graph* G. *Then*

$$Z(G) - 1 \leq Z(G/e) \leq Z(G) + 1. \quad \square$$

The following are examples of all three possible cases in Proposition 5.1.5.

- Decrease:

$$Z(G) = 2 \qquad\qquad\qquad Z(G/e) = 1$$
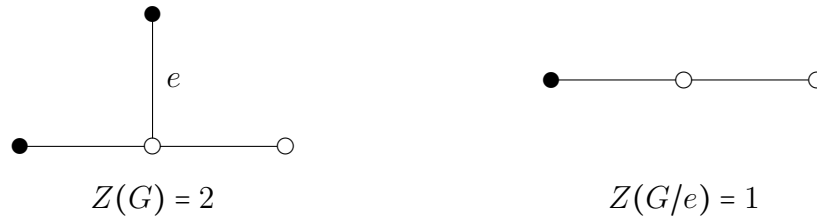
An example of a graph for which contracting an edge decreases the zero forcing number

Figure 5.7

- No change:

$$Z(G) = 1 \qquad\qquad Z(G/e) = 1$$

An example of a graph for which contracting an edge does not change the zero forcing number

Figure 5.8

- Increase:

$Z(G) = 2$ $Z(G/e) = 3$
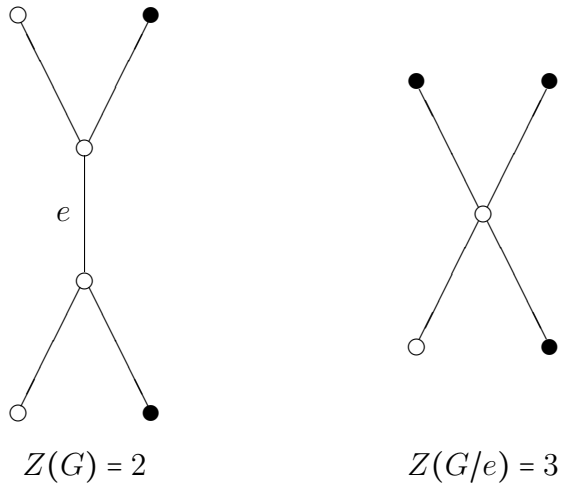
An example of a graph for which contracting an edge increases the zero forcing number

Figure 5.9

Subdividing an edge can only increase the zero forcing number by at most one.

Proposition 5.1.6 (see [32, Theorem 5.4]). *If H is obtained from G by subdividing an edge $e = uv$, then*

$$Z(G) \leq Z(H) \leq Z(G) + 1.$$

The following are examples of the two possible cases in Proposition 5.1.6.

- No change:

$$Z(G) = 1 \qquad Z(H) = 1$$

An example of a graph for which subdividing an edge does not change the zero forcing number

Figure 5.10

- Increase:

$$Z(G) = 2 \qquad Z(H) = 3$$

An example of a graph for which subdividing an edge increases the zero forcing number

Figure 5.11

In the next proposition we bound the variation of the zero forcing number of a specific type of graphs after adding edges and vertices to it. A graph G is called a *semi-complete* graph if all of its forcing chains in any minimal zero forcing process are pairwise adjacent. Note that such a graph is *contractable* to a $K_{Z(G)}$ and any complete graph is semi-complete.

Proposition 5.1.7. *Let G be a semi-complete graph and u and v be two nonadjacent*

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vertices of G. Let $G_1 = G + uv$ and G_2 be the graph obtained from G by adding a new vertex z to $V(G)$ that is connected to the vertices of G in any fashion. Then

1. $Z(G) \leq Z(G_1) \leq Z(G) + 1$.

2. $Z(G) \leq Z(G_2) \leq Z(G) + 1$.

Proof. Suppose that $Z(G_1) < Z(G)$. Let $\mathcal{P}$ be a collection of forcing chains of G_1 with $|\mathcal{P}| = Z(G_1)$. If in this zero forcing process neither u forces v nor v forces u, then $\mathcal{P}$ is a ZFS for G. Thus $Z(G) = Z(G_1 - uv) \leq Z(G_1)$ that implies $Z(G) < Z(G)$ which is a contradiction. Assume that there is a forcing chain in $\mathcal{P}$, say P, that has a force through uv, then $P - uv = P_1 \cup P_2$ where P_1 and P_2 are two non-adjacent forcing chains one of them starting with u (or v). Therefore by assigning a black colour to u (or v), $(\mathcal{P} \setminus \{P\}) \cup \{P_1, P_2\}$ is a minimal collection of forcing chains of G with two non-adjacent chains P_1 and P_2. This contradicts the fact that G is a semi-complete graph and therefore, proves the claim.

A similar argument applies for the second statement. $\qquad \square$

Proposition 5.1.7 and testing several examples motivates us to propose the following conjecture.

Conjecture 5.1.8. *If the graph G has a K_n minor, then G has a semi-complete subgraph, H, with $Z(H) \geq n - 1$.*

5.2   Vertex-sum of graphs

One of the most common graph operations is the vertex-sum of two graphs. The following theorem calculates the zero forcing number of the vertex-sum of two graphs.

Theorem 5.2.1. *Let G and H be two graphs each with a vertex labeled v. Then the following hold.*

1. *If v is in some minimal ZFS of H but not in any minimal ZFS of G, then*

$$Z(G \underset{v}{+} H) = Z(G) + Z(H).$$

2. *If v is in some minimal ZFS of H and some minimal ZFS of G, then*

$$Z(G \underset{v}{+} H) = Z(G) + Z(H) - 1.$$

3. *If v is neither in any minimal ZFS of H nor in any minimal ZFS of G, then*

$$Z(G \underset{v}{+} H) = Z(G) + Z(H) + 1.$$

Proof. Let Z_G be a minimal ZFS of G and Z_H be a minimal ZFS of H.

To verify the first case assume that $v \in Z_H$. By Theorem 4.4.1, if Z'_H is the reversal of Z_H, then Z'_H is a minimal ZFS in which v doesn't perform a force. Therefore starting with the initial black vertices in $Z'_H \cup Z_G$ and colouring H before G we can colour the entire graph $G \underset{v}{+} H$. Thus

$$Z(G \underset{v}{+} H) \leq Z(G) + Z(H).$$

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To prove the equality in the equation suppose that

$$Z(G \underset{v}{+} H) < Z(G) + Z(H).$$

Hence we can colour $G \underset{v}{+} H$ using $Z(G) + Z(H) - 1$ initial black vertices. No vertex of G, except v, can force any vertex of H in any zero forcing process. In addition, since v is not a member of any minimal ZFS of G, v is not an end-point of any zero forcing chain in G. Therefore v can not perform any force in H. Thus we need at least $Z(H)$ initial black vertices from $V(H)$ to colour H. Hence we have at most $Z(G) - 1$ black vertices of G along with v (that is already turned to black) to colour G. This means there is a minimal ZFS for G that contains v which contradicts the fact that v is not in any minimal ZFS of G.

To verify the second case we use a similar reasoning as in the first case. First using Z'_H we can colour all of H. If Z_G is a minimal ZFS for G that contains v then $Z'_H \cup (Z_G \backslash \{v\})$ forms a ZFS for $G \underset{v}{+} H$. Thus

$$Z(G \underset{v}{+} H) \le Z(G) + Z(H) - 1.$$

In order to prove the equality, suppose that

$$Z(G \underset{v}{+} H) < Z(G) + Z(H) - 1.$$

Therefore we can colour $G \underset{v}{+} H$ with an initial set of black vertices of size $Z(G) + Z(H) - 2$. Since at most one of the vertices of H, i.e. v, can be forced by a vertex of

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G, we need at least $Z(H) - 1$ black vertices from $V(H)$ to colour H. Therefore there are needed at most $Z(G) - 1$ initial black vertices from $V(G)$ to finish colouring of G. This contradicts the fact that $Z(G)$ is the zero forcing number of G.

To prove the third case let u be a vertex that is forced by v in a zero forcing process started with the vertices of Z_G. Then $Z_H \cup Z_G \cup \{u\}$ is a ZFS for $G \underset{v}{+} H$. Thus

$$Z(G \underset{v}{+} H) \leq Z(G) + Z(H) + 1.$$

Since v is neither in any minimal ZFS of H nor in any minimal ZFS of G, in all zero forcing processes starting with any minimal set of black vertices of $V(Z(G \underset{v}{+} H))$, v must perform a force both in G and in H. Therefore in any zero forcing process, v will have two white neighbours, one in G and one in H. Thus we can not complete the colouring of graph $G \underset{v}{+} H$ using at most $Z(G) + Z(H)$ initial black vertices and this completes the proof. $\qquad\square$

In order to find a connection between the zero forcing number and the path cover number of the vertex-sum of two graphs we try to calculate the path cover number of the vertex-sum of two graphs G and H in terms of those of G and H.

Proposition 5.2.2. *Let G and H be two graphs each with a vertex labeled v. Then the following hold.*

1. *If there is a minimal path covering of G in which v is a path of length zero and*

v in not an end-point in any minimal path covering of H, then

$$P(G \underset{v}{+} H) = P(G) + P(H) - 1.$$

2. If v is an end-point of some minimal path covering of G but no minimal path covering contains v as a path of length 1 and v is not an end-point in any minimal path covering of H, then

$$P(G \underset{v}{+} H) = P(G) + P(H).$$

3. If there are minimal path coverings of H and G in which v is an end-point of a path in both, then

$$P(G \underset{v}{+} H) = P(G) + P(H) - 1.$$

4. If v is neither an end-point of any minimal path covering of H nor the end-point in any minimal path covering of G, then

$$P(G \underset{v}{+} H) = P(G) + P(H) + 1.$$

Proof. To prove the second case, let $\mathcal{P}(G)$ and $\mathcal{P}(H)$ be two minimal path coverings of G and H respectively. Let $P \in \mathcal{P}(G)$ be the path for which v is an end-point and $P' = P - v$. Then $(\mathcal{P}(G) \backslash \{P\}) \cup \{P'\} \cup \mathcal{P}(H)$ is a path covering for $G \underset{v}{+} H$. Thus $P(G \underset{v}{+} H) \leq P(G) + P(H)$. To show the equality, suppose that $P(G \underset{v}{+} H) < P(G) + P(H)$. Hence there is a path covering for $G \underset{v}{+} H$ with size $P(G) + P(H) - 1$.

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Let P_v be the path in a path covering of $G \mathbin{\underset{v}{+}} H$ with size $P(G) + P(H) - 1$ that covers v. Three following cases are possible:

1. P_v is entirely in G.

2. P_v is entirely in H.

3. $P_v = P_1 \mathbin{\underset{v}{+}} P_2$ such that P_1 and P_2 are two paths in G and H respectively.

