

THE COMMUTATOR SUBGROUP OF THE HECKE GROUP G_5 IS NOT CONGRUENCE

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ABSTRACT. Let $q \geq 3$ be an integer and let G_q be the Hecke group associated with q . We prove that the power subgroup G_5^5 and the commutator subgroup G_5' are not congruence.

1. INTRODUCTION

1.1. Let $q \geq 3$ be a fixed integer. The (homogeneous) Hecke group H_q is defined to be the maximal discrete subgroup of $SL_2(\mathbb{R})$ generated by S and T , where $\lambda_q = 2\cos(\pi/q)$,

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}. \quad (1.1)$$

Let A be an ideal of $\mathbb{Z}[\lambda_q]$. The principal congruence subgroup of H_q of level A is defined to be

$$H(q, A) = \{(a_{ij}) \in H_q : a_{11} - 1, a_{22} - 1, a_{12}, a_{21} \in A\}. \quad (1.2)$$

Let $Z = \langle \pm I \rangle$. The (inhomogeneous) Hecke group and its principal congruence subgroup are defined as $G_q = H_q/Z$ and $G(q, A) = H(q, A)Z/Z$. A subgroup K of G_q is congruence if $G(q, A) \subseteq K$ for some A . Whether subgroups of finite indices are congruence have been studied extensively (see [F], [Lu], [S]). In the case $q = 3$, it is known that not every subgroup of finite index of the modular group G_3 is congruence and that the commutator subgroup G_3' is congruence of level 6. We suspect that $q = 3$ is the only case that G_q' is congruence (see Discussion 5.3). The main purpose of the present article is to show that

Proposition 5.2. *The subgroups G_5^5 and G_5' of the Hecke group G_5 are not congruence.*

Note that G_q^n is the subgroup of G_q generated by all the elements of the form $x^n \in G_q$. Note also that in the case $q \geq 3$ is a prime, these two groups G_q^q and G_q^2 are special in the sense that they are the only normal torsion subgroups of G_q .

1.2. Our proof of the above proposition is elementary and requires some basic facts about the fundamental domains of certain subgroups of G_5 . The following two facts about G_5 are essential in our proof as well.

- (i) If G_5^5 is congruence, then $G(5, 5) \subseteq G_5^5$ (Lemma 5.1).
- (ii) $G_5/G(5, 5) \cong E_{5^3}PSL(2, 5)$ does not possess subgroups of index 5 (Proposition 5.2), where E_{5^3} is an elementary abelian 5-group of order 5^3 . Note that the indices of G_q^q and G_q' in G_q are q and $2q$ respectively (Lemmas 3.3).

1.3. The rest of the article is organised as follows. In Sections 2 and 3, we study the geometric aspects of the Hecke group G_q , such study allows us to give the geometric invariants (index, number of elliptic elements, number of cusps, genus) of G_5^5 and G_5' . Section 4 lists all the known results which is necessary for our study of G_5^5 and G_5' . They are mainly results on the indices of the principal congruence subgroups of G_5 . Section 5 gives us the main result of the present article. The present article is part of our project on G_q . We have determined the normalisers (see [L]) and the indices (see [LL1], [LLT2]) for some congruence subgroups of G_5 . We are currently working on the index formula for $G(q, \pi)$, where $q \geq 7$.

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2. GEOMETRIC INVARIANTS

In [K], Kulkarni applied a combination of geometric and arithmetic methods to show that one can produce a set of independent generators in the sense of Rademacher for the congruence subgroups of the modular group, in fact for all subgroups of finite indices. His method can be generalised to all subgroups of finite indices of the Hecke groups G_q , where q is a prime. See [LLT1] for detail (Propositions 8-10 and section 3 of [LLT1]). In short, for each subgroup V of finite index of G_q , one can associate to V a set of Hecke-Farey symbols (HFS) $\{-\infty, x_0, x_1, \dots, x_n, \infty\}$, a special polygon (fundamental domain) Φ , and an additional structure on each consecutive pair of x_i 's of the three types described below :

$$x_i \underset{\circ}{\frown} x_{i+1}, x_i \underset{\bullet}{\frown} x_{i+1}, x_i \underset{a}{\frown} x_{i+1}.$$

where a is a nature number. Each nature number a occurs exactly twice or not at all. Similar to the modular group, the actual values of the a 's is unimportant: it is the pairing induced on the consecutive pairs that matters.

