Linearly Degenerate Hamiltonian PDEs and a New Class of Solutions to the WDVV Associativity Equations

B. A. Dubrovin, M. V. Pavlov, and S. A. Zykov

ABSTRACT. We define a new class of solutions to the WDVV associativity equations. This class is determined by the property that one of the commuting PDEs associated with such a WDVV solution is linearly degenerate. We reduce the problem of classifying such solutions of the WDVV equations to the particular case of the so-called algebraic Riccati equation and, in this way, arrive at a complete classification of irreducible solutions.

To the memory of V. I. Arnold

CONTENTS

1. Introduction

The Witten–Dijkgraaf–E. Verlinde–H. Verlinde (WDVV) system of associativity equations is the overdetermined system of partial differential equations

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\delta} = \frac{\partial^3 F}{\partial v^\delta \partial v^\beta \partial v^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial v^\mu \partial v^\gamma \partial v^\alpha}, \qquad \alpha, \beta, \gamma, \delta = 1, \ldots, n, \quad (1.1)$$

for a function $F = F(\mathbf{v})$, $\mathbf{v} = (v^1, \ldots, v^n)$, satisfying the conditions

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^1} = \eta_{\alpha\beta}.$$

Here $(\eta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n}$ and $(\eta^{\alpha\beta})_{1 \leq \alpha, \beta \leq n}$ are mutually inverse constant symmetric nonsingular matrices, that is, $\eta_{\alpha\lambda}\eta^{\lambda\beta} = \delta_\alpha^\beta$. Throughout this section summation over repeated Greek indices will be assumed.

Recall [5] that the solutions to the WDVV associativity equations are in one-to-one correspondence with the n-parameter families of n-dimensional commutative associative algebras

$$\mathscr{A}_\mathbf{v} = \mathrm{span}(e_1, \ldots, e_n)$$

with a unit $e = e_1$ equipped with a symmetric nondegenerate invariant bilinear form $(\ ,\ )$ such that the structure constants are expressed via the third derivatives of a function F, called the *potential*:

$$e_\alpha \cdot e_\beta = c_{\alpha\beta}^\gamma(\mathbf{v})e_\gamma, \qquad \alpha, \beta = 1, \ldots, n,$$
$$e_1 \cdot e_\alpha = e_\alpha \quad \text{for any } \alpha,$$
$$(e_\alpha, e_\beta) = \eta_{\alpha\beta},$$
$$(e_\alpha \cdot e_\beta, e_\gamma) = (e_\alpha, e_\beta \cdot e_\gamma) = \eta_{\gamma\lambda}c_{\alpha\beta}^\lambda(\mathbf{v}) = \frac{\partial^3 F(\mathbf{v})}{\partial v^\alpha \partial v^\beta \partial v^\gamma}.$$

If, in addition, the function F satisfies a certain quasi-homogeneity condition, then one arrives at a local description of *Frobenius manifolds* (see details in [5]). On these manifolds the natural metric

$$ds^2 = \eta_{\alpha\beta}\, dv^\alpha\, dv^\beta \tag{1.2}$$

(not necessarily positive definite) is defined. The variables $v^1, \ldots, v^n$ are *flat coordinates* for this metric. The algebra $\mathscr{A}_\mathbf{v}$ is identified with the tangent space to the manifold at the point $\mathbf{v}$:

$$e_\alpha \leftrightarrow \frac{\partial}{\partial v^\alpha};$$

see [5] for more details about the coordinate-free geometric description of Frobenius manifolds.

A solution to the associativity equations (1.1) is called *semisimple* if the algebra $\mathscr{A}_\mathbf{v}$ has no nilpotent elements for a generic point $\mathbf{v}$. It was proved in [4] that, in the semisimple case, there exist local *canonical coordinates* $u_i = u_i(\mathbf{v})$, $i = 1, \ldots, n$, such that the multiplication table takes the standard form

$$\frac{\partial}{\partial u_i} \cdot \frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_i}.$$

The metric (1.2) becomes diagonal in these canonical coordinates:

$$ds^2 = \sum_{i=1}^n h_i^2(\mathbf{u})\, du_i^2.$$

Moreover, this is a *Egorov* metric (see [7]), which means that the *rotation coefficients*

$$\gamma_{ij}(\mathbf{u}) = \frac{1}{h_j}\frac{\partial h_i}{\partial u_j} \tag{1.3}$$

are symmetric in i and j, i.e., $\gamma_{ji} = \gamma_{ij}$. They satisfy the following system of *Darboux–Egorov equations* [1]:

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj} \qquad \text{for distinct } i, j, k, \tag{1.4}$$

$$\sum_{k=1}^{n} \frac{\partial \gamma_{ij}}{\partial u_k} = 0 \qquad \text{for } i \neq j. \tag{1.5}$$

Any solution to the Darboux–Egorov equations comes from a semisimple solution to the WDVV associativity equations. The reconstruction procedure of the latter involves solutions to the following system of linear differential equations for a vector-function $\psi = (\psi_1(\mathbf{u}), \dots, \psi_n(\mathbf{u}))$:

$$\frac{\partial \psi_i}{\partial u_j} = \gamma_{ij}\psi_j, \qquad i \neq j, \tag{1.6}$$

$$\sum_{k=1}^{n} \frac{\partial \psi}{\partial u_k} = 0. \tag{1.7}$$

Let $\psi_{i\alpha} = \psi_{i\alpha}(\mathbf{u})$, $\alpha = 1, \dots, n$, be a system of n linearly independent solutions to system (1.6), (1.7). The reconstruction depends on a choice of one of these solutions to be identified with the Lamé coefficients of the invariant metric (1.2); suppose that the chosen solution corresponds to $\alpha = 1$, that is, $h_i = \psi_{i1}$. Then

