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TOPOLOGICAL QUASI-GROUP SHIFTS

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Abstract. In this work we characterize those shift spaces which can support a 1-block quasi-group operation and show the analogous of Kitchens result: any such shift is conjugated to a product of a full shift with a finite shift. Moreover, we prove that every expansive automorphism on a compact zero-dimensional quasi-group that verifies the medial property, commutativity and has period 2, is isomorphic to the shift map on a product of a finite quasi-group with a full shift.

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1. **Introduction.** One of the main questions concerning symbolic dynamics and algebraic structures was asked by R. Bowen: characterize group shifts, that is shifts supporting a group structure so that the shift map is an automorphism. This question was answered by B. Kitchens [3], who showed that any group shift is conjugated to the product of a full shift with a finite set. A more general case was studied by N.T. Sindhushayana, B. Marcus and M. Trott [5], who proved the analogous result for a homogeneous shift, that is a shift space X on the alphabet $\mathcal{A}$ for which there exist a group $P(\mathcal{A})$ of permutations of $\mathcal{A}$ and a group shift $Y \subseteq P(\mathcal{A})^{\mathbb{Z}}$, such that X is invariant under the action of any element of Y.

This work concentrates on quasigroups, often called cancellation semi-groups, thus with left and right cancellable operations. In §3 we present sufficient and necessary conditions to a compact zero-dimensional quasi-group $(X, *)$, where is defined an expansive automorphism $T : X \to X$, to be conjugated and isomorphic to a Markov shift with a 1-block operation. Furthermore, we give examples of zero-dimensional quasigroups which verify such conditions. These are quasi-group versions of results of [3], and their proofs use a quasi-group version of compact zero-dimensional groups ([4],Theorem 16, pg.77).

We will show that the unique shift spaces which can support a 1-block quasi-group operation are Markov shifts. So, §4 is dedicated to study the case when Λ

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is a Markov shift and $*$ is a 1-block operation. There, we characterize completely its structure by supplying a conjugacy with a product of a finite quasigroup with a full shift.

In the last section we use amalgamations and state splittings operations ([3],[1] and [7]), to characterize any isomorphism between two quasi-group shifts as in Kitchens [3].

2. **Background.** Let $\mathcal{A}$ be a finite alphabet and $\mathcal{A}^{\mathbb{Z}}$ be the *two-sided full shift* endowed with the product topology (it is a Hausdorff compact space) . Let $\Lambda \subseteq \mathcal{A}^{\mathbb{Z}}$ be a *Shift space*, that is a closed shift-invariant set, and denote by $L_\Lambda \subseteq \mathcal{A}$ the alphabet used by Λ.

Let $\mathcal{W}(\Lambda, n)$ be the set of all words or blocks with length n which are allowed in Λ (often we simply write $\mathcal{W}(n)$ instead of $\mathcal{W}(\Lambda, n)$) . Given $u = [u_1, \ldots, u_n] \in \mathcal{W}(\Lambda, n)$, we write $\mathcal{F}(\Lambda, u)$, or simply $\mathcal{F}(u)$, the follower set of u:

$$\mathcal{F}(\Lambda, u) = \{b \in \mathcal{A} : [u_1, \ldots, u_n, b] \in \mathcal{W}(\Lambda, n+1)\}.$$

In the same way, we define $\mathcal{P}(\Lambda, u)$, or simply $\mathcal{P}(u)$, the set of predecessors of u.

For $\mathbf{x} = (x_i)_{i\in\mathbb{Z}} \in \Lambda$, $m \leq n$, we denote $\mathbf{x}[m, n] := [x_m, x_{m+1}, \ldots, x_n] \in \mathcal{W}(n - m + 1)$.

Let σ_Λ be the *shift map* defined on Λ, when the context is clear we simply put σ instead of σ_Λ.

We say that Λ is a *shift of finite type* (SFT) if there exists $N \geq 0$ such that for any $\mathbf{x} \in \Lambda$ and for all $n \geq N$ we have $\mathcal{F}(\mathbf{x}[-n, 0]) = \mathcal{F}(\mathbf{x}[-N, 0])$. In this case we refer to Λ as a $(N + 1)$-step SFT.

If $\mathbf{A}$ is a transition matrix on the alphabet $\mathcal{A}$, denote by $\Sigma_{\mathbf{A}} := \left\{\mathbf{x} \in \mathcal{A}^{\mathbb{Z}} : \quad A_{x_i x_{i+1}} = 1\right\}$ the *two sided Markov shift* and by $L_{\mathbf{A}}$ the alphabet used by $\Sigma_{\mathbf{A}}$. Without lost of generality, we can assume that all rows and columns of $\mathbf{A}$ are not null, what is equivalent to say that $L_{\mathbf{A}} = \mathcal{A}$. A Markov shift is a 1-step SFT.

Let G be a set and $*$ a binary operation on G. We say that $(G, *)$ is a *quasigroup* if $*$ is left and right cancellable:

$$\forall a, b, c \in G, \quad a * b = a * c \quad (or \quad b * a = c * a) \Longleftrightarrow b = c$$

If, in addition, G is a topological space and $*$ is continuous with respect to topology of G, we say that $(G, *)$ is a *topological quasigroup*. When the context is clear, for $a, b \in (G, *)$, we write ab instead of $a * b$.

A partition $\mathcal{U} = \{U_i\}_{i\in I}$ of G is said to be compatible with $*$ if defining $U_i * U_j := \{a * b \in G : \quad a \in U_i, b \in U_j\}$, so for all $i, j \in I$ there exists $k \in I$ such that $U_i * U_j = U_k$, which is equivalent to say $(\mathcal{U}, *)$ is also a quasigroup.

Suppose that $(\Lambda, *)$ is a topological quasigroup. Then, the shift map is a continuous isomorphism if and only if $*$ is given by a $(\ell + r + 1)$-block local rule, i.e., there exists $\ell, r \geq 0$ and $\rho : \mathcal{W}(\ell + r + 1) \times \mathcal{W}(\ell + r + 1) \to \mathcal{A}$, such that

$$\forall \mathbf{x}, \mathbf{y} \in \Lambda, \forall j \in \mathbb{Z}, (x * y)_j = \rho(\mathbf{x}[j - \ell, j + r], \mathbf{y}[j - \ell, j + r]).$$

In this case, we say that ℓ is the memory and r the anticipation of $*$. When $\ell = r = 0$, $*$ is a 1-block operation.

(X, T) is a *topological dynamical system* if X is a compact space and $T : X \to X$ a homeomorphism. If there exists $x \in X$, such that $\{T^n(x) : n \geq 0\}$ is dense in X, (X, T) is said to be *transitive* or *irreducible*. Two topological dynamical systems (X, T) and (Y, S) are *topologically conjugated* if and only if there exists a homeomorphism $\zeta : X \to Y$, such that $\zeta \circ T = S \circ \zeta$.

If $(X, *)$ is a topological quasigroup and (X, T) a topological dynamical system, such that $T : X \to X$ is an automorphism for $*$, we will denote it by $(X, *, T)$.

We will say that $(X, *, T)$ and $(Y, *, S)$ are isomorphic if and only if there exists $\zeta : X \to Y$, which is both a topological conjugation between (X, T) and (Y, S), and an isomorphism between $(X, *)$ and $(Y, *)$.

If (X, T) is a topological dynamical system, then its topological entropy [6] will be denoted by $\mathbf{h}(T)$. When we refer to the entropy of a shift $(\Lambda, \sigma_\Lambda)$, we will write $\mathbf{h}(\Lambda)$.

3. **Expansive automorphisms on zero-dimensional quasi-groups.** In [3] Kitchens proved that if $(X, \star)$ is a topological group and $T : X \to X$ is an automorphism, such that:

(H1): X is compact (Hausdorff), zero-dimensional and has a numerable topological basis, that is, each element $\mathbf{a} \in X$ has a clopen fundamental neighborhood $\{V_n\}_{n \geq 1}$:

$$V_1 \supset V_2 \supset V_3 \supset \cdots \qquad \text{, and} \qquad \bigcap_{n=1}^{\infty} V_n = \{\mathbf{a}\}\,.$$

(H2): T is an expansive automorphism;

then,

- $(X, \star, T)$ is isomorphic by a 1-block code to $\left(\mathbb{F} \times \Sigma_n, \otimes, \sigma_{\mathbb{F}} \times \sigma_{\Sigma_n}\right)$, where $\mathbb{F}$ is a finite group with 1-block operation; Σ_n is a full n shift; and $\otimes$ is a k-block operation, with memory 0 and anticipation $k-1$.
- If $\mathbf{h}(T) = 0$, then $\Sigma_n = \{a\}$, that is, the full shift is trivial.
- If T is irreducible, then $\mathbb{F} = \{e\}$, that is, $\mathbb{F}$ is trivial.

Recall that expansivity means that *there exists $\mathcal{U}$, a partition of X by clopen sets (which is finite since X is compact), such that $\forall \mathbf{x}, \mathbf{y} \in X$, $\mathbf{x} \neq \mathbf{y}$, there exists $n \in \mathbb{Z}$ such that $T^n(\mathbf{x})$ and $T^n(\mathbf{y})$ belong to distinct sets of $\mathcal{U}$.*

Our aim is to extend the previous result to the case when $(X, \star)$ is a topological quasigroup. Now, since a quasigroup has fewer assumptions about its structure, we need some additional hypotheses on $(X, \star)$. In particular, it is reasonable to assume the following property:

(H3): $\forall \mathbf{x} \in X$: $\mathbf{x} \star X = X \star \mathbf{x} = X$.

(H3) is equivalent to: $\forall \mathbf{y}, \mathbf{z} \in X$, $\exists \mathbf{x_1}, \mathbf{x_2} \in X$, such that $\mathbf{x_1} \star \mathbf{y} = \mathbf{z}$ and $\mathbf{y} \star \mathbf{x_2} = \mathbf{z}$. Furthermore, since $\star$ is a quasigroup, the elements $\mathbf{x_1}$ and $\mathbf{x_2}$ are unique. Notice that if $(X, \star)$ is a finite quasigroup, then (H3) holds.

Under (H3) we can define on X the following quasi-group operations $\tilde{\star}$ and $\hat{\star}$:

$$\mathbf{x}\tilde{\star}\mathbf{y} = \mathbf{z} \qquad \Longleftrightarrow \qquad \mathbf{z} \star \mathbf{y} = \mathbf{x}$$

and

$$\mathbf{x}\hat{\star}\mathbf{y} = \mathbf{z} \qquad \Longleftrightarrow \qquad \mathbf{x} \star \mathbf{z} = \mathbf{y}$$

Also, for any $\mathbf{a} \in X$, we can define the functions $f_{\mathbf{a}} : X \to X$ and $f^{\mathbf{a}} : X \to X$ by $f_{\mathbf{a}}(\mathbf{x}) = \mathbf{a} \star \mathbf{x}$ and $f^{\mathbf{a}}(\mathbf{x}) = \mathbf{x} \star \mathbf{a}$. In the same way we define $\tilde{f}_{\mathbf{a}}$ and $\tilde{f}^{\mathbf{a}}$, using the operation $\tilde{\star}$, and the functions $\hat{f}_{\mathbf{a}}$ and $\hat{f}^{\mathbf{a}}$, using the operation $\hat{\star}$. It is easy to check that all of these functions are homeomorphisms.

