

Surface pole bracket polynomials of virtual knots and twisted knots

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Abstract

Dye and Kauffman defined surface bracket polynomials for virtual links by use of surface states, and found a relationship between the surface states and the minimal genus of a surface in which a virtual link diagram is realized. They and Miyazawa independently defined a multivariable polynomial invariant of virtual links. This invariant is deeply related to the surface states. In this paper, we introduce the notion of surface pole bracket polynomials for link diagrams in closed surfaces, as a generalization of surface bracket polynomials by Dye and Kauffman. The polynomials induce the invariant of twisted links defined by the author before as a generalization of Dye, Kauffman and Miyazawa's polynomial invariant. Furthermore we discuss a relationship between curves in surface pole states and variables of the polynomial invariant.

1 Introduction

Virtual knot theory is a generalization of knot theory which is based on Gauss chord diagrams and link diagrams on closed oriented surfaces [7]. Virtual links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over closed oriented surfaces (cf. [2, 6]). A twisted link defined by Bourgoin [1] is an extension of the notion of vital links. Twisted links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundle over closed surfaces which are possibly non-orientable surfaces [1].

A *virtual link diagram* is a link diagram which may have *virtual crossings*, which are encircled crossings without over-under information. A *virtual link*

is an equivalence class of a virtual link diagram by Reidemeister moves and virtual Reidemeister moves depicted in Figures 1 and 2. We call these moves *generalized Reidemeister moves*.

A *twisted link diagram* is a virtual link diagram which may have *bars* on arcs. A *twisted link* is an equivalence class of a twisted link diagram by Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves in Figures 1, 2 and 3. We call these moves *extended Reidemeister moves*.

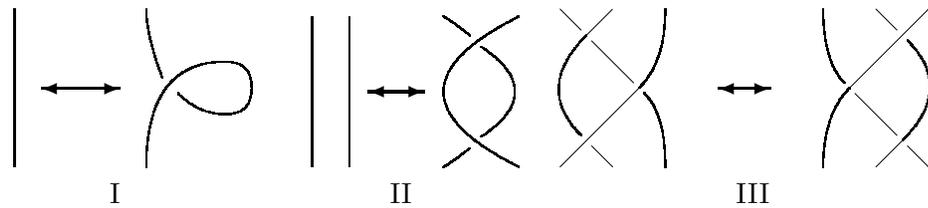


Figure 1: Reidemeister moves

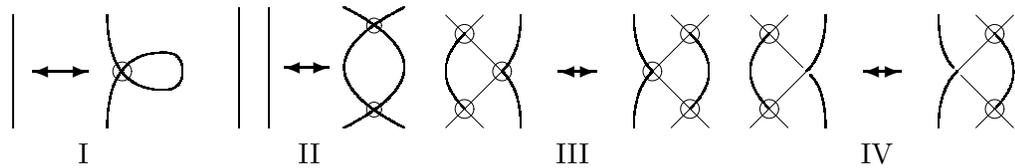


Figure 2: Virtual Reidemeister moves

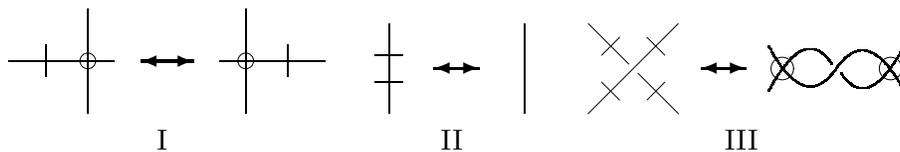


Figure 3: Twisted Reidemeister moves

Bourgoin introduced the Jones polynomials (f -polynomials) for twisted links and a group invariant called the twisted knot group [1]. The author introduced a twisted quandle for twisted links [5]. For a twisted link L , it is an interesting and important problem to determine an irreducible representative or to determine the minimum genus of a surface F in which a diagram

of L is realized. Surface bracket polynomials of virtual links are defined by Dye and Kauffman [3] by use of surface states which are obtained from a link diagram in a closed oriented surface in which a diagram of L is realized.