In the first case the vertices of $G \backslash P_v$ are covered by at least $P(G) - 1$ paths. On the other hand, since the number of paths in this covering is at most $P(G) + P(H) - 1$, the vertices of $H - v$ are covered by at most $P(H) - 1$ paths. Therefore there is a path covering of H of size $P(H)$ in which v is an end-point. In the second case the vertices of $H \backslash P_v$ are covered by at least $P(H) - 1$ paths. Since the number of paths in this covering is at most $P(G) + P(H) - 1$, the vertices of $G - v$ are covered by at most $P(G) - 1$ paths. Thus there is a path covering of G of size $P(G)$ in which there is a path of length 1 consisting of v. Similarly for the third case the vertices of $G \backslash P_v$ are covered by at least $P(G) - 1$ paths. On the other hand since the number of paths in this covering is at most $P(G) + P(H) - 1$, the vertices of $H - P_v$ are covered by at most $P(H) - 1$ paths. Therefore there is a path covering of H of size $P(H)$ in which v is an end-point. We have shown that in all cases we have a contradiction with the conditions in the original statement. Thus $P(G \mathbin{\underset{v}{+}} H) = P(G) + P(H)$.

Using similar arguments, it is not hard to prove the other cases. $\qquad\square$

In Proposition 4.4.5, it was shown that for trees, $P(T) = Z(T)$. Also for any cycle C_n we have $P(C_n) = Z(C_n) = 2$. In Corollary 4.5.4, we see that for any *unicycle* graph G, $P(G) = Z(G)$. On the other hand, any unicycle graph can be obtained by a sequence of vertex-sums of a cycle and a number of trees. Also it was shown that for block-cycle graphs G, $P(G) = Z(G)$. Furthermore, any block-cycle graph can be obtained by vertex-sums of a number of unicycle graphs. These observations along with Theorem 5.2.1, Proposition 5.2.2 and Proposition 4.4.5 lead us to the following conjecture:

Conjecture 5.2.3. *Let G and H be two graphs with $Z(G) = P(G)$ and $Z(H) = P(H)$. Then*

$$Z(G \underset{v}{+} H) = P(G \underset{v}{+} H).$$

Note that if we can show that the conditions in the cases of Theorem 5.2.1 and Proposition 5.2.2 coincide properly then we will prove the conjecture.

5.3   Join of graphs

Next we turn our attention to the zero forcing number of the *join* of two graphs and we give a formula to obtain the zero forcing number of the join of two graphs based on the zero forcing number of each of the initial graphs and their sizes. The join of two graphs is their union with all the edges gh where $g \in V(G)$ and $h \in V(H)$. The join of graphs G and H is denoted by $G \vee H$.

Theorem 5.3.1. *Let G and H be two connected graphs. Then*

$$Z(G \vee H) = \min\{|H| + Z(G), |G| + Z(H)\}.$$

Proof. Without loss of generality, assume that

$$\min\{|H| + Z(G), |G| + Z(H)\} = |H| + Z(G).$$

Let Z be a minimum zero forcing set for G. Clearly $V(H) \cup Z$ is a zero forcing set for $Z(G \vee H)$. Thus

$$Z(G \vee H) \leq \min\{|H| + Z(G), |G| + Z(H)\}.$$

We show that, in fact, it is not possible to colour $G \vee H$ with fewer vertices. Let B be a minimal ZFS for $G \vee H$. Let $u \rightarrow v$ be the first force in the chronological list of forces. If $u, v \in G$, then $V(H) \subset B$ otherwise, u can not perform any force. Also, since all the vertices of H are adjacent to all the vertices of G, a vertex of H can only force a vertex w of G if w is the last white vertex of G and no vertex of G forces w, which contradicts with G being connected. Therefore, in order to colour the vertices of G there are at least $Z(G)$ zero forcing chains in this zero forcing process starting with the vertices of a zero forcing set of G. Thus

$$Z(G \vee H) \geq |H| + Z(G).$$

If $u \in G$ and $v \in H$, then $V(H) \backslash \{v\} \subset B$. With a similar reasoning as above, no vertex of H forces any vertex of G. Therefore, in order to colour the vertices of G there are

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at least $Z(G)$ zero forcing chains in this zero forcing process starting with the vertices

of a zero forcing set of G. In addition, all neighbours of u in G are black. This implies

that $B \cap V(G)$ cannot be a minimal zero forcing set of G. Hence $|B \cap V(G)| > Z(G)$.

Therefore

$$Z(G \vee H) \geq |H| + Z(G).$$

A similar argument applies when $u \in H$. $\qquad \square$

Example. $K_1 \vee C_k$ with $k \geq 3$ is a wheel on $k + 1$ vertices. According to Theo-

rem 5.3.1,

$$Z(K_1 \vee C_k) = \min\{1 + 2, k + 1\} = 3.$$

Chapter 6

Positive Zero Forcing Sets

6.1 Positive zero forcing number

In this chapter we study the positive semidefinite zero forcing number which was introduced and its connection with the maximum positive semidefinite nullity was established in [1]. Also, we will establish some relations between the positive semidefinite zero forcing number and some graph parameters such as vertex connectivity and chromatic number.

The *positive semidefinite colour change rule* is as follows: let G be a graph and B be the set consisting of all the black vertices of G. Let $W_1, \ldots, W_k$ be the set of vertices of the k components of $G \backslash B$ with $k \geq 1$. If $u \in B$ and w is the only white neighbour of u in the graph induced by $V(W_i \cup B)$, then change the colour of w to black. The definitions and terminology for the positive zero forcing process, such as, colouring, derived set, etc., are similar to those for zero forcing number, but the

colour change rule is different.

A *positive zero forcing set* Z_p of G, is a subset of vertices of G such that if initially the vertices in Z_p are coloured black and the remaining vertices are coloured white, then the derived set of G is $V(G)$. The *positive zero forcing number* of G, denoted by $Z_+(G)$, is then defined to be the smallest size of a positive zero forcing set of G. We abbreviate the term positive zero forcing set as PZFS. Note that for a non-empty graph G,

$$1 \le Z_+(G) \le |V(G)| - 1.$$

Example 6.1.1. Let G be the graph in Figure 6.1. Then, Figure 6.2 illustrates why the set $Z_p = \{v_1, v_3, v_{10}\}$ is a PZFS for G. It is clear that this set is not a ZFS for G.
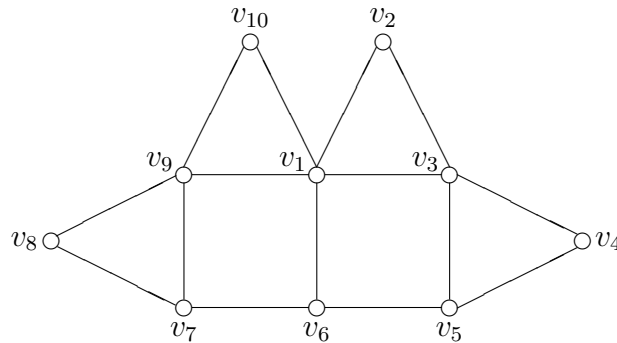
A   graph   for   which   we   want   to   find   a   PZFS

Figure 6.1

We easily deduce the following relationship between $Z_+(G)$ and $Z(G)$.
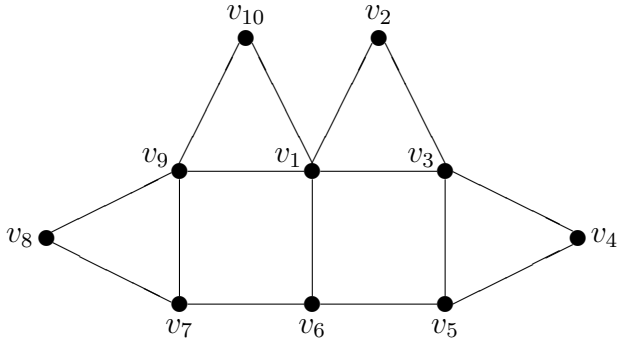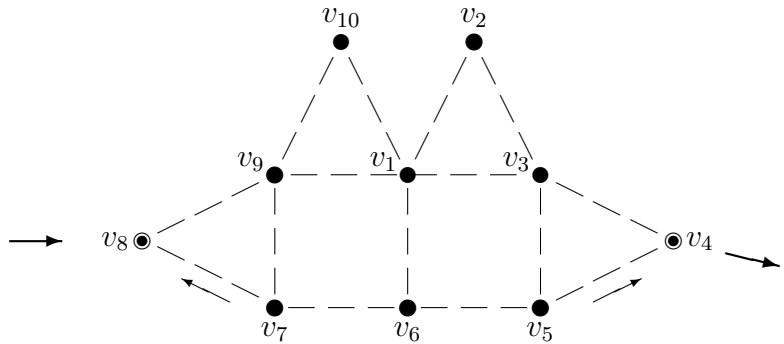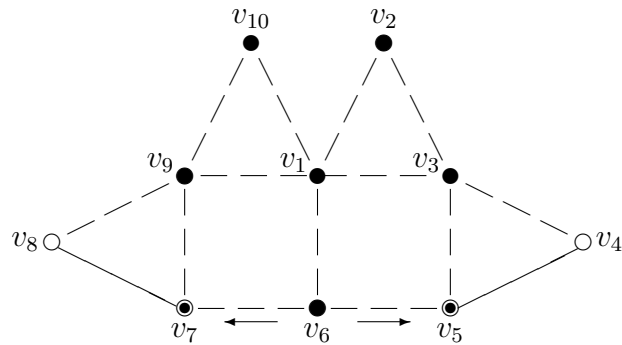
Finding a PZFS

Figure 6.2

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Observation 6.1.2. *Since every zero forcing set is a positive zero forcing set, $Z_+(G) \leq$*

$Z(G)$. □

Next we calculate the positive zero forcing number of trees. This result has been shown in [1].

