PROOF OF THE 1-FACTORIZATION AND HAMILTON DECOMPOSITION CONJECTURES II: THE BIPARTITE CASE

BÉLA CSABA, DANIELA KÜHN, ALLAN LO, DERYK OSTHUS AND ANDREW TREGLOWN

Abstract. In a sequence of four papers, we prove the following results (via a unified approach) for all sufficiently large n:

(i) [*1-factorization conjecture*] Suppose that n is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D-regular graph G on n vertices has a decomposition into perfect matchings. Equivalently, $\chi'(G) = D$.

(ii) [*Hamilton decomposition conjecture*] Suppose that $D \geq \lfloor n/2 \rfloor$. Then every D-regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.

(iii) [*Optimal packings of Hamilton cycles*] Suppose that G is a graph on n vertices with minimum degree $\delta \geq n/2$. Then G contains at least $\mathrm{reg}_{\mathrm{even}}(n, \delta)/2 \geq (n-2)/8$ edge-disjoint Hamilton cycles. Here $\mathrm{reg}_{\mathrm{even}}(n, \delta)$ denotes the degree of the largest even-regular spanning subgraph one can guarantee in a graph on n vertices with minimum degree δ.

According to Dirac, (i) was first raised in the 1950s. (ii) and the special case $\delta = \lceil n/2 \rceil$ of (iii) answer questions of Nash-Williams from 1970. All of the above bounds are best possible. In the current paper, we prove the above results for the case when G is close to a complete balanced bipartite graph.

Contents

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1. Introduction

The topic of decomposing a graph into a given collection of edge-disjoint subgraphs has a long history. Indeed, in 1892, Walecki [19] proved that every complete graph of odd order has a decomposition into edge-disjoint Hamilton cycles. In a sequence of four papers, we provide a unified approach towards proving three long-standing graph decomposition conjectures for all sufficiently large graphs.

1.1. The 1-factorization conjecture.

Vizing's theorem states that for any graph G of maximum degree Δ, its edge-chromatic number $\chi'(G)$ is either Δ or $\Delta+1$. However, the problem of determining the precise value of $\chi'(G)$ for an arbitrary graph G is NP-complete [8]. Thus, it is of interest to determine classes of graphs G that attain the (trivial) lower bound Δ – much of the recent book [28] is devoted to the subject. If G is a regular graph then $\chi'(G) = \Delta(G)$ precisely when G has a 1-factorization: a 1-*factorization* of a graph G consists of a set of edge-disjoint perfect matchings covering all edges of G. The 1-factorization conjecture states that every regular graph of sufficiently high degree has a 1-factorization. It was first stated explicitly by Chetwynd and Hilton [1, 2] (who also proved partial results). However, they state that according to Dirac, it was already discussed in the 1950s. We prove the 1-factorization conjecture for sufficiently large graphs.

Theorem 1.1. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ is even and $D \geq 2\lceil n/4 \rceil - 1$. Then every D-regular graph G on n vertices has a 1-factorization. Equivalently, $\chi'(G) = D$.*

The bound on the minimum degree in Theorem 1.1 is best possible. In fact, a smaller degree bound does not even ensure a single perfect matching. To see this, suppose first that $n = 2 \pmod 4$. Consider the graph which is the disjoint union of two cliques of order $n/2$ (which is odd). If $n = 0 \pmod 4$, consider the graph obtained from the disjoint union of cliques of orders $n/2-1$ and $n/2+1$ (both odd) by deleting a Hamilton cycle in the larger clique.

Perkovic and Reed [26] proved an approximate version of Theorem 1.1 (they assumed that $D \geq n/2 + \varepsilon n$). Recently, this was generalized by Vaughan [29] to multigraphs of bounded multiplicity, thereby proving an approximate version of a 'multigraph 1-factorization conjecture' which was raised by Plantholt and Tipnis [27]. Further related results and problems are discussed in the recent monograph [28].

1.2. **The Hamilton decomposition conjecture.** A *Hamilton decomposition* of a graph G consists of a set of edge-disjoint Hamilton cycles covering all the edges of G. A natural extension of this to regular graphs G of odd degree is to ask for a decomposition into Hamilton cycles and one perfect matching (i.e. one perfect matching M in G together with a Hamilton decomposition of $G-M$). Nash-Williams [23, 25] raised the problem of finding a Hamilton decomposition in an even-regular graph of sufficiently large degree. The following result completely solves this problem for large graphs.

Theorem 1.2. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Let $n, D \in \mathbb{N}$ be such that $n \geq n_0$ and $D \geq \lfloor n/2 \rfloor$. Then every D-regular graph G on n vertices has a decomposition into Hamilton cycles and at most one perfect matching.*

The bound on the degree in Theorem 1.2 is best possible (see Proposition 3.1 in [14] for a proof of this). Note that Theorem 1.2 does not quite imply Theorem 1.1, as the degree threshold in the former result is slightly higher.

Previous results include the following: Nash-Williams [22] showed that the degree bound in Theorem 1.2 ensures a single Hamilton cycle. Jackson [9] showed that one can ensure close to $D/2 - n/6$ edge-disjoint Hamilton cycles. More recently, Christofides, Kühn and Osthus [3] obtained an approximate decomposition under the assumption that $D \geq n/2 + \varepsilon n$. Finally, under the same assumption, Kühn and Osthus [16] obtained an exact decomposition (as a consequence of the main result in [15] on Hamilton decompositions of robustly expanding graphs).

1.3. **Packing Hamilton cycles in graphs of large minimum degree.** Dirac's theorem is best possible in the sense that one cannot lower the minimum degree condition. Remarkably though, the conclusion can be strengthened considerably: Nash-Williams [24] proved that every graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains $\lfloor 5n/224 \rfloor$ edge-disjoint Hamilton cycles. Nash-Williams [24, 23, 25] raised the question of finding the best possible bound on the number of edge-disjoint Hamilton cycles in a Dirac graph. This question is answered by Corollary 1.4 below.

In fact, we answer a more general form of this question: what is the number of edge-disjoint Hamilton cycles one can guarantee in a graph G of minimum degree δ?

Let $\mathrm{reg}_{\mathrm{even}}(G)$ be the largest degree of an even-regular spanning subgraph of G. Then let

$$\mathrm{reg}_{\mathrm{even}}(n, \delta) := \min\{\mathrm{reg}_{\mathrm{even}}(G) : |G| = n,\ \delta(G) = \delta\}.$$

Clearly, in general we cannot guarantee more than $\mathrm{reg}_{\mathrm{even}}(n, \delta)/2$ edge-disjoint Hamilton cycles in a graph of order n and minimum degree δ. The next result shows that this bound is best possible (if $\delta < n/2$, then $\mathrm{reg}_{\mathrm{even}}(n, \delta) = 0$).

Theorem 1.3. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices with minimum degree $\delta \geq n/2$. Then G contains at least $\mathrm{reg}_{\mathrm{even}}(n, \delta)/2$ edge-disjoint Hamilton cycles.*

Kühn, Lapinskas and Osthus [11] proved Theorem 1.3 in the case when G is not close to one of the extremal graphs for Dirac's theorem. An approximate version of Theorem 1.3 for $\delta \geq n/2 + \varepsilon n$ was obtained earlier by Christofides, Kühn and Osthus [3]. Hartke and Seacrest [7] gave a simpler argument with improved error bounds.

The following consequence of Theorem 1.3 answers the original question of Nash-Williams.

Corollary 1.4. *There exists an $n_0 \in \mathbb{N}$ such that the following holds. Suppose that G is a graph on $n \geq n_0$ vertices with minimum degree $\delta \geq n/2$. Then G contains at least $(n-2)/8$ edge-disjoint Hamilton cycles.*

See [14] for an explanation as to why Corollary 1.4 follows from Theorem 1.3 and for a construction showing the bound on the number of edge-disjoint Hamilton cycles in Corollary 1.4 is best possible (the construction is also described in Section 3.1).

1.4. **Overall structure of the argument.** For all three of our main results, we split the argument according to the structure of the graph G under consideration:

(i) G is close to the complete balanced bipartite graph $K_{n/2,n/2}$;
(ii) G is close to the union of two disjoint copies of a clique $K_{n/2}$;
(iii) G is a 'robust expander'.

Roughly speaking, G is a robust expander if for every set S of vertices, its neighbourhood is at least a little larger than $|S|$, even if we delete a small proportion of the edges of G. The main result of [15] states that every dense regular robust expander has a Hamilton decomposition. This immediately implies Theorems 1.1 and 1.2 in Case (iii). For Theorem 1.3, Case (iii) is proved in [11] using a more involved argument, but also based on the main result of [15].

Case (ii) is proved in [14, 12]. The current paper is devoted to the proof of Case (i). In [14] we derive Theorems 1.1, 1.2 and 1.3 from the structural results covering Cases (i)–(iii).

The arguments in the current paper for Case (i) as well as those in [14] for Case (ii) make use of an 'approximate' decomposition result proved in [4]. In both Case (i) and Case (ii) we use the main lemma from [15] (the 'robust decomposition lemma') when transforming this approximate decomposition into an exact one.

1.5. **Statement of the main results of this paper.** As mentioned above, the focus of this paper is to prove Theorems 1.1, 1.2 and 1.3 when our graph is *close* to the complete balanced bipartite graph $K_{n/2,n/2}$. More precisely, we say that a graph G on n vertices is *ε-bipartite* if there is a partition S_1, S_2 of $V(G)$ which satisfies the following:

- $n/2 - 1 < |S_1|, |S_2| < n/2 + 1$;
- $e(S_1), e(S_2) \leq \varepsilon n^2$.