The following conjecture is due to Kauffman and Przytycki.

Conjecture 1 *For a virtual knot L , if a diagram of L is realized in a surface of the minimal genus, then this fact is detected by the surface bracket polynomial.*

H. A. Dye and L. H. Kauffman [4], and Y. Miyazawa [9] independently, defined a multivariable polynomial invariant of virtual links, which we call the DKM polynomial. Dye and Kauffman showed that this invariant is deeply related to the surface states for link diagrams on closed oriented surfaces. In this paper, we introduce the notion of surface pole bracket polynomials for link diagrams in closed surfaces, as a generalization of surface bracket polynomials by Dye and Kauffman. The polynomials induce the invariant of twisted links defined by the author in [5] as a generalization of the DKM polynomial invariant. Then we discuss a relationship between curves in surface pole states and variables of the polynomial invariant.

2 Link diagram realizations of twisted links

An *abstract link diagram* is a pair (Σ, D_Σ) of a compact, possibly non-orientable surface Σ and a link diagram D_Σ in Σ such that $|D_\Sigma|$ is a deformation retract of Σ , where $|D_\Sigma|$ is the subset of Σ obtained from D_Σ by replacing each crossing with a 4-valent vertex. Two examples of abstract links are depicted in Figure 4. The surface Σ in (ii) of the figure is a non-orientable surface.

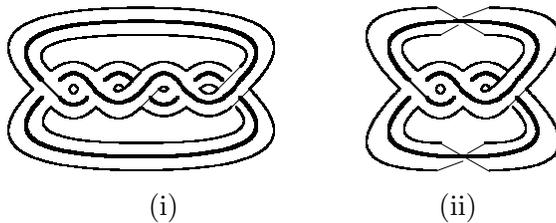


Figure 4: Examples of abstract link diagrams

Let (Σ_1, D_{Σ_1}) and (Σ_2, D_{Σ_2}) be abstract link diagrams, where Σ_i is a compact surface and D_{Σ_i} is a link diagram in Σ_i for each $i \in \{1, 2\}$. If there

are embeddings $f_1 : \Sigma_1 \rightarrow F$ and $f_2 : \Sigma_2 \rightarrow F$ to a common closed surface F such that $f_2(D_{\Sigma_2})$ is obtained from $f_1(D_{\Sigma_1})$ by a Reidemeister move in F , then we say that (Σ_2, D_{Σ_2}) is obtained from (Σ_1, D_{Σ_1}) by an *abstract Reidemeister move*. Two abstract link diagrams are said to be *equivalent* if they are related by a finite sequence of abstract Reidemeister moves. We call equivalence classes of abstract link diagrams *abstract links*. (This notion was introduced in [6] for the case where surfaces Σ 's and F 's are oriented. Note that, in this paper, we do not assume that these surfaces are orientable.)

Theorem 2 (Bourgoin [1]) *There is a map from the family of twisted link diagrams to that of abstract link diagrams such that it induces a bijection from the family of twisted links to that of abstract links.*

This map is depicted in Figure 5. We call the abstract link diagram obtained this way the *abstract link diagram associated with D* . For example, see Figure 6.

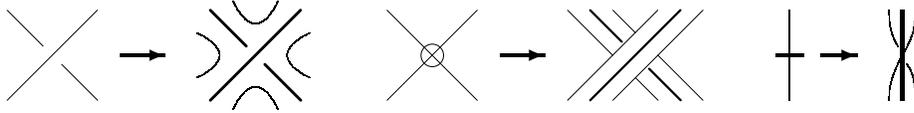


Figure 5: Twisted link diagram and abstract link diagram

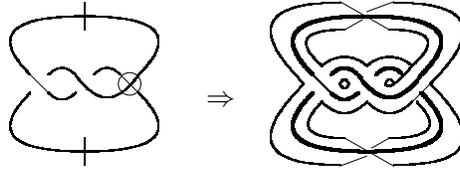


Figure 6: Abstract link diagram associated with twisted link diagram

A pair (F, D_F) of a closed surface F and a link diagram D_F in F is called a *link diagram realization* of a twisted link diagram D if there is an embedding $f : \Sigma \rightarrow F$ such that $f(D_{\Sigma}) = D_F$, where (Σ, D_{Σ}) is the abstract link diagram associated with D . For example, see Figure 7, where the link diagram realization depicted in the bottom right is a link diagram in a projective plane.