- (i) The side pairing \circ is an elliptic element of order 2 that pairs the even line (x_i, x_{i+1}) with itself. The trace of such an element is 0.
- (ii) The side pairing \bullet is an elliptic element of order q that pairs the odd line (x_i, x_{i+1}) . The absolute value of the trace of such an element is λ_q .
- (iii) The two sides with the label a are paired together by an element of infinite order.
- (iv) The special polygon associated to the HFS is a fundamental domain of V and the side pairings $I = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$ associated to the HFS is a set of independent generators of V (Theorem 7, Propositions 8-10 of [LLT1]).
- (v) The number d of special triangles (a special triangle is a fundamental domain of G_q) of the special polygon is the index of the subgroup.
- (vi) The set of independent generators consists of r matrices of infinite order, where r is the number of the nature number a 's in the Hecke-Farey symbols.
- (vii) The subgroup has v_2 (the number of the circles \circ in HFS) inequivalent classes of elliptic elements of order 2. Each class has exactly one representative in I .
- (viii) The subgroup has v_q (the number of the bullets \bullet in HFS) inequivalent classes of elliptic elements of order q . Each class has exactly one representative in I .
- (ix) The Hecke-Farey symbols can be partitioned into v_∞ classes under the action of the set of independent generators, which gives the number of cusps of the subgroup.
- (x) The genus g can be determined by the Riemann-Hurwitz formula.

$$(q-2)d = qv_2 + 2(q-1)v_q + 4qg + 2qv_\infty - 4q. \quad (2.1)$$

- (xi) The width of a cusp x , denoted by $w(x)$, is the number of even lines in Φ that comes into x . Algebraically, it is the smallest positive integer m such that $\pm T_q^m$ is conjugate in G_q to an element of K fixing x (keep in mind that a matrix is identified with its negative in G_q). The least common multiple N of the cusp widths of V is called the geometric width of V .

Discussion 2.1. The vertices of the Hecke-Farey symbols can be obtained by applying Lemma 3 of [LLT1] and the side pairings in (i)-(iii) of the above can be obtained by Propositions 8-10 of [LLT1].

3. SUBGROUPS OF SMALL INDICES, POWER SUBGROUPS

Let $q \geq 3$ be a prime and let K be a subgroup of G_q . It is clear that if K is of index 2, then the only possible Hecke-Farey symbols for K is $\{-\infty, 0, \infty\}$ with the set of independent generators $\{ST^{-1}, T^{-1}S\}$. The invariants of K is given by

$$d = 2, v_2 = 0, v_q = 2, v_\infty = 1, g = 0. \quad (3.1)$$

It is not clear that G_q cannot possess subgroups of indices between 3 and $q - 1$ from algebraic point of view. However, it is clear that there is no such Hecke-Farey symbols. As a consequence, we have the following :

Proposition 3.1. *Let K be a subgroup of G_q of index at most $q - 1$. Then K is generated by the set of independent generators $\{ST^{-1}, T^{-1}S\}$, where $o(ST^{-1}) = o(T^{-1}S) = q$. The invariants of K are $d = 2, v_2 = 0, v_q = 2, v_\infty = 1, g = 0$. Further, $[K : K'] = q^2$.*

Proof. Since $\{ST^{-1}, T^{-1}S\}$ is a set of independent generators and $o(ST^{-1}) = o(T^{-1}S) = q$, one must have $[K : K'] = q^2$. The rest of the proposition is clear. \square

Remark. Note that unlike G_q ($q \geq 5$), $G_3 = PSL_2(\mathbb{Z})$ does possess subgroups of all possible indices, which can be proved by investigation of the Hecke-Farey symbols.

3.1. Power subgroups of G_q . Denoted by G_q^n the subgroup of G_q generated by all the elements of the form x^n , where $x \in G_q$. It is clear that G_q^n is a characteristic subgroup of G_q . Since G_q is a free product of two elliptic elements of orders 2 and q respectively, G_q^n is a proper subgroup of G_q if and only if $\gcd(n, 2q) \neq 1$. The following are well known.

Lemma 3.2. *Let q be an odd prime. Then G_q^2 is the only subgroup of G_q of index 2. G_q^2 is a free product of two elliptic elements of order q . In particular, $[G_q^2 : [G_q^2, G_q^2]] = q^2$.*

Proof. Since $\{S, ST^{-1}\}$ is a set of independent generators of G_q , $o(S) = 2, o(ST^{-1}) = q$, one has $ST^{-1}, T^{-1}S \in G_q^2, S \notin G_q^2$. We may now complete the proof of the lemma by applying Proposition 3.1. \square

Lemma 3.3. *Let q be an odd prime. Then G_q^q is the only normal subgroup of G_q of index q . Further, G_q^q is a free product of q elliptic elements of order 2 and $[G_q^q : [G_q^q, G_q^q]] = 2^q$. The invariants of G_q^q are given by $d = q, v_2 = q, v_q = 0, v_\infty = 1, g = 0$.*