We recall the identity element plays a fundamental role in the study of zero-dimensional groups (see [3], and [4],Theorem 16, pg.77). In the case of zero-dimensional quasi-groups we will need the hypothesis (H3) to define a substitutive notion:

Definition 3.1. For an arbritarily fixed element $\mathbf{e} \in X$, given $\mathbf{a} \in X$ we define $\mathbf{a}^-$ and $\mathbf{a}^+$ as the unique elements in X (which there exist due (H3)), such that $\mathbf{a}^- \star \mathbf{a} = \mathbf{e}$ and $\mathbf{a} \star \mathbf{a}^+ = \mathbf{e}$. We say $\mathbf{a}^-$ and $\mathbf{a}^+$ are respectively the left and right inverses of $\mathbf{a}$ with respect to $\mathbf{e}$.

Notice that $\mathbf{a}^- = (f^{\mathbf{a}})^{-1}(\mathbf{e}) = \tilde{f}_{\mathbf{e}}(\mathbf{a})$ and $\mathbf{a}^+ = {f_{\mathbf{a}}}^{-1}(\mathbf{e}) = \hat{f}^{\mathbf{e}}(\mathbf{a})$. Moreover, we have that $(\mathbf{a}^-)^+ = (\mathbf{a}^+)^- = \mathbf{a}$ and the maps $\mathbf{a} \mapsto \mathbf{a}^-$ and $\mathbf{a} \mapsto \mathbf{a}^+$ are homeomorphisms.

In order to reach our goal, some additional hypotheses over $(X, \star)$ will be needed.

3.1. **Expansive automorphisms.** Assume that hypotheses (H1) and (H2) hold for $(X, \star, T)$. We shall prove that there exists a quasi-group shift $(\Lambda, *)$ such that $(X, \star, T)$ is isomorphic to $(\Lambda, *, \sigma)$.

Lemma 3.2. *Let $(X, \star, T)$ as above. Then $(X, \star, T)$ is isomorphic to $(\Lambda, *, \sigma)$, where Λ is a shift and $*$ is a k-block operation, for some $k \geq 1$.*

Proof. From expansibility and 0-dimensionality there exists a partition $\mathcal{U}$ of X, a shift space $\Lambda \subseteq \mathcal{U}^{\mathbb{Z}}$, and a homeomorphism $\zeta : X \to \Lambda$, which is a topological conjugacy between (X, T) and $(\Lambda, \sigma_\Lambda)$.

In Λ, we define the quasi-group operation $*$, given by:

$$\forall \mathbf{a}, \mathbf{b} \in \Lambda, \qquad \mathbf{a} * \mathbf{b} := \zeta\big(\zeta^{-1}(\mathbf{a}) \star \zeta^{-1}(\mathbf{b})\big)$$

We have that $(\Lambda, *)$ is isomorphic by ζ to $(X, \star)$. Furthermore, since σ_Λ is an automorphism for $*$, then $*$ is k-block, for some $k \geq 1$. $\square$

In particular we will be interested in the case $*$ being a 1-block operation. From the proof of Lemma 3.2 we deduce that $*$ is a 1-block operation if and only if the partition $\mathcal{U}$ is compatible with $\star$. For instance, if X is a shift space with a 1-block operation $\star$, then any partition $\mathcal{U}$ of X by cylinders defined by the same coordinates is compatible with $\star$ (which means $(X, \star, T)$ is isomorphic to $(\Lambda, *, \sigma)$, where $*$ is 1-block).

The natural problem consists in finding such partitions compatible with the operation for any topological quasigroup in which (H1), (H2) (and additionally (H3)) hold. This problem remain open. Therefore, we can ask for the kind of quasi-group structures allowing to obtain analogous results.

Suppose $(X, \star, T)$ is a topological quasigroup, verifying (H1), (H2), and such that the following properties hold:

(h1): $(X,\star)$ is commutative (that is $f_{\mathbf{a}} = f^{\mathbf{a}}$);
(h2): $\star$ has period 2, this means, $\forall \mathbf{a} \in X$, $f_{\mathbf{a}}$ has period 2;
(h3): The aforementioned element $\mathbf{e} \in X$ has a fundamental neighborhood system $(V_n)_{n\geq 1}$, such that

$$\forall n \geq 1, \qquad \mathbf{e}V_n \subseteq V_n$$

(h4): $(X,\star)$ has the medial property:

$$\forall \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in X, \qquad (\mathbf{a}\star\mathbf{b})\star(\mathbf{c}\star\mathbf{d}) = (\mathbf{a}\star\mathbf{c})\star(\mathbf{b}\star\mathbf{d}).$$

Notice that [(h1) and (h2)] is equivalent to [$\star$, $\tilde{\star}$ and $\hat{\star}$ are identical]. Thus, for all $\mathbf{a} \in X$: $\mathbf{a}^- = \mathbf{a}^+$. Moreover, these two hypotheses imply that (H3) holds and that the hypothesis (h3) is equivalent to $\forall n \geq 1$, $\mathbf{e}V_n = V_n\mathbf{e} = V_n$.

Furthermore, from (h3), $\mathbf{e}\star\mathbf{e} = \mathbf{e}$, which implies $\mathbf{e}^- = \mathbf{e} = \mathbf{e}^+$. We notice $\mathbf{e}$ is *not* an identity element, since in general $\mathbf{e}\star\mathbf{a} \neq \mathbf{a} \neq \mathbf{a}\star\mathbf{e}$.

Hence, by using (h4), we deduce that for all $\mathbf{a},\mathbf{b} \in X$, the left inverse with respect to $\mathbf{e}$ verifies:

$$(\mathbf{a}\star\mathbf{b})^- = (\mathbf{a}^-\star\mathbf{b}^-) \tag{1}$$

Example 3.3. Let $X = \{0,1\}^{\mathbb{Z}}\times\{0,1\}^{\mathbb{Z}}$ and let a 2-block operation $\star$ with local rule $\rho : \mathcal{W}(X,2)\times\mathcal{W}(X,2) \to L_X$ be defined for $\mathbf{x},\mathbf{y} \in \mathcal{W}(X,2)$, $\mathbf{x} = [(x_0^1,x_0^2),(x_1^1,x_1^2)]$ and $\mathbf{y} = [(y_0^1,y_0^2),(y_1^1,y_1^2)]$, by:

$$\rho(\mathbf{x},\mathbf{y}) = \begin{cases} (x_0^1+y_0^1, x_0^2+y_0^2+1) & if\ x_0^1 = x_1^1,\ y_0^1 = y_1^1 \\ \\ (x_0^1+y_0^1, x_0^2+y_0^2) & otherwise \end{cases},$$

where $+$ denote the sum *mod* 2.

Then, $(X,\star)$ verifies all of the previous hypotheses, but it is not a group.

The next result is a quasi-group version of a construction done on topological zero-dimensional groups ([4],Theorem 17, pg.77).

Theorem 3.4. *Let $(X,\star)$ be a topological quasigroup, such that (H1), (h1), (h2), (h3), (h4) hold. Then, given a neighborhood U of $\mathbf{e}$, there exists a clopen neighborhood $Q \subseteq U$ of $\mathbf{e}$, such that $\mathcal{Q} := \{\mathbf{a}Q : \mathbf{a} \in X\}$ is a finite partition of X compatible with $\star$.*

Proof. step 1: Let $(V_n)_{n\geq 1}$ be the neighborhood system over $\mathbf{e}$, given by (h3). We can suppose $V_{n+1} \subseteq V_n$, $\forall n \geq 1$.

Let $M \subseteq X$ be a compact subset such that $\mathbf{e} \in M$. We say that $\mathbf{a} \in M$ can be connected to $\mathbf{e}$ over M by a chain of order n, if and only if there exists a sequence $\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{a} \in M$, such that

$$\mathbf{a}_i^-\mathbf{a}_{i+1} \in V_n, \qquad 1 \leq i \leq k-1$$

Let M_n be the set of all points in M, which can be connected to $\mathbf{e}$ over M by a chain of order n. It is straightforward to see that $M_{n+1} \subseteq M_n$. Moreover, every point in M_n can be connected to $\mathbf{e}$ over M_n by a chain of order n. In fact, if $\mathbf{a} \in M_n$, then there exist $\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{a} \in M$ such that $\mathbf{a}_i^-\mathbf{a}_{i+1} \in V_n$,

$1 \leq i \leq k-1$. Hence, for any $j = 1, \ldots, k$, $\mathbf{a}_j$ can be connected to $\mathbf{e}$ over M by a chain of order n. Therefore, for any $j = 1, \ldots, k$, $\mathbf{a}_j \in M_n$. Thus, $\mathbf{a}$ can be connected to $\mathbf{e}$ over M_n by a chain of order n.

step 2: We will show that M_n is a relative open set of $\tau_M := \{A \cap M : A \subseteq X \textit{ is open}\}$ the induced topology in M.

Given $\mathbf{a} \in M_n$, we search for a relative open set of M, neighborhood of $\mathbf{a}$, that is a subset of M_n.

Let $V' := f_{\mathbf{a}^-}^{-1}(V_n) \cap M$, which is a relative open set of M, since the continuity of $f_{\mathbf{a}^-}$ implies that $f_{\mathbf{a}^-}^{-1}(V_n)$ is open. Moreover, $f_{\mathbf{a}^-}(f_{\mathbf{a}^-}^{-1}(V_n)) = \mathbf{a}^- \star f_{\mathbf{a}^-}^{-1}(V_n) = V_n$, which implies that $\mathbf{a} \in f_{\mathbf{a}^-}^{-1}(V_n)$. Now, let $\mathbf{b} \in V'$, and $\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{a} \in M$ be a chain of order n connecting $\mathbf{a}$ to $\mathbf{e}$ over M. Since $\mathbf{b} \in V'$, it is direct to see that $\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{a},\ \mathbf{a}_{k+1} = \mathbf{b} \in M$ is a chain of order n connecting $\mathbf{b}$ to $\mathbf{e}$ over M, because $\mathbf{a}_k^- \mathbf{a}_{k+1} = \mathbf{a}^- \mathbf{b} = f_{\mathbf{a}^-}(\mathbf{b}) \in V_n$. Thus, $\mathbf{b} \in M_n$, so $V' \subseteq M_n$.

step 3: We need to prove that M_n is a closed set of X or equivalently $M_n^c := X \setminus M_n$ is open. Since $M_n^c = M \setminus M_n \cup M^c$, and M^c is open, it is sufficient to show that $M \setminus M_n$ is open.

Let $\mathbf{a} \in M \setminus M_n$, put $V' := (f^{\mathbf{a}})^{-1}(V_n)$, which is an open set since $f^{\mathbf{a}}$ is continuous. Notice that $f^{\mathbf{a}}(V') = V' \star \mathbf{a} = V_n$ and, since $\mathbf{e} \in V_n$, it implies that $\mathbf{a}^- \in V'$. Let $V'' := (V')^+ := \left\{\mathbf{v}'' \in X : \mathbf{v}'' = \mathbf{v'}^+,\ \mathbf{v}' \in V'\right\}$, which is an open set containing $\mathbf{a}$ because $\mathbf{a}^- \in V'$, and $(\mathbf{a}^-)^+ = \mathbf{a}$.