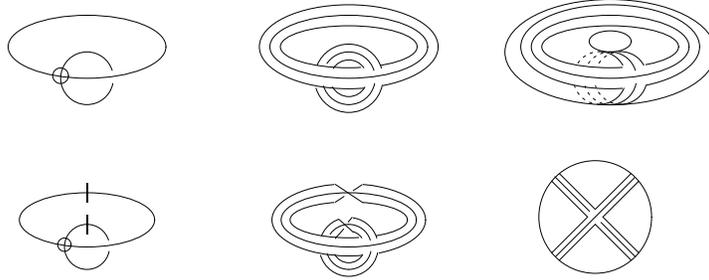


Figure 7: Examples of link diagram realizations

3 Surface pole states

A *pole curve* in a surface F is a simple closed curve with (or without) poles in F as in Figure 8, where we call the pole on the left side of the figure a *sink pole* (or an *I-pole*) and the one on the right side a *source pole* (or an *O-pole*). A collection of mutually disjoint pole curves in a surface F is called a *pole curve link* in F .

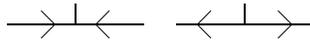


Figure 8: A sink pole (I-pole) and a source pole (O-pole)

The local moves depicted in Figure 9 are called *pole reductions*.



Figure 9: Pole reductions

A pole curve is said to be *irreducible* if one cannot apply any pole reduc-

tion to it. An irreducible pole curve looks like as in Figure 10, where there is no pole in the broken line of the curve. A pole curve link is said to be *irreducible* if every component is irreducible.

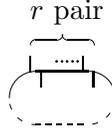


Figure 10: An irreducible pole curve

For a pole curve l in F , applying pole reductions, we obtain an irreducible pole curve in F . We denote it by \bar{l} . The *index* of l is the half of the numbers of poles of the irreducible pole curve \bar{l} obtained from l , which is denoted by $\iota(l)$.

Let D be a twisted link diagram and (F, D_F) a link diagram realization of D in a surface F . The local replacement at a crossing of a link diagram in F illustrated in Figure 11 is called an *A-splice* or a *B-splice*.

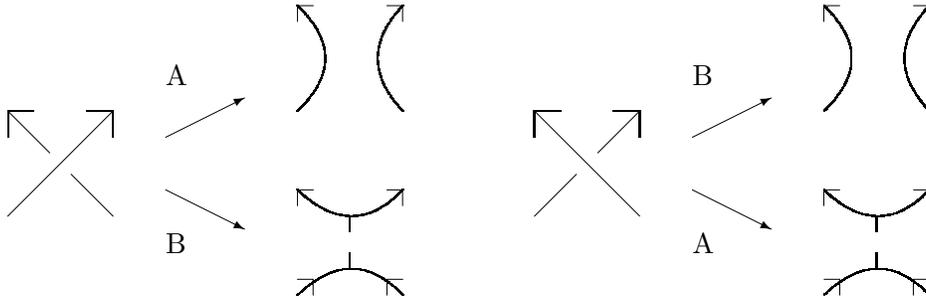


Figure 11: A-splices and B-splices

Applying A- or B-splice at each crossing of D_F , we have a pole curve link in F , say s . The pair (F, s) or s is called a *surface pole state* of (F, D_F) . Moreover, applying pole reductions of all pole curves in s , we have an irreducible pole curve link in F . We denote it by \bar{s} , and call the pair (F, \bar{s}) an *irreducible pole state* associated with (F, s) . (An irreducible pole curve link

\bar{s} obtained from s is determined uniquely up to equivalence in the sense of Definition 9 (Remark 10).)

