On Cox-Kemperman moment inequalities for independent centered random variables

P.S.Ruzankin¹

Abstract

In 1983 Cox and Kemperman proved that $\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \leq \mathbf{E}f(\xi + \eta)$ for all functions f, such that f(0) = 0 and the second derivative f''(y) is convex, and all independent centered random variables ξ and η satisfying certain moment restrictions. We show that the minimal moment restrictions are sufficient for the inequality to be valid, and write out a less restrictive condition on f for the inequality to hold.

Besides, Cox and Kemperman (1983) found out the optimal constants A_{ρ} and B_{ρ} for the inequalities $A_{\rho}(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \leq \mathbf{E}|\xi + \eta|^{\rho} \leq B_{\rho}(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho})$, where $\rho \geq 1$, ξ and η are independent centered random variables. We write out similar sharp inequalities for symmetric random variables.

Keywords: Cox-Kemperman inequalities, moment inequalities, centered random variable, symmetric random variable, two-point distribution.

1. Introduction and formulation of the results

Cox and Kemperman have proved the following theorem:

Theorem A [Cox and Kemperman, 1983].

Let random variables ξ and η be such that

$$\mathbf{E}(\xi|\eta) = 0, \quad \mathbf{E}(\eta|\xi) = 0 \quad a.s. \tag{1}$$

Then, for each $\rho \geq 1$, the following inequalities hold:

$$2^{\rho-2}(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \le \mathbf{E}|\xi + \eta|^{\rho} \le \left(\max_{0 \le z \le 1} \psi(\rho, z)\right) (\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \quad \text{if } 1 \le \rho \le 2, \tag{2}$$

$$\left(\min_{0 \le z \le 1} \psi(\rho, z)\right) \left(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}\right) \le \mathbf{E}|\xi + \eta|^{\rho} \le 2^{\rho - 2} \left(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}\right) \quad \text{if } 2 \le \rho \le 3,\tag{3}$$

$$\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho} < \mathbf{E}|\xi + \eta|^{\rho} < 2^{\rho-2}(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \quad \text{if } \rho > 3$$

$$\tag{4}$$

whenever $\mathbf{E}|\xi|^{\rho} < \infty$ and $\mathbf{E}|\eta|^{\rho} < \infty$, where

$$\psi(\rho, z) = 2^{\rho - 1} (z + z^{\rho - 1} + (1 - z)^{\rho}) / ((1 + z)(1 + z^{\rho - 1}).$$

All the estimates in (2) – (4) for $\mathbf{E}|\xi+\eta|^{\rho}$ are sharp in the sense that, for each inequality, there exist distributions of ξ and η , such that $\xi \neq 0$ and the inequality turns into equality for independent ξ and η with these distributions.

This theorem does not consider the case $0 < \rho < 1$ because in the case the sharp inequalities are trivial ones: $0 \le \mathbf{E}|\xi + \eta|^{\rho} \le \mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}$.

Besides, Cox and Kemperman (1983) noted that, for i.i.d. ξ and η having a symmetric two-point distribution,

$$\mathbf{E}|\xi + \eta|^{\rho} = 2^{\rho - 2} (\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \quad \text{for all } \rho > 0.$$

It is also known (e.g. see Rosenthal (1972)) that, for symmetric independent random variables ξ and η ,

$$\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho} \le \mathbf{E}|\xi + \eta|^{\rho} \quad \text{if } \rho \ge 2 \tag{6}$$

when $\mathbf{E}|\xi|^{\rho} < \infty$ and $\mathbf{E}|\eta|^{\rho} < \infty$.

¹Sobolev Institute of Mathematics, Novosibirsk State University, pr. Ak. Koptyuga, 4, Novosibirsk, Russia, ruzankin@math.nsc.ru

Besides of estimates for expectations of power functions, there have been obtained certain inequalities for expectations for some other classes of functions.

Theorem B [Cox and Kemperman, 1983]. Let a function f on the real line be such that $f(0) \leq 0$ and the second derivative f''(y) exists for all y and is convex. Let random variables ξ and η satisfy condition (1) and be such that

$$\mathbf{E}|f'(\xi)| < \infty, \quad \mathbf{E}|f'(\eta)| < \infty.$$
 (7)

Then

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \le \mathbf{E}f(\xi + \eta).$$

Note that by Theorem E below condition (7) can be omitted for independent ξ and η .