The following result implies Theorems 1.1 and 1.2 in the case when our given graph is close to $K_{n/2,n/2}$.

Theorem 1.5. *There are $\varepsilon_{\rm ex} > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that $D \geq (1/2 - \varepsilon_{\rm ex})n$ and D is even and suppose that G is a D-regular graph on $n \geq n_0$ vertices which is $\varepsilon_{\rm ex}$-bipartite. Then G has a Hamilton decomposition.*

The next result implies Theorem 1.3 in the case when our graph is close to $K_{n/2,n/2}$.

Theorem 1.6. *For each $\alpha > 0$ there are $\varepsilon_{\rm ex} > 0$ and $n_0 \in \mathbb{N}$ such that the following holds. Suppose that F is an $\varepsilon_{\rm ex}$-bipartite graph on $n \geq n_0$ vertices with $\delta(F) \geq (1/2 - \varepsilon_{\rm ex})n$. Suppose that F has a D-regular spanning subgraph G such that $n/100 \leq D \leq (1/2 - \alpha)n$ and D is even. Then F contains $D/2$ edge-disjoint Hamilton cycles.*

Note that Theorem 1.5 implies that the degree bound in Theorems 1.1 and 1.2 is not tight in the almost bipartite case (indeed, the extremal graph is close to being the union of two cliques). On the other hand, the extremal construction for Corollary 1.4 is close to bipartite (see Section 3.1 for a description). So it turns out that the bound on the number of edge-disjoint Hamilton cycles in Corollary 1.4 is best possible in the almost bipartite case but not when the graph is close to the union of two cliques.

In Section 3 we give an outline of the proofs of Theorems 1.5 and 1.6. The results from Sections 4 and 5 are used in both the proofs of Theorems 1.5 and 1.6. In Sections 6 and 7 we build up machinery for the proof of Theorem 1.5. We then prove Theorem 1.6 in Section 8 and Theorem 1.5 in Section 9.

2. Notation and Tools

2.1. **Notation.** Unless stated otherwise, all the graphs and digraphs considered in this paper are simple and do not contain loops. So in a digraph G, we allow up to two edges between any two vertices; at most one edge in each direction. Given a graph or digraph G, we write $V(G)$ for its vertex set, $E(G)$ for its edge set, $e(G) := |E(G)|$ for the number of its edges and $|G| := |V(G)|$ for the number of its vertices.

Suppose that G is an undirected graph. We write $\delta(G)$ for the minimum degree of G and $\Delta(G)$ for its maximum degree. Given a vertex v of G and a set $A \subseteq V(G)$, we write $d_G(v, A)$ for the number of neighbours of v in G which lie in A. Given $A, B \subseteq V(G)$, we write $E_G(A)$ for the set of all those edges of G which have both endvertices in A and $E_G(A, B)$ for the set of all those edges of G which have one endvertex in A and its other endvertex in B. We also call the edges in $E_G(A, B)$ *AB-edges* of G. We let $e_G(A) := |E_G(A)|$ and $e_G(A, B) := |E_G(A, B)|$. We denote by $G[A]$ the subgraph of G with vertex set A and edge set $E_G(A)$. If $A \cap B = \emptyset$, we denote by $G[A,B]$ the bipartite subgraph of G with vertex classes A and B and edge set $E_G(A,B)$. If $A=B$ we define $G[A,B]:=G[A]$. We often omit the index G if the graph G is clear from the context. A spanning subgraph H of G is an *r-factor* of G if every vertex has degree r in H.

Given a vertex set V and two multigraphs G and H with $V(G), V(H) \subseteq V$, we write $G+H$ for the multigraph whose vertex set is $V(G)\cup V(H)$ and in which the multiplicity of xy in $G+H$ is the sum of the multiplicities of xy in G and in H (for all $x,y \in V(G)\cup V(H)$). We say that a graph G has a *decomposition* into $H_1,\dots,H_r$ if $G = H_1+\dots+H_r$ and the H_i are pairwise edge-disjoint.

If G and H are simple graphs, we write $G\cup H$ for the (simple) graph whose vertex set is $V(G)\cup V(H)$ and whose edge set is $E(G)\cup E(H)$. Similarly, $G\cap H$ denotes the graph whose vertex set is $V(G)\cap V(H)$ and whose edge set is $E(G)\cap E(H)$. We write $G-H$ for the subgraph of G which is obtained from G by deleting all the edges in $E(G)\cap E(H)$. Given $A\subseteq V(G)$, we write $G-A$ for the graph obtained from G by deleting all vertices in A.

A *path system* is a graph Q which is the union of vertex-disjoint paths (some of them might be trivial). We say that P is a *path in Q* if P is a component of Q and, abusing the notation, sometimes write $P\in Q$ for this.

If G is a digraph, we write xy for an edge directed from x to y. A digraph G is an *oriented graph* if there are no $x,y\in V(G)$ such that $xy, yx\in E(G)$. Unless stated otherwise, when we refer to paths and cycles in digraphs, we mean directed paths and cycles, i.e. the edges on these paths/cycles are oriented consistently. If x is a vertex of a digraph G, then $N^+_G(x)$ denotes the *outneighbourhood* of x, i.e. the set of all those vertices y for which $xy\in E(G)$. Similarly, $N^-_G(x)$ denotes the *inneighbourhood* of x, i.e. the set of all those vertices y for which $yx\in E(G)$. The *outdegree* of x is $d^+_G(x) := |N^+_G(x)|$ and the *indegree* of x is $d^-_G(x) := |N^-_G(x)|$. We write $\delta(G)$ and $\Delta(G)$ for the minimum and maximum degrees of the underlying simple undirected graph of G respectively.

For a digraph G, whenever $A,B\subseteq V(G)$ with $A\cap B=\emptyset$, we denote by $G[A,B]$ the bipartite subdigraph of G with vertex classes A and B whose edges are all the edges of G directed from A to B, and let $e_G(A,B)$ denote the number of edges in $G[A,B]$. We define $\delta(G[A,B])$ to be the minimum degree of the underlying undirected graph of $G[A,B]$ and define $\Delta(G[A,B])$ to be the maximum degree of the underlying undirected graph of $G[A,B]$. A spanning subdigraph H of G is an *r-factor* of G if the outdegree and the indegree of every vertex of H is r.

If P is a path and $x,y\in V(P)$, we write xPy for the subpath of P whose endvertices are x and y. We define xPy similarly if P is a directed path and x precedes y on P.

In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results have to be chosen from right to left. More precisely, if we claim that a result holds whenever $0 < 1/n \ll a \ll b \ll c \le 1$ (where n is the order of the graph or digraph), then this means that there are non-decreasing functions $f:(0,1]\to(0,1]$, $g:(0,1]\to(0,1]$ and $h:(0,1]\to(0,1]$ such that the result holds for all $0<a,b,c\le 1$ and all $n\in\mathbb{N}$ with $b\le f(c)$, $a\le g(b)$

and $1/n \leq h(a)$. We will not calculate these functions explicitly. Hierarchies with more constants are defined in a similar way. We will write $a = b \pm c$ as shorthand for $b - c \leq a \leq b + c$.

2.2. **ε-regularity.** If $G = (A, B)$ is an undirected bipartite graph with vertex classes A and B, then the *density* of G is defined as

$$d(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

For any $\varepsilon > 0$, we say that G is *ε-regular* if for any $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ we have $|d(A', B') - d(A, B)| < \varepsilon$. We say that G is *$(\varepsilon, \geq d)$-regular* if it is ε-regular and has density d' for some $d' \geq d - \varepsilon$.

We say that G is *$[\varepsilon, d]$-superregular* if it is ε-regular and $d_G(a) = (d \pm \varepsilon)|B|$ for every $a \in A$ and $d_G(b) = (d \pm \varepsilon)|A|$ for every $b \in B$. G is *$[\varepsilon, \geq d]$-superregular* if it is *$[\varepsilon, d']$-superregular* for some $d' \geq d$.

Given disjoint vertex sets X and Y in a digraph G, recall that $G[X, Y]$ denotes the bipartite subdigraph of G whose vertex classes are X and Y and whose edges are all the edges of G directed from X to Y. We often view $G[X, Y]$ as an undirected bipartite graph. In particular, we say $G[X, Y]$ is *ε-regular*, *$(\varepsilon, \geq d)$-regular*, $[\varepsilon, d]$-superregular or $[\varepsilon, \geq d]$-superregular if this holds when $G[X, Y]$ is viewed as an undirected graph.

We often use the following simple proposition which follows easily from the definition of (super-)regularity. We omit the proof, a similar argument can be found e.g. in [15].

Proposition 2.1. *Suppose that $0 < 1/m \ll \varepsilon \leq d' \ll d \leq 1$. Let G be a bipartite graph with vertex classes A and B of size m. Suppose that G' is obtained from G by removing at most $d'm$ vertices from each vertex class and at most $d'm$ edges incident to each vertex from G. If G is $[\varepsilon, d]$-superregular then G' is $[2\sqrt{d'}, d]$-superregular.*

We will also use the following simple fact.