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i\beta},$$

$$dv_\alpha = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i1}\, du_i,$$

$$\frac{\partial^3 F}{\partial v^\alpha \partial v^\beta \partial v^\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}.$$

We also mention the following formula for the differentials of the second derivatives

$$\Omega_{\alpha\beta} = \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} \tag{1.8}$$

of the potential F:

$$d\Omega_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha}\psi_{i\beta}\, du_i. \tag{1.9}$$

As shown in [3], the Darboux–Egorov system (1.4)–(1.5) can be identified with a special reduction of the n-wave system well known in the theory of integrable PDEs and written in the form suggested in [2]. It can also be embedded in the framework of the nKP system (see, e.g., [10]). All known particular solutions to the associativity equations correspond to further reductions of the n-wave system to a system of ODEs. For example, the semisimple Frobenius manifolds are determined by the homogeneity condition on the rotation coefficients, or the *scaling* reduction

$$\sum_{k=1}^{n} u_k \frac{\partial \gamma_{ij}}{\partial u_k} = -\gamma_{ij}, \qquad i \neq j.$$

This condition corresponds to the quasi-homogeneity axiom of the theory of Frobenius manifolds (see [4] and [5]). Other particular classes of solutions (such as solitons, algebro-geometric solutions, and degenerate Frobenius manifolds) also naturally arise in the framework of the n-wave system.

In this paper we introduce another class of solutions to the WDVV equations. Before describing this class, we recall the connection between the associativity equations and integrable hierarchies. Let $\theta = \theta(\mathbf{v})$ be a solution to the system of linear differential equations

$$\frac{\partial^2 \theta}{\partial v^\alpha \partial v^\beta} = c_{\alpha\beta}^\gamma \frac{\partial^2 \theta}{\partial v^1 \partial v^\gamma}, \qquad \alpha, \beta = 1, \ldots, n. \tag{1.10}$$

Consider the following system of first-order quasilinear PDEs for the vector-function $\mathbf{v} = \mathbf{v}(x, t)$:

$$\mathbf{v}_t = [\nabla \theta(\mathbf{v})]_x. \tag{1.11}$$

This is a Hamiltonian PDE with Hamiltonian $H = \int \theta(\mathbf{v}) \, dx$ and Poisson bracket $\{v^\alpha(x), v^\beta(y)\} = \eta^{\alpha\beta} \delta'(x - y)$ (see [6]). All Hamiltonian systems of the form (1.10), (1.11) pairwise commute. Moreover, Hamiltonians (1.10) satisfy certain *completeness* conditions (see [11]). Thus, any such system (1.11) can be considered as a completely integrable Hamiltonian system of PDEs.

In the semisimple case all such PDEs diagonalize in the canonical coordinates, i.e.,

$$\mathbf{u}_t = \Lambda(\mathbf{u})\mathbf{u}_x, \qquad \Lambda(\mathbf{u}) = \mathrm{diag}(\lambda_1(\mathbf{u}), \ldots, \lambda_n(\mathbf{u})). \tag{1.12}$$

Thus, the canonical coordinates are *Riemann invariants* for the quasilinear systems (1.11). For a generic solution to (1.10), the characteristic velocities are pairwise distinct, i.e.,

$$\lambda_i(\mathbf{u}) \neq \lambda_j(\mathbf{u}), \qquad i \neq j, \tag{1.13}$$

at a generic point $\mathbf{u}$.

DEFINITION 1.1. A semisimple solution $F(\mathbf{v})$ to the WDVV associativity equations is called *linearly degenerate* if among the commuting PDEs (1.10)–(1.12) there exists at least one satisfying (1.13) along with the condition

$$\frac{\partial \lambda_i(\mathbf{u})}{\partial u_i} = 0, \qquad i = 1, \ldots, n.$$

The motivation for our terminology is that one of the quasilinear systems of the commuting family (1.10)–(1.12) is linearly degenerate, i.e., the ith characteristic velocity λ_i does not depend on the ith Riemann invariant u_i for every i from $i = 1$ to $i = n$.

The main goal of the present paper is to classify linearly degenerate solutions to the WDVV associativity equations. Such a solution is called *reducible* if, for some i, one has $\gamma_{ij}(\mathbf{u}) \equiv 0$ for all $j \neq i$. Otherwise it will be called *irreducible*. It suffices to classify irreducible linearly degenerate solutions.

THEOREM 1.2. *The rotation coefficients of an irreducible linearly degenerate solution to the WDVV associativity equations has the form*

$$\gamma_{ij}(u) = \frac{[G(1 - \frac{1}{\rho} \tanh \rho U \cdot G)^{-1}]_{ij}}{\cosh \rho u_i \cosh \rho u_j}, \qquad i, j = 1, \ldots, n, \ i \neq j, \tag{1.14}$$

where $U = \mathrm{diag}(u_1, \ldots, u_n)$ and G is a symmetric matrix satisfying the condition $G^2 = \rho^2 \cdot 1$, in which ρ is an arbitrary complex parameter.

For $\rho = 0$, the above formulas are considered in the sense of the limits

$$\frac{1}{\rho} \tanh \rho U \to U, \qquad \cosh \rho u_i \to 1.$$

The paper is organized as follows. In Section 2 we recall the necessary constructions of the theory of the WDVV associativity equations and derive the basic system of differential equations (2.6) of the theory of linearly degenerate solutions to the WDVV equations. In Section 3 we solve the basic system and describe its symmetry group acting by fractional linear transformations. In Section 4 we select those solutions to the basic system that give rise to the WDVV equations and derive the matrix algebraic Riccati equation. Using the symmetries of this equation, we classify all irreducible linearly degenerate solutions to the WDVV associativity equations.

Acknowledgments

2. Linearly Degenerate Solutions to the WDVV Associativity Equations

Let $\Gamma = (\gamma_{ij}(\mathbf{u}))_{1 \leq i,j \leq n}$ be the symmetric matrix of rotation coefficients[1] (1.3) of a linearly degenerate irreducible solution to the associativity equations.