The set V'' cannot intersect M_n. In fact, if the contrary $M_n \cap V'' \neq \emptyset$ holds, then it is possible to take $\mathbf{b} \in M_n \cap V''$ such that $\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{b} \in M$ is a chain of order n which connects $\mathbf{b}$ to $\mathbf{e}$ over M. But $\mathbf{b} \in V''$, so $\mathbf{b}^- \in V'$ and then $\mathbf{b}^-\mathbf{a} \in V_n$. Since $\mathbf{a} \in M$, $\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{b},\ \mathbf{a}_{k+1} = \mathbf{a} \in M$ is a chain of order n which connects $\mathbf{a}$ to $\mathbf{e}$ over M. Hence, $\mathbf{a} \in M_n$, a contradiction with the assumption $\mathbf{a} \in M \setminus M_n$.

Since $V'' \subseteq M \setminus M_n$ is an open neighborhood of $\mathbf{a}$, we deduce that $M \setminus M_n$ is an open set.

step 4: Let $M^* := \bigcap_{n \geq 1} M_n$. Since $\{M_n\}_{n \geq 1}$ is a collection of closed sets and $\forall n \geq 1$, $\mathbf{e} \in M_n$, we have that $\mathbf{e} \in M^*$.

Let us show that $M^* = \{\mathbf{e}\}$. It is sufficient to show that $\forall \mathbf{b} \in M$, $\mathbf{b} \neq \mathbf{e}$, there exists $t \geq 1$, such that $\mathbf{b} \notin M_t$, which implies $\mathbf{b} \notin M^*$.

In fact, given $\mathbf{b} \in M$, and since M is a closed set and X is a zero-dimensional Hausdorff set, we can write $M = A \cup B$, a disjoint union of closed sets where $\mathbf{e} \in A$ and $\mathbf{b} \in B$. Since A and B are compacta, we have that $A^- \star B$ is also compact, so it is a closed set. Furthermore, $\mathbf{e} \notin A^- \star B$ because A and B are disjoint. Then, we can take V_t a neighborhood of $\mathbf{e}$, such that $V_t \cap (A^- \star B) = \emptyset$.

We have that $\mathbf{b} \notin M_t$. In fact, if the contrary $\mathbf{b} \in M_t$ holds, there would be a chain of order t connecting $\mathbf{b}$ to $\mathbf{e}$ over M

$$\mathbf{a}_1 = \mathbf{e},\ \mathbf{a}_2,\ \ldots,\ \mathbf{a}_k = \mathbf{b} \in M, \qquad \mathbf{a}_i^- \mathbf{a}_{i+1} \in V_t, \qquad 1 \leq i \leq k-1$$

This would imply that there exists j such that $\mathbf{a}_j \in A$ and $\mathbf{a}_{j+1} \in B$, and so $\mathbf{a}_j^- \mathbf{a}_{j+1} \in V_t \cap (A^- \star B)$.

step 5: Let $U \subseteq X$ be a neighborhood of $\mathbf{e}$. Since $\star$ is continuous, there exists $V \subseteq U$, open neighborhood of $\mathbf{e}$, such that $V \star V \subseteq U$.

We put $M := \overline{U}$ (the closure of U) and since $M^* = \{\mathbf{e}\}$, there exists $t \geq 1$, such that $M_t \subseteq V$. In fact, if there did not exist such t, then for each $n \geq 1$ the set $M_n \cap V^c$ would be closed and not empty. Then, $\bigcap_{n\geq 1}(M_n \cap V^c)$ would be not empty, what is a contradiction with $M^* = \{\mathbf{e}\}$.

M_t is a relative open set of M, so there exists W an open set of X, such that $M_t = M \cap W = \overline{U} \cap W$. Since $M_t \subseteq V \subseteq U$, we have that $M_t = M_t \cap W \subseteq V \cap W \subseteq U \cap W \subseteq \overline{U} \cap W = M_t$, that is, $M_t = V \cap W$ is an intersection of two open sets, so itself is an open set.

Let us show that $M_t \star M_t = M_t$. By construction, $M_t \star M_t \subseteq V \star V \subseteq M$. If $\mathbf{c} \in M_t \star M_t$, then $\mathbf{c} = \mathbf{ab}$, with $\mathbf{a}, \mathbf{b} \in M_t$, and there exist two chains of order t, $(\mathbf{a}_i)_{1\leq i\leq k}$ and $(\mathbf{b}_j)_{1\leq j\leq m}$ which connect respectively $\mathbf{a}$ and $\mathbf{b}$ to $\mathbf{e}$ over M_t.

Thus, we can take the chain $(\mathbf{c}_i)_{1\leq i\leq k+m-1}$:

$$\begin{array}{ll} \mathbf{c}_i = \mathbf{a}_i\mathbf{e} & , \quad if\ 1 \leq i \leq k \\ \mathbf{c}_i = \mathbf{a}\mathbf{b}_{i-k+1} & , \quad if\ k+1 \leq i \leq k+m-1 \end{array}$$

Using the medial property and $\mathbf{e}V_t = V_t\mathbf{e} = V_t$ we get that for all $i \in \{1, \ldots,\ k+m-1\}$ follows $\mathbf{c}_i^- \mathbf{c}_{i+1} \in V_t$. In fact,

$$\mathbf{c}_i^- \mathbf{c}_{i+1} = \begin{cases} (\mathbf{a}_i^- \mathbf{e})(\mathbf{a}_{i+1}\mathbf{e}) = (\mathbf{a}_i^- \mathbf{a}_{i+1})(\mathbf{ee}) \in V_t\mathbf{e} & , \quad if\ 1 \leq i < k \\ (\mathbf{a}^-\mathbf{e})(\mathbf{ae}) = (\mathbf{a}^-\mathbf{a})(\mathbf{ee}) \in V_t & , \quad if\ i = k \\ (\mathbf{a}^-\mathbf{b}_{i-k+1}^-)(\mathbf{ab}_{i-k+2}) = (\mathbf{a}^-\mathbf{a})(\mathbf{b}_{i-k+1}^-\mathbf{b}_{i-k+2}) \in \mathbf{e}V_t & , \quad otherwise \end{cases}$$

Then, this is a chain of order t connecting $\mathbf{c}$ to $\mathbf{e}$ over M, and then $\mathbf{c} \in M_t$.

We have proved $M_t \star M_t \subseteq M_t$. Since $\star$, $\tilde{\star}$ and $\hat{\star}$ are identical, it follows that

$$M_t \star M_t = M_t \qquad \text{(which implies } M_t^- = M_t = M_t^+\text{)} \tag{2}$$

step 6: Let $Q := M_t$, we will show that $\mathcal{Q} := \{\mathbf{a}Q : \mathbf{a} \in X\}$ is a partition of X compatible with $\star$. It follows straightforwardly from hypothesis (H3) that $\mathcal{Q}$ is an open cover of X and, by medial property, $\mathbf{a}Q \star \mathbf{b}Q = (\mathbf{a} \star \mathbf{b})Q$. Then we only need to prove that $\mathcal{Q}$ is a partition of X. To do this, we introduce the following relation over X:

$$\mathbf{a} \sim \mathbf{b} \qquad \Longleftrightarrow \qquad \mathbf{a}^-\mathbf{b} \in Q \tag{3}$$

We claim that $\sim$ is an equivalence relation. Clearly $\sim$ is reflexive. To prove that $\sim$ is symmetric and transitive, we use that

$$\mathbf{a} \sim \mathbf{b} \qquad \Longleftrightarrow \qquad \mathbf{a}Q = \mathbf{b}Q \tag{4}$$

($\Longrightarrow$) In fact, $\mathbf{a} \sim \mathbf{b}$ is equivalent to saying that $\mathbf{a}^- \star \mathbf{b} \in Q$. So, for all $\mathbf{q} \in Q$, let $\mathbf{x} \in X$ be such that $\mathbf{a} \star \mathbf{q} = \mathbf{b} \star \mathbf{x}$. By multiplying this equation by the left by $(\mathbf{a}^- \star \mathbf{q}^-)$, and by using the medial property, we get $\mathbf{e} = (\mathbf{a}^- \star \mathbf{b}) \star (\mathbf{q}^- \star \mathbf{x})$. Therefore, since $(\mathbf{a}^- \star \mathbf{b}) \in Q$, we deduce that $(\mathbf{q}^- \star \mathbf{x}) \in Q$ and so $\mathbf{x} \in Q$. Thus, we can conclude that $\mathbf{a}Q \subseteq \mathbf{b}Q$. By symmetric reasoning $\mathbf{a}Q \supseteq \mathbf{b}Q$; hence $\mathbf{a}Q = \mathbf{b}Q$.

($\Longleftarrow$) If $\mathbf{a}Q = \mathbf{b}Q$, then for all $\mathbf{q}_1 \in Q$, there exists $\mathbf{q}_2 \in Q$, such that $\mathbf{a} \star \mathbf{q}_1 = \mathbf{b} \star \mathbf{q}_2$. Again, multiplying this equation by the left by $(\mathbf{a}^- \star \mathbf{q}_1^-)$, and by using the medial property, we deduce that $\mathbf{a}^- \star \mathbf{b} \in Q$.

It is not hard to see that $\mathcal{Q} = \{\mathbf{a}Q : \mathbf{a} \in X\} = \{[\mathbf{b}] : \mathbf{b} \in X\}$, where $[\mathbf{b}] := \{\mathbf{c} \in X : \mathbf{c} \sim \mathbf{b}\}$ is the equivalence class of $\mathbf{b}$. Then, $\mathcal{Q}$ is a partition of X into equivalence classes. Moreover, since X is compact, $\mathcal{Q}$ is finite. $\square$

Using last theorem, the following result has a proof similar to the one of Proposition 2 at [3].

Proposition 3.5. *Let $(X, \star, T)$ be a topological quasigroup, such that all hypotheses of Theorem 3.4 hold, and $T : X \to X$ is an expansive automorphism (that is, (H2) holds). Then $(X, \star, T)$ is isomorphic to $(\Lambda, *, \sigma)$, where Λ is a shift and $*$ is a 1-block operation.*

Remark 3.6. The hypotheses (h1)-(h4) are strongly restrictive. In fact, for a finite quasigroup $(X, \star)$, Dénes-Keedwell ([2], Theorem 2.2.2, p.70) showed that the medial property implies the quasi-group operation comes from a Abelian group operation, that is, there exist an Abelian group operation $+$ on X, two automorphisms η and ρ on X, and $c \in X$, such that $a \star b = \eta(a) + \rho(b) + c$ for all $a, b \in X$. For our case of zero-dimensional quasigroups, Theorem 3.4, and propositions 3.5 and 3.7, allow us to get an analogous result whenever there exists an expansive automorphism (or endomorphism) on X.

3.2. **1-block quasi-group shifts.** Let $\Lambda \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift space. Suppose that $(\Lambda, *)$ is a quasigroup where $*$ is a 1-block operation. In particular, since $*$ is 1-block, σ is an automorphism over $(\Lambda, *)$.