Fact 2.2. *Let $\varepsilon > 0$. Suppose that G is a bipartite graph with vertex classes of size n such that $\delta(G) \geq (1 - \varepsilon)n$. Then G is $[\sqrt{\varepsilon}, 1]$-superregular.*

2.3. **A Chernoff-Hoeffding bound.** We will often use the following Chernoff-Hoeffding bound for binomial and hypergeometric distributions (see e.g. [10, Corollary 2.3 and Theorem 2.10]). Recall that the binomial random variable with parameters (n, p) is the sum of n independent Bernoulli variables, each taking value 1 with probability p or 0 with probability $1 - p$. The hypergeometric random variable X with parameters (n, m, k) is defined as follows. We let N be a set of size n, fix $S \subseteq N$ of size $|S| = m$, pick a uniformly random $T \subseteq N$ of size $|T| = k$, then define $X := |T \cap S|$. Note that $\mathbb{E}X = km/n$.

Proposition 2.3. *Suppose X has binomial or hypergeometric distribution and $0 < a < 3/2$. Then $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-\frac{a^2}{3}\mathbb{E}X}$.*

3. Overview of the proofs of Theorems 1.5 and 1.6

Note that, unlike in Theorem 1.5, in Theorem 1.6 we do not require a complete decomposition of our graph F into edge-disjoint Hamilton cycles. Therefore, the proof of Theorem 1.5 is considerably more involved than the proof of Theorem 1.6. Moreover, the ideas in the proof of Theorem 1.6 are all used in the proof of Theorem 1.5 too.

3.1. **Proof overview for Theorem 1.6.** Let F be a graph on n vertices with $\delta(F) \geq (1/2 - o(1))n$ which is close to the balanced bipartite graph $K_{n/2,n/2}$. Further, suppose that G is a D-regular spanning subgraph of F as in Theorem 1.6. Then there is a partition A, B of $V(F)$ such that A and B are of roughly equal size and most edges in F go between A and B. Our ultimate aim is to construct $D/2$ edge-disjoint Hamilton cycles in F.

Suppose first that, in the graph F, both A and B are independent sets of equal size. So F is an almost complete balanced bipartite graph. In this case, the densest spanning even-regular subgraph G of F is also almost complete bipartite. This means that one can extend existing techniques (developed e.g. in [3, 5, 6, 7, 21]) to find an approximate Hamilton decomposition. This is achieved in [4] and is more than enough to prove Theorem 1.6 in this case. (We state the main result from [4] as Lemma 8.1 in the current paper.) The real difficulties arise when

(i) F is unbalanced;
(ii) F has vertices having high degree in both A and B (these are called exceptional vertices).

To illustrate (i), consider the following example due to Babai (which is the extremal construction for Corollary 1.4). Consider the graph F on $n = 8k + 2$ vertices consisting of one vertex class A of size $4k + 2$ containing a perfect matching and no other edges, one empty vertex class B of size $4k$, and all possible edges between A and B. Thus the minimum degree of F is $4k + 1 = n/2$. Then one can use Tutte's factor theorem to show that the largest even-regular spanning subgraph G of F has degree $D = 2k = (n - 2)/4$. Note that to prove Theorem 1.6 in this case, each of the $D/2 = k$ Hamilton cycles we find must contain exactly two of the $2k + 1$ edges in A. In this way, we can 'balance out' the difference in the vertex class sizes.

More generally we will construct our Hamilton cycles in two steps. In the first step, we find a path system J which balances out the vertex class sizes (so in the above example, J would contain two edges in A). Then we extend J into a Hamilton cycle using only AB-edges in F. It turns out that the first step is the difficult one. It is easy to see that a path system J will balance out the sizes of A and B (in the sense that the number of uncovered vertices in A and B is the same) if and only if

$$e_J(A) - e_J(B) = |A| - |B|. \tag{3.1}$$

Note that any Hamilton cycle also satisfies this identity. So we need to find a set of $D/2$ path systems J satisfying (3.1) (where D is the degree of G). This is achieved (amongst other things) in Sections 5.2 and 5.3.

As indicated above, our aim is to use Lemma 8.1 in order to extend each such J into a Hamilton cycle. To apply Lemma 8.1 we also need to extend the balancing path systems J into 'balanced exceptional (path) systems' which contain all the exceptional vertices from (ii). This is achieved in Section 5.4. Lemma 8.1 also assumes that the path systems are 'localized' with respect to a given subpartition of A, B (i.e. they are induced by a small number of partition classes). Section 5.1 prepares the ground for this.

Finding the balanced exceptional systems is extremely difficult if G contains edges between the set A_0 of exceptional vertices in A and the set B_0 of exceptional vertices in B. So in a preliminary step, we find and remove a small number of edge-disjoint Hamilton cycles covering all A_0B_0-edges in Section 4. We put all these steps together in Section 8. (Sections 6, 7 and 9 are only relevant for the proof of Theorem 1.5.)

3.2. **Proof overview for Theorem 1.5.** The main result of this paper is Theorem 1.5. Suppose that G is a D-regular graph satisfying the conditions of that theorem. Using the approach of the previous subsection, one can obtain an approximate decomposition of G, i.e. a set of edge-disjoint Hamilton cycles covering almost all edges of G. However, one does not have any control over the 'leftover' graph H, which makes a complete decomposition seem infeasible. This problem was overcome in [15] by introducing the concept of a 'robustly decomposable graph' $G^{\rm rob}$. Roughly speaking, this is a sparse regular graph with the following property: given *any* very sparse regular graph H with $V(H) = V(G^{\rm rob})$ which is edge-disjoint from $G^{\rm rob}$, one can guarantee that $G^{\rm rob} \cup H$ has a Hamilton decomposition. This leads to the following strategy to obtain a decomposition of G:

(1) find a (sparse) robustly decomposable graph $G^{\rm rob}$ in G and let G' denote the leftover;
(2) find an approximate Hamilton decomposition of G' and let H denote the (very sparse) leftover;
(3) find a Hamilton decomposition of $G^{\rm rob} \cup H$.

It is of course far from obvious that such a graph $G^{\rm rob}$ exists. By assumption our graph G can be partitioned into two classes A and B of almost equal size such that almost all the edges in G go between A and B. If both A and B are independent sets of equal size then the 'robust decomposition lemma' of [15] guarantees our desired subgraph $G^{\rm rob}$ of G. Of course, in general our graph G will contain edges in A and B. Our aim is therefore to replace such edges with 'fictive edges' between A and B, so that we can apply the robust decomposition lemma (which is introduced in Section 7).

More precisely, similarly as in the proof of Theorem 1.6, we construct a collection of localized balanced exceptional systems. Together these path systems contain all the edges in $G[A]$ and $G[B]$. Again, each balanced exceptional system balances out the sizes of A and B and covers the exceptional vertices in G (i.e. those vertices having high degree into both A and B).

By replacing edges of the balanced exceptional systems with fictive edges, we obtain from G an auxiliary (multi)graph G^* which only contains edges between A and B and which does not contain the exceptional vertices of G. This will allow us to apply the robust decomposition lemma. In particular this ensures that each Hamilton cycle obtained in G^* contains a collection of fictive edges corresponding to a single balanced exceptional system (the set-up of the robust decomposition lemma does allow for this). Each such Hamilton cycle in G^* then corresponds to a Hamilton cycle in G.

We now give an example of how we introduce fictive edges. Let m be an integer so that $(m-1)/2$ is even. Set $m' := (m-1)/2$ and $m'' := (m+1)/2$. Define the graph G as follows: Let A and B be disjoint vertex sets of size m. Let A_1, A_2 be a partition of A and B_1, B_2 be a partition of B such that $|A_1| = |B_1| = m''$. Add all edges between A and B. Add a matching $M_1 = \{e_1, \dots, e_{m'/2}\}$ covering precisely the vertices of A_2 and add a matching $M_2 = \{e'_1, \dots, e'_{m'/2}\}$ covering precisely the vertices of B_2. Finally add a vertex v which sends an edge to every vertex in $A_1 \cup B_1$. So G is $(m+1)$-regular (and v would be regarded as a exceptional vertex).

Now pair up each edge e_i with the edge e'_i. Write $e_i = x_{2i-1}x_{2i}$ and $e'_i = y_{2i-1}y_{2i}$ for each $1 \le i \le m'/2$. Let $A_1 = \{a_1, \dots, a_{m''}\}$ and $B_1 = \{b_1, \dots, b_{m''}\}$ and write $f_i := a_i b_i$ for all $1 \le i \le m''$. Obtain G^* from G by deleting v together with the edges in $M_1 \cup M_2$ and by adding the following fictive edges: add f_i for each $1 \le i \le m''$ and add $x_j y_j$ for each $1 \le j \le m'$. Then G^* is a balanced bipartite $(m+1)$-regular multigraph containing only edges between A and B.

First, note that any Hamilton cycle C^* in G^* that contains precisely one fictive edge f_i for some $1 \le i \le m''$ corresponds to a Hamilton cycle C in G, where we replace the fictive edge f_i with $a_i v$ and $b_i v$. Next, consider any Hamilton cycle C^* in G^* that contains precisely three fictive edges; f_i for some $1 \le i \le m''$ together with $x_{2j-1}y_{2j-1}$ and $x_{2j}y_{2j}$ for some $1 \le j \le m'/2$. Further suppose C^* traverses the vertices $a_i, b_i, x_{2j-1}, y_{2j-1}, x_{2j}, y_{2j}$ in this order. Then C^* corresponds to a Hamilton cycle C in G, where we replace the fictive edges with $a_i v, b_i v, e_j$ and e'_j (see Figure 1). Here the path system J formed by the edges $a_i v, b_i v, e_j$ and e'_j is an example of a balanced exceptional system. The above ideas are formalized in Section 6.