LEMMA 2.1. *The matrix-valued function $\Gamma = \Gamma(\mathbf{u})$ satisfies the differential equations*

$$\frac{\partial \Gamma}{\partial u_k} = \Gamma E_k \Gamma + \sigma_k(u_k) E_k, \qquad k = 1, \ldots, n, \tag{2.1}$$

with some functions $\sigma_1(u_1), \ldots, \sigma_n(u_n)$. Here E_k is a matrix with only one nonzero entry, namely,

$$(E_k)_{ij} = \delta_{ik}\delta_{jk}. \tag{2.2}$$

PROOF. Proof By construction the equations

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj} \tag{2.3}$$

hold true for distinct values of the indices i, j, and k. Let us first prove that (2.3) also holds when $k = i$ or $k = j$ and $i \neq j$ or when $i = j$ but $k \neq i$.

According to [4], the characteristic velocities $\lambda_k(\mathbf{u})$ of the commuting PDEs (1.10)–(1.12) can be represented in the form

$$\lambda_k(\mathbf{u}) = \frac{\phi_k(\mathbf{u})}{h_k(\mathbf{u})}, \qquad k = 1, \ldots, n,$$

where the vector-function $\phi = (\phi_1(\mathbf{u}), \ldots, \phi_n(\mathbf{u}))$ satisfies the system of linear differential equations

$$\frac{\partial \phi_i}{\partial u_j} = \gamma_{ij}\phi_j, \qquad i \neq j. \tag{2.4}$$

---

[1]Actually, in the differential geometry of curvilinear orthogonal coordinate systems only the off-diagonal entries of the matrix Γ are called rotation coefficients. However, in our case it will be convenient to add the diagonal entries $\gamma_{ii} = \partial \log h_i / \partial u_i$.

In particular, $\phi_k = h_k$ is one of the solutions to (2.4). Let ϕ be the solution to (2.4) corresponding to a linearly degenerate member of the commuting family (1.10)–(1.12). Differentiating the equation

$$\frac{\partial}{\partial u_k}\left(\frac{\phi_k}{h_k}\right) = 0$$

in u_i with $i \neq k$, we obtain the equation

$$\frac{h_i}{h_k}(\lambda_i - \lambda_k)\,\gamma_{ik}\,\frac{\partial}{\partial u_k}[\log\gamma_{ik} - \log h_k] = 0.$$

Due to the assumptions of irreducibility and (1.13), we arrive at the equation

$$\frac{\partial \log\gamma_{ik}}{\partial u_k} = \frac{\partial \log h_k}{\partial u_k} = \gamma_{kk}.$$

This proves (2.3) for the case where $k = j$ and $i \neq j$. Next, assuming that $k \neq i$, one has

$$\frac{\partial\gamma_{ii}}{\partial u_k} = \frac{\partial}{\partial u_i}\frac{\partial \log h_i}{\partial u_k} = \frac{\partial}{\partial u_i}\left(\gamma_{ik}\frac{h_k}{h_i}\right) = \gamma_{ik}^2.$$

Thus, Eq. (2.3) with $i = j$ and $k \neq i$ is also verified. The last step is to verify that the difference $\sigma_i := \partial\gamma_{ii}/\partial u_i - \gamma_{ii}^2$ depends only on u_i. Indeed, for $k \neq i$,

$$\frac{\partial}{\partial u_k}\left(\frac{\partial\gamma_{ii}}{\partial u_i} - \gamma_{ii}^2\right) = \frac{\partial}{\partial u_i}\frac{\partial\gamma_{ii}}{\partial u_k} - 2\gamma_{ii}\gamma_{ik}^2 = \frac{\partial\gamma_{ik}^2}{\partial u_i} - 2\gamma_{ii}\gamma_{ik}^2 = 0. \qquad \square$$

$$\square$$

Now, let us describe a class of transformations

$$u_k \mapsto \tilde{u}_k, \qquad \gamma_{ij} \mapsto \tilde{\gamma}_{ij}$$

which leave system (2.1) invariant.

LEMMA 2.2. *The substitution*

$$\tilde{u}_k = f_k(u_k), \qquad\qquad\qquad k = 1, \ldots, n,$$

$$\tilde{\gamma}_{ij} = \frac{\gamma_{ij}}{\sqrt{f_i'(u_i)f_j'(u_j)}} - \frac{f_i''(u_i)}{2[f_i'(u_i)]^2}\delta_{ij}, \qquad i, j = 1, \ldots, n, \qquad (2.5)$$

with arbitrary nonconstant smooth functions $f_1(u_1), \ldots, f_n(u_n)$ leaves invariant the form of
Eqs. (2.1), which transform into

$$\frac{\partial\widetilde{\Gamma}}{\partial\tilde{u}_k} = \widetilde{\Gamma}E_k\widetilde{\Gamma} + \tilde{\sigma}_k(\tilde{u}_k)E_k, \qquad k = 1, \ldots, n,$$

with $f_k'^2\tilde{\sigma}_k = \sigma_k - \frac{1}{2}S_{u_k}(f_k)$. Here $S_u(f)$ is the Schwarzian derivative of a function $f = f(u)$, that is,

$$S_u(f) = \frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2}.$$

This lemma is proved by a straightforward calculation. $\qquad\qquad\square$

COROLLARY 2.3. *A suitable transformation of the form* (2.5) *reduces system* (2.1) *to the form*

$$\frac{\partial \widetilde{\Gamma}}{\partial \tilde{u}_k} = \widetilde{\Gamma} E_k \widetilde{\Gamma}, \qquad k = 1, \ldots, n. \tag{2.6}$$

PROOF. Proof The needed transformation $\tilde{u}_k = f_k(u_k)$ is determined from the Schwarzian equations

$$S_{u_k}(f_k) = 2\sigma_k(u_k), \qquad k = 1, \ldots, n. \qquad\qquad \square$$

$$\square$$

Recall that the solution to the general Schwarzian equation $S_u(f(u)) = 2\sigma(u)$ can be represented as the ratio of two solutions to the linear second-order equation

$$y'' + \sigma(u)y = 0.$$

REMARK 2.4. System (2.6) was studied in [8] in the investigation of the so-called multi-flow cold gas reductions of the nonlocal kinetic equation derived as the thermodynamical limit of the averaged multi-phase solutions of the KdV equation by the Whitham method.