Proposition 3.7. *If $(\Lambda, *)$ is as above and in addition (H3) holds, then:*

i. *There exists an operation $\bullet$ over L_Λ, such that $(L_\Lambda, \bullet)$ is a quasigroup which induces $(\Lambda, *)$;*

ii. *$\forall k \geq 1$, $\forall g, h \in \mathcal{W}(\Lambda, k)$, we have $|\mathcal{F}(g)| = |\mathcal{F}(h)|$. Furthermore, if $a \in \mathcal{F}(g)$, then $a\mathcal{F}(h) = \mathcal{F}(g \bullet h) = \mathcal{F}(g) \bullet \mathcal{F}(h)$ and $F(h)a = \mathcal{F}(h \bullet g) = \mathcal{F}(h) \bullet \mathcal{F}(g)$;*

iii. *Λ is a SFT. Moreover $(\Lambda, *, \sigma)$ is isomorphic to a Markov shift with a 1-block operation.*

Proof. i. Since $*$ is a 1-block operation, there exists $\rho : L_\Lambda \times L_\Lambda \to L_\Lambda$ a local rule of $*$. For $a, b \in L_\Lambda$, put $a \bullet b := \rho(a, b)$.

Let us show that $(L_\Lambda, \bullet)$ is a quasigroup. Since L_Λ is finite, this property is equivalent to the fact that for all $a, b \in L_\Lambda$ there exist $c, c' \in L_\Lambda$ which are the unique solutions of $c \bullet a = b$ and $a \bullet c' = b$.

In fact, if we take $\mathbf{y}, \mathbf{z} \in \Lambda$, such that $y_0 = a$, $z_0 = b$, there exists $\mathbf{x} \in \Lambda$ a unique solution of $\mathbf{x} * \mathbf{y} = \mathbf{z}$. So $x_0 \bullet y_0 = z_0$, which means that $c := x_0$ is solution of $c \bullet a = b$. Since L_Λ is finite, if we fix a, for each b there exists a distinct solution c. So, we can deduce that $\bullet$ is right permutative. Using the same argument we also deduce the left permutativity. Then, $(L_\Lambda, \bullet)$ is a quasigroup.

ii. The proof of this fact uses similar arguments as in the proofs of Proposition 4.2 and Claim 4.3 after.

iii. Fix $\mathbf{u} = (u_i)_{i \in \mathbb{Z}} \in \Lambda$. We have that $L_\Lambda \supseteq \mathcal{F}(\mathbf{u}[0, 0]) \supseteq \mathcal{F}(\mathbf{u}[-1, 0]) \cdots \supseteq \mathcal{F}(\mathbf{u}[-n, 0]) \supseteq \mathcal{F}(\mathbf{u}[-n-1, 0]) \neq \emptyset$. Since L_Λ is finite, there exists N, such that $\mathcal{F}(\mathbf{u}[-n, 0]) = \mathcal{F}(\mathbf{u}[-N, 0])$, for all $n \geq N$.

Furthermore, if $[g_0,\dots,g_N] \in \mathcal{W}(\Lambda, N+1)$ and $[h_1,\dots,h_k,g_0,\dots,g_N] \in \mathcal{W}(\Lambda, N+k+1)$, then $\mathcal{F}([h_1,\dots,h_k,g_0,\dots,g_N]) \subseteq \mathcal{F}([g_0,\dots,g_N])$ and they are both cosets of $\mathcal{F}(\mathbf{u}[-N,0])$ (by part (ii)). Then, $\mathcal{F}([h_1,\dots,h_k,g_0,\dots,g_N]) = \mathcal{F}([g_0,\dots,g_N])$ and we deduce that Λ is a $(N+1)$-step SFT.

To conclude the proof, we define $\Sigma_{\mathbf{A}}$ as the $(N+1)$-block presentation of Λ, and consider the 1-block quasi-group operation induced by Λ. □

We notice that there is no evidence about existence of quasigroups $(\Lambda, *)$ such that $*$ is a 1-block operation but (H3) does not hold.

The previous result implies that if (H3) holds, then $(\Lambda, *)$ is a subquasigroup of $(\mathcal{A}^{\mathbb{Z}}, *)$. The following proposition reproduce the result of Proposition 3.7 using the hypothesis that $(\Lambda, *)$ is a subquasigroup:

Proposition 3.8. *If $(\Lambda, *)$ is a subquasigroup of $(\mathcal{A}^{\mathbb{Z}}, *)$, where $*$ is a 1-block operation, then there exists an operation $\bullet$ over L_Λ, such that $(L_\Lambda, \bullet)$ is a quasigroup which induces $(\Lambda, *)$.*

Proof. From the fact of $(\mathcal{A}^{\mathbb{Z}}, *)$ is quasigroup follows that for any constant sequences $\mathbf{a}, \mathbf{b} \in \mathcal{A}^{\mathbb{Z}}$, $\mathbf{a} = (\dots, a, a, a, \dots)$ and $\mathbf{b} = (\dots, b, b, b, \dots)$, there exist unique $\mathbf{c}, \mathbf{c}' \in \mathcal{A}^{\mathbb{Z}}$, $\mathbf{c} = (\dots, c, c, c, \dots)$ and $\mathbf{c}' = (\dots, c', c', c', \dots)$ solutions of the equations

$$\mathbf{a} * \mathbf{c} = \mathbf{b}, \qquad \mathbf{c}' * \mathbf{a} = \mathbf{b}.$$

Hence, denoting the local rule of $*$ as $\bullet$, for any $a, b \in L_\Lambda$ the equations $a \bullet c = b$ and $c' \bullet a = b$ also have unique solutions, which implies $(L_{\mathbf{A}}, \bullet)$ is quasigroup. □

4. **Topological 1-block quasi-group Markov shifts.** In this section we consider the case of subquasigroups $(\Sigma_{\mathbf{A}}, *, \sigma) \subseteq (\mathcal{A}^{\mathbb{Z}}, *, \sigma)$, where $\Sigma_{\mathbf{A}}$ is a topological Markov shift, and $*$ is a 1-block quasi-group operation. According to the previous section, the operation $*$ over $\Sigma_{\mathbf{A}}$ is canonically induced by a quasi-group operation $\bullet$ over $L_{\mathbf{A}}$, such that for any $a, b, a', b' \in L_{\mathbf{A}}$,

$$\begin{array}{c} a \in \mathcal{F}(b), a' \in \mathcal{F}(b') \Longrightarrow (a \bullet a') \in \mathcal{F}(b \bullet b') \\ \text{and} \\ a \in \mathcal{P}(b), a' \in \mathcal{P}(b') \Longrightarrow (a \bullet a') \in \mathcal{P}(b \bullet b') \end{array} \tag{5}$$

4.1. **Elementary properties.**

Claim 4.1. *Let $K \subseteq L_{\mathbf{A}}$. Then $\forall g \in L_{\mathbf{A}}$, $|gK| = |Kg| = |K|$.*

Proof. It follows from the bipermutativity of $*$. □

Proposition 4.2. *Let $(\Sigma_{\mathbf{A}}, *)$ be a 1-block quasi-group shift. Then,*

i. *$\forall g, h \in L_{\mathbf{A}}$, $|\mathcal{F}(g)| = |\mathcal{F}(h)|$ and $|\mathcal{P}(g)| = |\mathcal{P}(h)|$*
ii. *If $s \in \mathcal{F}(r)$, $s \in \mathcal{P}(t)$, then*

$$s\mathcal{F}(h) = \mathcal{F}(r \bullet h),\ \mathcal{F}(h)s = \mathcal{F}(h \bullet r),$$

$$s\mathcal{P}(h) = \mathcal{P}(t \bullet h),\ \mathcal{P}(h)s = \mathcal{P}(h \bullet t).$$

Proof. i. Since $\bullet$ is bipermutative, $\exists r \in L_{\mathbf{A}}$, such that $r \bullet h = g$. Let $s \in \mathcal{F}(r)$, for all $h' \in \mathcal{F}(h)$ we have $s \bullet h' \in \mathcal{F}(r \bullet h) = \mathcal{F}(g)$. Then,

$$s\mathcal{F}(h) \subseteq \mathcal{F}(g),$$

and from Claim 4.1, we deduce $|F(h)| \leq |F(g)|$.

Now, let f_r be the permutation over $L_{\mathbf{A}}$, defined by $f_r(a) = r \bullet a$. There exists $k \in \mathbb{N}$, such that $f_r^k(h) = h$, so $h = f_r^{k-1}(g)$.

Thus, for $[r,s] \in \mathcal{W}(2)$, $\forall g' \in \mathcal{F}(g)$, we have

$$\underbrace{[r,s] * (\ldots([r,s] * [g,g']))}_{[r,s]\ appears\ k-1\ times} = [f_r^{k-1}(g), f_s^{k-1}(g')] = [h, f_s^{k-1}(g')] \in \mathcal{W}(2)$$

Then, $\forall g' \in \mathcal{F}(g)$, we have $f_s^{k-1}(g') \in \mathcal{F}(h)$ and so $f_s^{k-1}(\mathcal{F}(g)) \subseteq \mathcal{F}(h)$. From Claim 4.1, we have

$$\left|f_s^{k-1}(\mathcal{F}(g))\right| = \underbrace{|s * (s * (\ldots(s * \mathcal{F}(g))))|}_{s\ appears\ k-1\ times} = |\mathcal{F}(g)| \leq |\mathcal{F}(h)|$$

We conclude the aimed equality for the follower sets. Using similar arguments we deduce the similar equality for the predecessor sets.

ii. It is straightforward from part i. and fact (5). □

Claim 4.3. *For any $a, b \in L_{\mathbf{A}}$ we have that $\mathcal{F}(a) \bullet \mathcal{F}(b) = \mathcal{F}(a \bullet b)$ and $\mathcal{P}(a) \bullet \mathcal{P}(b) = \mathcal{P}(a \bullet b)$*

Proof.

$$\mathcal{F}(a) \bullet \mathcal{F}(b) = \bigcup_{a' \in \mathcal{F}(a)} a'\mathcal{F}(b) =_{(1)} \bigcup_{a' \in \mathcal{F}(a)} \mathcal{F}(a \bullet b) = \mathcal{F}(a \bullet b),$$

where $=_{(1)}$ follows from part (ii) of Proposition 4.2.

For the predecessor sets, we use the same argument. □

Definition 4.4. Let $\Sigma_{\mathbf{A}}$ and $(L_{\mathbf{A}}, \bullet)$ be as before and define

- $L_{\bar{\mathbf{A}}} := \{\mathcal{F}(a) : a \in L_{\mathbf{A}}\}$
- $L_{\underline{\mathbf{A}}} := \{\mathcal{P}(a) : a \in L_{\mathbf{A}}\}$

Notice that $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$ are both covers of $L_{\mathbf{A}}$. On $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$ we consider the operation canonically defined from the operation on $L_{\mathbf{A}}$ which will be also denoted by $\bullet$. The Claim 4.3 guarantees that $\bullet$ is closed in $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$.