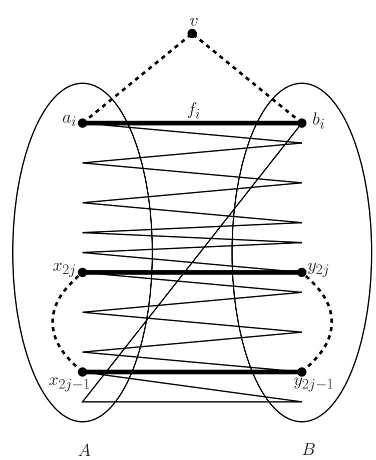


FIGURE 1. Transforming the problem of finding a Hamilton cycle in G into finding a Hamilton cycle in the balanced bipartite graph G^*

We can now summarize the steps leading to proof of Theorem 1.5. In Section 4, we find and remove a set of edge-disjoint Hamilton cycles covering all edges in $G[A_0, B_0]$. We can then find the localized balanced exceptional systems in Section 5. After this, we need to extend and combine them into certain path systems and factors (which contain fictive edges) in Section 6, before we can use them as an 'input' for the robust decomposition lemma in Section 7. Finally, all these steps are combined in Section 9 to prove Theorem 1.5.

4. Eliminating edges between the exceptional sets

Suppose that G is a D-regular graph as in Theorem 1.5. The purpose of this section is to prove Corollary 4.13. Roughly speaking, given $K \in \mathbb{N}$, this corollary states that one can delete a small number of edge-disjoint Hamilton cycles from G to obtain a spanning subgraph G' of G and a partition A, A_0, B, B_0 of $V(G)$ such that (amongst others) the following properties hold:

- almost all edges of G' join $A \cup A_0$ to $B \cup B_0$;
- $|A| = |B|$ is divisible by K;
- every vertex in A has almost all its neighbours in $B \cup B_0$ and every vertex in B has almost all its neighbours in $A \cup A_0$;
- $A_0 \cup B_0$ is small and there are no edges between A_0 and B_0 in G'.

We will call (G', A, A_0, B, B_0) a framework. (The formal definition of a framework is stated before Lemma 4.12.) Both A and B will then be split into K clusters of equal size. Our assumption that G is $\varepsilon_{\rm ex}$-bipartite easily implies that there is such a partition A, A_0, B, B_0 which satisfies all these properties apart from the property that there are no edges between A_0 and B_0. So the main part of this section shows that we can cover the collection of all edges between A_0 and B_0 by a small number of edge-disjoint Hamilton cycles.

Since Corollary 4.13 will also be used in the proof of Theorem 1.6, instead of working with regular graphs we need to consider so-called balanced graphs. We also need to find the above Hamilton cycles in the graph $F \supseteq G$ rather than in G itself (in the proof of Theorem 1.5 we will take F to be equal to G).

More precisely, suppose that G is a graph and that A', B' is a partition of $V(G)$, where $A' = A_0 \cup A$, $B' = B_0 \cup B$ and A, A_0, B, B_0 are disjoint. Then we say that G is *D-balanced (with respect to (A, A_0, B, B_0))* if

(B1) $e_G(A') - e_G(B') = (|A'| - |B'|)D/2$;
(B2) all vertices in $A_0 \cup B_0$ have degree exactly D.

Proposition 4.1 below implies that whenever A, A_0, B, B_0 is a partition of the vertex set of a D-regular graph H, then H is D-balanced with respect to (A, A_0, B, B_0). Moreover, note that if G is D_G-balanced with respect to (A, A_0, B, B_0) and H is a spanning subgraph of G which is D_H-balanced with respect to (A, A_0, B, B_0), then $G - H$ is $(D_G - D_H)$-balanced with respect to (A, A_0, B, B_0). Furthermore, a graph G is D-balanced with respect to (A, A_0, B, B_0) if and only if G is D-balanced with respect to (B, B_0, A, A_0).

Proposition 4.1. *Let H be a graph and let A', B' be a partition of $V(H)$. Suppose that A_0, A is a partition of A' and that B_0, B is a partition of B' such that $|A| = |B|$. Suppose that $d_H(v) = D$ for every $v \in A_0 \cup B_0$ and $d_H(v) = D'$ for every $v \in A \cup B$. Then $e_H(A') - e_H(B') = (|A'| - |B'|)D/2$.*

Proof. Note that

$$\sum_{x \in A'} d_H(x, B') = e_H(A', B') = \sum_{y \in B'} d_H(y, A').$$

Moreover,

$$2e_H(A') = \sum_{x \in A_0} (D - d_H(x, B')) + \sum_{x \in A} (D' - d_H(x, B')) = D|A_0| + D'|A| - \sum_{x \in A'} d_H(x, B')$$

and

$$2e_H(B') = \sum_{y \in B_0} (D - d_H(y, A')) + \sum_{y \in B} (D' - d_H(y, A')) = D|B_0| + D'|B| - \sum_{y \in B'} d_H(y, A').$$

Therefore

$$2e_H(A') - 2e_H(B') = D(|A_0| - |B_0|) + D'(|A| - |B|) = D(|A_0| - |B_0|) = D(|A'| - |B'|),$$

as desired. □

The following observation states that balancedness is preserved under suitable modifications of the partition.

Proposition 4.2. *Let H be D-balanced with respect to (A, A_0, B, B_0). Suppose that A'_0, B'_0 is a partition of $A_0 \cup B_0$. Then H is D-balanced with respect to (A, A'_0, B, B'_0).*

Proof. Observe that the general result follows if we can show that H is D-balanced with respect to (A, A'_0, B, B'_0), where $A'_0 = A_0 \cup \{v\}$, $B'_0 = B_0 \setminus \{v\}$ and $v \in B_0$. (B2) is trivially satisfied in this case, so we only need to check (B1) for the new partition. For this, let $A' := A_0 \cup A$ and $B' := B_0 \cup B$. Now note that (B1) for the original partition implies that

$$\begin{aligned} e_H(A'_0 \cup A) - e_H(B'_0 \cup B) &= e_H(A') + d_H(v, A') - (e_H(B') - d_H(v, B')) \\ &= (|A'| - |B'|)D/2 + D = (|A'_0 \cup A| - |B'_0 \cup B|)D/2. \end{aligned}$$

Thus (B1) holds for the new partition. □

Suppose that G is a graph and A', B' is a partition of $V(G)$. For every vertex $v \in A'$ we call $d_G(v, A')$ the *internal degree of* v in G. Similarly, for every vertex $v \in B'$ we call $d_G(v, B')$ the *internal degree of* v in G.

Given a graph F and a spanning subgraph G of F , we say that (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$*-weak framework* if the following holds, where $A' := A_0 \cup A$, $B' := B_0 \cup B$ and $n := |G| = |F|$:

(WF1) A, A_0, B, B_0 forms a partition of $V(G) = V(F)$;
(WF2) G is D-balanced with respect to (A, A_0, B, B_0);
(WF3) $e_G(A'), e_G(B') \leq \varepsilon n^2$;

(WF4) $|A| = |B|$ is divisible by K. Moreover, $a + b \leq \varepsilon n$, where $a := |A_0|$ and $b := |B_0|$;

(WF5) all vertices in $A \cup B$ have internal degree at most $\varepsilon' n$ in F;

(WF6) any vertex v has internal degree at most $d_G(v)/2$ in G.

Throughout the paper, when referring to internal degrees without mentioning the partition, we always mean with respect to the partition A', B', where $A' = A_0 \cup A$ and $B' = B_0 \cup B$. Moreover, a and b will always denote $|A_0|$ and $|B_0|$.

We say that (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-*pre-framework* if it satisfies (WF1)–(WF5). The following observation states that pre-frameworks are preserved if we remove suitable balanced subgraphs.

Proposition 4.3. *Let $\varepsilon, \varepsilon' > 0$ and $K, D_G, D_H \in \mathbb{N}$. Let (F, G, A, A_0, B, B_0) be an $(\varepsilon, \varepsilon', K, D_G)$-pre framework. Suppose that H is a D_H-regular spanning subgraph of F such that $G \cap H$ is D_H-balanced with respect to (A, A_0, B, B_0). Let $F' := F - H$ and $G' := G - H$. Then (F', G', A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D_G - D_H)$-pre framework.*

Proof. Note that all required properties except possibly (WF2) are not affected by removing edges. But G' satisfies (WF2) since $G \cap H$ is D_H-balanced with respect to (A, A_0, B, B_0). □

Lemma 4.4. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon', 1/K \ll 1$ and let $D \geq n/200$. Suppose that F is a graph on n vertices which is ε-bipartite and that G is a D-regular spanning subgraph of F. Then there is a partition A, A_0, B, B_0 of $V(G) = V(F)$ so that (F, G, A, A_0, B, B_0) is an $(\varepsilon^{1/3}, \varepsilon', K, D)$-weak framework.*

Proof. Let S_1, S_2 be a partition of $V(F)$ which is guaranteed by the assumption that F is ε-bipartite. Let S be the set of all those vertices $x \in S_1$ with $d_F(x, S_1) \geq \sqrt{\varepsilon} n$ together with all those vertices $x \in S_2$ with $d_F(x, S_2) \geq \sqrt{\varepsilon} n$. Since F is ε-bipartite, it follows that $|S| \leq 4\sqrt{\varepsilon} n$.