When D is a virtual knot and (F, D_F) is a link diagram realization of D in a closed oriented surface F , the notions of pole diagrams and surface pole states are essentially due to Dye and Kauffman [4].

Theorem 3 (Dye and Kauffman [4]) *Let D be a virtual link diagram and (F, D_F) a link diagram realization of D in a closed oriented surface F . Let l be a pole curve of a surface pole state (F, s) of (F, D_F) . If the index $\iota(l)$ is positive, then l is an essential curve in F .*

Theorem 3 holds for twisted links.

Theorem 4 *Let D be a twisted link diagram and (F, D_F) a link diagram realization of D . Let l be a pole curve of a surface pole state (F, s) of (F, D_F) . If the index $\iota(l)$ is positive, then l is not a separating curve in F .*

Note that if a simple closed curve in F is not a separating curve in F then it is an essential curve in F . From this theorem, we have the following.

Corollary 5 *Let D be a virtual link diagram and (F, D_F) a link diagram realization of D . Let l be a pole curve of a surface pole state (F, s) of (F, D_F) . If the index $\iota(l)$ is positive, then l is not a separating curve in F .*

4 Proof of Theorem 4

Lemma 6 *For a pole curve l in F , the number of I-poles on l is equal to that of O-poles.*

Proof. Since I-poles and O-poles appear adjacently on l , the number of I-poles on l is equal to that of O-poles. ■

Lemma 7 *Let D be a twisted link diagram and (F, D_F) a link diagram realization of D . For a surface pole state (F, s) of (F, D_F) , let R be a connected component of $F \setminus s$. The number of I-poles on s in R is equal to that of O-poles.*

Proof. By a splice yielding a pair of poles, two poles come up in the same component of $F \setminus s$. Since one is an I-pole and the other is an O-pole in such a pair, the number of I-poles in R is equal to that of O-poles. ■

By a pole reduction, a pair of an I-pole and an O-pole is reduced in the same component of $F \setminus s$. Therefore we have the following from the above lemma.

Lemma 8 *Let D be a twisted link diagram and (F, D_F) a link diagram realization of D . Let (F, \bar{s}) be an irreducible surface pole state obtained from a surface pole state (F, s) of (F, D_F) , and let R be a connected component of $F \setminus \bar{s}$ ($= F \setminus s$). The number of I-poles on \bar{s} in R is equal to that of O-poles.*

Proof of Theorem 4. Let (F, \bar{s}) be an irreducible surface pole state of a surface pole state (F, s) . Let \bar{l} be the irreducible pole curve of (F, \bar{s}) obtained from a pole curve l of s such that $\iota(l)$ is positive. Assume that $F \setminus \bar{l}$ consists of two components, say R_1 and R_2 , and we show a contradiction. Let l_1, \dots, l_m be irreducible pole curves of (F, \bar{s}) in R_1 and $n_1(I), \dots, n_m(I)$ (or $n_1(O), \dots, n_m(O)$, resp.) be the numbers of I-poles (or O-poles, resp.) on l_1, \dots, l_m (see Figure 12). Let $n^i(I)$ (or $n^i(O)$, resp.) be the number of I-poles (or O-poles, resp.) on \bar{l} in R_i for $i = 1, 2$. By Lemma 6, we have $n_k(I) = n_k(O)$ for $k = 1, \dots, m$ and $n^1(I) + n^2(I) = n^1(O) + n^2(O)$. On the other hand, considering all connected components of $F \setminus \bar{s}$ contained in R_1 , we see by Lemma 8 that

$$\sum_{k=1}^m n_k(I) + n^1(I) = \sum_{k=1}^m n_k(O) + n^1(O).$$

Thus we have that $n^1(I) = n^1(O)$ and $n^2(I) = n^2(O)$. Since $\iota(l)$ is positive, all O-poles (or all I-poles, resp.) on \bar{l} are in R_1 and all I-poles (or all O-poles, resp.) on \bar{l} are in R_2 (cf. Figure 13). This implies that $n^1(O)$ is positive and $n^1(I) = 0$ (or $n^1(I)$ is positive and $n^1(O) = 0$, resp.). It contradicts $n^1(I) = n^1(O)$. ■