Proof. It is clear that $S \in G_q^q, ST^{-1} \notin G_q^q$. Hence G_q^q is a proper subgroup that contains all the elliptic elements of order 2 (G_q^q is normal). Let K be the subgroup of G_q with Hecke-Farey symbols

$$\{-\infty = x_0 \underset{\circ}{\cup} x_1 \underset{\circ}{\cup} x_2, \dots, x_{q-1/2} \underset{\circ}{\cup} x_{q+1/2}, \dots, x_{q-2} \underset{\circ}{\cup} x_{q-1} \underset{\circ}{\cup} x_q = \infty\},$$

where the x_i ' are the vertices of an ideal q -gon of depth 1 (see Discussion 2.1 of Section 2). It follows that $[G_q : K] = q$ and that a set of independent generators of K is given by $\{g_1, g_2, \dots, g_q\}$, where $o(g_i) = 2$ for all i . Since G_q^q contains all the elliptic elements of order 2, we conclude that K is a subgroup of G_q^q . An easy study of the indices implies that $G_q^q = K$. Since $G_q^q = K$ is generated by q independent generators of order 2, $[G_q^q : [G_q^q, G_q^q]] = 2^q$. Let I be a normal subgroup of index q of G_q . Since q is an odd prime, $S = S^q \in I$. Since I is normal, I contains all the elliptic elements of order 2. Hence $G_q^q \subseteq I$. Since they have the same index, one must have $I = G_q^q$. This implies that G_q^q is the only normal subgroup of index q of G_q . \square

Example 3.4. The side pairings associated with the Hecke-Farey symbols of G_5^5 is given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & -1 \\ \lambda+2 & -\lambda \end{pmatrix}, \begin{pmatrix} 2\lambda+1 & -2\lambda-2 \\ \lambda+2 & -2\lambda-1 \end{pmatrix}, \begin{pmatrix} 2\lambda+1 & -\lambda-2 \\ 2\lambda+2 & -2\lambda-1 \end{pmatrix}, \begin{pmatrix} \lambda & -\lambda-2 \\ 1 & -\lambda \end{pmatrix}.$$

4. KNOWN RESULTS ABOUT G_5

Applying the main results in [LL1] and [LL2] (Section 7 of [LL1] and Theorem 4.1 of [LL2]), we have the following.

- (i) $G_5/G(5, 5) \cong G(5, \lambda + 2)/G(5, 5) \cdot G_5/G(5, \lambda + 2) \cong E_{5^3}PSL(2, 5)$, where $G(5, \lambda + 2)/G(5, 5) \cong E_{3^5} \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ is the elementary abelian group of order 5^3 and $G_5/G(5, \lambda + 2) \cong PSL(2, 5) \cong A_5$.
- (ii) Let V be a congruence subgroup of G_5 . Suppose that the geometric level of V is r where r is odd (see (xi) for the definition of the geometric level), then $G(5, r) \subseteq V$.

5. G_5^5 AND G'_5 ARE NOT CONGRUENCE

It is well known that the commutator subgroup of $\Gamma = G_3$ is congruence. The main purpose of this section is to show that the commutator subgroup G'_5 of G_5 is not congruence.

Lemma 5.1. *If G_5^5 is congruence, then $G(5, 5) \subseteq G_5^5$.*

Proof. By Lemma 3.3, the geometric level (see (xi) for the definition of the geometric level) of G_5^5 is 5. By (ii) of Section 4, $G(5, 5) \subseteq G_5^5$. \square

5.1. The group structure of $G(5, \lambda + 2)/G(5, 5)$. Recall first that $5 = \lambda^{-2}(\lambda + 2)^2$. By Example 3 of [LLT1],

$$a = \begin{pmatrix} -11\lambda - 6 & 10\lambda + 5 \\ 4\lambda + 3 & -4\lambda - 2 \end{pmatrix} = T^{-2} \begin{pmatrix} 3\lambda + 2 & -2\lambda - 3 \\ 4\lambda + 3 & -4\lambda - 2 \end{pmatrix} \in G(5, \lambda + 2) - G(5, 5) \quad (5.1)$$

By (i) of Section 4, $G(5, \lambda + 2)/G(5, 5)$ is elementary abelian of order 5^3 . It follows that $G(5, \lambda + 2)/G(5, 5)$ can be generated by $\Delta = \{a, b = SaS^{-1}, c = JaJ^{-1}\}$ (see (5.3) for the definition of J). Note that Δ modulo $G(5, 5)$ is given by

$$a \equiv I + (\lambda + 2) \begin{pmatrix} 4 & 0 \\ 4 & 1 \end{pmatrix}, b \equiv I + (\lambda + 2) \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, c \equiv I + (\lambda + 2) \begin{pmatrix} 1 & 4 \\ 0 & 4 \end{pmatrix}. \quad (5.2)$$