In the next section we shall solve system (2.6).

3. Basic System

In this section we shall describe solutions to the basic system

$$\frac{\partial \Gamma}{\partial u_k} = \Gamma E_k \Gamma, \qquad k = 1, \ldots, n. \tag{3.1}$$

Here

$$\Gamma = (\gamma_{ij}(\mathbf{u}))_{1 \leq i,j \leq n}$$

is a symmetric matrix (the tildes used in the previous section are omitted). The compatibility conditions

$$\frac{\partial}{\partial u_l} \frac{\partial \Gamma}{\partial u_k} = \frac{\partial}{\partial u_k} \frac{\partial \Gamma}{\partial u_l}$$

for any k and l can be readily verified. So, locally, any solution to (3.1) is uniquely determined by the initial data

$$\Gamma^0 = \Gamma(\mathbf{u}^0).$$

Here $\mathbf{u}^0$ is any point in the space of independent variables. Therefore, the space of solutions to the system (3.1) has dimension $n(n+1)/2$.

Without loss of generality, one can assume that $\mathbf{u}^0 = 0$. The solution to system (3.1) with given initial data at the point $\mathbf{u} = 0$ can be written explicitly.

PROPOSITION 3.1. *The solution* $\Gamma = \Gamma(\mathbf{u})$ *to the basic system* (3.1) *with initial data*

$$\Gamma(0) = G,$$

where $G = (g_{ij})$ *is a given symmetric matrix, is determined by the formula*

$$\Gamma = G(1 - UG)^{-1}, \tag{3.2}$$

where 1 is the $n \times n$ *identity matrix and* $U = \mathrm{diag}(u_1, \ldots, u_n)$.

PROOF. Proof The symmetry of the matrix (3.2) is tantamount to the relation

$$G(1 - UG)^{-1} = (1 - GU)^{-1}G.$$

To prove this relation, we multiply it by $1 - GU$ on the left and by $1 - UG$ on the right and arrive at the obvious identity $(1 - GU)G = G(1 - UG) = G - GUG$. Clearly, $\Gamma(0) = G$. The proof of the proposition is completed by applying the well-known rule

$$\frac{\partial \Gamma}{\partial u_k} = -G(1 - UG)^{-1}\frac{\partial(1 - UG)}{\partial u_k}(1 - UG)^{-1} = G(1 - UG)^{-1}E_kG(1 - UG)^{-1} = \Gamma E_k\Gamma$$

for differentiating inverse matrices. $\square$ $\square$

EXAMPLE 3.2. For a matrix $g_{ij} = \omega_i\omega_j$ of rank 1, one obtains the following solution to the basic system:

$$\gamma_{ij} = \frac{\omega_i\omega_j}{1 - \sum_{k=1}^n \omega_k^2 u_k}. \tag{3.3}$$

Now, let us describe a subclass of transformations (2.5) leaving invariant the basic system (3.1).

PROPOSITION 3.3. *The basic system* (3.1) *is invariant with respect to transformations* (2.5) *if and only if* $f_k(u_k)$ *for every* $k = 1, \ldots, n$ *is a fractional linear transformation*

$$f_k(u_k) = \frac{a_k u_k + b_k}{c_k u_k + d_k}, \qquad a_k d_k - b_k c_k = 1.$$

PROOF. Proof It is well known that the general solution to the homogeneous Schwarzian equation

$$\frac{f'''}{f'} - \frac{3}{2}\frac{f''^2}{f'^2} = 0$$

is given by a fractional linear function. $\square$ $\square$

COROLLARY 3.4. *The basic system* (3.1) *is invariant with respect to the transformations*

$$\tilde{u}_k = \frac{a_k u_k + b_k}{c_k u_k + d_k}, \qquad \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \in SL_2(\mathbb{R}), \ k = 1, \ldots, n,$$
$$\tilde{\gamma}_{ij} = (c_i u_i + d_i)(c_j u_j + d_j)\gamma_{ij} + c_i(c_i u_i + d_i)\delta_{ij}, \qquad i, j = 1, \ldots, n. \tag{3.4}$$

The matrix version of transformation (3.4) is

$$\widetilde{U} = (AU + B)(CU + D)^{-1}, \qquad \widetilde{\Gamma} = (CU + D)\Gamma(CU + D) + C(CU + D), \tag{3.5}$$

where $A = \text{diag}(a_1, \ldots, a_n)$, $B = \text{diag}(b_1, \ldots, b_n)$, $C = \text{diag}(c_1, \ldots, c_n)$, $D = \text{diag}(d_1, \ldots, d_n)$, and $AD - BC = 1$.

EXAMPLE 3.5. The substitution

$$\tilde{u}_k = \omega_k^2 u_k, \qquad \tilde{\gamma}_{ij} = \frac{\gamma_{ij}}{\omega_i\omega_j}$$

reduces solution (3.3) to the standard form

$$\tilde{\gamma}_{ij} = \frac{1}{1 - \sum_{k=1}^n \tilde{u}_k}, \qquad i, j = 1, \ldots, n.$$

The action of the $[SL_2(\mathbb{R})]^n$ transformations (3.4) on solutions (3.2) is given by the following analogue of Siegel modular transformations.