Appendix A below contains a simple proof of this theorem proposed by Borisov. (The proof is valid under certain moment restrictions which can also be omitted for independent ξ and η by Theorem E.)

Note that the function $f(y) = |y|^{\rho}$ satisfies the conditions of this theorem only if $\rho = 2$ or $\rho \geq 3$.

Utev has obtained the following corresponding result:

Theorem C [Utev, 1985]. Let a function f on the real line have f''(y) for all y. Then the following three conditions are equivalent

- 1). f'' is convex.
- 2). For all independent symmetric random variables ξ and η and for all y, the inequality

$$\mathbf{E}f(y+\xi) + \mathbf{E}f(y+\eta) \le f(y) + \mathbf{E}f(y+\xi+\eta)$$

holds whenever these expectations exist.

3). For all independent bounded random variables ξ and η , such that $\mathbf{E}\xi = \mathbf{E}\eta = 0$, and for all y, the inequality

$$\mathbf{E}f(y+\xi) + \mathbf{E}f(y+\eta) \le f(y) + \mathbf{E}f(y+\xi+\eta)$$

holds.

In fact, Utev formulated and proved this theorem for Hilbert space - valued random variables. However such spaces will not be discussed in the present paper.

Note that the restriction that the random variables ξ and η in condition 3) be bounded can be omitted by Theorem E

Another class of functions is considered in the following statement.

Theorem D [Figiel, Hitczenko, Johnson, Schechtman, and Zinn, 1997]. Let a function $f, f(0) \leq 0$, be even and such that the function $y \mapsto f(\sqrt{|y|})$ is convex. Then

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \le \mathbf{E}f(\xi + \eta)$$

for all random variables ξ and η , such that the conditional distribution of η under the condition $\xi = x$ is symmetric for all x.

Note that the function $|y|^{\rho}$ satisfies conditions of this theorem only if $\rho > 2$.

We will call a function f on the real line twice differentiable if f'(y) exists for all y, f''(y) exists for almost all (with respect to the Lebesgue measure) y, and, for all a < b, $f'(b) - f'(a) = \int_a^b f''(y) dy$. **Theorem 1.** Let a function f be twice differentiable, $f(0) \le 0$, and the function f''(t) + f''(-t) be nondecreasing

for t > 0.

Then, for all independent symmetric random variables ξ and η ,

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \le \mathbf{E}f(\xi + \eta)$$

whenever the expectations exist.

A useful corollary of the theorem is the following one.

Corollary 1. For independent symmetric random variables ξ and η , the following inequalities are valid:

$$2^{\rho-2}(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \le \mathbf{E}|\xi + \eta|^{\rho} \le \mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho} \quad \text{if } 0 < \rho \le 2, \tag{8}$$

$$\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho} \le \mathbf{E}|\xi + \eta|^{\rho} \le 2^{\rho - 2}(\mathbf{E}|\xi|^{\rho} + \mathbf{E}|\eta|^{\rho}) \quad \text{if } \rho \ge 2$$
(9)

when $\mathbf{E}|\xi|^{\rho} < \infty$ and $\mathbf{E}|\eta|^{\rho} < \infty$. The four estimates for $\mathbf{E}|\xi+\eta|^{\rho}$ are sharp in the sense that, for each inequality, there exist distributions of ξ and η , such that $\xi \neq 0$ and the inequality turns into equality for these distributions.

Note that (9) and (8) for $\rho = 1$ follow directly from Theorem A and relations (5) and (6). Besides, the left inequality in (8) for $1 \le \rho \le 2$ follows directly from Theorem A and relation (5).

Note also that the right inequality in (8) for $0 < \rho \le 1$ is trivial because $|\alpha + \beta|^{\rho} \le |\alpha|^{\rho} + |\beta|^{\rho}$ for any real numbers α and β , $0 < \rho \le 1$.

Theorem 2. Let a function f be twice differentiable and such that $f(0) \leq 0$ and

$$f''(-\alpha) + f''(\beta) \ge f''(-\alpha + \gamma) + f''(\beta - \gamma),\tag{10}$$

for any $\alpha > 0$, $\beta > 0$, $0 < \gamma < \alpha + \beta$ such that f'' is defined at the points $-\alpha$, β , $-\alpha + \gamma$ and $\beta - \gamma$.