Proposition 5.2. *G_5^5 and G'_5 are not congruence.*

Proof. Since G_5/G'_5 is abelian of order 10 and G_5^5 is the only normal subgroup of G_5 of index 5 (Lemma 3.3), $G'_5 \subseteq G_5^5$. To prove our assertion, it suffices to show that G_5^5 is not congruence. Suppose that G_5^5 is congruence. By Lemma 5.1, $G(5, 5) \subseteq G_5^5$. Since $G_5/G(5, \lambda + 2) \cong A_5$ has no normal subgroup of index 5 and G'_5 has index 5 in G_5 , $G(5, \lambda + 2)$ is not a subgroup of G_5^5 . This implies that $G_5^5 G(5, \lambda + 2) = G_5$. By Second Isomorphism Theorem, $|G(5, \lambda + 2)/(G_5^5 \cap G(5, \lambda + 2))| = 5$ and $|[G_5^5 \cap G(5, \lambda + 2)]/G(5, 5)| = 5^2$. Note that $E_{5^3}A_5 \cong G_5/G(5, 5)$ acts on $D = [G_5^5 \cap G(5, \lambda + 2)]/G(5, 5) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ by conjugation. Note also that D is a subgroup of $\langle \Delta \rangle$. Recall that

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \text{Aut } G_5. \quad (5.3)$$

Since $[G(5, 5) \cap G(5, \lambda + 2)]/G(5, 5) = D$ is invariant under the conjugation of $E_{5^3}A_5 \cong G_5/G(5, 5)$, D is invariant under the conjugation of J and every element of G_5 (in particular, S and T). However, one sees by direct calculation that the only subgroup of $\langle \Delta \rangle$ invariant under J , S , and T is $\langle \Delta \rangle$ itself (see Appendix A). A contradiction. Hence G_5^5 is not congruence. \square

Discussion 5.3. A key step in the proof of G'_3 is congruence is that $G_3/G(3, 3) \cong A_4 \cong E_4\mathbb{Z}_3$ has a normal subgroup of index 3 (see Lemma 3.7). This fact is no longer true if $q = 5$ as $G_5/G(5, 5)$ possesses no normal subgroups of index 5. As this may be true for all $q \geq 5$, we therefore suggest that G'_q is not congruence if $q \geq 5$.

6. APPENDIX A

Lemma A1. *Let $\pi = \lambda + 2$ and let $\Delta = \{a, b, c\}$, where a, b, c are given as in (5.2). Then the only nontrivial subgroup of $\langle \Delta \rangle$ invariant under the action of S, T and J is $\langle \Delta \rangle$.*

Proof. Since $(I + \pi U)(I + \pi V) \equiv I + \pi(U + V) \pmod{5}$, multiplication of $(I + \pi U)(I + \pi V)$ can be transformed into addition of U and V . This makes the multiplication of matrices a , b , and c easy. Consequently, one has

$$r = (ac)(ab) \equiv I + \pi \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, s = (ac)(ab)^{-1} \equiv I + \pi \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, t = bc \equiv I + \pi \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}.$$

It is clear that $\langle \Delta \rangle = \langle a, b, c \rangle = \langle r, s, t \rangle$. Let $A, B \in G_5$. Set $A^B = BAB^{-1}$. Direct calculation shows that

$$r^S = s^{-1}, r^T = rs^{-1}t^2, r^J = s, s^S = r^{-1}, s^T = s, s^J = r, t^S = t^{-1}, t^T = st, t^J = t^{-1}. \quad (A1)$$

Denoted by M a nontrivial subgroup of $\langle r, s, t \rangle$ that is invariant under the conjugation of J , S and T . Let $1 \neq \sigma = r^i s^j t^k \in M$. One sees easily that

- (i) If $k \not\equiv 0 \pmod{5}$, without loss of generality, we may assume that $k = 1$. Then $\sigma^J \sigma^S = t^{-2} \in M$. It follows that $t \in M$. Hence $t^T = st \in M$. Consequently, $s \in M$. This implies $s^S = r^{-1} \in M$. In summary, $r, s, t \in M$.
- (ii) If $k \equiv 0 \pmod{5}$, then σ takes the form $r^i s^j$. Suppose that $i \equiv 0 \pmod{5}$. Then $1 \neq s^j \in M$. It follows that $s \in M$. Consequently, $r = s^T \in M$. Hence $rs^{-1}t^2 = r^T \in M$. As a consequence, $t \in M$. In summary, $r, s, t \in M$. In the case $i \not\equiv 0 \pmod{5}$, we may assume that $i = 1$. Hence $rs^j \in M$. It follows that $(rs^j)^T (rs^j)^{-1} = s^{-1}t^2 \in M$. Consequently, $(s^{-1}t^2)^T = st^2 \in M$. This implies that $(s^{-1}t^2)(st^2) = t^4 \in M$. Hence $t \in M$. One now sees easily that $r, s, t \in M$.

Hence the only nontrivial subgroup of $\langle \Delta \rangle$ invariant under J , S and T is $\langle \Delta \rangle$. □

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