Theorem 6.1.3. *Let T be a tree. Then $Z_+(T) = 1$ and any vertex $v \in V(T)$ forms a PZSF.*
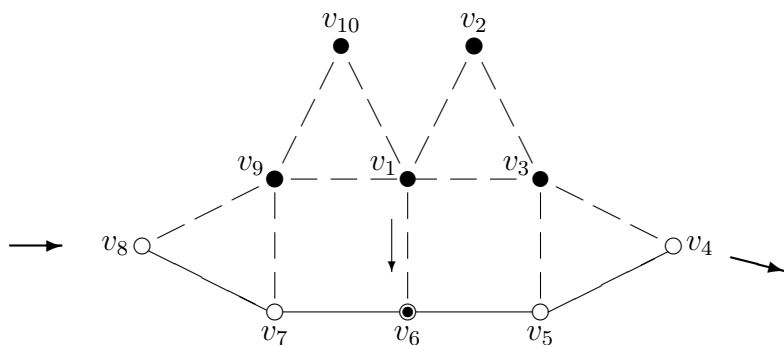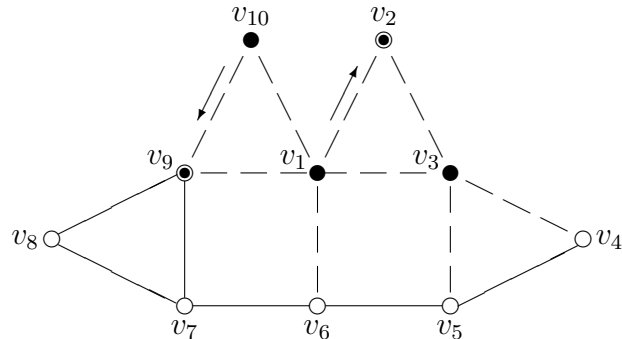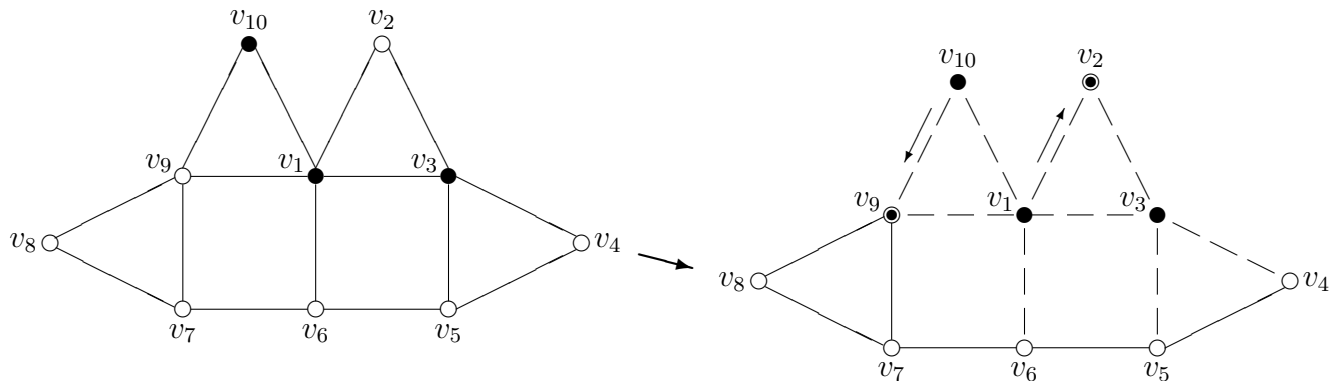
Proof. The proof is by induction on $|V(T)|$. It is clearly true for $|V(T)| = 1$. Assume that the statement is true for any tree with $|V(T)| < n$. Let T be a tree on n vertices and let v be any vertex in T. If v is a leaf, it forces its neighbour; if not, deleting v creates components such that v has only one white neighbour in each component. In either case, smaller tree(s) are obtained for which the induction hypothesis is applied. □

The following theorem shows that the maximum positive semidefinite nullity of a graph is bounded above by the positive zero forcing number of the graph [1, Theorem 3.5].

Theorem 6.1.4. *For any graph G, $M_+(G) \leq Z_+(G)$.*

Proof. Let $A \in \mathcal{H}_+(G)$ with $\mathrm{null}(A) = M_+(G)$. Let $\mathbf{x} = [x_i]$ be a nonzero vector in $\mathrm{null}(A)$. Define B to be the set of indices u such that $x_u = 0$ and let $W_1, \ldots, W_k$ be the vertex sets of the k components of $G \backslash B$. We claim that in the graph induced

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by $V(B \cup W_i)$, any vertex $w \in W_i$ can not be the unique neighbour of any vertex $u \in B$. Once the claim is established, if X is a positive semidefinite zero forcing set for G, then the only vector in null(A) with zeros in positions indexed by X is the zero vector, and by Proposition 4.3.1, $M_+(G) \le Z_+(G)$. To establish the claim, renumber the vertices so that the vertices of B are last, the vertices of W_1 are first, followed by the vertices of W_2, etc. Then A has the block form

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 & C_1^* \\ 0 & A_2 & \dots & 0 & C_2^* \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & A_k & C_k^* \\ C_1 & C_2 & \dots & C_k & D \end{pmatrix}.$$

Partition vector $\mathbf{x}$ as $\mathbf{x} = [\mathbf{x}_1^*, \dots, \mathbf{x}_k^*, 0]^\top$, and note that all entries of $\mathbf{x}_i$ are nonzero with $i = 1, \dots, k$. Then $A\mathbf{x} = 0$ implies $A_i \mathbf{x}_i = 0$ with $i = 1, \dots, k$. Since A is positive semidefinite, each column in C_i^* is in the span of the columns of A_i by the column inclusion property of Hermitian positive semidefinite matrices [26]. That is, for $i = 1, \dots, k$, there exists Y_i such that $C_i^* = A_i Y_i$. Thus $C_i \mathbf{x}_i = Y_i^* A_i \mathbf{x}_i = 0$, and $w \in W_i$ can not be the unique neighbour in W_i of any vertex $u \in B$. Otherwise $\mathbf{x}_i$ would be a length 1 vector so $A_i \mathbf{x}_i = 0$ implies $\mathbf{x}_i = 0$. $\qquad\square$

It has long been known (see [14]) that for any tree T, $M_+(T) = 1$, but the use

of the positive zero forcing number provides an easy proof of this result; indeed by Theorem 6.1.4 $M_+(T) = 1$ is an immediate consequence of $Z_+(T) = 1$.

The following lemma shows how it is possible to replace a member of a PZFS by a new vertex which is not in the PZSF.

Lemma 6.1.5. *Let B be a PZFS for a graph G whose deletion leaves components $C_1, \ldots, C_k$. Then there is at least one vertex in each C_i for $i = 1, \ldots, k$, which is the only white neighbour in C_i of at least one vertex of B. Also, if $v \in C_i$ is the only white neighbour of $u \in B$ in C_i, then $B' = (B \backslash \{u\}) \cup \{v\}$ is a PZFS for G.*

Proof. There is at least one vertex in each C_i with $i = 1, \ldots, k$, that is the only white neighbour in C_i of some vertex of B because if there is no such a vertex in B for a component, the derived set won't include the vertices of the component which contradicts the fact that B is a PZFS for G.

Assume $v \in C_i$ is the only white neighbour of $u \in B$ in C_i. Set $B' = (B \backslash \{u\}) \cup \{v\}$. Then two cases are possible:

(a) u has no neighbour in $G \backslash B$ except v; in this case after deletion of B' we get the component $\{u\}$ as an isolated vertex. Thus v forces u and the rest of the positive zero forcing process will be the same as the positive zero forcing process initiated by the vertices in B after forcing v.

(b) u has some other neighbours in $G \backslash B$ other than v. Clearly these neighbours are

not in C_i. Let C_{n_i} be the components with

$$(N(u)\setminus(B\cup\{v\})) \subseteq \bigcup_{i=1}^{k} C_{n_i}.$$

Thus in the forcing process initiated by the black vertices in B', u is the only white neighbour of v in the component induced by $V(\cup_{i=1}^{k} C_{n_i}) \cup \{u\}$. Hence v forces u and the rest of the positive zero forcing process will be the same as the positive zero forcing process initiated by the vertices in B after forcing v. $\qquad\square$

6.2 Forcing trees

We start this section by defining a concept analogous to zero forcing chains. Let G be a graph and Z_p be a positive zero forcing set of G. Construct the derived set, recording the forces in the order in which they are performed; this is the chronological list of forces. Note that two or more vertices can perform forces at the same time by applying the colour change rule once and a vertex can perform two or more forces at the same time (for example see step 4 in Figure 6.2).

For any chronological list of forces, as above, a *multi-forcing chain* is an induced rooted tree, T_r, formed by a sequence of sets of vertices $(r, X_1, \ldots, X_k)$, where $r \in Z_P$ is the root and the vertices in X_i are at distance i from r in the tree. The vertices of X_i for $i = 1, \ldots, k$, are forced by applying the positive semidefinite colour change rule to the vertices in X_{i-1}; so for any $v \in X_i$ there is a $u \in X_{i-1}$, such that u forces v if and only if v is a neighbour of u. A multi-forcing chain is also called a *forcing tree*.
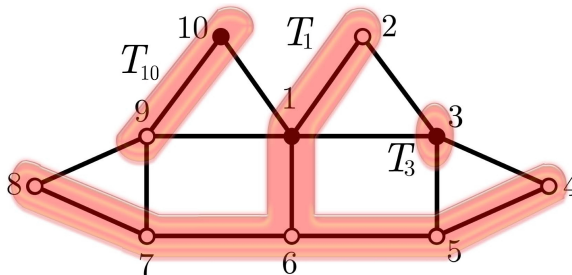
Graph G Figure 6.3

Also, the set of all multi-forcing chains in a positive zero forcing process is called the *set of forcing trees*.

In a forcing tree, the vertices in X_i are said to be the vertices of the i-th level in the tree. Note that the vertices in a specific level may have been forced in different steps of the positive semidefinite colour change procedure and they may also perform forces in different steps.

Example 6.2.1. The positive zero forcing number of graph G in Figure 6.3 is 3 and $\{1, 3, 10\}$ is a PZFS for the graph. The forcing trees in this colouring procedure (as depicted in Figure 6.4) are as follows:

$T_1 = \{1, X_1, X_2, X_3\}$, where $X_1 = \{2, 6\}$, $X_2 = \{5, 7\}$, $X_3 = \{4, 8\}$,

$T_3 = \{3\}$,

$T_{10} = \{10, X_1\}$, where $X_1 = \{9\}$.

We now define the tree cover number which is a very interesting and useful graph
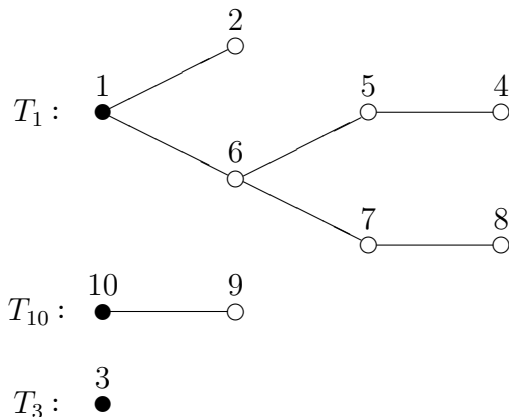
Forcing trees of G Figure 6.4

parameter. Given a graph G, the *tree cover number*, $T(G)$, is the smallest positive integer m such that there are m vertex-disjoint induced trees in G that cover all the vertices of G. In Example 6.3, the trees T_1, T_{10} and T_3 form a tree covering of the graph. Also it can be shown that the tree cover number of the graph in Example 6.3 is three. Note that any set of forcing trees corresponding to a minimal PZFS is of size $Z_+(G)$ and covers all vertices of the graph. This implies the following.