Then, for all independent centered random variables ξ and η ,

$$\mathbf{E}f(\xi) + \mathbf{E}f(\eta) \le \mathbf{E}f(\xi + \eta)$$

whenever the expectations exist.

Note that if f'' is convex then it satisfies the condition (10). But the class of functions subject to condition (10) is wider than the class of functions with convex second derivative.

For instance, if $f''(y) = \lfloor h(y) \rfloor$, where h(y) is nonnegative and convex, h(0) = 0, then f(y) satisfies condition (10). $\lfloor \cdot \rfloor$ denotes integer part of a number.

As another example, we can take f''(y) = -y if y < 1, $f''(y) = \lfloor y \rfloor - (y - \lfloor y \rfloor)$ if $y \ge 1$. Such f(y) also satisfies (10).

Remark. In Theorem 2, if $\xi + \eta \in [-B, C]$ a.s. then it is sufficient to require that the function f satisfy condition (10) only for α , β lying in (-B, C).

In Theorem 1, if $\xi + \eta \leq C$ a.s. then it is sufficient to require that f''(t) + f''(-t) be nondecreasing for 0 < t < C. The proof of Theorem 1 is based on the fact that any symmetric distribution can be "decomposed" into a mixture of symmetric distributions (e.g. see Figiel, Hitczenko et al. (1997)). As for Theorem 2, any centered distribution can be "decomposed" into a mixture of two-point centered distributions (e.g. see Pinelis (2009) and references therein). So one can prove the corresponding inequalities for two-point distributions only:

Theorem E. Let a function g of m + n arguments, $m \ge 0$ and $n \ge 0$, be such that

$$\mathbf{E}g(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n) \ge 0$$

for all independent random variables ξ_j and η_j , where each of the random variables ξ_j has a centered two-point distribution or equals zero, and each of η_j has a symmetric two-point distribution or equals zero.

Then, for all independent random variables $\xi_1, ..., \xi_m, \eta_1, ..., \eta_n$, where the random variables ξ_j are centered and η_j are symmetric, the following inequality is valid:

$$\mathbf{E}g(\xi_1, ..., \xi_m, \eta_1, ..., \eta_n) \ge 0$$

whenever the expectation exists.

2. Proofs

2.1. Proof of Theorem E

For the sake of convenience we give here the proof of Theorem E, but for the case m = 2, n = 0 only. The case of arbitrary m and n can be considered analogously.

Denote $\xi = \xi_1$, $\eta = \xi_2$. If ξ and η have centered two-point distributions, ξ takes values -a, b and η takes values -c, d then

$$P(\xi = -a) = b/(a+b), P(\xi = b) = a/(a+b), P(\eta = -c) = d/(c+d), P(\xi = d) = c/(c+d).$$

Thus we have

$$\mathbf{E}f(\xi,\eta) = \frac{1}{(a+b)(c+d)} \left(bd \, f(-a,-c) + bc \, f(-a,d) + ad \, f(b,-c) + ac \, f(b,d) \right) \ge 0 \tag{11}$$

for all a, b, c, d > 0.

Now let ξ and η have arbitrary centered distributions such that $\mathbf{P}(\xi \neq 0) = \mathbf{P}(\eta \neq 0) = 1$. Put

$$p = \mathbf{P}(\xi > 0), \quad F_{\xi}(u) = \mathbf{P}(\xi < u) - (1 - p), \quad G_{\xi}(u) = \mathbf{P}(-\xi < u) - p.$$

Put also

$$s(y) = \int_0^y F_{\xi}^{(-1)}(u)du, \quad t(x) = \int_0^x G_{\xi}^{(-1)}(u)du,$$

where $F_{\xi}^{(-1)}(u) := \sup\{x : F_{\xi}(x) < u\}$ is the quantile transformation of F_{ξ} , Then s(y) and t(x) are (strictly) increasing continuous functions on [0, p] and [0, 1-p], respectively, and

$$s(p) = \mathbf{E} \max\{0, \xi\} = \mathbf{E} \max\{0, -\xi\} = t(1-p).$$

Put

$$z(y) = t^{-1}(s(y)).$$

We have t(z(y)) = s(y), hence dt(z(y)) = ds(y) which can be rewritten as

$$G_{\xi}^{(-1)}(z(y)) dz(y) = F_{\xi}^{(-1)}(y) dy.$$

For a function h,

$$\mathbf{E}h(\xi) = \int_0^p h(F_{\xi}^{(-1)}(y)) dy + \int_0^{1-p} h(-G_{\xi}^{(-1)}(x)) dx.$$