Proposition 4.5. $(L_{\bar{\mathbf{A}}}, \bullet)$ *and* $(L_{\underline{\mathbf{A}}}, \bullet)$ *are quasi-groups.*

Proof. We will only show the result for $(L_{\bar{\mathbf{A}}}, \bullet)$, because the case $(L_{\underline{\mathbf{A}}}, \bullet)$ is entirely analogous.

Since $L_{\bar{\mathbf{A}}}$ is finite, to prove that $(L_{\bar{\mathbf{A}}}, \bullet)$ is right and left cancellable, is equivalent to prove for all $\mathcal{F}_1, \mathcal{F}_2 \in L_{\bar{\mathbf{A}}}$, there exist $\mathcal{F}_i, \mathcal{F}_j \in L_{\bar{\mathbf{A}}}$, that verify $\mathcal{F}_1 \bullet \mathcal{F}_i = \mathcal{F}_2$ and $\mathcal{F}_j \bullet \mathcal{F}_1 = \mathcal{F}_2$.

We have that $\mathcal{F}_1 = \mathcal{F}(a)$ and $\mathcal{F}_2 = \mathcal{F}(b)$, for some $a, b \in L_{\mathbf{A}}$. By bipermutativity in $L_{\mathbf{A}}$, there exist $x, x' \in L_{\mathbf{A}}$ such that $a \bullet x = b$ and $x' \bullet a = b$. Then, $\mathcal{F}_i := \mathcal{F}(x)$ and $\mathcal{F}_j := \mathcal{F}(x')$ are the solutions for above equations. □

Corolary 4.6. *The Markov shift $\Sigma_{\mathbf{A}}$ has disjoint follower (and predecessor) sets, i.e., $\mathcal{F}(a) \cap \mathcal{F}(b) \neq \emptyset$ if and only if $\mathcal{F}(a) = \mathcal{F}(b)$.*

Proof. Suppose $\mathcal{F}(a) \cap \mathcal{F}(b) \neq \emptyset$. Let $r \in \mathcal{F}(a) \cap \mathcal{F}(b)$ and $c \in L_{\mathbf{A}}$. We have

$$\mathcal{F}(a) \bullet \mathcal{F}(c) = \mathcal{F}(a \bullet c) =_{(*)} r\mathcal{F}(c) = \mathcal{F}(b \bullet c) = \mathcal{F}(b) \bullet \mathcal{F}(c),$$

where $=_{(*)}$ is by Proposition 4.2(ii).

Since $(L_{\bar{\mathbf{A}}}, \bullet)$ is bipermutative we conclude that $\mathcal{F}(a) = \mathcal{F}(b)$. □

Corolary 4.7. *$(L_{\bar{\mathbf{A}}}, \bullet)$ and $(L_{\underline{\mathbf{A}}}, \bullet)$ are isomorphic. In particular, for any $a, b \in L_{\mathbf{A}}$ we have $|\mathcal{F}(a)| = |\mathcal{P}(b)|$.*

Proof. Let $\tau : L_{\bar{\mathbf{A}}} \to L_{\underline{\mathbf{A}}}$ defined by $\tau(\mathcal{F}_1) = \mathcal{P}(b)$, where $b \in \mathcal{F}_1$ is an arbitrary element. Let us show that τ is well defined, i.e., it depends not on the choice of b. In fact,

$$b, b' \in \mathcal{F}_1 = \mathcal{F}(a) \iff \exists a \in \mathcal{P}(b) \cap \mathcal{P}(b') \iff \mathcal{P}(b) \cap \mathcal{P}(b') \neq \emptyset \iff_{(*)} \mathcal{P}(b) = \mathcal{P}(b'),$$

where $(*)$ is by Corollary 4.6.

Also from above expressions it is direct that τ is one-to-one. On the other hand, is easy to see that τ is onto.

Now, given $\mathcal{F}_1, \mathcal{F}_2 \in L_{\bar{\mathbf{A}}}$, let $b_1 \in \mathcal{F}_1$ and $b_2 \in \mathcal{F}_2$. We have that $b_1 \bullet b_2 \in \mathcal{F}_1 \bullet \mathcal{F}_2$ and

$$\tau(\mathcal{F}_1 \bullet \mathcal{F}_2) = \mathcal{P}(b_1 \bullet b_2) = \mathcal{P}(b_1) \bullet \mathcal{P}(b_2) = \tau(\mathcal{F}_1) \bullet \tau(\mathcal{F}_2).$$

To conclude, notice that this isomorphism implies that $|L_{\bar{\mathbf{A}}}| = \left|L_{\underline{\mathbf{A}}}\right|$. Since $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$ are both partitions of $L_{\mathbf{A}}$, each of them containing sets with the same cardinality (Proposition 4.2), we deduce that $|\mathcal{F}(a)| = |\mathcal{P}(b)|$, $\forall a, b \in L_{\mathbf{A}}$. □

Example 4.8. Suppose that $*$ is a group operation. In this case, if we denote $e \in L_{\mathbf{A}}$ as the identity element, we have that $\mathcal{F}(e) = \mathcal{F}(e) \bullet \mathcal{F}(e)$ and $\mathcal{P}(e) = \mathcal{P}(e) \bullet \mathcal{P}(e)$ which implies that $(\mathcal{F}(e), \bullet)$ and $(\mathcal{P}(e), \bullet)$ are subgroups of $(L_{\mathbf{A}}, \bullet)$. Moreover, since $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$ are the sets of cosets of these subgroups, and $(L_{\bar{\mathbf{A}}}, \bullet)$ and $(L_{\underline{\mathbf{A}}}, \bullet)$ are also groups, we conclude that $\mathcal{F}(e)$ and $\mathcal{P}(e)$ are normal subgroups.

Definition 4.9. Given $a \in L_{\mathbf{A}}$, let $\mathcal{F}(r) \in L_{\bar{\mathbf{A}}}$ and $\mathcal{P}(t) \in L_{\underline{\mathbf{A}}}$ be such that $a \in \mathcal{F}(r) \cap \mathcal{P}(t)$. We define $\mathcal{H}_a := \mathcal{F}(r) \cap \mathcal{P}(t)$ and denote $L_{\hat{\mathbf{A}}} := \{\mathcal{H}_a : a \in L_{\mathbf{A}}\}$.

Notice that $\mathcal{H}_a$ is well defined because for each $a \in L_{\mathbf{A}}$ there exists a unique $\mathcal{F}(r) \in L_{\bar{\mathbf{A}}}$ and $\mathcal{P}(t) \in L_{\underline{\mathbf{A}}}$ satisfying $a \in \mathcal{F}(r)$ and $a \in \mathcal{P}(t)$. Moreover, we can write $L_{\hat{\mathbf{A}}} = \{\mathcal{F}(r) \cap \mathcal{P}(t) : \quad r, t \in L_{\mathbf{A}}\}$.

Consider the operation $\bullet$ over $L_{\hat{\mathbf{A}}}$ as in $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$. The Claim 4.10 give us that $\bullet$ is closed in $L_{\hat{\mathbf{A}}}$.

Claim 4.10. *$\forall \mathcal{H}_1, \mathcal{H}_2 \in L_{\mathbf{A}}$, $(\mathcal{H}_1 \bullet \mathcal{H}_2) \in L_{\hat{\mathbf{A}}}$. Moreover, $\mathcal{H}_{a \bullet b} = \mathcal{H}_a \bullet \mathcal{H}_b = a \bullet \mathcal{H}_b = \mathcal{H}_a \bullet b$ and for all $a \in L_{\mathbf{A}}$, $|\mathcal{H}_a| \left|L_{\hat{\mathbf{A}}}\right| = |L_{\mathbf{A}}|$.*

Proof. Suppose $\mathcal{H}_1 = \mathcal{F}(r) \cap \mathcal{P}(t)$ and $\mathcal{H}_2 = \mathcal{F}(s) \cap \mathcal{P}(u)$. Thus,

$$\begin{aligned}\mathcal{H}_1 \bullet \mathcal{H}_2 &= (\mathcal{F}(r) \cap \mathcal{P}(t)) \bullet (\mathcal{F}(s) \cap \mathcal{P}(u)) = \textstyle\bigcup_{g \in (\mathcal{F}(r) \cap \mathcal{P}(t))} g\big(\mathcal{F}(s) \cap \mathcal{P}(u)\big) \\ &= \textstyle\bigcup_{g \in (\mathcal{F}(r) \cap \mathcal{P}(t))} \big(g\mathcal{F}(s) \cap g\mathcal{P}(u)\big) =_{(1)} \mathcal{F}(r \bullet s) \cap \mathcal{P}(t \bullet u),\end{aligned} \tag{6}$$

where $=_{(1)}$ comes from Proposition 4.2(ii).

Since $\mathcal{H}_1 \bullet \mathcal{H}_2$ is a non-empty intersection of sets in $L_{\bar{\mathbf{A}}}$ and $L_{\underline{\mathbf{A}}}$, we deduce that it lies in $L_{\hat{\mathbf{A}}}$.

Moreover, from definition of $\mathcal{H}_a$ and $\mathcal{H}_b$ it follows that $(a \bullet b) \in (\mathcal{H}_a \bullet \mathcal{H}_b)$. Then $\mathcal{H}_{a \bullet b} = \mathcal{H}_a \bullet \mathcal{H}_b$. On the other hand, from equation (6), we get $\mathcal{H}_a \bullet \mathcal{H}_b = a \bullet \mathcal{H}_b = \mathcal{H}_a \bullet b$. These last equalities implies that any element of $L_{\hat{\mathbf{A}}}$ can be written as the product of any other element of $L_{\hat{\mathbf{A}}}$ by some element of $L_{\mathbf{A}}$, which implies $|\mathcal{H}_a| \left|L_{\hat{\mathbf{A}}}\right| = |L_{\mathbf{A}}|$ for any $a \in L_{\mathbf{A}}$. □

Proposition 4.11. $(L_{\hat{\mathbf{A}}}, \bullet)$ *is a quasigroup.*

Proof. Use the same argument as in Proposition 4.5. □

Corolary 4.12. $\forall \mathcal{H}_1, \mathcal{H}_2 \in L_{\hat{\mathbf{A}}}$, $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset \Longleftrightarrow \mathcal{H}_1 = \mathcal{H}_2$.

Proof. The relation $(\Longleftarrow)$ is obvious. For the other one $(\Longrightarrow)$, put $\mathcal{H}_1 = \mathcal{F}(r) \cap \mathcal{P}(t)$ and $\mathcal{H}_2 = \mathcal{F}(s) \cap \mathcal{P}(u)$. Notice that $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset$ implies $\mathcal{F}(r) \cap \mathcal{F}(s) \neq \emptyset$ and $\mathcal{P}(t) \cap \mathcal{P}(u) \neq \emptyset$. Thus, by Corollary 4.6, we have $\mathcal{F}(r) = \mathcal{F}(s)$ and $\mathcal{P}(t) = \mathcal{P}(u)$, and the result follows. □

4.2. **Homomorphisms and isomorphisms.** Fix $e \in L_{\mathbf{A}}$ and let $\mathcal{H} := \mathcal{H}_e = \mathcal{F}(\bar{x}) \cap \mathcal{P}(\bar{y})$, where $\bar{x} \in \mathcal{P}(e)$ and $\bar{y} \in \mathcal{F}(e)$. Given $a \in L_{\mathbf{A}}$, define a^- as the element in $L_{\mathbf{A}}$ that verifies $a^- \bullet a = e$.