Proposition 6.2.2. *For any graph G, it holds that $T(G) \le Z_+(G)$.* $\square$

Based on Proposition 6.2.2, the tree cover number is a lower bound for the positive zero forcing number. This bound is clearly tight for trees. But there are graphs, such as complete bipartite graphs, for which the discrepancy between these parameters is relatively large. It is, hence, an interesting question to ask for which families of

graphs the equality between these two parameters holds. One way to approach this problem is to find graph operations which preserve the equality in graphs for which these parameters agree.

We have the following fact for the vertex-sum of any graph with any tree.

Theorem 6.2.3. *Let G be any graph and T be a tree both with a vertex labeled v then the following equalities hold:*

(a) $Z_+(G \underset{v}{+} T) = Z_+(G)$;

(b) $\mathrm{T}(G \underset{v}{+} T) = \mathrm{T}(G)$.

Proof. For (a) assume that B is a PZFS for G. After applying the colour change rule with B the set of initial black vertices, all vertices in G including v get forced. According to Theorem 6.1.3, B is therefore a PZFS for T. Thus $Z_+(G \underset{v}{+} T) \leq Z_+(G)$. To show that the reverse inequality also holds, let B' be a PZFS for $Z_+(G \underset{v}{+} T)$. If all members of B' belong to G, then we are done. Otherwise at most one of the vertices of B', say u, is in T. Let T_u be the forcing tree started by u and X_i be the first level of T_u in which the vertices of G appeared. Since v is the only vertex in the intersection set of $V(G)$ and $V(T)$, $X_i \cap V(G) = \{v\}$. Thus

$$Z_+(G \underset{v}{+} T) \geq Z_+(G).$$

The statement (b) is trivial. □

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6.3 Positive zero forcing number and graph operations

In this section we study the effects of the graph operations considered in Section 5.1 on the positive zero forcing number and show that it is not monotone on subgraphs. These results are in fact analogous to what we have already shown in Section 5.1 (for the proofs, see [13, Proposition 5.3, Proposition 5.14, Proposition 5.22, Theorem 5.23]).

First, we consider vertex deletion.

Proposition 6.3.1. *Let G be a graph and v be a vertex in $V(G)$. Then*

$$Z_+(G) + \deg(v) - 1 \geq Z_+(G - v) \geq Z_+(G) - 1. \quad \square$$

Next, we consider edge deletion.

Proposition 6.3.2. *Let G be a graph and e be an edge in G. Then*

$$Z_+(G) - 1 \leq Z_+(G - e) \leq Z_+(G) + 1. \quad \square$$

Edge contraction has the following effect on the positive zero forcing number.

Proposition 6.3.3. *Let $e = uv$ be an edge in a graph G. Then*

$$Z_+(G) - 1 \leq Z_+(G/e) \leq Z_+(G) + 1. \quad \square$$

And finally the following result is about the effect of edge subdivision on the positive zero forcing number.

Proposition 6.3.4. *Let H be the graph obtained from G by subdividing an edge $e \in E(G)$. Then $Z_+(H) = Z_+(G)$ and any positive semidefinite zero forcing set for G is a positive semidefinite zero forcing set for H.* □

Because of the strong relationship between the tree cover number and the positive zero forcing number and Proposition 6.3.4, we turn our attention to the changes of the tree cover number of a graph after subdividing an edge.

Theorem 6.3.5. *Let H be the graph obtained from G by subdividing an edge. Then*

$$T(G) = T(H).$$

Proof. Assume H is obtained by subdividing the edge $e = uv \in E(G)$. Hence uw and wv are the new edges in H. Let $\mathcal{T}_G$ be a minimal tree covering for G. If e is an edge of one of the trees, T, in $\mathcal{T}_G$, then subdividing e in T produces a new tree that covers w. Next consider when v and u are covered by two different trees, say T_1 and T_2 respectively. In this case by extending T_1 to include the edge vw we are able to cover w. Alternatively we could extend T_2 by $\{u, w\}$. Thus $T(H) \leq T(G)$. To establish the equality, suppose that $T(H) < T(G)$. Let $\mathcal{T}_H$ be a minimal tree covering for H. By contracting uw we obtain a tree covering for G with the same size as $\mathcal{T}_H$; this contradicts $T(G)$ being the tree cover number of G and this completes the proof. □

Combining Proposition 6.3.4 and Theorem 6.3.5 we have the following corollary.

Corollary 6.3.6. *Let G be a graph with $Z_+(G) = T(G)$ and H be a graph obtained from G by subdividing some edges of G. Then*

$$Z_+(H) = T(H). \quad \square$$

Corollary 6.3.6 is a useful tool to generate more families of graphs satisfying $Z_+(G) = T(G)$ out of graphs which satisfy this equality.

6.4 Outerplanar graphs

Recall that a graph is a partial 2-tree if it does not have a K_4 minor and that outerplanar graphs are exactly the graphs with no K_4 and $K_{2,3}$ minors. In [4] it is shown that $M_+(G) = T(G)$ for any outerplanar graph. This is extended to $Z_+(G) = M_+(G) = T(G)$ for any partial 2-tree. This implies that $Z_+(G) = M_+(G) = T(G)$ holds for any outerplanar graph. It is easy to see that every outerplanar graph is a partial 2-tree. In this section we give a different proof of the fact that the positive zero forcing number of an outerplanar simple graph agrees with its tree cover number. Moreover we show that any minimal tree covering of an outerplanar graph, G, coincides with a collection of forcing trees that contains exactly $Z_+(G)$ trees.

6.4.1 Double paths and trees

In this section we introduce two families of graphs: double paths and double trees. We establish some properties for the PZFS of these graphs which we use to prove some

important results in the next section. If the vertices of an outerplanar graph with more than one face can be covered with two induced paths, then the graph is called a *double path*. These paths are called the covering paths. Similarly, if the vertices of an outerplanar graph with more than two faces can be covered with two induced trees, then the graph is called a *double tree*. Note that the subgraph H of a double tree G which is induced by the vertices on the boundary of the outer face of G is a double path. Also $G \backslash H$ is a forest $\mathcal{T}_H = \{T_1, T_2, \ldots, T_k\}$. In this case there is a subset $\{v_1, v_2, \ldots, v_k\}$ of $V(H)$, such that

$$G = (\ldots((H \underset{v_1}{+} T_1) \underset{v_2}{+} T_2)\ldots) \underset{v_k}{+} T_k.$$

Since any vertex of a tree is a minimal PZFS of the tree, It is not hard to observe the following.

Observation 6.4.1. *Let G and H be as above and u be any vertex in $V(T_i)\backslash\{v_i\}$ with, $1 \le i \le k$. Assume $u \in B$. Then B is a minimal PZFS of G if and only if $(B\backslash\{u\}) \cup \{v_i\}$ is a minimal PZFS for G.* □

Now we show that for any double path starting with any vertex in one of the covering paths there is always another vertex in the other path such that these two vertices form a PZFS for the graph, and, using Theorem 6.1.3, the positive zero forcing number of a double path is 2.

In the following three lemmas we assume that G is a double path with covering paths P_1 and P_2. Also we focus on a specific planar embedding of G. Once we have a

planar embedding of G we can refer to the end points of a covering path as the right end point and the left end point.

Lemma 6.4.2. *If u and v are both right (left) end points of P_1 and P_2, respectively, then $\{u, v\}$ is a positive zero forcing set of G.*

Proof. We prove this lemma by induction on the number of vertices. This is clearly true for C_3 which is a double path with fewest number of vertices. Assume that it is true for all graphs H with $|V(H)| < n$. Let G be a graph of size n. Assume that u and v are left end points of P_1 and P_2, respectively. By assigning the colour black to each of these vertices we claim that $\{u, v\}$ is a PZFS of G. If u is a pendant vertex, then it forces its only neighbour, say w, which is a left end point of a covering path in $G \backslash u$. Thus by the induction hypothesis $\{w, v\}$ is a PZSF of G. Similarly if v is a pendant vertex, using a similar reasoning, the lemma follows. If neither u nor v are pendant then u and v are adjacent and since both are on the same side, at least one of them, say u, is of degree two. Let w be the only neighbour of u in P_1. Thus u can force w (its only white neighbour) and again by the induction hypothesis $\{w, v\}$ is a PZSF of G. The same reasoning applies when u and v are the right end points of P_1 and P_2, respectively. □

Lemma 6.4.3. *If u and v are two vertices of P_1 and P_2, respectively, which form a cut set for G, then $\{u, v\}$ is a positive zero forcing set for G.*

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Proof. Let $W_1 \subseteq V(G)$ and $W_2 \subseteq V(G)$ be the vertices of the left hand side and the right hand side components of $G\backslash\{u,v\}$ respectively and let G_1 and G_2 be the subgraphs induced by $\{u,v\} \cup W_1$ and $\{u,v\} \cup W_2$ respectively. Then according to Lemma 6.4.2, $\{u,v\}$ is a PZFS for both G_1 and G_2 and this completes the proof. $\square$

Lemma 6.4.4. *If u is a vertex in P_1 which is not an end point, then there always is a vertex v in P_2 such that $\{u,v\}$ is a cut set of G.*

Proof. Suppose there is a vertex u in P_1 for which there is no vertex v in P_2 such that $\{u,v\}$ is a cut set for G. Since G is an outerplanar graph, it has an embedding in the plane such that all vertices are on the same face. Let v be any non pendant vertex of P_2. Obviously u is a cut vertex of P_1 and v is a cut vertex of P_2. Since $\{u,v\}$ is not a cut set of G, there is a vertex in the left hand side (or right hand side) of u that is adjacent to a vertex in the right hand side (or left hand side) of v. Assume that l is the farthest vertex from v in P_2 having this property. Since l is the farthest vertex from v with the described property and G is an outerplanar graph, $\{u,l\}$ is a cut set of G which contradicts the fact that, there is no vertex in P_2 that forms a cut set along with u for G. $\square$

Combining Lemmas 6.4.2, 6.4.3 and 6.4.4 along with the fact that in the proof of all these three lemmas forces are performed along covering paths, we have the following.

Corollary 6.4.5. *Let G be a double path with covering paths P_1 and P_2. Then for any vertex v in P_1, there is always another vertex u in P_2 such that $\{u, v\}$ is a positive zero forcing set for G. Moreover the two paths P_1 and P_2 are a minimal set of forcing trees in G.* $\qquad\square$

The following corollary is a consequence of Observation 6.4.1 and Corollary 6.4.5.