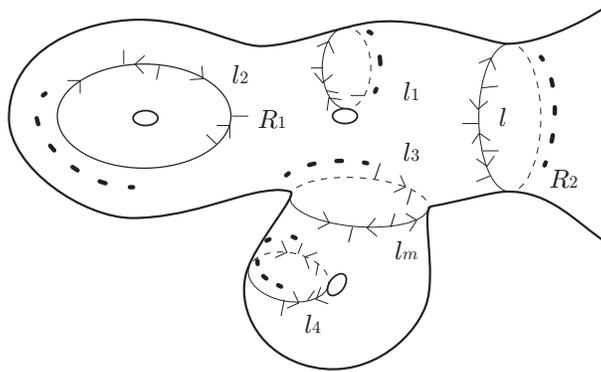


Figure 12: Irreducible surface pole state

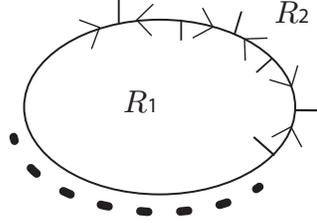


Figure 13: Irreducible pole curve whose index is positive

5 Surface pole bracket polynomials

In this section we introduce the notion of a surface pole bracket polynomial.

Let F be a closed, possibly non-orientable, surface.

Definition 9 Two pole links s and s' in F are *equivalent* if s' is obtained from s by an ambient isotopy of F and by changing the orientations of some (or none) of curves without poles.

Remark 10 Let l be a pole curve in F with some poles. If l has some poles and $\iota(l) = 0$, then by pole reductions we obtain a pole curve without poles, i.e., a simple loop in F . It is an irreducible pole curve \bar{l} obtained from l . The orientation of \bar{l} is not determined uniquely from l . Actually, changing how to apply pole reductions, we may obtain a simple loop with the opposite orientation. Thus, when we consider \bar{l} for a pole curve l with $\iota(l) = 0$, it is more natural to ignore the orientation of \bar{l} . By introducing Definition 9, we may say that for a pole curve l in F , an irreducible pole curve \bar{l} obtained from l by pole reductions is uniquely determined up to equivalence, and that for a pole curve link s in F , an irreducible pole curve link \bar{s} obtained from s is uniquely determined up to equivalence.

Let $P(F)$ be the family of all equivalence classes of pole curve links in F . (We allow the empty set as a pole curve link in F .) Let $\mathbb{Z}[A, A^{-1}]P(F)$ be the free module generated by $P(F)$ over the Laurent polynomial ring $\mathbb{Z}[A, A^{-1}]$.

For a pole curve link s in F , we denote by $\text{ess}(s)$ (or by $\text{iness}(s)$, resp.) the subset of s consisting of pole curves that are essential loops in F (or inessential loops in F , resp.). Note that $s = \text{ess}(s) \amalg \text{iness}(s)$. By \bar{s} we denote an irreducible pole curve link obtained from s by pole reductions.

Definition 11 For a pair (F, D_F) of a closed surface F and a link diagram D_F in F , the *surface pole bracket polynomial* $\langle (F, D_F) \rangle$ of (F, D_F) is defined by

$$\langle (F, D_F) \rangle = \sum_s A^{\natural(s)} (-A^2 - A^{-2})^{\sharp(\text{iness}(s))} \overline{[\text{ess}(s)]} \in \mathbb{Z}[A, A^{-1}]P(F),$$

where in the summation s runs all over surface pole states (F, s) of (F, D_F) , $\natural(s)$ is the number of A-splices minus that of B-splices obtaining the state (F, s) , $\sharp(\text{iness}(s))$ is the number of components of $\text{iness}(s)$, and $\overline{[\text{ess}(s)]}$ is the equivalence class of an irreducible pole curve link $\text{ess}(s)$ obtained from $\text{ess}(s)$ by pole reductions.