Substituting x = z(y) into the last integral yields

$$\mathbf{E}h(\xi) = \int_0^p \left(h(F_{\xi}^{(-1)}(y)) + h(-G_{\xi}^{(-1)}(z(y)) \frac{F_{\xi}^{(-1)}(y)}{G_{\xi}^{(-1)}(z(y))} \right) dy. \tag{12}$$

Let us introduce the same notations for η . Put

$$q = \mathbf{P}(\eta > 0), \quad F_{\eta}(t) = \mathbf{P}(\eta < u) - (1 - q), \quad G_{\eta}(t) = \mathbf{P}(-\eta < u) - q,$$

and let w(v) be defined by the relations

$$w(0) = 0,$$
 $G_{\eta}^{(-1)}(w(v))dw(v) = F_{\eta}^{(-1)}(v)dv.$

Using the above notations we can write

$$\mathbf{E}g(\xi,\eta) = \int_0^q \int_0^p \psi(y,v)dy\,dv,$$

where

$$\begin{split} \psi(y,v) &:= g\big(F_{\xi}^{(-1)}(y),F_{\eta}^{(-1)}(v)\big) + g\big(F_{\xi}^{(-1)}(y),-G_{\eta}^{(-1)}(w(v))\big) \frac{F_{\eta}^{(-1)}(v)}{G_{\eta}^{(-1)}(w(v))} + \\ g\big(-G_{\xi}^{(-1)}(z(y)),F_{\eta}^{(-1)}(v)\big) \frac{F_{\xi}^{(-1)}(y)}{G_{\varepsilon}^{(-1)}(z(y))} + g\big(-G_{\xi}^{(-1)}(z(y)),-G_{\eta}^{(-1)}(w(v))\big) \frac{F_{\xi}^{(-1)}(y)}{G_{\varepsilon}^{(-1)}(z(y))} \frac{F_{\eta}^{(-1)}(v)}{G_{\eta}^{(-1)}(w(v))}, \end{split}$$

and $\psi(y, v) \ge 0$ by relation (11).

We have proved the statement of the theorem for the case $\mathbf{P}(\xi \neq 0) = \mathbf{P}(\eta \neq 0) = 1$. The case $\mathbf{P}(\xi = 0) > 0$ or $P(\eta = 0) > 0$ can be easily dealt with using mixtures of zero and nonzero distributions.

2.2. Proof of Theorem 2

Without loss of generality we can assume f(0) = 0.

By Theorem E it is sufficient to prove the statement of Theorem 2 for all ξ and η with centered two-point distributions.

Take $\xi \in \{-a, b\}, \eta \in \{-c, d\}$, where a, b, c, d > 0. We have

$$\mathbf{E}f(\xi+\eta) - \mathbf{E}f(\xi) - \mathbf{E}f(\eta) = \frac{1}{(a+b)(c+d)}\phi(a,b,c,d),$$

where

$$\begin{array}{rcl} \phi(r,s,t,u) & = & su & \left(f(-r-t) - f(-r) - f(-t) \right) \\ & + & st & \left(f(-r+u) - f(-r) - f(u) \right) \\ & + & ru & \left(f(s-t) - f(s) - f(-t) \right) \\ & + & rt & \left(f(s+u) - f(s) - f(u) \right). \end{array}$$

Note that $\phi(r, s, t, u) = 0$ if r = 0 or s = 0 or t = 0 or u = 0. Moreover,

$$\frac{\partial^4}{\partial r \partial s \partial t \partial u} \phi(r, s, t, u) = f''(-r - t) + f''(s + u) - f''(-r + u) - f''(s - t) \ge 0$$

for positive r, s, t, u by condition (10).