Corollary 6.4.6. *Let G be a double tree with covering trees T_1 and T_2. Then for any vertex v in T_1, there is always another vertex u in T_2 such that $\{u, v\}$ is a positive zero forcing set for G. Moreover $\{T_1, T_2\}$ coincides with a minimal collection of forcing trees in G.* $\qquad\square$

6.4.2 Pendant trees

In this section we show that for any outerplanar graph there is a minimal tree covering in which there is a *pendant tree.*

In a tree covering of a graph, two trees, T_1 and T_2, are said to be *adjacent* if there is at least one edge $uv \in E(G)$ such that $v \in V(T_1)$ and $u \in V(T_2)$. In Figure 6.3, T_1 is adjacent to T_{10} and T_3 but T_{10} and T_3 are not adjacent. A tree is called *pendant* if it is adjacent to only one other tree. In Example 6.3, T_{10} and T_3 are two pendant trees.

Let G be an outerplanar graph with a planar embedding in which all the vertices are on the same face. Let $\mathcal{T}(G)$ be a minimal tree covering for G. Define $H_{\mathcal{T}}$ to be

the graph whose vertices correspond to the elements of $\mathcal{T}(G)$ in which two vertices are adjacent if there is an outer edge between their corresponding trees in the graph G. Two trees of $\mathcal{T}(G)$ are called *consecutive*, if their corresponding vertices in $H_{\mathcal{T}}$ are adjacent vertices each of degree two.

Theorem 6.4.7. *Let G be an outerplanar graph and $\mathcal{T}(G)$ a minimum tree covering for G. If there is no pendant tree in $\mathcal{T}(G)$, then there are at least two consecutive trees in $\mathcal{T}(G)$.*

Proof. Assume that $\mathcal{T}(G)$ is a minimum tree covering for G in which there is no pendant tree. Then, two cases are possible:

Case 1. There is no tree in $\mathcal{T}(G)$ with at least one of the inner edges of G in its edge set (see Figure 6.5 for an example of such a graph). Therefore $H_{\mathcal{T}}$ is a cycle. Accordingly, any adjacent pair of trees in $\mathcal{T}(G)$ are consecutive.
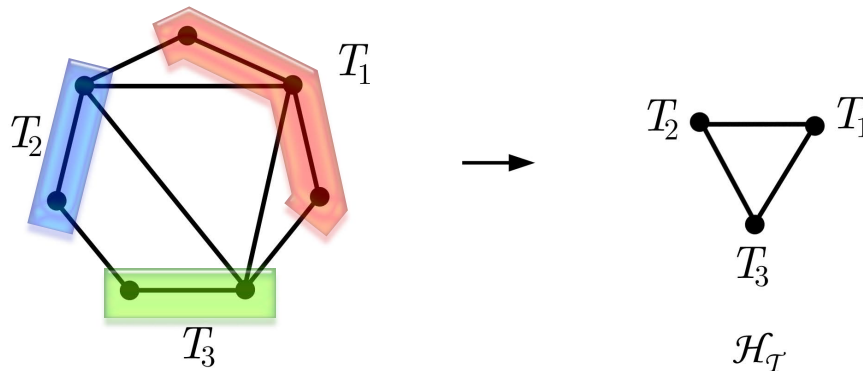
Forcing trees with no inner edge in their edge set
Figure 6.5
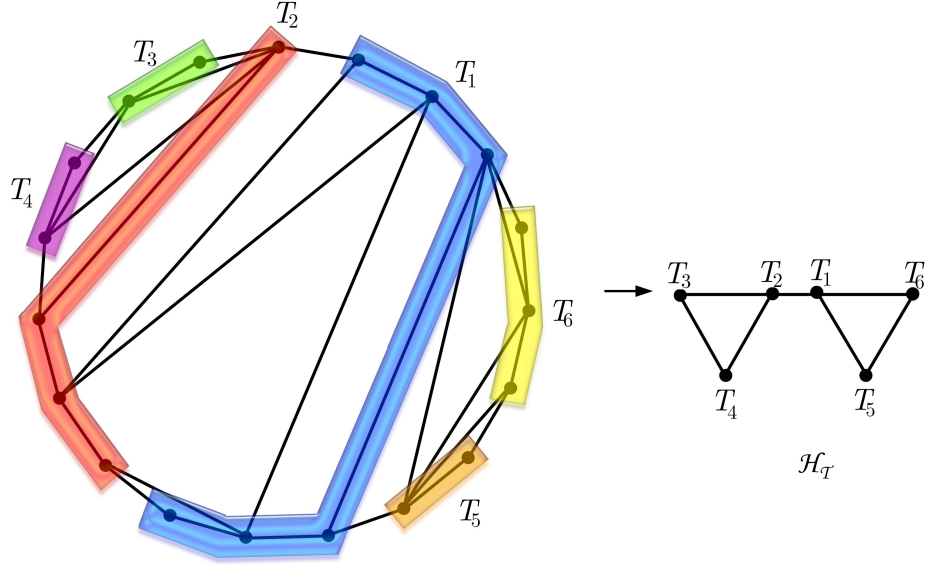
T_1 is a forcing tree with an inner edge in its edge set

Figure 6.6

Case 2. There is at least one tree in $\mathcal{T}(G)$ having at least one of the inner edges of G in its edge set. Any inner edge $e = uv \in E(G)$ partitions the plane into two parts and, consequently, the set $V(G)\backslash\{u,v\}$ into two subsets V_u and V_v. For any inner edge uv, let V_f be the subset of all vertices in $V(G)\backslash\{u,v\}$ which are not end-points of an inner edge that is included in a tree of $\mathcal{T}(G)$. Since G is outerplanar and finite, there exists an edge, $e = uv$, such that at least one of V_u or V_v is in V_f. Therefore there are at least two consecutive trees in $\mathcal{T}(G)$ covering the vertices of V_f. In Figure 6.6 the pairs T_3, T_4 and T_5, T_6 are examples of consecutive pairs of trees in $\mathcal{T}(G)$. $\qquad\square$

The following theorem plays a key role in the proof of the fact that outerplanar

graphs satisfy $Z_+(G) = T(G)$.

Theorem 6.4.8. *Let G be an outerplanar graph. Then there is a minimum tree covering for G in which there is a pendant tree.*

Proof. Since G is an outerplanar graph, it has an embedding in the plane such that all of its vertices are on the same face. Assume that $\mathcal{T}(G)$ is a minimum tree covering for G in which there is no pendant tree. We use Theorem 6.4.7 to construct a new tree covering $\mathcal{T}'(G)$ of $\mathcal{T}(G)$ with $|\mathcal{T}'(G)| = |\mathcal{T}(G)|$ in which there is a pendant tree.

Consider two trees T_1 and T_2 in $\mathcal{T}(G)$, which are consecutive and assume that T_1 is in the right hand side of T_2. Let H be the graph induced by $V(T_1) \cup V(T_2)$ (Figure 6.7). There are two outer edges in H that have an end-point from each of trees T_1 and T_2. One of these outer edges, call it $e = uv$, is an inner edge in G with $u \in T_1$ and $v \in T_2$. If v has any other neighbour in T_1, then u has no other neighbour in T_2; otherwise there will be either two crossing edges in the planar embedding of G or an edge beyond e; either case is a contradiction. Thus u along with all vertices on its right hand side in T_1 and all the vertices in T_2 induce a new tree T_i and the vertices in the left hand side of u in T_1 induce another new tree T_k which is a subtree of T_1. Now we have a new tree covering $\mathcal{T}'(G)$ for G with $|\mathcal{T}'(G)| = |\mathcal{T}(G)|$ in which T_k is a pendant tree.

A similar argument applies when u has another neighbour in T_2. If neither u nor v has any other neighbour in T_1 and T_2, respectively, then either cases mentioned

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above are applicable. $\qquad\qquad$ □

6.4.3   Outerplanar graphs satisfy $Z_+(G) = T(G)$

We now turn our attention to the positive zero forcing number of outerplanar graphs and show that this parameter for any graph among this family of graphs is equal to the tree cover number of the graph, moreover any minimal tree covering of such a graph coincides with a minimal collection of forcing trees.

Theorem 6.4.9. *Let G be an outerplanar graph. Then*

$$Z_+(G) = \mathrm{T}(G).$$

Also any minimal tree covering of the graph $\mathcal{T}(G)$ coincides with a collection of forcing trees with $|\mathcal{T}(G)| = Z_+(G)$.

Proof. We prove the claim by induction on the tree cover number and using Theorem 6.4.8. It is clearly true for $\mathrm{T}(G) = 1$. Assume that it is true for any outerplanar graph G' with $\mathrm{T}(G') < k$. Now let G be an outerplanar graph with $\mathrm{T}(G) = k$. By Proposition 6.2.2, we have $Z_+(G) \geq \mathrm{T}(G)$. According to Theorem 6.4.8 there is a minimum tree covering of G, $\mathcal{T}(G)$, containing $T_1, T_2, \ldots, T_k$ in which there is a pendant tree, say T_k. Let T_{k-1} be the only neighbour of T_k. Let $G' = G \backslash V(T_k)$ then the induction hypothesis holds, so $T(G') = Z_+(G') = k - 1$. Further, $T_1, T_2, \ldots, T_{k-1}$ are forcing trees in a positive zero forcing process whose initial set of black vertices, Z'_p,

has a vertex from each tree in $\mathcal{T}(G)\backslash T_k$. A zero forcing process in G starting with the black vertices in Z_p' can proceed as it does in G' until the first vertex of T_{k-1}, say x, that is adjacent to some vertex in T_k gets forced. Since the graph induced by $V(T_{k-1}) \cup V(T_k)$ is a double tree, according to Corollary 6.4.6, the vertex x determines a vertex y in T_k such that $\{x, y\}$ is a PZFS for the subgraph, W, induced by $V(T_{k-1}) \cup V(T_k)$. Since the induction hypothesis holds for W, T_{k-1} is a forcing tree in W too. Thus the vertices of T_{k-1} get forced in the same order as they get forced in G'. Therefore we can complete colouring the graph G by adding the black vertex y to the initial set of black vertices. Thus, $Z_p = Z_p' \cup \{y\}$ is a PZFS of G with $T_1, T_2 \ldots, T_k$ as the forcing trees in this positive zero forcing process. Thus, $Z_+(G) = \mathrm{T}(G)$.