Given a partition X, Y of $V(F)$, we say that $v \in X$ is *bad for* X, Y if $d_G(v, X) > d_G(v, Y)$ and similarly that $v \in Y$ is *bad for* X, Y if $d_G(v, Y) > d_G(v, X)$. Suppose that there is a vertex $v \in S$ which is bad for S_1, S_2. Then we move v into the class which does not currently contain v to obtain a new partition S_1', S_2'. We do not change the set S. If there is a vertex $v' \in S$ which is bad for S_1', S_2', then again we move it into the other class.

We repeat this process. After each step, the number of edges in G between the two classes increases, so this process has to terminate with some partition A', B' such that $A' \triangle S_1 \subseteq S$ and $B' \triangle S_2 \subseteq S$. Clearly, no vertex in S is now bad for A', B'. Also, for any $v \in A' \setminus S$ we have

$$d_G(v, A') \leq d_F(v, A') \leq d_F(v, S_1) + |S| \leq \sqrt{\varepsilon} n + 4\sqrt{\varepsilon} n < \varepsilon' n < D/2 = d_G(v)/2. \tag{4.1}$$

Similarly, $d_G(v, B') < \varepsilon' n < d_G(v)/2$ for all $v \in B' \setminus S$. Altogether this implies that no vertex is bad for A', B' and thus (WF6) holds. Also note that $e_G(A', B') \geq e_G(S_1, S_2) \geq e(G) - 2\varepsilon n^2$. So

$$e_G(A'), e_G(B') \leq 2\varepsilon n^2. \tag{4.2}$$

This implies (WF3).

Without loss of generality we may assume that $|A'| \geq |B'|$. Let A'_0 denote the set of all those vertices $v \in A'$ for which $d_F(v, A') \geq \varepsilon' n$. Define $B'_0 \subseteq B'$ similarly. We will choose sets $A \subseteq A' \setminus A'_0$ and $A_0 \supseteq A'_0$ and sets $B \subseteq B' \setminus B'_0$ and $B_0 \supseteq B'_0$ such that $|A| = |B|$ is divisible by K and so that A, A_0 and B, B_0 are partitions of A' and B' respectively. We obtain such sets by moving at most $||A' \setminus A'_0| - |B' \setminus B'_0|| + K$ vertices from $A' \setminus A'_0$ to A'_0 and at most $||A' \setminus A'_0| - |B' \setminus B'_0|| + K$ vertices from $B' \setminus B'_0$ to B'_0. The choice of A, A_0, B, B_0 is such that (WF1) and (WF5) hold. Further, since $|A| = |B|$, Proposition 4.1 implies (WF2).

In order to verify (WF4), it remains to show that $a + b = |A_0 \cup B_0| \leq \varepsilon^{1/3} n$. But (4.1) together with its analogue for the vertices in $B' \setminus S$ implies that $A'_0 \cup B'_0 \subseteq S$. Thus $|A'_0| + |B'_0| \leq |S| \leq 4\sqrt{\varepsilon} n$. Moreover, (WF2), (4.2) and our assumption that $D \geq n/200$ together imply that

$$|A'| - |B'| = (e_G(A') - e_G(B'))/(D/2) \leq 2\varepsilon n^2/(D/2) \leq 800\varepsilon n.$$

So altogether, we have

$$\begin{aligned} a + b &\leq |A'_0 \cup B'_0| + 2\left||A' \setminus A'_0| - |B' \setminus B'_0|\right| + 2K \\ &\leq 4\sqrt{\varepsilon} n + 2\left||A'| - |B'| - (|A'_0| - |B'_0|)\right| + 2K \\ &\leq 4\sqrt{\varepsilon} n + 1600\varepsilon n + 8\sqrt{\varepsilon} n + 2K \leq \varepsilon^{1/3} n. \end{aligned}$$

Thus (WF4) holds. □

Throughout this and the next section, we will often use the following result, which is a simple consequence of Vizing's theorem and was first observed by McDiarmid and independently by de Werra (see e.g. [30]).

Proposition 4.5. *Let H be a graph with maximum degree at most Δ. Then $E(H)$ can be decomposed into $\Delta+1$ edge-disjoint matchings $M_1, \dots, M_{\Delta+1}$ such that $||M_i| - |M_j|| \leq 1$ for all $i, j \leq \Delta + 1$.*

Our next goal is to cover the edges of $G[A_0, B_0]$ by edge-disjoint Hamilton cycles. To do this, we will first decompose $G[A_0, B_0]$ into a collection of matchings. We will then extend each such matching into a system of vertex-disjoint paths such that altogether these paths cover every vertex in $G[A_0, B_0]$, each path has its endvertices in $A \cup B$ and the path system is 2-balanced. Since our path system will only contain a small number of nontrivial paths, we can then extend the path system into a Hamilton cycle (see Lemma 4.10).

We will call the path systems we are working with A_0B_0-path systems. More precisely, an *A_0B_0-path system (with respect to (A, A_0, B, B_0))* is a path system Q satisfying the following properties:

- Every vertex in $A_0 \cup B_0$ is an internal vertex of a path in Q.
- $A \cup B$ contains the endpoints of each path in Q but no internal vertex of a path in Q.

The following observation (which motivates the use of the word 'balanced') will often be helpful.

Proposition 4.6. *Let A_0, A, B_0, B be a partition of a vertex set V. Then an A_0B_0-path system Q with $V(Q) \subseteq V$ is 2-balanced with respect to (A, A_0, B, B_0) if and only if the number of vertices in A which are endpoints of nontrivial paths in Q equals the number of vertices in B which are endpoints of nontrivial paths in Q.*

Proof. Note that by definition any A_0B_0-path system satisfies (B2), so we only need to consider (B1). Let n_A be the number of vertices in A which are endpoints of nontrivial paths in Q and define n_B similarly. Let $a := |A_0|$, $b := |B_0|$, $A' := A \cup A_0$ and $B' := B \cup B_0$. Since $d_Q(v) = 2$ for all $v \in A_0$ and since every vertex in A is either an endpoint of a nontrivial path in Q or has degree zero in Q, we have

$$2e_Q(A') + e_Q(A', B') = \sum_{v \in A'} d_Q(v) = 2a + n_A.$$

So $n_A = 2(e_Q(A') - a) + e_Q(A', B')$, and similarly $n_B = 2(e_Q(B') - b) + e_Q(A', B')$. Therefore, $n_A = n_B$ if and only if $2(e_Q(A') - e_Q(B') - a + b) = 0$ if and only if Q satisfies (B1), as desired. □

The next observation shows that if we have a suitable path system satisfying (B1), we can extend it into a path system which also satisfies (B2).

Lemma 4.7. *Let $0 < 1/n \ll \alpha \ll 1$. Let G be a graph on n vertices such that there is a partition A', B' of $V(G)$ which satisfies the following properties:*

(i) *$A' = A_0 \cup A$, $B' = B_0 \cup B$ and A_0, A, B_0, B are disjoint;*
(ii) *$|A| = |B|$ and $a + b \leq \alpha n$, where $a := |A_0|$ and $b := |B_0|$;*
(iii) *if $v \in A_0$ then $d_G(v, B) \geq 4\alpha n$ and if $v \in B_0$ then $d_G(v, A) \geq 4\alpha n$.*

Let $Q' \subseteq G$ be a path system consisting of at most αn nontrivial paths such that $A \cup B$ contains no internal vertex of a path in Q' and $e_{Q'}(A') - e_{Q'}(B') = a - b$. Then G contains a 2-balanced A_0B_0-path system Q (with respect to (A, A_0, B, B_0)) which extends Q' and consists of at most $2\alpha n$ nontrivial paths. Furthermore, $E(Q) \setminus E(Q')$ consists of A_0B- and AB_0-edges only.

Proof. Since $A \cup B$ contains no internal vertex of a path in Q' and since Q' contains at most αn nontrivial paths, it follows that at most $2\alpha n$ vertices in $A \cup B$ lie on nontrivial paths in Q'. We will now extend Q' into an A_0B_0-path system Q consisting of at most $a + b + \alpha n \leq 2\alpha n$ nontrivial paths as follows:

- for every vertex $v \in A_0$, we join v to $2 - d_{Q'}(v)$ vertices in B;
- for every vertex $v \in B_0$, we join v to $2 - d_{Q'}(v)$ vertices in A.

Condition (iii) and the fact that at most $2\alpha n$ vertices in $A \cup B$ lie on nontrivial paths in Q' together ensure that we can extend Q' in such a way that the endvertices in $A \cup B$ are distinct for different paths in Q. Note that $e_Q(A') - e_Q(B') = e_{Q'}(A') - e_{Q'}(B') = a - b$. Therefore, Q is 2-balanced with respect to (A, A_0, B, B_0). □

The next lemma constructs a small number of 2-balanced A_0B_0-path systems covering the edges of $G[A_0, B_0]$. Each of these path systems will later be extended into a Hamilton cycle.

Lemma 4.8. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon', 1/K \ll \alpha \ll 1$. Let F be a graph on n vertices and let G be a spanning subgraph of F. Suppose that (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-weak framework with $\delta(F) \geq (1/4+\alpha)n$ and $D \geq n/200$. Then for some $r^* \leq \varepsilon n$ the graph G contains r^* edge-disjoint 2-balanced A_0B_0-path systems $Q_1, \dots, Q_{r^*}$ which satisfy the following properties:*

(i) *Together $Q_1, \dots, Q_{r^*}$ cover all edges in $G[A_0, B_0]$;*
(ii) *For each $i \leq r^*$, Q_i contains at most $2\varepsilon n$ nontrivial paths;*
(iii) *For each $i \leq r^*$, Q_i does not contain any edge from $G[A, B]$.*

Proof. (WF4) implies that $|A_0| + |B_0| \leq \varepsilon n$. Thus, by Proposition 4.5, there exists a collection $M'_1, \dots, M'_{r^*}$ of r^* edge-disjoint matchings in $G[A_0, B_0]$ that together cover all the edges in $G[A_0, B_0]$, where $r^* \leq \varepsilon n$.