Therefore

$$0 \le \int_0^d \int_0^c \int_0^b \int_0^a \frac{\partial^4}{\partial r \partial s \partial t \partial u} \phi(r, s, t, u) \, dr \, ds \, dt \, du = \sum_{c \in \{0, a\}, s \in \{0, b\}, t \in \{0, c\}, u \in \{0, d\}} (-1)^{\operatorname{sgn} r + \operatorname{sgn} s + \operatorname{sgn} t + \operatorname{sgn} u} \phi(r, s, t, u) = \phi(a, b, c, d),$$

where $\operatorname{sgn} r = 0$ if r = 0 and $\operatorname{sgn} r = 1$ if r > 0. Thus

$$\phi(a, b, c, d) \ge 0,$$

and hence the theorem is proved.

2.3. Proof of Theorem 1

The proof is analogous to that of Theorem 2.

Without loss of generality we can assume f(0) = 0.

By Theorem E it is sufficient to prove the statement of Theorem 1 for all ξ and η with symmetric two-point distributions.

Take $\xi \in \{-a, a\}, \eta \in \{-b, b\}$, where a, b > 0. We have

$$\mathbf{E}f(\xi + \eta) - \mathbf{E}f(\xi) - \mathbf{E}f(\eta) = \frac{1}{4}\phi(a, b),$$

where

$$\phi(r,s) = (f(-r-s) - f(-r) - f(-s)) + (f(-r+s) - f(-r) - f(s)) + (f(r-t) - f(r) - f(-s)) + (f(r+s) - f(r) - f(s)).$$

Further,

$$\frac{\partial^2}{\partial r \partial s} \phi(r,s) = f''(-r-s) + f''(r+s) - f''(-r+s) - f''(r-s) \ge 0$$

for positive r, s because f''(t) + f''(-t) is nondecreasing for positive t.

Thus

$$0 \le \int_0^b \int_0^a \frac{\partial^2}{\partial r \partial s} \phi(r, s) \, dr \, ds = \phi(a, b) - \phi(a, 0) - \phi(0, b) + \phi(0, 0) = \phi(a, b).$$

Therefore $\phi(a,b) \geq 0$, and hence the theorem is proved.

2.4. Proof of Corollary 1

In the case $1 < \rho < 2$ the function $f(y) = -|y|^{\rho}$ satisfies the conditions of Theorem 1. Hence, (8) is valid for $1 < \rho < 2$.

It remains to show that the left inequality in (8) holds for $0 < \rho < 1$. By Theorem E it suffices to show that, for any a and b, 0 < a < b,

$$\phi(a,b) := (a+b)^{\rho} + (b-a)^{\rho} - 2^{\rho-1}a^{\rho} - 2^{\rho-1}b^{\rho} > 0.$$

We have $\phi(a, b) = a^{\rho}h(z)$, where z = b/a,

$$h(z) = (1+z)^{\rho} + (z-1)^{\rho} - 2^{\rho-1} - 2^{\rho-1}z^{\rho},$$

Thus $h(z) \ge 0$ for $z \ge 1$ because h(1) = 0, h'(z) > 0 for $z \ge 1$.

Hence, (8) is valid for $0 < \rho < 1$.

The corollary is proved.

Appendix A. A simple proof of Theorem B

I.S. Borisov in an oral conversation has proposed the following proof of Theorem B. We have

$$f(\xi + \eta) = f(\xi) + f'(\xi)\eta + \eta^2 \int_0^1 (1 - z)f''(\xi + z\eta)dz.$$

Now let us consider the expectation of the last integral. By convexity of f'',

$$\mathbf{E}\left(\eta^{2} \int_{0}^{1} (1-z)f''(\xi+z\eta)dz \mid \eta\right) \geq \eta^{2} \int_{0}^{1} (1-z)f''(z\eta)dz = f(\eta) - f(0) - f'(o)\eta.$$

Thus

$$\mathbf{E}f(\xi + \eta) \ge \mathbf{E}f(\xi) + \mathbf{E}f(\eta) + \mathbf{E}f'(\xi)\eta - \mathbf{E}f'(0)\eta.$$

The restriction of this proof is that all the needed moments, such as $\mathbf{E}\eta^2 f''(\xi + z\eta)$ for 0 < z < 1, must exist. For, roughly speaking, "regular" functions f(y) growing not faster than $e^{c|y|}$, c = const, the existence of the moments follows from the monotonicity of the functions f''(y), f'(y), f(y) for sufficiently large y and for sufficiently large -y. As was noted above, for independent ξ and η , these moment restrictions can be omitted by virtue of Theorem E.

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