Now we need to show that the original minimal tree covering of the graph G also coincides with a collection of forcing trees. Note that in the procedure of constructing a pendant tree in the minimal tree covering of G, we modified two consecutive trees T_1 and T_2 to obtain a pendant tree T_k in a new minimal forcing tree covering. Assume that $v \in T_2$ has a neighbour, other than u, in T_1 as is shown in Figure 6.7.
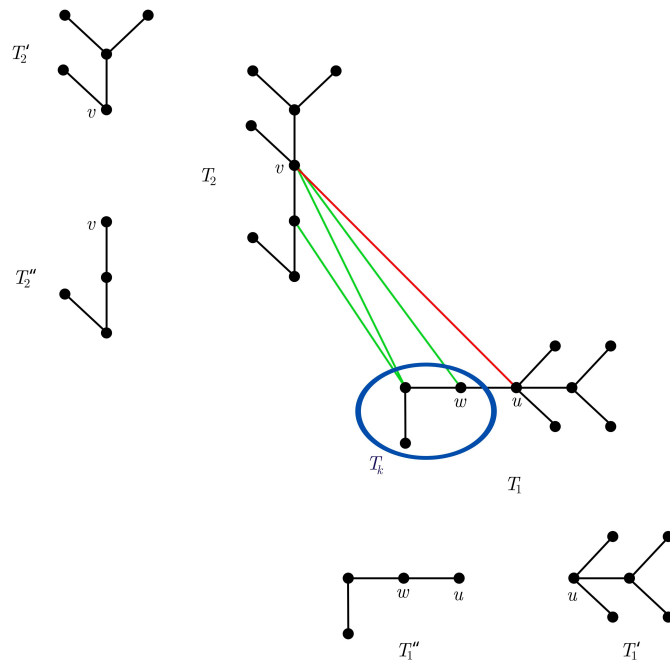
Construction of a pendant tree in an outerplanar graph

Figure 6.7

Let $T_1 = T_1' \underset{u}{+} T_1''$, where T_1'' involves all the vertices of T_1 that have some neighbours in T_2. Similarly, let $T_2 = T_2' \underset{v}{+} T_2''$, where T_2'' involves all the vertices of T_2 that have some neighbours in T_1. Then the tree adjacent to T_k is the tree

$$T_{k-1} = T_2 \underset{vu}{+} T_1'.$$

The tree T_k is obtained by removing u from T_1''. In other words $T_1 = T_k \underset{wu}{+} T_1'$ with $w \in V(T_k)$.

Now we show that, one can change the direction of forces in this positive zero

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forcing process, so that instead of forcing along the trees T_k and T_{k-1}, the forces are performed along the original trees T_1 and T_2; consequently, we show that the original tree covering of the graph is in fact a set of forcing trees for the graph.

Consider the positive zero forcing process starting with the vertices of Z_p'. Let x be the first black vertex of T_{k-1} that is adjacent to a vertex in T_k.

If $x \in T_2$, then according to Corollary 6.4.6, the vertex x determines a (not necessarily unique) vertex y in T_k such that $Z_p' \cup \{y\}$ is a PZFS for G. In this case the level of vertex v is lower than the level of vertex u in the forcing tree T_{k-1}. Thus v forces u. The set $\{x, y\}$ is a PZFS for the graph $H = G[V(T_2'') \cup V(T_1'')]$ which is a double tree. Hence by the induction hypothesis $\{T_1'', T_2''\}$ is a set of forcing trees for H. Thus the colouring of vertex u can be performed by vertex w instead of vertex v. From this we have that $T_1' \underset{u}{+} T_1'' = T_1$ with the root y and T_2 with the root x are two forcing trees in a positive zero forcing process of G.

Next, assume that x belongs to T_1'. In this case according to Lemma 6.4.2 and Observation 6.4.1, the set $Z_p' \cup \{w\}$ is a PZFS for G. Also in this case the level of vertex u is lower than the level of vertex v in the forcing tree T_{k-1}. Thus u forces v. But since u has only two white neighbours in the component $G[V(T_k) \cup V(T_2'')]$ which are w and v, we can replace w by v in the PZFS of G. Then u forces w and the rest of the forces will be performed along T_1 and T_2. Thus T_1 with the root x and T_2 with the root v are two forcing trees in a positive zero forcing process of G. $\qquad\square$

The following is a consequence of combining Theorem 6.4.9 and Corollary 6.3.6.

Corollary 6.4.10. *Any subdivision of an outerplanar graph (which may no longer be outerplanar) satisfies*

$$Z_+(G) = T(G). \quad \square$$

It seems that the scope of graphs satisfying $Z_+(G) = T(G)$ goes well beyond outerplanar graphs and even 2-trees. The graph in Figure 6.8 is an example of a graph that is not a partial 2-tree but still satisfies $Z_+(G) = T(G)$. This graph includes a K_4 as a subgraph therefore $Z_+(G) \geq 3$. But the set $\{1, 2, 3\}$ is a PZFS for G. Thus $Z_+(G) = 3$. It is not hard to verify that the tree cover number of this graph is no less than 3 and since the positive zero forcing number is an upper bound for the tree cover number we have $T(G) = 3$ as well.
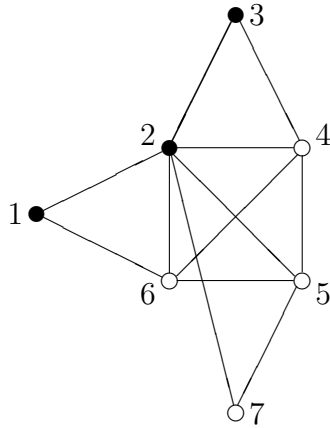
A non-outerplanar graph with $Z_+(G) = T(G)$

Figure 6.8

6.5 Positive zero forcing and the tree cover number

Since the positive zero forcing number and tree cover number are very nicely related it seems reasonable to study other families of graphs satisfying $Z_+(G) = T(G)$. In this section we try to find other graph operations that preserve the equivalency of these parameters for the graphs in which these parameters already agree. Moreover we will compare these parameters for k-trees.

6.5.1 Vertex-sum of two graphs

Theorem 6.5.1. *For any graphs G and H with a vertex labeled v*

$$T(G \underset{v}{+} H) = T(G) + T(H) - 1,$$

and

$$Z_+(G \underset{v}{+} H) = Z_+(G) + Z_+(H) - 1.$$

Proof. In order to prove the first equality let $\mathcal{T}_G$ and $\mathcal{T}_H$ be the minimal tree coverings of G and H, respectively, and $T_1 \in \mathcal{T}_G$ and $T_2 \in \mathcal{T}_H$ be the trees covering v. Let $T_v = T_1 \underset{v}{+} T_2$. Observe that T_v is an induced tree in $G \underset{v}{+} H$ that covers v. Therefore, $((\mathcal{T}_G \cup \mathcal{T}_H) \setminus \{T_1, T_2\}) \cup T_v$ is a tree covering of $G \underset{v}{+} H$.

Next we show, in fact, the vertices of $G \underset{v}{+} H$ can not be covered with fewer trees. Suppose that the vertices of $G \underset{v}{+} H$ can be covered with $T(G) + T(H) - 2$ induced disjoint trees. Let T be the tree that covers v in such a tree covering of $G \underset{v}{+} H$. At least $T(G) - 1$ trees are needed to cover the vertices of $G \setminus T$. Since the number of trees in the covering of $G \underset{v}{+} H$ is $T(G) + T(H) - 2$, the vertices of $H \setminus T$ are covered by at most $(T(H) - 2)$ trees, but this is a contradiction with the fact that $T(H)$ is the least number of trees that cover the vertices of H. Thus

$$T(G \underset{v}{+} H) = T(G) + T(H) - 1.$$

To prove the second equality, let $\mathcal{T}_G$ and $\mathcal{T}_H$ be the sets of forcing trees for a minimal PZFS in G and H respectively. Let $T_1 \in \mathcal{T}_G$ and $T_2 \in \mathcal{T}_H$ be the trees that contain

v. Then $T_1 \underset{v}{+} T_2$ is a forcing tree in $G \underset{v}{+} H$ covering v. Then a similar reasoning as above applies. $\qquad\square$

Corollary 6.5.2. *For any graphs G and H with a vertex labeled v if $Z_+(G) = T(G)$ and $Z_+(H) = T(H)$, then*

$$Z_+(G \underset{v}{+} H) = T(G \underset{v}{+} H). \quad \square$$

In [7], it has been shown that

$$M_+(G \underset{v}{+} H) = M_+(G) + M_+(H) - 1.$$

Therefore we have the following as another consequence of Theorem 6.5.1.

Corollary 6.5.3. *Let G and H be two graphs both with a vertex labeled v. If $Z_+(G) = M_+(G)$, then*

$$Z_+(G \underset{v}{+} H) = M_+(G \underset{v}{+} H). \quad \square$$

6.5.2 Coalescence of graphs

In Section 6.5.1, the positive zero forcing number and the tree cover number of the vertex-sum of two graphs were studied. This provides the motivation for verifying the variations of these parameters in a more general way of merging two graphs.

Let H be a graph that is a subgraph of both the graphs G_1 and G_2, then the *coalescence* of G_1 and G_2 on H, denoted by $G_1 \underset{H}{+} G_2$, is obtained by the process of

merging these graphs through the subgraph H. The coalescence of two graphs G_1 and G_2 on H is obtained by identifying the subgraph H in these two graphs. As an example see Figure 6.9.
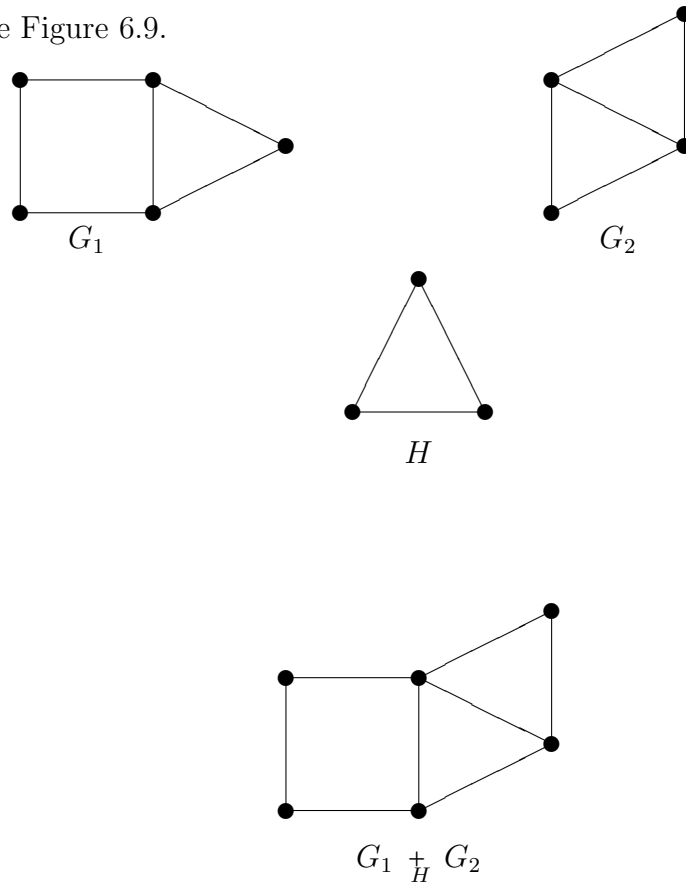
Coalescence of two graphs   Figure 6.9

The following theorem gives a lower bound for the tree cover number of the coalescence of a graph G and itself on an induced subgraph H.