We may assume that $a \geq b$ (the case when $b > a$ follows analogously). We will use edges in $G[A']$ to extend each M'_i into a 2-balanced A_0B_0-path system. (WF2) implies that $e_G(A') \geq (a-b)D/2$. Since $d_G(v) = D$ for all $v \in A_0 \cup B_0$ by (WF2), (WF5) and (WF6) imply that $\Delta(G[A']) \leq D/2$. Thus Proposition 4.5 implies that $E(G[A'])$ can be decomposed into $\lfloor D/2 \rfloor + 1$ edge-disjoint matchings $M_{A,1}, \dots, M_{A,\lfloor D/2 \rfloor +1}$ such that $||M_{A,i}| - |M_{A,j}|| \leq 1$ for all $i, j \leq \lfloor D/2 \rfloor + 1$.

Notice that at least εn of the matchings $M_{A,i}$ are such that $|M_{A,i}| \geq a-b$. Indeed, otherwise we have that

$$\begin{aligned} (a-b)D/2 \leq e_G(A') &\leq \varepsilon n(a-b) + (a-b-1)(D/2+1-\varepsilon n) \\ &= (a-b)D/2 + a - b - D/2 - 1 + \varepsilon n \\ &< (a-b)D/2 + 2\varepsilon n - D/2 < (a-b)D/2, \end{aligned}$$

a contradiction. (The last inequality follows since $D \geq n/200$.) In particular, this implies that $G[A']$ contains r^* edge-disjoint matchings $M''_1, \dots, M''_{r^*}$ that each consist of precisely $a-b$ edges.

For each $i \leq r^*$, set $M_i := M'_i \cup M''_i$. So for each $i \leq r^*$, M_i is a path system consisting of at most $b + (a-b) = a \leq \varepsilon n$ nontrivial paths such that $A \cup B$ contains no internal vertex of a path in M_i and $e_{M_i}(A') - e_{M_i}(B') = e_{M''_i}(A') = a-b$.

Suppose for some $0 \leq r < r^*$ we have already found a collection $Q_1, \dots, Q_r$ of r edge-disjoint 2-balanced A_0B_0-path systems which satisfy the following properties for each $i \leq r$:

$(\alpha)_i$ Q_i contains at most $2\varepsilon n$ nontrivial paths;
$(\beta)_i$ $M_i \subseteq Q_i$;
$(\gamma)_i$ Q_i and M_j are edge-disjoint for each $j \leq r^*$ such that $i \neq j$;
$(\delta)_i$ Q_i contains no edge from $G[A, B]$.

(Note that $(\alpha)_0$–$(\delta)_0$ are vacuously true.) Let G' denote the spanning subgraph of G obtained from G by deleting the edges lying in $Q_1 \cup \dots \cup Q_r$. (WF2), (WF4) and (WF6) imply that, if $v \in A_0$, $d_{G'}(v, B) \geq D/2 - \varepsilon n - 2r \geq 4\varepsilon n$ and if $v \in B_0$ then $d_{G'}(v, A) \geq 4\varepsilon n$. Thus Lemma 4.7 implies that G' contains a 2-balanced A_0B_0-path system Q_{r+1} that satisfies $(\alpha)_{r+1}$–$(\delta)_{r+1}$.

So we can proceed in this way in order to obtain edge-disjoint 2-balanced A_0B_0-path systems $Q_1, \dots, Q_{r^*}$ in G such that $(\alpha)_i$–$(\delta)_i$ hold for each $i \le r^*$. Note that (i)–(iii) follow immediately from these conditions, as desired. □

The next lemma (Corollary 5.4 in [13]) allows us to extend a 2-balanced path system into a Hamilton cycle. Corollary 5.4 concerns so-called '(A, B)-balanced'-path systems rather than 2-balanced A_0B_0-path systems. But the latter satisfies the requirements of the former by Proposition 4.6.

Lemma 4.9. *Let $0 < 1/n \ll \varepsilon' \ll \alpha \ll 1$. Let F be a graph and suppose that A_0, A, B_0, B is a partition of $V(F)$ such that $|A| = |B| = n$. Let H be a bipartite subgraph of F with vertex classes A and B such that $\delta(H) \ge (1/2 + \alpha)n$. Suppose that Q is a 2-balanced A_0B_0-path system with respect to (A, A_0, B, B_0) in F which consists of at most $\varepsilon' n$ nontrivial paths. Then F contains a Hamilton cycle C which satisfies the following properties:*

- *$Q \subseteq C$;*
- *$E(C) \setminus E(Q)$ consists of edges from H.*

Now we can apply Lemma 4.9 to extend a 2-balanced A_0B_0-path system in a pre-framework into a Hamilton cycle.

Lemma 4.10. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon', 1/K \ll \alpha \ll 1$. Let F be a graph on n vertices and let G be a spanning subgraph of F. Suppose that (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-pre-framework, i.e. it satisfies (WF1)–(WF5). Suppose also that $\delta(F) \ge (1/4 + \alpha)n$. Let Q be a 2-balanced A_0B_0-path system with respect to (A, A_0, B, B_0) in G which consists of at most $\varepsilon' n$ nontrivial paths. Then F contains a Hamilton cycle C which satisfies the following properties:*

(i) *$Q \subseteq C$;*
(ii) *$E(C) \setminus E(Q)$ consists of AB-edges;*
(iii) *$C \cap G$ is 2-balanced with respect to (A, A_0, B, B_0).*

Proof. Note that (WF4), (WF5) and our assumption that $\delta(F) \ge (1/4 + \alpha)n$ together imply that every vertex $x \in A$ satisfies

$$d_F(x, B) \ge d_F(x, B') - |B_0| \ge d_F(x) - \varepsilon' n - |B_0| \ge (1/4 + \alpha/2)n \ge (1/2 + \alpha/2)|B|.$$

Similarly, $d_F(x, A) \ge (1/2 + \alpha/2)|A|$ for all $x \in B$. Thus, $\delta(F[A, B]) \ge (1/2 + \alpha/2)|A|$. Applying Lemma 4.9 with $F[A, B]$ playing the role of H, we obtain a Hamilton cycle C in F that satisfies (i) and (ii). To verify (iii), note that (ii) and the 2-balancedness of Q together imply that

$$e_{C\cap G}(A') - e_{C\cap G}(B') = e_Q(A') - e_Q(B') = a - b.$$

Since every vertex $v \in A_0 \cup B_0$ satisfies $d_{C\cap G}(v) = d_Q(v) = 2$, (iii) holds. □

We now combine Lemmas 4.8 and 4.10 to find a collection of edge-disjoint Hamilton cycles covering all the edges in $G[A_0, B_0]$.

Lemma 4.11. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon', 1/K \ll \alpha \ll 1$ and let $D \geq n/100$. Let F be a graph on n vertices and let G be a spanning subgraph of F. Suppose that (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-weak framework with $\delta(F) \geq (1/4+\alpha)n$. Then for some $r^* \leq \varepsilon n$ the graph F contains edge-disjoint Hamilton cycles $C_1, \dots, C_{r^*}$ which satisfy the following properties:*

(i) *Together $C_1, \dots, C_{r^*}$ cover all edges in $G[A_0, B_0]$;*
(ii) *$(C_1 \cup \dots \cup C_{r^*}) \cap G$ is $2r^*$-balanced with respect to (A, A_0, B, B_0).*

Proof. Apply Lemma 4.8 to obtain a collection of $r^* \leq \varepsilon n$ edge-disjoint 2-balanced A_0B_0-path systems $Q_1, \dots, Q_{r^*}$ in G which satisfy Lemma 4.8(i)–(iii). We will extend each Q_i to a Hamilton cycle C_i.

Suppose that for some $0 \leq r < r^*$ we have found a collection $C_1, \dots, C_r$ of r edge-disjoint Hamilton cycles in F such that the following holds for each $0 \leq i \leq r$:

$(\alpha)_i$ $Q_i \subseteq C_i$;
$(\beta)_i$ $E(C_i) \setminus E(Q_i)$ consists of AB-edges;
$(\gamma)_i$ $G \cap C_i$ is 2-balanced with respect to (A, A_0, B, B_0).

(Note that $(\alpha)_0$–$(\gamma)_0$ are vacuously true.) Let $H_r := C_1 \cup \dots \cup C_r$ (where $H_0 := (V(G), \emptyset)$). So H_r is $2r$-regular. Further, since $G \cap C_i$ is 2-balanced for each $i \leq r$, $G \cap H_r$ is $2r$-balanced. Let $G_r := G - H_r$ and $F_r := F - H_r$. Since (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-pre-framework, Proposition 4.3 implies that $(F_r, G_r, A, A_0, B, B_0)$ is an $(\varepsilon, \varepsilon', K, D - 2r)$-pre-framework. Moreover, $\delta(F_r) \geq \delta(F) - 2r \geq (1/4 + \alpha/2)n$. Lemma 4.8(iii) and $(\beta)_1$–$(\beta)_r$ together imply that Q_{r+1} lies in G_r. Therefore, Lemma 4.10 implies that F_r contains a Hamilton cycle C_{r+1} which satisfies $(\alpha)_{r+1}$–$(\gamma)_{r+1}$.