When F is orientable, this notion is essentially introduced to Dye and Kauffman [4]. The surface pole bracket polynomial is not an invariant of a link in the thickened surface. However it might be useful for study such links, virtual links and twisted links as discussed in [4].

Example 12 Let F be a torus and let D_F be a diagram in F illustrated in the top right of Figure 14. It has 3 crossings and there are 8 surface pole states as depicted in the figure. In the figure, for each state s , $A^{\natural(s)}$ is indicated.

Let s_1, \dots, s_8 be the states depicted in (1)–(8) of Figure 14 and let L_0, \dots, L_3 be pole curve links depicted in (i)–(iv) of Figure 15, respectively. Then we have

$$\begin{aligned} \natural(s_1) &= 3, & \sharp(\text{iness}(s_1)) &= 0, & \overline{[\text{ess}(s_1)]} &= [L_1], \\ \natural(s_2) &= 1, & \sharp(\text{iness}(s_2)) &= 1, & \overline{[\text{ess}(s_2)]} &= [L_0], \\ \natural(s_3) &= 1, & \sharp(\text{iness}(s_3)) &= 1, & \overline{[\text{ess}(s_3)]} &= [L_0], \\ \natural(s_4) &= 1, & \sharp(\text{iness}(s_4)) &= 1, & \overline{[\text{ess}(s_4)]} &= [L_0], \\ \natural(s_5) &= -3, & \sharp(\text{iness}(s_5)) &= 1, & \overline{[\text{ess}(s_5)]} &= [L_2], \\ \natural(s_6) &= -1, & \sharp(\text{iness}(s_6)) &= 2, & \overline{[\text{ess}(s_6)]} &= [L_0], \\ \natural(s_7) &= -1, & \sharp(\text{iness}(s_7)) &= 0, & \overline{[\text{ess}(s_7)]} &= [L_2], \\ \natural(s_8) &= -1, & \sharp(\text{iness}(s_8)) &= 0, & \overline{[\text{ess}(s_8)]} &= [L_2]. \end{aligned}$$

Thus $\langle (F, D_F) \rangle = (3A(-A^2 - A^{-2}) + A^{-1}(-A^2 - A^{-2})^2)[L_0] + A^3[L_1] + A^{-3}(-A^2 - A^{-2})[L_2] = (2A - A^{-3})(-A^2 - A^{-2})[L_0] + A^3[L_1] + A^{-3}(-A^2 - A^{-2})[L_2]$.

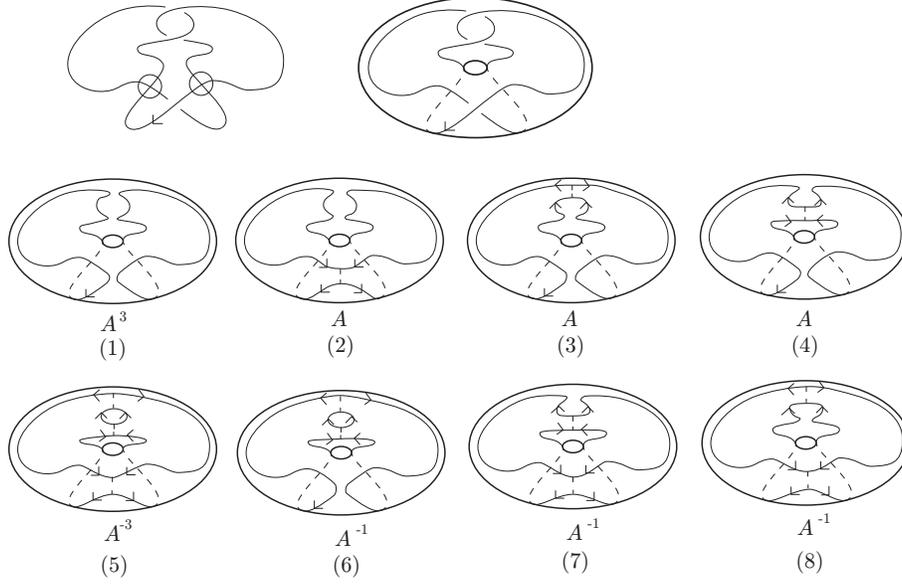


Figure 14: Example of a surface pole bracket polynomial

Definition 13 Let (F, D_F) be a pair of a closed surface F and a link diagram D_F in F . The *double bracket polynomial* $\langle\langle (F, D_F) \rangle\rangle$ of (F, D_F) is defined by