Definition 4.13. Let $S : L_{\hat{\mathbf{A}}} \to L_{\mathbf{A}}$ be an arbitrary section of $L_{\hat{\mathbf{A}}}$, i.e., an arbitrary map such that $\forall \mathcal{H}_1 \in L_{\hat{\mathbf{A}}}$, $S(\mathcal{H}_1) \in \mathcal{H}_1$.

Notice that $\forall \mathcal{H}_1 \in L_{\hat{\mathbf{A}}}$, $\mathcal{H}_{S(\mathcal{H}_1)} = \mathcal{H}_1$.

Claim 4.14. $\forall a \in L_{\mathbf{A}}$, $(S(\mathcal{H}_a)^- \bullet a) \in \mathcal{H}$.

Proof. By definition of S we have $\mathcal{H}_{S(\mathcal{H}_a)} = \mathcal{H}_a$. Then,

$$\begin{aligned} \mathcal{H}_{S(\mathcal{H}_a)^- \bullet a} &=_{(1)} \mathcal{H}_{S(\mathcal{H}_a)^-} \bullet \mathcal{H}_a =_{(2)} \mathcal{H}_{S(\mathcal{H}_a)^-} \bullet \mathcal{H}_{S(\mathcal{H}_a)} \\ &=_{(1)} \mathcal{H}_{S(\mathcal{H}_a)^- \bullet S(\mathcal{H}_a)} =_{(3)} \mathcal{H}_e = \mathcal{H}, \end{aligned}$$

where $=_{(1)}$ is by Claim 4.10, $=_{(2)}$ follows from definition of S, and $=_{(3)}$ follows from definition of $^-$. □

Proposition 4.15. *The map* $\phi : L_{\mathbf{A}} \to L_{\hat{\mathbf{A}}} \times \mathcal{H}$ *given by* $\phi(a) = \big(\mathcal{H}_a, S(\mathcal{H}_a)^- \bullet a\big)$ *is a bijection. Moreover,* $\phi^{-1} : L_{\hat{\mathbf{A}}} \times \mathcal{H} \to L_{\mathbf{A}}$ *is given by* $\phi^{-1}(\mathcal{H}_a, h) = g$, *where* $g \in L_{\mathbf{A}}$ *is the unique element such that* $S(\mathcal{H}_a)^- \bullet g = h_a$.

Proof. To check ϕ is one-to-one let $a, b \in L_{\mathbf{A}}$, then

$$\begin{aligned} \phi(a) = \phi(b) &\iff \big(\mathcal{H}_a, S(\mathcal{H}_a)^- \bullet a\big) = \big(\mathcal{H}_b, S(\mathcal{H}_b)^- \bullet b\big) \\ &\iff \mathcal{H}_a = \mathcal{H}_b \ \text{ and } \ S(\mathcal{H}_a)^- \bullet a = S(\mathcal{H}_b)^- \bullet b \Longleftrightarrow a = b \end{aligned}$$

Since $L_{\mathbf{A}}$ and $L_{\hat{\mathbf{A}}} \times \mathcal{H}$ are both finite sets with the same cardinality, by Claim 4.10, ϕ is also onto.

Moreover, given $(\mathcal{H}_a, h) \in L_{\hat{\mathbf{A}}} \times \mathcal{H}$, let $g \in L_{\mathbf{A}}$ be the unique element such that $h = S(\mathcal{H}_a)^- \bullet g$. We have that

$$\begin{aligned}\mathcal{H}_{S(\mathcal{H}_a)^-} \bullet \mathcal{H}_{S(\mathcal{H}_a)} &=_{(1)} \mathcal{H}_{S(\mathcal{H}_a)^- \bullet S(\mathcal{H}_a)} = \mathcal{H} \\ &=_{(2)} \mathcal{H}_h = \mathcal{H}_{S(\mathcal{H}_a)^- \bullet g} =_{(1)} \mathcal{H}_{S(\mathcal{H}_a)^-} \bullet \mathcal{H}_g,\end{aligned}$$

where $=_{(1)}$ is by Claim 4.10, and $=_{(2)}$ is because $h \in \mathcal{H}$.

Hence, by Proposition 4.11, we get $\mathcal{H}_g = \mathcal{H}_{S(\mathcal{H}_a)} = \mathcal{H}_a$. Then, $\phi(g) = (\mathcal{H}_g, S(\mathcal{H}_g)^- \bullet g) = (\mathcal{H}_a, S(\mathcal{H}_a)^- \bullet g) = (\mathcal{H}_a, h)$. ☐

Definition 4.16. Define in $L_{\hat{\mathbf{A}}} \times \mathcal{H}$ the operation $\diamond$, given by

$$(\mathcal{H}_1, h_1) \diamond (\mathcal{H}_2, h_2) := \phi\big[\phi^{-1}(\mathcal{H}_1, h_1) \bullet \phi^{-1}(\mathcal{H}_2, h_2)\big]$$

Notice that alternatively we can write

$$(\mathcal{H}_1, h_1) \diamond (\mathcal{H}_2, h_2) = \big(\mathcal{H}_1 \bullet \mathcal{H}_2, S(\mathcal{H}_1 \bullet \mathcal{H}_2)^- \bullet (g_1 \bullet g_2)\big),$$

where $g_1, g_2 \in L_{\mathbf{A}}$ are the unique elements which verify

$$\begin{aligned}S(\mathcal{H}_1)^- \bullet g_1 = h_1,\\ S(\mathcal{H}_2)^- \bullet g_2 = h_2.\end{aligned}$$

Notice that on the first coordinate $\diamond$ coincides with $\bullet$ on $L_{\hat{\mathbf{A}}}$.

Proposition 4.17. *We can identify* $(L_{\mathbf{A}}, \bullet)$ *to* $(L_{\hat{\mathbf{A}}} \times \mathcal{H}, \diamond)$.

Proof. It follows straightforward from the definition of $\diamond$ that ϕ is an isomorphism between $(L_{\mathbf{A}}, \bullet)$ and $(L_{\hat{\mathbf{A}}} \times \mathcal{H}, \diamond)$. ☐

Definition 4.18. Define the Markov Shift $\Sigma_{\hat{\mathbf{A}}}$ on the alphabet $L_{\hat{\mathbf{A}}}$, given by transitions:

$$\mathcal{H}_0 \to \mathcal{H}_1 \iff \mathcal{H}_1 \subseteq \mathcal{F}(\mathcal{H}_0)$$

The transitions in Definition 4.18 can be defined by $\mathcal{H}_1 \subseteq \mathcal{F}(a)$ for any $a \in \mathcal{H}_0$. In fact, if $\mathcal{H}_0 = \mathcal{F}(w) \cap \mathcal{P}(z)$, then for all $a \in \mathcal{H}_0$,

$$\mathcal{F}(\mathcal{H}_0) = \bigcup_{a' \in \mathcal{H}_0 = \mathcal{F}(w) \cap \mathcal{P}(z)} \mathcal{F}(a') =_{(1)} \mathcal{F}(a), \tag{7}$$

where $=_{(1)}$ is due to the fact that for every $a' \in \mathcal{P}(z)$, we have $z \in \mathcal{F}(a')$, hence $\mathcal{F}(a') = \mathcal{F}(a)$ because the follower sets partition $L_{\mathbf{A}}$, by Corollary 4.6.

Now, consider the map $(a_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathbf{A}} \mapsto \big(\phi(a_i)\big)_{i\in\mathbb{Z}} = \big(\mathcal{H}_{a_i}, S(\mathcal{H}_{a_i})^- \bullet a_i\big)_{i\in\mathbb{Z}} \in \Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}$, which is also denoted as ϕ.

We shall check that $\phi : \Sigma_{\mathbf{A}} \to \Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}$ is well defined, i.e., for every $(a_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathbf{A}}$ we have $\big(\phi(a_i)\big)_{i\in\mathbb{Z}} \in \Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}$. Since $\phi\big((a_i)_{i\in\mathbb{Z}}\big) = \big(\phi(a_i)\big)_{i\in\mathbb{Z}} = \big(\mathcal{H}_{a_i}, S(\mathcal{H}_{a_i})^- \bullet a_i\big)_{i\in\mathbb{Z}}$, and for all $i \in \mathbb{Z}$ we have $(S(\mathcal{H}_{a_i})^- \bullet a_i) \in \mathcal{H}$ by Claim 4.14, it suffices to verify $(\mathcal{H}_{a_i})_{i\in\mathbb{Z}} \in \Sigma_{\hat{\mathbf{A}}}$. This last property is fulfilled because, if $a_i \in \mathcal{F}(a_{i-1})$ and $a_i \in \mathcal{P}(a_{i+1})$, then $\mathcal{H}_{a_i} = \mathcal{F}(a_{i-1}) \cap \mathcal{P}(a_{i+1}) \subseteq \mathcal{F}(a_{i-1}) =_{(*)} \mathcal{F}(\mathcal{H}_{a_{i-1}})$, where $=_{(*)}$ is by equation (7).

Proposition 4.19. *We can identify* $(\Sigma_{\mathbf{A}}, *, \sigma)$ *to* $(\Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}, \star, \sigma)$*, where* $\star$ *is the 1-block operation induced by* $\diamond$.

Proof. Let $\phi : \Sigma_{\mathbf{A}} \to \Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}$ be the previous map.

$\phi : \Sigma_{\mathbf{A}} \to \Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}$ is one-to-one because its local rule is (Proposition 4.17).

On the other hand if $(\mathcal{H}_i, h_i)_{i\in\mathbb{Z}} \in \Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}$, from Proposition 4.17 we get $(\mathcal{H}_i, h_i)_{i\in\mathbb{Z}} = \big(\mathcal{H}_{a_i}, S(\mathcal{H}_{a_i})^{-} \bullet a_i\big)_{i\in\mathbb{Z}}$. So, to deduce that ϕ is onto and ϕ^{-1} is 1-block, it is sufficient to show that $(a_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathbf{A}}$. Now, by definition of $\Sigma_{\hat{\mathbf{A}}}$, we have $\forall i \in \mathbb{Z}$, $\mathcal{H}_{a_i} \subseteq \mathcal{F}(a_{i-1})$, and so $a_i \in \mathcal{F}(a_{i-1})$.

Therefore, $(\Sigma_{\mathbf{A}}, *, \sigma)$ is isomorphic to $(\Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}, \star, \sigma)$. ☐

Corolary 4.20. $(\Sigma_{\hat{\mathbf{A}}}, *)$ *is a quasigroup, where* $*$ *is the operation induced by* $\bullet$ *over* $L_{\hat{\mathbf{A}}}$.