So we can proceed in this way in order to obtain r^* edge-disjoint Hamilton cycles $C_1, \dots, C_{r^*}$ in F such that for each $i \leq r^*$, $(\alpha)_i$–$(\gamma)_i$ hold. Note that this implies that (ii) is satisfied. Further, the choice of $Q_1, \dots, Q_{r^*}$ ensures that (i) holds. □

Given a graph G, we say that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-*framework* if the following holds, where $A' := A_0 \cup A$, $B' := B_0 \cup B$ and $n := |G|$:

(FR1) A, A_0, B, B_0 forms a partition of $V(G)$;
(FR2) G is D-balanced with respect to (A, A_0, B, B_0);
(FR3) $e_G(A'), e_G(B') \leq \varepsilon n^2$;
(FR4) $|A| = |B|$ is divisible by K. Moreover, $b \leq a$ and $a + b \leq \varepsilon n$, where $a := |A_0|$ and $b := |B_0|$;
(FR5) all vertices in $A \cup B$ have internal degree at most $\varepsilon' n$ in G;
(FR6) $e(G[A_0, B_0]) = 0$;
(FR7) all vertices $v \in V(G)$ have internal degree at most $d_G(v)/2 + \varepsilon n$ in G.

Note that the main differences to a weak framework are (FR6) and the fact that a weak framework involves an additional graph F. In particular (FR1)–(FR4) imply (WF1)–(WF4). Suppose that $\varepsilon_1 \geq \varepsilon$, $\varepsilon'_1 \geq \varepsilon'$ and that K_1 divides K. Then note that every $(\varepsilon, \varepsilon', K, D)$-framework is also an $(\varepsilon_1, \varepsilon'_1, K_1, D)$-framework.

Lemma 4.12. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon', 1/K \ll \alpha \ll 1$ and let $D \geq n/100$. Let F be a graph on n vertices and let G be a spanning subgraph of F. Suppose that*

(F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-weak framework. Suppose also that $\delta(F) \geq (1/4+\alpha)n$ and $|A_0| \geq |B_0|$. Then the following properties hold:

(i) *there is an $(\varepsilon, \varepsilon', K, D_{G'})$-framework (G', A, A_0, B, B_0) such that G' is a spanning subgraph of G with $D_{G'} \geq D - 2\varepsilon n$;*

(ii) *there is a set of $(D - D_{G'})/2 \leq \varepsilon n$ edge-disjoint Hamilton cycles in $F - G'$ containing all edges of $G - G'$. In particular, if D is even then $D_{G'}$ is even.*

Proof. Lemma 4.11 implies that there exists some $r^* \leq \varepsilon n$ such that F contains a spanning subgraph H satisfying the following properties:

(a) H is $2r^*$-regular;

(b) H contains all the edges in $G[A_0, B_0]$;

(c) $G \cap H$ is $2r^*$-balanced with respect to (A, A_0, B, B_0);

(d) H has a decomposition into r^* edge-disjoint Hamilton cycles.

Set $G' := G - H$. Then (G', A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D_{G'})$-framework where $D_{G'} := D - 2r^* \geq D - 2\varepsilon n$. Indeed, since (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-weak framework, (FR1) and (FR3)–(FR5) follow from (WF1) and (WF3)–(WF5). Further, (FR2) follows from (WF2) and (c) while (FR6) follows from (b). (WF6) implies that all vertices $v \in V(G)$ have internal degree at most $d_G(v)/2$ in G. Thus all vertices $v \in V(G')$ have internal degree at most $d_G(v)/2 \leq (d_{G'}(v) + 2r^*)/2 \leq d_{G'}(v)/2 + \varepsilon n$ in G'. So (FR7) is satisfied. Hence, (i) is satisfied.

Note that by definition of G', H contains all edges of $G - G'$. So since $r^* = (D - D_{G'})/2 \leq \varepsilon n$, (d) implies (ii). □

The following result follows immediately from Lemmas 4.4 and 4.12.

Corollary 4.13. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon^* \ll \varepsilon', 1/K \ll \alpha \ll 1$ and let $D \geq n/100$. Suppose that F is an ε-bipartite graph on n vertices with $\delta(F) \geq (1/4+\alpha)n$. Suppose that G is a D-regular spanning subgraph of F. Then the following properties hold:*

(i) *there is an $(\varepsilon^*, \varepsilon', K, D_{G'})$-framework (G', A, A_0, B, B_0) such that G' is a spanning subgraph of G, $D_{G'} \geq D - 2\varepsilon^{1/3}n$ and such that F satisfies (WF5) (with respect to the partition A, A_0, B, B_0);*

(ii) *there is a set of $(D - D_{G'})/2 \leq \varepsilon^{1/3}n$ edge-disjoint Hamilton cycles in $F - G'$ containing all edges of $G - G'$. In particular, if D is even then $D_{G'}$ is even.*

5. Finding path systems which cover all the edges within the classes

The purpose of this section is to prove Corollary 5.11 which, given a framework (G, A, A_0, B, B_0), guarantees a set $\mathcal{C}$ of edge-disjoint Hamilton cycles and a set $\mathcal{J}$ of suitable edge-disjoint 2-balanced A_0B_0-path systems such that the graph G^* obtained from G by deleting the edges in all these Hamilton cycles and path systems is bipartite with vertex classes A' and B' and $A_0 \cup B_0$ is isolated in G^*. Each of the path systems in $\mathcal{J}$ will later be extended into a Hamilton cycle by adding suitable edges between A and B. The path systems in $\mathcal{J}$ will need to be 'localized' with respect to a given partition. We prepare the ground for this in the next subsection.

Throughout this section, given sets $S, S' \subseteq V(G)$ we often write $E(S)$, $E(S, S')$, $e(S)$ and $e(S, S')$ for $E_G(S)$, $E_G(S, S')$, $e_G(S)$ and $e_G(S, S')$ respectively.

5.1. **Choosing the partition and the localized slices.** Let $K, m \in \mathbb{N}$ and $\varepsilon > 0$. A (K, m, ε)-*partition* of a set V of vertices is a partition of V into sets $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ such that $|A_i| = |B_i| = m$ for all $1 \le i \le K$ and $|A_0 \cup B_0| \le \varepsilon|V|$. We often write V_0 for $A_0 \cup B_0$ and think of the vertices in V_0 as 'exceptional vertices'. The sets $A_1, \dots, A_K$ and $B_1, \dots, B_K$ are called *clusters* of the (K, m, ε_0)-partition and A_0, B_0 are called *exceptional sets*. Unless stated otherwise, when considering a (K, m, ε)-partition $\mathcal{P}$ we denote the elements of $\mathcal{P}$ by $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ as above. Further, we will often write A for $A_1 \cup \dots \cup A_K$ and B for $B_1 \cup \dots \cup B_K$.

Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and that $\varepsilon_1, \varepsilon_2 > 0$. We say that $\mathcal{P}$ is a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-*partition for* G if $\mathcal{P}$ satisfies the following properties:

- (P1) $\mathcal{P}$ is a (K, m, ε)-partition of $V(G)$ such that the exceptional sets A_0 and B_0 in the partition $\mathcal{P}$ are the same as the sets A_0, B_0 which are part of the framework (G, A, A_0, B, B_0). In particular, $m = |A|/K = |B|/K$;
- (P2) $d(v, A_i) = (d(v, A) \pm \varepsilon_1 n)/K$ for all $1 \le i \le K$ and $v \in V(G)$;
- (P3) $e(A_i, A_j) = 2(e(A) \pm \varepsilon_2 \max\{n, e(A)\})/K^2$ for all $1 \le i < j \le K$;
- (P4) $e(A_i) = (e(A) \pm \varepsilon_2 \max\{n, e(A)\})/K^2$ for all $1 \le i \le K$;
- (P5) $e(A_0, A_i) = (e(A_0, A) \pm \varepsilon_2 \max\{n, e(A_0, A)\})/K$ for all $1 \le i \le K$;
- (P6) $e(A_i, B_j) = (e(A, B) \pm 3\varepsilon_2 e(A, B))/K^2$ for all $1 \le i, j \le K$;

and the analogous assertions hold if we replace A by B (as well as A_i by B_i etc.) in (P2)–(P5).

Our first aim is to show that for every framework we can find such a partition with suitable parameters (see Lemma 5.2). To do this, we need the following lemma.