$$\langle\langle (F, D_F) \rangle\rangle = \sum_s A^{\natural(s)} (-A^2 - A^{-2})^{\sharp(\text{iness}(s))} M^{\sharp(\text{non-ori}(s))} \prod_{l \in s} d_{\iota(l)} \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \dots],$$

where in the summation s runs all over surface pole states (F, s) of (F, D_F) , $\natural(s)$ is the number of A-splices minus that of B-splices obtaining the state (F, s) , $\sharp(\text{iness}(s))$ is the number of components of $\text{iness}(s)$, $\sharp(\text{non-ori}(s))$ is the number of pole curves of s whose regular neighborhoods are Möbius bands, and $\iota(l)$ is the index of l , and we set $d_0 = 1$.

The *normalized double bracket polynomial* $R_{(F, D_F)}$ is defined by

$$R_{(F, D_F)} = ((-A)^{-3\omega(D)}) \langle\langle (F, D_F) \rangle\rangle \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \dots],$$

where $\omega(D)$ is the writhe of D .

For a twisted link diagram D , let (F, D_F) be a link diagram realization. Then the double bracket polynomial $\langle\langle (F, D_F) \rangle\rangle$ and the normalized double bracket polynomial $R_{(F, D_F)}$ of (F, D_F) are exactly equal to the double

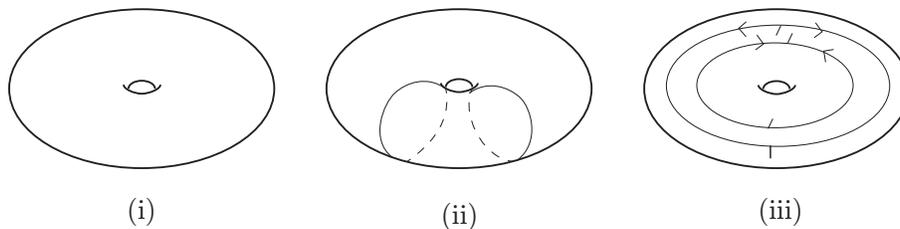


Figure 15: Pole curve links

bracket polynomial $\langle\langle D \rangle\rangle$ and the normalized double bracket polynomial R_D of D defined in [5], respectively.

Theorem 14 ([5]) *The normalized double bracket polynomial R_D is an invariant of a twisted link.*

For a twisted link L , we denote by R_L the polynomial R_D , where D is a twisted link diagram representing L .

By Theorem 14, the following is a corollary to Theorem 4.

Corollary 15 *Let L be a twisted link. Suppose that there is at least one term which includes $d_1^{k_1} \dots d_m^{k_m}$ in the polynomial invariant R_L of L . For any link diagram realization (F, D_F) of a link diagram D of L , there is a surface pole state (F, s) of (F, D_F) such that there is a pole curve of s which is not a separating curve in F .*

Proof. By Theorem 14, for any link diagram realization (F, D_F) of a twisted link diagram D representing L , the double bracket polynomial $\langle\langle (F, D_F) \rangle\rangle$ of (F, D_F) has at least one term which includes $d_1^{k_1} \dots d_m^{k_m}$. Thus there is a state (F, s) such that $\prod_{l \in s} d_{\iota(l)} \neq 1$. Therefore there is a pole curve l in s with $\iota(l) > 0$. By Theorem 4, the pole curve l is not a separating curve in F . ■

Acknowledgements

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