PROPOSITION 3.6. *Let the symmetric matrix G satisfy the condition* $\det(A+BG) \neq 0$. *Then transformation* (3.4) *transforms the solution* $\Gamma(\mathbf{u})$ *with initial data* $\Gamma(0) = G$ *into*

$$\widetilde{\Gamma} = \widetilde{G}(1 - \widetilde{U}\widetilde{G})^{-1}$$

with

$$\widetilde{G} = (C + DG)(A + BG)^{-1}. \tag{3.6}$$

PROOF. Proof An easy calculation employing (3.5) yields

$$\widetilde{\Gamma} = (-C\widetilde{U} + A)^{-1}G[A + BG - \widetilde{U}(C + DG)]^{-1} + C(-C\widetilde{U} + A)^{-1}.$$

Computing the initial data of this solution at $\tilde{\mathbf{u}} = 0$, we arrive at $\widetilde{\Gamma}(0) = \widetilde{G}$ with the matrix $\widetilde{G}$ given by (3.6). $\qquad\square\qquad\qquad\qquad\qquad\square$

DEFINITION 3.7. Two solutions Γ and $\tilde{\Gamma}$ to the basic system are called *equivalent* if they are related by a symmetry transformation of the form (3.5). Two symmetric matrices G and $\widetilde{G}$ related by transformation (3.6) will also be called equivalent.

Note that the useful identity

$$(C + DG)(A + BG)^{-1} = (A + GB)^{-1}(C + GD) \tag{3.7}$$

is equivalent to the symmetry of the matrix G.

4. From Solutions of the Basic System to Linearly Degenerate Solutions of the Associativity Equations

In this section we address the problem of selecting those solutions to the basic system (3.1) that come from a linearly degenerate solution to the associativity equations.

Given a symmetric matrix-valued function $\Gamma(\mathbf{u})$ satisfying (3.1), we look for a substitution of the form (2.5) such that the transformed matrix $\widetilde{\Gamma}$ satisfies also the last equation (1.5) of the Darboux–Egorov system, that is,

$$\sum_{k=1}^{n} \frac{\partial \widetilde{\Gamma}}{\partial \tilde{u}_k} \quad \text{is a diagonal matrix.} \tag{4.1}$$

Recall that the equations

$$\frac{\partial \tilde{\gamma}_{ij}}{\partial \tilde{u}_k} = \tilde{\gamma}_{ik}\tilde{\gamma}_{kj} \quad \text{for distinct } i, j, \text{ and } k,$$

which are the first part of this system (Eqs. (1.4)), follow from the basic system by Lemma 2.2.

Applying Lemma 2.2, we arrive at the following simple statement.

PROPOSITION 4.1. *Let* $\Gamma(\mathbf{u})$ *be a solution to the basic system* (3.1). *Suppose that the functions* $f_1(u_1), \ldots, f_n(u_n)$ *are chosen in such a way that the transformed matrix* (2.5) *satisfies* (4.1). *Then the off-diagonal entries of the transformed matrix* $\widetilde{\Gamma}$ *are the rotation coefficients of some Egorov metric.*

We introduce the diagonal matrices

$$S = \mathrm{diag}(s_1, \ldots, s_n), \qquad s_i = \frac{1}{f_i'}, \tag{4.2}$$

$$S' = \mathrm{diag}(s_1', \ldots, s_n'), \qquad s_i' = \frac{ds_i}{du_i} = -\frac{f_i''}{[f_i']^2}. \tag{4.3}$$

Here and in the sequel we use the short notation

$$f_i' = f_i'(u_i), \quad f_i'' = f_i''(u_i), \quad \text{etc.}$$

In this notation the transformation law (2.5) reads

$$\widetilde{\Gamma} = S^{1/2} \Gamma S^{1/2} + \tfrac{1}{2} S'.$$

Thus, condition (4.1) can be represented in the form

$$\Gamma S \Gamma + \tfrac{1}{2} S' \Gamma + \tfrac{1}{2} \Gamma S' + P = 0 \tag{4.4}$$

for some diagonal matrix P.

DEFINITION 4.2. A solution Γ is called *reducible* if, for some i,

$$\gamma_{ij} \equiv 0 \quad \text{for any } j \neq i.$$

Otherwise it is called *irreducible*.

A reducible solution is expressed in terms of functions depending on a smaller number of variables.

THEOREM 4.3. *For an irreducible solution*

$$\Gamma = G(1 - UG)^{-1} = (1 - GU)^{-1} G,$$

a transformation of the form (2.5) *satisfying* (4.1) *exists if and only if the matrix G satisfies the quadratic equation*

$$GRG + QG + GQ + P = 0 \tag{4.5}$$

for some constant diagonal matrices

$$P = \mathrm{diag}(p_1, \ldots, p_n), \quad Q = \mathrm{diag}(q_1, \ldots, q_n), \quad R = \mathrm{diag}(r_1, \ldots, r_n).$$

The transformation in question is determined by

$$\frac{d\tilde{u}_i}{du_i} = \frac{1}{p_i u_i^2 + 2 q_i u_i + r_i}, \quad i = 1, \ldots, n.$$

PROOF. Proof Differentiating (4.4) in u_i and using (3.1) and the obvious formulas

$$\frac{\partial S}{\partial u_i} = s_i' E_i, \qquad \frac{\partial S'}{\partial u_i} = s_i'' E_i,$$

etc., one obtains

$$\left(\frac{1}{2} s_i'' - p_i \right) (\Gamma E_i + E_i \Gamma) + \frac{\partial P}{\partial u_i} = 0. \tag{4.6}$$