Proof. $(\Sigma_{\hat{\mathbf{A}}}, *)$ is a quasigroup because it is a factor of $(\Sigma_{\hat{\mathbf{A}}} \times \mathcal{H}^{\mathbb{Z}}, \star)$, which is itself a quasigroup because by Proposition 4.19 says it is isomorphic to $(\Sigma_{\mathbf{A}}, *)$. ☐

Claim 4.21. *The shift* $\Sigma_{\hat{\mathbf{A}}}$ *verifies* $\mathcal{F}(\mathcal{H}_{\bar{x}}) \cap \mathcal{P}(\mathcal{H}_{\bar{y}}) = \{\mathcal{H}\}$*, for all* $\bar{x} \in \mathcal{P}(e)$ *and* $\bar{y} \in \mathcal{F}(e)$. .

Proof. Since $[\bar{x}, e, \bar{y}] \in \mathcal{W}(\Sigma_{\mathbf{A}}, 3)$, we have $[\mathcal{H}_{\bar{x}}, \mathcal{H}, \mathcal{H}_{\bar{y}}] \in \mathcal{W}(\Sigma_{\hat{\mathbf{A}}}, 3)$. Then, $\mathcal{H} \in \mathcal{F}(\mathcal{H}_{\bar{x}}) \cap \mathcal{P}(\mathcal{H}_{\bar{y}})$.

If $\mathcal{H}_1 \in \mathcal{F}(\mathcal{H}_{\bar{x}}) \cap \mathcal{P}(\mathcal{H}_{\bar{y}})$ then $[\mathcal{H}_{\bar{x}}, \mathcal{H}_1, \mathcal{H}_{\bar{y}}] \in \mathcal{W}(\Sigma_{\hat{\mathbf{A}}}, 3)$. Let $a \in \mathcal{H}_1$, so that $\mathcal{H}_1 = \mathcal{H}_a$. By definition of $\Sigma_{\hat{\mathbf{A}}}$, we have $\mathcal{H}_a \subseteq \mathcal{F}(\bar{x})$, and so $a \in \mathcal{F}(\bar{x})$.

On the other hand, also from definition of $\Sigma_{\hat{\mathbf{A}}}$, it follows that $\mathcal{H}_{\bar{y}} \subseteq \mathcal{F}(a)$. Then, $\bar{y} \in \mathcal{F}(a)$, which is equivalent to $a \in \mathcal{P}(\bar{y})$.

We deduce $a \in \mathcal{H} = \mathcal{F}(\bar{x}) \cap \mathcal{P}(\bar{y})$, and so we conclude $\mathcal{H}_a = \mathcal{H}$. ☐

Definition 4.22. Define the shift $\Sigma_{\bar{\mathbf{A}}}$ with alphabet $L_{\bar{\mathbf{A}}}$, and whose transitions are given by:

$$\mathcal{F}_1 \to \mathcal{F}_2 \iff \exists g \in \mathcal{F}_1, \text{ such that } \mathcal{F}(g) = \mathcal{F}_2$$

Let $\theta : L_{\mathbf{A}} \to L_{\bar{\mathbf{A}}}$ be the map defined by $\theta(a) = \mathcal{F}(a)$. It is an onto homomorphism from $(L_{\mathbf{A}}, \bullet)$ to $(L_{\bar{\mathbf{A}}}, \bullet)$, by Proposition 3.7(ii).

Let $(\Sigma_{\bar{\mathbf{A}}}, *)$ be the quasigroup, with the operation $*$ on $\Sigma_{\bar{\mathbf{A}}}$, induced by the operation $\bullet$ on $L_{\bar{\mathbf{A}}}$.

We also denote by θ the map $(a_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathbf{A}} \mapsto \big(\mathcal{F}(a_i)\big)_{i\in\mathbb{Z}} \in \Sigma_{\bar{\mathbf{A}}}$. Let us show that this map is well defined. Let $(a_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathbf{A}}$, then $\forall i \in \mathbb{Z}, a_i \in \mathcal{F}(a_{i-1})$. Thus, $\mathcal{F}(a_{i-1}) \to \mathcal{F}(a_i)$, i.e., $\theta\big((a_i)_{i\in\mathbb{Z}}\big) = \big(\mathcal{F}(a_i)\big)_{i\in\mathbb{Z}} \in \Sigma_{\bar{\mathbf{A}}}$.

Proposition 4.23. *With the notations above:*

i. $\theta : \Sigma_{\mathbf{A}} \to \Sigma_{\bar{\mathbf{A}}}$ *is a homomorphism from* $(\Sigma_{\mathbf{A}}, *)$ *onto* $(\Sigma_{\bar{\mathbf{A}}}, *)$ *;*

ii. *If* $\mathcal{H} = \{e\}$*, then the element* g *appearing in the definition of* $\Sigma_{\bar{\mathbf{A}}}$ *is unique. In such case,* θ *is an isomorphism between* $(\Sigma_{\mathbf{A}}, *, \sigma)$ *and* $(\Sigma_{\bar{\mathbf{A}}}, *, \sigma)$.

Proof. i. $\theta : \Sigma_{\mathbf{A}} \to \Sigma_{\bar{\mathbf{A}}}$ is a homomorphism because its local rule is a homomorphism from $(L_{\mathbf{A}}, \bullet)$ to $(L_{\bar{\mathbf{A}}}, \bullet)$.

Let us check that θ is onto. Let $(\mathcal{F}_i)_{i\in\mathbb{Z}} \in \Sigma_{\bar{\mathbf{A}}}$, and notice that from definition of $\Sigma_{\bar{\mathbf{A}}}$, $\forall i \in \mathbb{Z}$, $\exists a_i \in \mathcal{F}_{i-1}$, such that $\mathcal{F}(a_i) = \mathcal{F}_i$. Then, $\exists (a_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathbf{A}}$, verifying $\theta((a_i)_{i\in\mathbb{Z}}) = (\mathcal{F}(a_i))_{i\in\mathbb{Z}} = (\mathcal{F}_i)_{i\in\mathbb{Z}}$.

ii. Suppose $\mathcal{H} = \{e\}$. Let $\mathcal{F}_1 \to \mathcal{F}_2$ and $g_1, g_2 \in \mathcal{F}_1$ be such that $\mathcal{F}(g_1) = \mathcal{F}(g_2) = \mathcal{F}_2$. Let $a \in L_{\mathbf{A}}$ be such that $\mathcal{F}_1 = \mathcal{F}(a)$, and let $h \in \mathcal{F}_2 = \mathcal{F}(g_1) = \mathcal{F}(g_2)$. Then,

$$[a, g_1, h], [a, g_2, h] \in \mathcal{W}(\Sigma_{\mathbf{A}}, 3)$$

This implies that $g_1, g_2 \in \mathcal{F}(a) \cap \mathcal{P}(h) = \mathcal{H}_1$. Since $\mathcal{H}_1 = b \bullet \mathcal{H}$ for some $b \in L_{\mathbf{A}}$, and since $\mathcal{H}$ is unitary, we deduce that $g_1 = g_2$. In this case, it is trivial to see that there exists θ^{-1}. □

Remark 4.24. If $\mathcal{H} = \{e\}$, then θ^{-1} is a 2-block code, with memory 1:

$$\forall (\mathcal{F}_i)_{i \in \mathbb{Z}} \in \Sigma_{\bar{\mathbf{A}}}, \qquad \theta^{-1}\big((\mathcal{F}_i)_{i\in\mathbb{Z}}\big) = (g_i)_{i \in \mathbb{Z}},$$

where for all $i \in \mathbb{Z}$, $g_i \in \mathcal{F}_{i-1}$ is the unique element such that $\mathcal{F}(g_i) = \mathcal{F}_i$.

The following theorems are the analogous statements for quasi-groups as those of theorems stated in [3]. From our previous results on quasi-groups these theorems have similar proof than those in [3].

Theorem 4.25. *Let $(\Sigma_{\mathbf{A}}, *)$ be a quasigroup, where $\Sigma_{\mathbf{A}}$ is a Markov shift and $*$ is a 1-block operation. Then,*

i. *$(\Sigma_{\mathbf{A}}, *, \sigma)$ is isomorphic by a 1-block code to $\big(\mathbb{F} \times \Sigma_n, \otimes, \sigma_{\mathbb{F}} \times \sigma_{\Sigma_n}\big)$, where $\mathbb{F}$ is a finite quasigroup with 1-block operation; Σ_n is a full n shift; and $\otimes$ is a k-block quasi-group operation, with memory k-1 and anticipation 0.*
ii. *$\mathbf{h}(\Sigma_{\mathbf{A}}) = 0$ if and only if $\Sigma_n = \{(\ldots, a, a, a, \ldots)\}$ (i.e., the full shift is trivial).*
iii. *$\Sigma_{\mathbf{A}}$ is irreducible and has a constant sequence if and only if $\mathbb{F} = \{e\}$ (i.e., $\mathbb{F}$ is unitary).*

Theorem 4.26. *Let $(\Sigma_{\mathbf{A}}, *)$ be an irreducible Markov shift, such that $*$ is a 1-block quasi-group operation. Let $\mathbf{h}(\Sigma_{\mathbf{A}}) = \log(N)$, where $N = p_1^{q_1} \cdots p_r^{q_r}$ is the prime decomposition of N. Then $(\Sigma_{\mathbf{A}}, *, \sigma)$ is isomorphic to $(\mathbb{F} \times \Sigma_N, \otimes, \sigma_{\mathbb{F}} \times \sigma_{\Sigma_N})$, where Σ_N is the full N shift and $\otimes$ is at most $(q_1 + \cdots + q_r)$-block, with anticipation 0.*

Proposition 4.27. *Let $(\mathcal{A}^{\mathbb{Z}}, *)$ be a quasigroup, where $*$ is induced by a 1-block operation $\bullet$ on $\mathcal{A}$. Let $\Sigma_{\mathbf{A}} \subset \mathcal{A}^{\mathbb{Z}}$ be a topological Markov chain. Define $\theta : L_{\mathbf{A}} \to L_{\bar{\mathbf{A}}}$ by $\theta(a) = \mathcal{F}(a)$, as before.*

Then $\Sigma_{\mathbf{A}}$ is closed under $*$ if and only if $\theta : L_{\mathbf{A}} \to L_{\bar{\mathbf{A}}}$ is an onto homomorphism. Furthermore, $(\Sigma_{\mathbf{A}}, *)$ is irreducible (transitive) if and only if there exists $a \in L_{\mathbf{A}}$ such that $\mathcal{F}^k(a) = L_{\mathbf{A}}$ for some $k \geq 0$, where $\mathcal{F}^k(a)$ is defined inductively by $\mathcal{F}^{n+1}(a) = \bigcup_{h \in \mathcal{F}^n(a)} \mathcal{F}(h)$.

Remark 4.28. We can define the shift $\Sigma_{\underline{\mathbf{A}}}$, in the same way than Definition 4.22:

$$\mathcal{P}_1 \to \mathcal{P}_2 \iff \exists g \in \mathcal{P}_2, \text{ such that } \mathcal{P}(g) = \mathcal{P}_1.$$

If we consider $\Sigma_{\underline{\mathbf{A}}}$ instead of $\Sigma_{\bar{\mathbf{A}}}$ in Proposition 4.23, we obtain analogous results, but $\theta^{-1} : \Sigma_{\underline{\mathbf{A}}} \to \Sigma_{\mathbf{A}}$ will be a 2-block code with anticipation 1.

Moreover, in the Theorems 4.25 and 4.26, $\otimes$ will be a k-block operation with memory 0 and anticipation $k-1$.