Theorem 6.5.4. *Let H be a subgraph of G. Then*

$$T(G \underset{H}{+} G) \geq 2T(G) - k,$$

where k is the maximum number of trees that cover the vertices of H in any minimal tree covering of G.

Proof. Suppose that $T(G \underset{H}{+} G) < 2T(G) - k$. Thus we can cover the vertices of $G \underset{H}{+} G$ with $|\mathcal{T}| = 2T(G) - k - 1$ trees. In this covering assume that the vertices of H have been covered by m $(m \le k)$ trees. To cover the vertices of $G \backslash H$ at least $T(G) - m$ trees are needed, where $m \le k$. On the other hand since the entire graph has been covered by $2T(G) - k - 1$ trees, $G \backslash H$ must be covered by at most $T(G) - k - 1$ trees, which contradicts the definition of k and the fact that $T(G)$ is the least number of trees that cover the vertices of G. $\square$

Recall that the positive zero forcing number of a graph is in fact the number of forcing trees in a minimal positive zero forcing process of G. Using similar reasoning as in Theorem 6.5.4 we obtain the following.

Theorem 6.5.5. *Let H be a subgraph of G. Then*

$$Z_+(G \underset{H}{+} G) \ge 2Z_+(G) - k,$$

where k is the maximum number of forcing trees that cover the vertices of H in any minimal positive zero forcing process of G. $\square$

The following is easy to prove.

Lemma 6.5.6. *Let K_n be a subgraph of G. In any positive zero forcing process of G at most two of the vertices of K_n are in the same forcing tree.*

Corollary 6.5.7. *If G is a graph that contains a complete graph on n vertices as a subgraph, then*

$$T(G \underset{K_n}{+} G) = 2T(G) - k,$$

where k is the maximum number of trees that cover the vertices of K_n in any minimal tree covering of G.

Proof. To prove this, we show that, in fact, there is a tree covering of size $2T(G) - k$ for $G \underset{K_n}{+} G$. Let $\mathcal{T}$ be a minimal tree covering for G with $|\mathcal{T}| = T(G)$ in which k trees cover all the vertices of K_n. Clearly, a vertex v of K_n is either covered in a tree in $T \in \mathcal{T}$ that covers no other vertices of K_n , or only one other vertex of K_n say u is covered in the same tree too. In the first case $T \underset{v}{+} T$ is a tree and in the second case $T \underset{vu}{+} T$ is a tree formed by coalescencening the covering trees of $\mathcal{T}$. Thus there is a tree covering of $G \underset{K_n}{+} G$ of size $2T(G) - k$.

$\square$

If we replace trees in a minimal tree covering of the graph by forcing trees in a minimal positive zero forcing process of the graph in Corollary 6.5.7 and combine this with Lemma 6.5.6, we obtain the following corollary.

Corollary 6.5.8. *If G is a graph that has K_n as a subgraph, then*

$$Z_+(G \underset{K_n}{+} G) = 2Z_+(G) - k,$$

where k is the maximum number of forcing trees that cover the vertices of K_n in any minimal positive zero forcing process of G. □

The following is a consequence of combining Corollaries 6.5.7 and 6.5.8.

Theorem 6.5.9. *If the graph G satisfies $Z_+(G) = T(G)$ and has K_n as a subgraph and any minimal family of forcing trees coincides with a collection of forcing trees, then*

$$Z_+(G \underset{K_n}{+} G) = T(G \underset{K_n}{+} G). \quad \square$$

6.5.3 k-trees

Being motivated by the fact that all partial 2-trees (including 2-trees) satisfy $Z_+(G) = T(G)$, in this section we try to track the variations between the positive zero forcing number and the tree cover number in k-trees with $k > 2$. Our results in this section demonstrate that 2-trees are rather special when it comes to comparing $Z_+(G)$ and $T(G)$. In particular, we show that these two parameters need not agree for "k-trees" with $k > 2$.

A *k-tree* is constructed inductively by starting with K_{k+1} and at each step a new vertex is added to the graph and this vertex is adjacent to the vertices of an existing K_k. Based on this definition we can deduce that the minimum degree of a k-tree is k (the degree of the last vertex added). We will consider a special subfamily of k-trees that we call a *cluster*.

Let G be a k-tree that is constructed in the following way: start with a $H = K_{k+1}$. Add new vertices that are adjacent to a sub-clique K_k of H. Let S be the set of all distinct subcliques K_k of H such that $K_k \cup v$ is a clique of size $k + 1$ in G for some $v \in V(G)\backslash V(H)$. Therefore for each vertex not in H there is exactly one vertex in H that is not adjacent to it. We call this type of k-tree a *cluster*. Obviously the vertices of H form a PZFS for a cluster.

Theorem 6.5.10. *Let G be a k-tree that is also a cluster.*

1. *If $|S| \geq 3$, then $Z_+(G) = k + 1$.*

2. *If $|S| < 3$, then $Z_+(G) = k$.*

3. *If $|S| = k + 1$ and k is even, then $T(G) = \lceil \frac{k+1}{2} \rceil + 1$.*

4. *If $|S| < k + 1$ and k is even, then $T(G) = \lceil \frac{k+1}{2} \rceil$.*

Proof. Since the minimum degree of G is k by Corollary 6.6.2, $Z_+(G) \geq k$. Suppose H is the first K_{k+1}.

To prove the first statement suppose B is a PZFS of G with $|B| = k$. First assume that $B \subset V(H)$. Let u and v be a black and a white vertex of H, respectively. According to the definition of the graph G, along with the fact that $|S| \geq 3$, v and u have at least one common neighbour, say w, in $V(G)\backslash V(H)$. Thus u has two white neighbours v and w in the component consisting of v and w. This is true for all vertices of B which contradicts the fact that B is a PZFS for G. Second assume $B \nsubseteq V(H)$.

Then there is a vertex $z \in B$ such that $z \notin H$. If z has only one white neighbour, then z can force it and we return to the first case. If z has more white neighbours, then no vertex of B can perform any force and again we reach a contradiction. Obviously, the set $V(H)$ is a PZFS for G. Thus $Z_+(G) = k + 1$.

To verify the second statement observe that if $|S| < 3$, then there is at least one vertex in $V(H)$, call it v, that has at most one neighbour in $V(G) \backslash V(H)$, call it u. Thus u along with all its neighbours except v forms a PZFS for G. If v has no neighbour in $V(G) \backslash V(H)$, then any subset of size k of the vertices of H including v is a PZFS for G.

For the third statement assume k is even and $|S| = k + 1$. Since K_{k+1} is a subgraph of G, the tree cover number of G is no less than $\lceil \frac{k+1}{2} \rceil$. Suppose that $\mathcal{T}$ is a minimal tree covering for G with $|\mathcal{T}| = \lceil \frac{k+1}{2} \rceil$. No tree in $\mathcal{T}$ can contain more than two vertices of H. Also since $T(G) = \lceil \frac{k+1}{2} \rceil$ there is no more than one tree in $\mathcal{T}$ that covers only one vertex of H. Each tree in $\mathcal{T}$ except one, say T_1, contains exactly two vertices of H and T_1 contains only a single vertex of H. Since $|S| = k + 1$, for any vertex, $w \in V(H)$, there is a corresponding vertex in $V(G) \backslash V(H)$ which is adjacent to all of the vertices of H except w. In particular, if v is the single vertex of H in the tree T_1, then there is a vertex u which is adjacent to all vertices in H except v. Since $N_G(u) = V(H) \backslash \{v\}$, the vertex u can not be covered by extending any of the trees of $\mathcal{T}$ and as u is adjacent to both vertices in any tree from $\mathcal{T}$. This contradicts $\mathcal{T}$ being

a minimal tree covering for G. Thus

$$T(G) \geq \left\lceil \frac{k+1}{2} \right\rceil + 1.$$

Finally we will show that we can construct a tree covering of this size. Let $\mathcal{T}'$ be a minimal tree covering for H. Thus $|\mathcal{T}'| = \lceil \frac{k+1}{2} \rceil$. By considering the construction of G for any $u \in V(G) \backslash V(H)$ except one, call it v, there is a tree T in $\mathcal{T}'$ that is an edge, say xy, such that $xu \in E(G)$ and $yu \notin E(G)$. Hence we can extend T to cover u. Thus $\mathcal{T}' \cup \{v\}$ is a tree covering of size $\lceil \frac{k+1}{2} \rceil + 1$ of G. Therefore $T(G) = \lceil \frac{k+1}{2} \rceil + 1$.

Finally for the last statement assume $|S| < k + 1$ and k is even. Thus there is a vertex, v, in H that is adjacent to all of the vertices in the graph G. If we consider a tree covering $\mathcal{T}'$ for K_{k+1} in which $T = \{v\}$ is a covering tree. we can extend T to cover all the vertices of $V(G) \backslash V(H)$. Thus $T(G) = |\mathcal{T}'| = \lceil \frac{k+1}{2} \rceil$, which proves the last statement. $\qquad \square$

Note that if in Theorem 6.5.10 we have $k = 2$, then the positive zero forcing number and the tree cover number coincide.

Theorem 6.5.11. *Let G be a k-tree for some odd k. Then $T(G) = \frac{k+1}{2}$.*

Proof. The proof is by induction on the number of vertices. It is clearly true for K_4 (the smallest 3-tree). Assume that the statement is true for all G' with $|V(G')| < n$. Let G be a k-tree with $|V(G)| = n$. Let $v \in V(G)$ be a vertex of degree k. Thus by the induction hypothesis $T(G \backslash v) = \frac{k+1}{2}$. Let $\mathcal{T}$ be a tree covering of $G \backslash v$ with

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$|\mathcal{T}| = T(G\backslash v)$. By the definition of a k-tree, the neighbours of v form a clique on k vertices. The neighbours of v form a clique, so any tree in $\mathcal{T}$ covers at most two of the neighbours of v. Since v has an odd number of neighbours, there is at least (in fact exactly) one tree, say T, in $\mathcal{T}$ that covers only one of the neighbours of v. Therefore we can extend T to cover v. Thus $T(G) = \frac{k+1}{2}$. $\qquad\square$

The equality between the positive zero forcing number and the tree cover number of 2-trees along with our approach in the proof of Theorem 6.5.10 and Theorem 6.5.11 leads us to the following conjecture.

Conjecture 6.5.12. *If a k-tree G, has t clusters as subgraphs, each with $|S| \geq 3$, such that each pair of them share at most $k-1$ cliques of size $k+1$ then $Z_+(G) = k+t$.*

6.6 The positive zero forcing number and other graph parameters

This section is devoted to comparing the positive zero forcing number and other combinatorial graph parameters such as the chromatic number and the independence number.