All entries of the matrix $\Gamma E_i + E_i \Gamma$ vanish, except the ith row and the ith column, which coincide with $(\gamma_{1i}, \ldots, \gamma_{ni})$. Due to the irreducibility assumption, it follows from (4.6) that

$$p_i = \frac{1}{2} s_i''. \tag{4.7}$$

Substituting this into (4.6) yields

$$\frac{\partial P}{\partial u_i} = 0.$$

Repeating this procedure for every $i = 1, \ldots, n$, one proves that the matrix P is constant. Using (4.7), we conclude that $s_i = s_i(u_i)$ is a quadratic polynomial, i.e., $s_i = p_i u_i^2 + 2 q_i u_i + r_i$. Finally, multiplying Eq. (4.4) by $1 - GU$ on the left and by $1 - UG$ on the right, we arrive at the quadratic equation (4.5).   □   □

DEFINITION 4.4. A symmetric matrix G is called *admissible* if it satisfies the matrix quadratic equation (4.5). A solution of the form $\Gamma = G(1 - UG)^{-1}$ is called *admissible* if the parameter matrix G is admissible.

The matrix quadratic equation (4.5) for the symmetric matrix G is a particular case of the so-called *algebraic Riccati equation* (see, e.g., [9]). The class of such equations is invariant with respect to fractional linear transformations, as the following lemma shows.

LEMMA 4.5. *If a symmetric matrix G satisfies the matrix quadratic equation*

$$GRG + QG + GQ + P = 0$$

with some diagonal matrices P, Q, and R, then the equivalent matrix $\widetilde{G} = (C + DG)(A + BG)^{-1}$ satisfies an equation of the same form

$$\widetilde{G}\widetilde{R}\widetilde{G} + \widetilde{Q}\widetilde{G} + \widetilde{G}\widetilde{Q} + \widetilde{P} = 0$$

with

$$
\begin{aligned}
\widetilde{P} &= D^2 P - 2CDQ + C^2 R, \\
\widetilde{Q} &= -BDP + (AD + BC)Q - ACR, \\
\widetilde{R} &= B^2 P - 2ABQ + A^2 R.
\end{aligned}
\tag{4.8}
$$

The proof of this lemma is straightforward and uses identity (3.7).   □

COROLLARY 4.6. *The class of admissible solutions to the basic system (3.1) is invariant with respect to the $[SL_2]^n$ action (3.5).*

The entries $\Delta_1, \ldots, \Delta_n$ of the diagonal matrix

$$\Delta = Q^2 - PR \tag{4.9}$$

are invariants of the $[SL_2]^n$ action (4.8).

The next step is to parameterize linearly degenerate solutions to the associativity equations by solutions to the algebraic Riccati equation (4.5) with prescribed coefficients satisfying the condition

$$|p_i|^2 + |q_i|^2 + |r_i|^2 \neq 0, \qquad i = 1, \ldots, n.$$

Let us first simplify the matrix quadratic equation by means of transformations (4.8).

LEMMA 4.7. *(1) For an irreducible admissible matrix G, the matrix quadratic equation (4.5) is equivalent, up to transformations (4.8), to*

$$G^2 = \Delta, \tag{4.10}$$

where Δ is given by (4.9).

(2) For an admissible irreducible G, the matrix Δ is proportional to the identity matrix, i.e.,

$$\Delta_1 = \cdots = \Delta_n =: \rho^2.$$

PROOF. Proof If all entries of the matrix R are different from zero, then Eq. (4.5) can be reduced to the canonical form (4.10) by a transformation of the form

$$G \mapsto AGA + B$$

with suitable diagonal matrices A and B. This is a particular class of transformation (4.8). If $r_i = 0$ for some i, then one can assume that $p_i \neq 0$. Let us apply the fractional linear transformation of the form (3.6) with $A = 1 - E_i$, $B = -E_i$, $C = E_i$, and $D = 1 - E_i$, that is, $G \mapsto \widetilde{G} = [G + E_i(1 - G)][1 - E_i(1 + G)]^{-1}$, where the matrix E_i is of the form (2.2). Such a transformation is applicable only if the matrix $1 - E_i(1 + G)$ is nonsingular. It is easy to see that the determinant of this matrix is equal to $\pm g_{ii} = \gamma_{ii}(0)$. If $g_{ii} = 0$ but the solution is irreducible, then one can perform a shift $\mathbf{u} \mapsto \mathbf{u} + \mathbf{u}^0$ to obtain a matrix $G' = \Gamma(\mathbf{u}^0)$ with $g'_{ii} \neq 0$. After the transformation, one obtains $\tilde{r}_i = p_i \neq 0$.

To prove the second part of the lemma, it suffices to observe that any eigenvector f of the matrix G with eigenvalue λ is an eigenvector of G^2 with eigenvalue λ^2. So, if e_i and e_j are the ith and jth basic vectors and $\Delta_i \neq \Delta_j$, then these vectors belong, respectively, to the sums of root subspaces $R(\sqrt{\Delta_i}) \oplus R(-\sqrt{\Delta_i})$ and $R(\sqrt{\Delta_j}) \oplus R(-\sqrt{\Delta_j})$ of the matrix G. Such root subspaces of symmetric matrices are orthogonal; hence the matrix G must have block-diagonal form in the same basis. $\qquad\qquad\square\qquad\qquad\qquad\square$

The main Theorem 1.2 readily follows from the above considerations.

Recall that the reconstruction of the solution to the associativity equations with given rotation coefficients (1.14) depends on the choice of a solution to the linear system (1.6), (1.7). Below we apply this procedure to produce examples of linearly degenerate WDVV solutions. It is convenient to separately consider the cases $\rho \neq 0$ and $\rho = 0$.