5. **Amalgamation and state splitting.** Let $(\Sigma_{\mathbf{A}}, *)$ be a Markov shift with a 1-block quasi-group operation. As before, denote by $\bullet$ the quasi-group operation on $L_{\mathbf{A}}$ induced by $*$. We define the four elementary isomorphisms as in [7]:

- State splitting by successors: Given $a \in L_{\mathbf{A}}$, let $\mathcal{H} \subseteq \mathcal{F}(a)$ be a subset such that $L_{\mathbf{A}}/_{\mathcal{H}} := \{g\mathcal{H} : \quad g \in L_{\mathbf{A}}\}$ is a partition of $L_{\mathbf{A}}$ compatible with $\bullet$. Define

$$L_{\tilde{\mathbf{A}}} := \{(g, \mathcal{H}_h) : \quad \mathcal{H}_h \subseteq \mathcal{F}(g)\} \subseteq L_{\mathbf{A}} \times L_{\mathbf{A}}/_{\mathcal{H}},$$

where $\mathcal{H}_h$ denotes the coset of $L_{\mathbf{A}}/_{\mathcal{H}}$ containing h.

Consider on $L_{\tilde{\mathbf{A}}}$ the operation coinciding with $\bullet$ in each coordinate, and let $\Sigma_{\tilde{\mathbf{A}}}$ be the shift defined by the following transitions:

$$(g, \mathcal{H}_h) \to (g', \mathcal{H}_{h'}) \qquad \Longleftrightarrow \qquad g' \in \mathcal{H}_h,$$

which is considered with the operation canonically induced by $L_{\tilde{\mathbf{A}}}$.

The state splitting is the 2-block code, $\varphi : \Sigma_{\mathbf{A}} \to \Sigma_{\tilde{\mathbf{A}}}$, defined by:

$$[g, h] \in \mathcal{W}(\Sigma_{\mathbf{A}}, 2) \qquad \mapsto \qquad (g, \mathcal{H}_h) \in L_{\tilde{\mathbf{A}}}.$$

Notice that φ^{-1} is a 1-block code given by $(g, \mathcal{H}_h) \in L_{\tilde{\mathbf{A}}} \mapsto g \in L_{\mathbf{A}}$.

The state splitting is an isomorphism between $\Sigma_{\mathbf{A}}$ and $\Sigma_{\tilde{\mathbf{A}}}$.

- State splitting by predecessors: It is defined as in the previous case, but using $\mathcal{P}(a)$ instead of $F(a)$.

- Amalgamation by common predecessors and disjoint successors: Given $a \in L_{\mathbf{A}}$, let $\mathcal{H} \subseteq \mathcal{F}(a)$ be a subset, such that $L_{\mathbf{A}}/_{\mathcal{H}} := \{g\mathcal{H} : \quad g \in L_{\mathbf{A}}\}$ is a partition of $L_{\mathbf{A}}$ compatible with the operation $\bullet$. Moreover, suppose that $\forall x \in L_{\mathbf{A}}$, we have $\mathcal{H} \cap \mathcal{P}(x)$ is either empty or unitary.

Define $L_{\tilde{\mathbf{A}}} := L_{\mathbf{A}}/_{\mathcal{H}}$, where it is considered the operation induced by $\bullet$. Let $\Sigma_{\tilde{\mathbf{A}}}$ be the shift given by transitions:

$$\mathcal{H}_g \to \mathcal{H}_{g'} \qquad \Longleftrightarrow \qquad \exists h \in \mathcal{H}_g : \quad \mathcal{H}_{g'} \subseteq \mathcal{F}(h),$$

where is defined the operation induced by $L_{\tilde{\mathbf{A}}}$.

The amalgamation is the 1-block code, $\varphi : \Sigma_{\mathbf{A}} \to \Sigma_{\tilde{\mathbf{A}}}$, given by

$$g \in L_{\mathbf{A}} \qquad \mapsto \qquad \mathcal{H}_g \in L_{\tilde{\mathbf{A}}}.$$

Notice that φ^{-1} is a 2-block code given by $[\mathcal{H}_g, \mathcal{H}_{g'}] \in \mathcal{W}(\Sigma_{\tilde{\mathbf{A}}}, 2) \mapsto h \in L_{\mathbf{A}}$, where h is the unique element belonging to $\mathcal{H}_g$, such that $\mathcal{H}_{g'} \subseteq \mathcal{F}(h)$.

It is straightforward to see that the amalgamation is an isomorphism between $\Sigma_{\mathbf{A}}$ and $\Sigma_{\tilde{\mathbf{A}}}$.

- Amalgamation by common successors and disjoint predecessors: It is defined in the same way than the previous case, but changing the roles of the predecessor and the follower sets.

Theorem 5.1. *Two quasi-group SFTs, each of them with 1-block quasi-group operation, are isomorphic if and only if it is possible to go from one to other by a finite sequence of elementary isomorphisms.*

Proof. Let $(\Sigma_{\mathbf{A}}, \sigma, *)$ and $(\Sigma_{\bar{\mathbf{A}}}, \sigma, \bar{*})$ be both quasi-group shifts with 1-block operations. Let $\phi : \Sigma_{\mathbf{A}} \to \Sigma_{\bar{\mathbf{A}}}$ be an isomorphism between them.

Without lost of generality we can consider $\Sigma_{\mathbf{A}}$ a Markov shift and ϕ a 1-block code (in fact, we can take the N-block presentation of $\Sigma_{\mathbf{A}}$, with N sufficiently large). Furthermore, $(L_{\mathbf{A}}, \bullet)$ and $(L_{\bar{\mathbf{A}}}, \bar{\bullet})$ are both quasi-groups which induce, respectively, the operations $*$ and $\bar{*}$.

We have that $\phi : L_{\mathbf{A}} \to L_{\bar{\mathbf{A}}}$ is an onto homomorphism between $(L_{\mathbf{A}}, \bullet)$ and $(L_{\bar{\mathbf{A}}}, \bar{\bullet})$ (notice that the local rule of the code is also denoted by ϕ).

Define $L_{\mathbf{A}}/_{\phi^{-1}} := \left\{\phi^{-1}(\{\bar{a}\}) : \quad \bar{a} \in L_{\bar{\mathbf{A}}}\right\}$, which is a partition of $L_{\mathbf{A}}$ compatible with $\bullet$. This property also holds when we consider for $n \geq 1$, $\phi : \mathcal{W}(\Sigma_{\mathbf{A}}, n) \to \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, n)$, that is

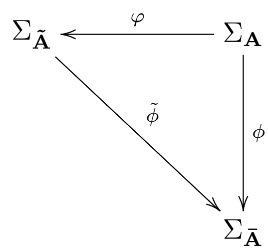
$$\mathcal{W}(\Sigma_{\mathbf{A}}, n)/_{\phi^{-1}} = \left\{\phi^{-1}(\{[\bar{a}_1, \ldots, \bar{a}_n]\}) : \quad [\bar{a}_1, \ldots, \bar{a}_n] \in \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, n)\right\},$$

which is a partition of $\mathcal{W}(\Sigma_{\mathbf{A}}, n)$ compatible with $\bullet$.

Since ϕ^{-1} is a N-block code, there exists m , $1 \leq m \leq N$, such that given $[\bar{a}_1, \ldots, \bar{a}_N] \in \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, N)$, $\forall [a_1, \ldots, a_N], [a'_1, \ldots, a'_N] \in \phi^{-1}(\{[\bar{a}_1, \ldots, \bar{a}_N]\})$ we have $a_m = a'_m$.

Fix $[\bar{x}_1, \ldots, \bar{x}_N] \in \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, N)$, $[x_1, \ldots, x_N] \in \phi^{-1}(\{[\bar{x}_1, \ldots, \bar{x}_N]\})$ and put $\mathcal{H} := \mathcal{F}(x_m) \cap \phi^{-1}(\bar{x}_{m+1})$. It follows that $\mathcal{H} \subseteq \mathcal{F}(x_m)$ and $L_{\mathbf{A}}/_{\mathcal{H}}$ is a partition of $L_{\mathbf{A}}$, compatible with $\bullet$. Furthermore, for every $b \in L_{\mathbf{A}}$ the set $\mathcal{H} \cap \mathcal{P}(b)$ has at most one element. In fact, if there was more than one element in $\mathcal{H} \cap \mathcal{P}(b)$, then we could find two distinct sequences in $\Sigma_{\mathbf{A}}$ with the same image by ϕ, what is a contradiction about injectivity of this map.

Denote by $\varphi : \Sigma_{\mathbf{A}} \to \Sigma_{\tilde{\mathbf{A}}}$ the amalgamation by common predecessors and disjoint successors, where $\Sigma_{\tilde{\mathbf{A}}} := L_{\mathbf{A}}/_{\mathcal{H}}$. We recall φ is a 1-block code and φ^{-1} is a 2-block code. We define $\tilde{\phi} : \Sigma_{\tilde{\mathbf{A}}} \to \Sigma_{\bar{\mathbf{A}}}$ the 1-block map which has local rule (which we will also denote by $\tilde{\phi}$), given by $\tilde{\phi}(\mathcal{H}_g) := \phi(g')$ for any $g' \in \mathcal{H}_g$.



Now, we have that given $[\bar{a}_1, \ldots, \bar{a}_N] \in \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, N)$, for all $[\tilde{a}_1, \ldots, \tilde{a}_N], [\tilde{a}'_1, \ldots, \tilde{a}'_N] \in \tilde{\phi}^{-1}(\{[\bar{a}_1, \ldots, \bar{a}_N]\})$ follows that $\tilde{a}_m = \tilde{a}'_m$ and $\tilde{a}_{m+1} = \tilde{a}'_{m+1}$.

We repeat the above process until we get a 1-block isomorphism, $\tilde{\phi} : \Sigma_{\tilde{\mathbf{A}}} \to \Sigma_{\bar{\mathbf{A}}}$, such that for any $[\bar{a}_1, \ldots, \bar{a}_N] \in \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, N)$, for all $[\tilde{a}_1, \ldots, \tilde{a}_N], [\tilde{a}'_1, \ldots, \tilde{a}'_N] \in \tilde{\phi}^{-1}(\{[\bar{a}_1, \ldots, \bar{a}_N]\})$ follows $\tilde{a}_i = \tilde{a}'_i$, $m \leq i \leq N$.

To conclude, we come back and applying the amalgamation by common successors and disjoint predecessors from the entry $(m-1)$ until the first entry. We obtain that $\tilde{\phi} : \Sigma_{\tilde{\mathbf{A}}} \to \Sigma_{\bar{\mathbf{A}}}$ is a 1-block isomorphism, and for all $[\bar{a}_1, \ldots, \bar{a}_N] \in \mathcal{W}(\Sigma_{\bar{\mathbf{A}}}, N)$ we have that $\tilde{\phi}^{-1}(\{[\bar{a}_1, \ldots, \bar{a}_N]\})$ contains a unique N-block of $\Sigma_{\tilde{\mathbf{A}}}$. This implies that $\Sigma_{\tilde{\mathbf{A}}}$ and $\Sigma_{\bar{\mathbf{A}}}$ are identical. $\square$

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