6.6.1 The positive zero forcing number and the chromatic number

Again it seems logical to ask if there is any relationship between the positive zero forcing number of a graph and the chromatic number of the graph since both parameters arise from a type of graph colouring. This, along with the last question of Section 4.7 motivates us to prove an inequality between the positive zero forcing number and the chromatic number of a graph. According to Conjecture 4.7.4 and inequalities (6.1.4) and (4.8), a weaker comparison can be stated as follows:

The following inequality holds for any graph G,

$$\chi(G) \leq Z_+(G) + 1.$$

This section is devoted to verifying the above inequality for any graph G.

Theorem 6.6.1. *Let G be a graph. Then we have*

$$Z_+(G) \geq \max\{\delta(H') \mid H' \text{ is an induced subgraph of } G\}.$$

Proof. Let H be an induced subgraph of G such that

$$\delta(H) = \max\{\delta(H') \mid H' \text{ is an induced subgraph of } G\}.$$

Also assume Z_p is a PZFS of G of size $Z_+(G)$. Having Z_p as a PZFS results in a set of covering forcing trees $\mathcal{T}$, with $|\mathcal{T}| = Z_+(G)$, the roots of which are the distinct members of Z_p.

We construct a new graph G' which is spanned by H and for which $Z_+(G') \leq Z_+(G)$. Define G' as follows. Fix a forcing tree T in $\mathcal{T}$; assume that l is the first level of T having some vertices of H and let $S = \{v_1, \ldots, v_k\}$ be the set of vertices of H appearing in level l of T. Set X_T'' to be the subgraph of G induced by $V(H) \cap V(T)$. Add an edge between $u, v \in V(X_T'')$, if u is in a level in T lower than the level of v in T and the (unique) path in T between u and v, neither contains a vertex of H nor a vertex in a level lower than the level of u, call this new graph X_T'. Finally define X_T to be the graph obtained from X_T' by adding the edges connecting v_1 to v_i, for $i = 2, \ldots, k$. Then set G' to be the subgraph of G induced by

$$\bigcup_{T \in \mathcal{T}} X_T.$$

Note that X_T will be a rooted tree with the root v_1.

We claim that $\mathcal{T}' = \{X_T \mid T \in \mathcal{T}\}$ is a set of forcing trees in G'. It is enough to show that each force in X_T is allowed in G', for any $X_T \in \mathcal{T}'$. First, note that v_1 can force all the vertices in $S \backslash v_1$ since after removing the vertices of $G \backslash H$ from T, the vertices of S will be in different components. Also $u \in V(G')$ can force $v \in V(G')$ in G' if they are adjacent in X_T and u lies in a higher level than v. To see this, let $u, u_1, \ldots, u_s, v$ be the only path connecting u to v in T. Since u forces u_1 in T, u_1 must be the only white neighbour of u in its component. Therefore by removing the vertices $u_1, \ldots, u_s$ and adding the edge uv to the graph, v will be the only white neighbour of u in its

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component. Thus u forces v in X_T. We, therefore, have

$$|\mathcal{T}'| \geq Z_+(G');$$

and since H spans G', we have $\delta(G') \geq \delta(H)$. Thus

$$Z_+(G) = |\mathcal{T}| \geq |\mathcal{T}'| \geq Z_+(G') \geq \delta(H),$$

which proves the desired inequality. □

Using this result we deduce the following.

Corollary 6.6.2. *For any graph G we have*

$$Z_+(G) \geq \delta(G). \quad \square$$

Note that, the bound in the above inequality is tight for some graphs such as trees, complete graphs and even cycles.

Using Theorem 6.6.1 and Lemma 4.8.2, we can deduce the following.

Corollary 6.6.3. *For any graph G we have*

$$\chi(G) \leq Z_+(G) + 1. \quad \square \tag{6.12}$$

Observe that Corollary 6.6.3 is stronger than Corollary 4.8.3 as

$$Z_+(G) \leq Z(G).$$

In 1943 Hadwiger proposed the following conjecture [19]:

"Every n-chromatic graph contains a K_n minor, a complete graph on n points".

By establishing Inequality (6.12), it seems tempting to verify Hadwiger's conjecture for $Z_+(G)$ i.e., to verify the truth of the following statement: "If $Z_+(G) \geq n - 1$ then G contains a K_n minor." It turns out that this equality fails for outerplanar graphs. According to Corollary 6.6.3 we have $Z_+(G) = T(G)$ if G is an outerplanar graph. The tree cover number and consequently the positive zero forcing number of an outerplanar graph, G, can be arbitrarily large while G does not contain even a K_4 minor.

6.6.2 The positive zero forcing number and the independence number

An *independent set*, or *coclique*, is a set of vertices of a graph G of which no pair is adjacent. The *independence number*, which is denoted by $\alpha(G)$, of G is the size of the largest independent set of G. Based on the definition of the positive zero forcing number of a graph G it seems reasonable that it should be related to the independence number of G. In this section we establish a lower bound and an upper bound for the positive zero forcing number of a graph G using the independence number of G.

Lemma 6.6.4. *Let G be a graph with chromatic number $\chi(G)$ and independence number $\alpha(G)$. Then*

$$\chi(G) \geq \left\lceil \frac{|G|}{\alpha(G)} \right\rceil.$$

Proof. The inequality follows from the fact that in a proper colouring of G every colour can be used at most $\alpha(G)$ times. $\square$

Theorem 6.6.5. *Let $Z_+(G)$ and $\alpha(G)$ be the positive zero forcing number and the independence number of the graph G respectively. Then,*

$$\left\lceil \frac{|G|}{\alpha(G)} \right\rceil - 1 \le Z_+(G) \le |G| - \alpha(G).$$

Proof. The first inequality follows from combining Corollary 6.6.3 and Lemma 6.6.4. To show the second inequality, let H be a set of vertices in G, no two of which are adjacent with $|H| = \alpha(G)$. Let $L = V(G) \backslash V(H)$. Then L is a positive zero forcing set for G, since removing L from G results in having $\alpha(G)$ isolated vertices which can be coloured black by the vertices in L. Thus $Z_+(G) \le |L|$. $\square$

Note that in Theorem 6.6.5 both upper and lower bounds can be tight. For example the lower bound holds with equality for odd cycles and the graphs $C_{2k+1} \underset{v}{+} P_{2k'+1}$, where $k, k' \ge 1$ and v is an endpoint of the path. On the other hand, complete graphs and complete bipartite graphs are examples for which the upper bound is tight.

If $Z_+(G) = n - 2$, then according to Theorem 6.6.5, $\alpha(G) \le 2$. If $\alpha(G) = 1$, then G is a complete graph, hence $Z_+(G) = n - 1$. Thus $\alpha(G) = 2$. Therefore graphs with $Z_+(G) = n - 2$ are other examples for which the second inequality in Theorem 6.6.5 is tight.

Chapter 7

Conclusion

The graph parameters zero forcing number and positive zero forcing number, offer a wide range of prospective research. It is very appealing to try to find bounds or the exact value of the maximum nullity and minimum rank of graphs without going through direct algebraic calculations. Therefore, discovering more families of graphs satisfying $M(G) = Z(G)$ or $M_+(G) = Z_+(G)$, is an interesting project.

Problem 1. *For which families of graphs G do we have $M(G) = Z(G)$ or $M_+(G) = Z_+(G)$?*

In Section 4.5, we introduced families of graphs for which the zero forcing number and the path cover number coincide. In fact, we showed that for the family of block-cycle graphs this is true. It seems, however, that there are more families for which the equality holds between these two parameters. For example, the graph $G = K_4 - e$, where e is an edge of K_4, has $Z(G) = P(G)$. It is, therefore, natural to propose the

following.

Problem 2. *Characterize all the graphs G for which $Z(G) = P(G)$.*

We showed in Section 4.1 that for trees, any minimal path cover coincides with a collection of forcing chains. It seems that this is also the case for the block-cycle graphs. Note that, if this is true for any graph G satisfying $Z(G) = P(G)$, then Conjecture 5.2.3 follows from Theorem 5.2.1 and Observation 5.2.2. Hence, we state the following.

Problem 3. *Let G be a graph with $Z(G) = P(G)$. Is it true that any minimal path cover of G coincides with a collection of forcing chains of G?*

In Section 6.2, we proved the equality $Z_+(G) = T(G)$ where G is an outerplanar graph. The structure of a planar embedding of outerplanar graphs was the key point to establish the equality. It looks like there are some non-outerplanar graphs whose pattern follows the same structure. Hence an approach consisting of discovering more graphs satisfying $Z_+(G) = T(G)$, seems promising. We, therefore, propose the following problem.

Problem 4. *Characterize all the graphs G for which $Z_+(G) = T(G)$.*

In Chapter 4 we discussed the zero forcing number of the corona of graphs as well as the join of graphs. In addition, in Chapter 6 we studied the problem of finding the positive zero forcing number for the coalescence of graphs. Thus the problem of

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evaluating the (positive) zero forcing number for some other graph operations such as the Cartesian product, the direct product, the strong product or the complement of graphs may also be interesting.

Problem 5. *What can we say about the (positive) zero forcing number of graph operations such as different products and complements of graphs?*

Recall from Section 4.6 that the equality $M(G) = Z(G)$ does not hold for all chordal graphs. On the other hand, however, we showed that for some families of chordal graphs, the equality does indeed hold. It might be possible to describe the conditions under which a chordal graph satisfies the equality $M(G) = Z(G)$. We phrase this problem as follows.

Problem 6. *For which chordal graphs G does the equality $M(G) = Z(G)$ hold?*

Recall from Subsection 6.5.3 that the equality $Z_+(G) = T(G)$ holds for 2-trees but fails to hold for general k-trees. Recall, also, that we evaluated the positive zero forcing number and the tree cover number for some families of k-trees, namely the clusters. We are, therefore, interested in finding the positive zero forcing number and the tree cover number of the general k-trees. Hence solving Conjecture 6.5.12 is one of the most interesting future works in this area.

Problem 7. *If a k-tree G, has t clusters as subgraphs, each with $|S| \geq 3$, such that each pair of them share at most $k-1$ clique(s) of size $k+1$ then does the equality*

$Z_+(G) = k + t$ *hold?*

Finally, note that the parameters μ and ν discussed in Chapter 4 are of great importance because of their minor monotonicity property. Studying Conjecture 4.7.3 and Conjecture 4.7.4 is yet another interesting direction to follow. In particular, a follow up project could consist of investigating the Hadwiger conjecture which is one of the deepest unsolved problems in graph theory and would directly verify Conjecture 4.7.3 and Conjecture 4.7.4.

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