Case 1. The eigenvalues of a symmetric matrix G satisfying $G^2 = \rho^2 \cdot 1$ are equal to $\pm\rho$. Let k denote the number of eigenvalues equal to $-\rho$. We consider the case $k = 1$ in more detail. It is more convenient to deal with the matrix $\widetilde{G} = G - \rho \cdot 1$, which satisfies the equation $\widetilde{G}^2 + 2\rho\widetilde{G} = 0$. In the case $k = 1$, this matrix can be represented in the form

$$\widetilde{G} = (\omega_i\omega_j), \qquad \sum_{i=1}^n \omega_i^2 = -2\rho.$$

To this matrix there corresponds a family of solutions of the form (3.3). The substitution $\tilde{u}_k = -\log[\omega_k^2(u_k - u_k^0)]$, $k = 1, \ldots, n$, with arbitrary constants u_k^0 satisfying $\sum_{k=1}^n u_k^0 = 0$ yields the following rotation coefficients satisfying the Darboux–Egorov equations:

$$\tilde{\gamma}_{ij} = \frac{e^{-(\tilde{u}_i + \tilde{u}_j)/2}}{\sum_{k=1}^n e^{-\tilde{u}_k}}, \qquad i \neq j.$$

In the sequel, we omit the tildes. System (1.6)–(1.7) can be easily solved:

$$\psi_{ii} = \frac{2e^{-u_i}}{D} - 1, \quad \psi_{ij} = \frac{2e^{-(u_i+u_j)/2}}{D}, \quad i \neq j, \quad \text{where } D = \sum_{k=1}^n e^{-u_k}.$$

The calculation of the quadratures (1.9) gives the following expression for the matrix Ω of the second derivatives of the potential (see (1.8)):

$$\Omega_{ij} = u_i \delta_{ij} + \frac{4e^{-(u_i+u_j)/2}}{D} . \tag{4.11}$$

Flat coordinates are obtained by choosing a linear combination of the columns of this matrix. The choice of the first column yields the Egorov metric

$$ds^2 = \left(1 - 4\frac{e^{-u_1}}{D}\right) du_1^2 + 4 \sum_{i=1}^{n} \frac{e^{-u_1-u_i}}{D^2} du_i^2$$

with the flat coordinates

$$v_1 = u_1 + \frac{4e^{-u_1}}{D}, \qquad v_i = \frac{4e^{-(u_1+u_i)/2}}{D} \quad \text{for } i \neq 1.$$

Solving these equations for the canonical coordinates u_i, we obtain

$$u_1 = v_1 - \sqrt{4-\sigma} - 2, \qquad u_i = v_1 - \sqrt{4-\sigma} - 2 + 2\log\frac{2+\sqrt{4-\sigma}}{v_i} \quad \text{for } i \neq 1$$

with $\sigma = \sum_{k=2}^{n} v_k^2$, and integrating quadratures (4.11), we arrive at the following expression for the potential being the corresponding linearly degenerate solution to the WDVV associativity equations:

$$F = \frac{1}{6} v_1^3 + \frac{1}{2} v_1 \sigma - \sum_{k=2}^{n} v_k^2 \log v_k - \frac{1}{3}(2+\sigma)\sqrt{4-\sigma} + \sigma \log(2+\sqrt{4-\sigma}). \tag{4.12}$$

One can also obtain an explicit realization of the integrable hierarchy associated, in the sense of [4], with (4.12). Recall that the hierarchy is an infinite family of commuting flows labeled by pairs (α, p), $\alpha = 1, \ldots, n$, $p = 0, 1, 2, \ldots$. The flows have the form

$$\frac{\partial v^\gamma}{\partial t^{\alpha,p}} = \partial_x(\nabla^\gamma \theta_{\alpha,p+1}(v)).$$

The generating functions

$$\theta_\alpha(v, z) = \sum_{p=0}^{\infty} \theta_{\alpha,p}(v) z^p$$

of $\theta_{\alpha,p}(v)$ (*deformed flat coordinates*) can be found in quadratures; we have

$$d\theta_\alpha(v, z) = \sum_{i=1}^{n} h_i \Psi_{i\alpha} du_i, \qquad \alpha = 1, \ldots, n,$$

where the $\Psi_{i\alpha}(v, z)$, $\alpha = 1, \ldots, n$, form a basis for the "wave functions" determined by the system

$$\frac{\partial \Psi_i}{\partial u_j} = \gamma_{ij} \Psi_j, \qquad i \neq j,$$

$$\sum_{k=1}^{n} \frac{\partial \Psi_i}{\partial u_k} = z\Psi_i.$$

The basis $\Psi_{i\alpha}$ can be conveniently orthonormalized by the conditions

$$\sum_{\alpha=1}^{n} \Psi_{i\alpha}(v, -z)\Psi_{j\alpha}(v, z) = \delta_{ij}.$$

In our case the normalized wave functions have the form

$$\Psi_{i\alpha} = \frac{2e^{z\,u_\alpha}}{\sqrt{1 - 4z^2}}\left[\left(z - \frac{1}{2}\right)\delta_{i\alpha} + \frac{e^{-\frac{u_i + u_\alpha}{2}}}{D}\right].$$

This gives

$$\theta_\alpha = \frac{1}{\sqrt{1 - 4z^2}}\left\{\left[\frac{1}{z}(e^{zu_1} - 1) - e^{zu_1}(u_1 + 2) + 2\right]\delta_{\alpha 1} + v_\alpha e^{zu_\alpha}\right\}, \qquad \alpha = 1, \ldots, n.$$

Case 2. Now, consider the second type of solutions, namely, those parametrized by symmetric matrices G satisfying $G^2 = 0$. In this case, one again obtains a solution to the WDVV equations which satisfies the quasihomogeneity condition.

All eigenvalues of G are equal to 0. All Jordan blocks are of order 1 or 2. Consider the simplest case of only one block of order 2. The entries of the matrix $G = (g_{ij})$ can be written in the form

$$g_{ij} = \omega_i\omega_j, \qquad \sum_{i=1}^{n}\omega_i^2 = 0.$$

The corresponding solution to the WDVV system can be obtained from the trivial (i.e., cubic) solution

$$F(v) = \frac{1}{6}\sum_{i,j,k}c_{ijk}v^iv^jv^k$$

by applying the inversion symmetry described in [5] (see Appendix B and Proposition 3.14 in [5]). Here the c_{ijk} are the structure constants of the semisimple Frobenius algebra

$$\mathscr{A} = \mathrm{span}(e_1, \ldots, e_n), \quad \langle e_i \cdot e_j, e_k\rangle = c_{ijk}, \quad \langle e_i, e_j\rangle = \delta_{i+j,n+1}$$

with a unit e_1 and trivial grading $\deg e_i = 0$ for all i. Recall that the structure constants can be represented in the form

$$c_{ijk} = \sum_{s=1}^{n}\frac{a_{si}a_{sj}a_{sk}}{a_{s1}},$$

where the matrix (a_{ij}) satisfies the condition

$$\sum_{s=1}^{n}a_{si}a_{sj} = \delta_{i+j,n+1}.$$

For our construction, we can choose the matrix in such a way that

$$a_{i1} = \omega_i, \qquad i = 1, \ldots, n.$$

After the substitution of

$$\hat{v}^1 = \frac{1}{2}\frac{v_\alpha v^\alpha}{v^n},$$
$$\hat{v}^\alpha = \frac{v^\alpha}{v^n}, \qquad \alpha \neq 1, n,$$
$$\hat{v}^n = -\frac{1}{v^n}$$

one obtains the needed solution $\widehat{F}$ to the WDVV equations in the form

$$\widehat{F}(\hat{v}) = \frac{1}{2}\hat{v}^1\hat{v}_\alpha\hat{v}^\alpha + (\hat{v}^n)^2 F(v) = \frac{1}{2}(\hat{v}^1)^2\hat{v}^n + \frac{1}{2}\sum_{\alpha=2}^{n-1}\hat{v}^1\hat{v}^\alpha\hat{v}^{n-\alpha+1} + \frac{P(\hat{v}^2,\ldots,\hat{v}^{n-1})}{\hat{v}^n}. \quad (4.13)$$

Here[2] $P(\hat{v}^2,\ldots,\hat{v}^{n-1})$ is a certain polynomial of degree 4. The potential $\widehat{F}$ satisfies the quasihomogeneity condition

$$\widehat{E}\widehat{F} = \widehat{F}, \qquad \widehat{E} = \hat{v}^1\frac{\partial}{\partial\hat{v}^1} - \hat{v}^n\frac{\partial}{\partial\hat{v}^n}.$$

References

[1] G. Darboux, Leçons sur systèmes orthogonaux et les coordonnées curvilignes Paris, (1910)

[2] B. A. Dubrovin, "Completely integrable Hamiltonian systems associated with matrix finite-gap operators and Abelian varieties", *Funkts. Anal. Prilozhen.*, **11** :4, (1977) p. 28–41. , English transl.: *Functional Anal. Appl.*, **11** :4, (1977) p. 265–277.

[3] B. A. Dubrovin, "On differential geometry of strongly integrable systems of hydrodynamic type", *Funkts. Anal. Prilozhen.*, **24** :4, (1990) p. 25–30. , English transl.: *Functional Anal. Appl.*, **24** :4, (1990) p. 280–285.

[4] B. Dubrovin, "Integrable systems in topological field theory", *Nucl. Phys. B*, **379** :3, (1992) p. 627–689.

[5] B. Dubrovin, "Geometry of 2D topological field theories",   *in the book* Integrable Systems and Quantum Groups, Montecatini, Terme, 1993 Springer-Verlag, Berlin Lecture Notes in Math. **1620** (1996) p. 120–348.

[6] B. A. Dubrovin and S.P. Novikov, "The Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogoliubov–Whitham averaging method", *Dokl. Akad. Nauk SSSR*, **270** (1983) p. 781–785. , English transl.: *Sov. Math. Doklady*, **27** (1983) p. 665–669.

[7] D. F. Egorov, "A class of orthogonal systems", *Uch. Zap. Moskov. Univ., Sec. Fiz.-Mat.*, **18** (1901) p. 1–239.

[8] G. A. El, A. M. Kamchatnov, M. V. Pavlov and S. A. Zykov, "Kinetic equation for a soliton gas and its hydrodynamic reductions", *J. Nonlinear Sci.*, **21** :2, p. 151–191. (2011)

[9] P. Lancaster, L. Rodman, Algebraic Riccati Equations Clarendon Press, Oxford University Press, Oxford (1995)

[10] J. W. van de Leur, R. Martini, "The construction of Frobenius manifolds from KP tau-functions", *Comm. Math. Phys.*, **205** :3, (1999) p. 587–616.

[11] S. P. Tsarev, "Geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method.", *Izv. Akad. Nauk SSSR, Ser. Mat.*, **54** :5, (1991) p. 1048–1068. , English transl.: *Math. USSR Izv.*, **37** (1991) p. 397–419.

---

[2]This example was considered in [10] in a different context. Our formula (4.13) differs from that given in [10].

B. A. Dubrovin, Scuola Internazionale Superiore di Studi Avanzati, Trieste; Moscow State University, Laboratory of Geometric methods in Mathematical Physics; V. A. Steklov Mathematical Institute, Moscow

E-mail address: `dubrovin@sissa.it`

M. V. Pavlov, Moscow State University, Laboratory of Geometric methods in Mathematical Physics;, P. N. Lebedev Physical Institute of RAS

E-mail address: `maxim@math.sinica.edu.tw`

S. A. Zykov, University of Salento, Lecce,Department of Physics;, Institute of Metal Physics, Ural branch of RAS, Ekaterinburg

E-mail address: `zykov@le.infn.it`