Lemma 5.1. *Suppose that $0 < 1/n \ll \varepsilon, \varepsilon_1 \ll \varepsilon_2 \ll 1/K \ll 1$, that $r \le 2K$, that $Km \ge n/4$ and that $r, K, n, m \in \mathbb{N}$. Let G and F be graphs on n vertices with $V(G) = V(F)$. Suppose that there is a vertex partition of $V(G)$ into $U, R_1, \dots, R_r$ with the following properties:*

- *$|U| = Km$.*
- *$\delta(G[U]) \ge \varepsilon n$ or $\Delta(G[U]) \le \varepsilon n$.*
- *For each $j \le r$ we either have $d_G(u, R_j) \le \varepsilon n$ for all $u \in U$ or $d_G(x, U) \ge \varepsilon n$ for all $x \in R_j$.*

Then there exists a partition of U into K parts $U_1, \dots, U_K$ satisfying the following properties:

- (i) *$|U_i| = m$ for all $i \le K$.*
- (ii) *$d_G(v, U_i) = (d_G(v, U) \pm \varepsilon_1 n)/K$ for all $v \in V(G)$ and all $i \le K$.*
- (iii) *$e_G(U_i, U_{i'}) = 2(e_G(U) \pm \varepsilon_2 \max\{n, e_G(U)\})/K^2$ for all $1 \le i \ne i' \le K$.*
- (iv) *$e_G(U_i) = (e_G(U) \pm \varepsilon_2 \max\{n, e_G(U)\})/K^2$ for all $i \le K$.*
- (v) *$e_G(U_i, R_j) = (e_G(U, R_j) \pm \varepsilon_2 \max\{n, e_G(U, R_j)\})/K$ for all $i \le K$ and $j \le r$.*
- (vi) *$d_F(v, U_i) = (d_F(v, U) \pm \varepsilon_1 n)/K$ for all $v \in V(F)$ and all $i \le K$.*

Proof. Consider an equipartition $U_1, \dots, U_K$ of U which is chosen uniformly at random. So (i) holds by definition. Note that for a given vertex $v \in V(G)$, $d_G(v, U_i)$ has the hypergeometric distribution with mean $d_G(v, U)/K$. So if $d_G(v, U) \ge \varepsilon_1 n/K$, Proposition 2.3 implies that

$$\mathbb{P}\left(\left|d_G(v,U_i)-\frac{d_G(v,U)}{K}\right|\geq\frac{\varepsilon_1 d_G(v,U)}{K}\right)\leq 2\exp\left(-\frac{\varepsilon_1^2 d_G(v,U)}{3K}\right)\leq\frac{1}{n^2}.$$

Thus we deduce that for all $v\in V(G)$ and all $i\leq K$,

$$\mathbb{P}\left(|d_G(v,U_i)-d_G(v,U)/K|\geq\varepsilon_1 n/K\right)\leq 1/n^2.$$

Similarly,

$$\mathbb{P}\left(|d_F(v,U_i)-d_F(v,U)/K|\geq\varepsilon_1 n/K\right)\leq 1/n^2.$$

So with probability at least 3/4, both (ii) and (vi) are satisfied.

We now consider (iii) and (iv). Fix $i,i'\leq K$. If $i\neq i'$, let $X:=e_G(U_i,U_{i'})$. If $i=i'$, let $X:=2e_G(U_i)$. For an edge $f\in E(G[U])$, let E_f denote the event that $f\in E(U_i,U_{i'})$. So if $f=xy$ and $i\neq i'$, then

$$\mathbb{P}(E_f)=2\mathbb{P}(x\in U_i)\mathbb{P}(y\in U_{i'}\mid x\in U_i)=2\frac{m}{|U|}\cdot\frac{m}{|U|-1}. \tag{5.1}$$

Similarly, if f and f' are disjoint (that is, f and f' have no common endpoint) and $i\neq i'$, then

$$\mathbb{P}(E_{f'}\mid E_f)=2\frac{m-1}{|U|-2}\cdot\frac{m-1}{|U|-3}\leq 2\frac{m}{|U|}\cdot\frac{m}{|U|-1}=\mathbb{P}(E_{f'}). \tag{5.2}$$

By (5.1), if $i\neq i'$, we also have

$$\mathbb{E}(X)=2\frac{e_G(U)}{K^2}\cdot\frac{|U|}{|U|-1}=\left(1\pm\frac{2}{|U|}\right)\frac{2e_G(U)}{K^2}=(1\pm\varepsilon_2/4)\frac{2e_G(U)}{K^2}. \tag{5.3}$$

If $f=xy$ and $i=i'$, then

$$\mathbb{P}(E_f)=\mathbb{P}(x\in U_i)\mathbb{P}(y\in U_i\mid x\in U_i)=\frac{m}{|U|}\cdot\frac{m-1}{|U|-1}. \tag{5.4}$$

So if $i=i'$, similarly to (5.2) we also obtain $\mathbb{P}(E_{f'}\mid E_f)\leq\mathbb{P}(E_f)$ for disjoint f and f' and we obtain the same bound as in (5.3) on $\mathbb{E}(X)$ (recall that $X=2e_G(U_i)$ in this case).

Note that if $i\neq i'$ then

$$\begin{aligned}\mathrm{Var}(X) &= \sum_{f\in E(U)}\sum_{f'\in E(U)}\big(\mathbb{P}(E_f\cap E_{f'})-\mathbb{P}(E_f)\mathbb{P}(E_{f'})\big)\\ &= \sum_{f\in E(U)}\mathbb{P}(E_f)\sum_{f'\in E(U)}\big(\mathbb{P}(E_{f'}\mid E_f)-\mathbb{P}(E_{f'})\big)\\ &\overset{(5.2)}{\leq} \sum_{f\in E(U)}\mathbb{P}(E_f)\cdot 2\Delta(G[U])\overset{(5.3)}{\leq}\frac{3e_G(U)}{K^2}\cdot 2\Delta(G[U])\leq e_G(U)\Delta(G[U]).\end{aligned}$$

Similarly, if $i=i'$ then

$$\mathrm{Var}(X)=4\sum_{f\in E(U)}\sum_{f'\in E(U)}\big(\mathbb{P}(E_f\cap E_{f'})-\mathbb{P}(E_f)\mathbb{P}(E_{f'})\big)\leq e_G(U)\Delta(G[U]).$$

Let $a := e_G(U)\Delta(G[U])$. In both cases, from Chebyshev's inequality, it follows that

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \sqrt{a/\varepsilon^{1/2}}\right) \leq \varepsilon^{1/2}.$$

Suppose that $\Delta(G[U]) \leq \varepsilon n$. If we also have have $e_G(U) \leq n$, then $\sqrt{a/\varepsilon^{1/2}} \leq \varepsilon^{1/4} n \leq \varepsilon_2 n/2K^2$. If $e_G(U) \geq n$, then $\sqrt{a/\varepsilon^{1/2}} \leq \varepsilon^{1/4} e_G(U) \leq \varepsilon_2 e_G(U)/2K^2$.

If we do not have $\Delta(G[U]) \leq \varepsilon n$, then our assumptions imply that $\delta(G[U]) \geq \varepsilon n$. So $\Delta(G[U]) \leq n \leq \varepsilon e_G(G[U])$ with room to spare. This in turn means that $\sqrt{a/\varepsilon^{1/2}} \leq \varepsilon^{1/4} e_G(U) \leq \varepsilon_2 e_G(U)/2K^2$. So in all cases, we have

$$\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \frac{\varepsilon_2 \max\{n, e_G(U)\}}{2K^2}\right) \leq \varepsilon^{1/2}. \tag{5.5}$$

Now note that by (5.3) we have

$$\left|\mathbb{E}(X) - \frac{2e_G(U)}{K^2}\right| \leq \frac{\varepsilon_2 e_G(U)}{2K^2}. \tag{5.6}$$

So (5.5) and (5.6) together imply that for fixed i, i' the bound in (iii) fails with probability at most $\varepsilon^{1/2}$. The analogue holds for the bound in (iv). By summing over all possible values of $i, i' \leq K$, we have that (iii) and (iv) hold with probability at least $3/4$.

A similar argument shows that for all $i \leq K$ and $j \leq r$, we have

$$\mathbb{P}\left(\left|e_G(U_i, R_j) - \frac{e_G(U, R_j)}{K}\right| \geq \frac{\varepsilon_2 \max\{n, e_G(U, R_j)\}}{K}\right) \leq \varepsilon^{1/2}. \tag{5.7}$$

Indeed, fix $i \leq K$, $j \leq r$ and let $X := e_G(U_i, R_j)$. For an edge $f \in G[U, R_j]$, let E_f denote the event that $f \in E(U_i, R_j)$. Then $\mathbb{P}(E_f) = m/|U| = 1/K$ and so $\mathbb{E}(X) = e_G(U, R_j)/K$. The remainder of the argument proceeds as in the previous case (with slightly simpler calculations).

So (v) holds with probability at least $3/4$, by summing over all possible values of $i \leq K$ and $j \leq r$ again. So with positive probability, the partition satisfies all requirements. □

Lemma 5.2. *Let* $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll 1/K \ll 1$. *Suppose that* (G, A, A_0, B, B_0) *is an* $(\varepsilon, \varepsilon', K, D)$*-framework with* $|G| = n$ *and* $\delta(G) \geq D \geq n/200$. *Suppose that* F *is a graph with* $V(F) = V(G)$. *Then there exists a partition* $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ *of* $V(G)$ *so that*

(i) $\mathcal{P}$ *is a* $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$*-partition for* G.
(ii) $d_F(v, A_i) = (d_F(v, A) \pm \varepsilon_1 n)/K$ *and* $d_F(v, B_i) = (d_F(v, B) \pm \varepsilon_1 n)/K$ *for all* $1 \leq i \leq K$ *and* $v \in V(G)$.

Proof. In order to find the required partitions $A_1, \dots, A_K$ of A and $B_1, \dots, B_K$ of B we will apply Lemma 5.1 twice, as follows. In the first application we let $U := A$, $R_1 := A_0$, $R_2 := B_0$ and $R_3 := B$. Note that $\Delta(G[U]) \leq \varepsilon' n$ by (FR5) and $d_G(u, R_j) \leq |R_j| \leq \varepsilon n \leq \varepsilon' n$ for all $u \in U$ and $j = 1, 2$ by (FR4). Moreover, (FR4) and (FR7) together imply that $d_G(x, U) \geq D/3 \geq \varepsilon' n$ for each $x \in R_3 = B$. Thus we can apply Lemma 5.1 with ε' playing the role of ε to obtain a partition $U_1, \dots, U_K$ of U. We let $A_i := U_i$ for all $i \leq K$. Then the A_i satisfy (P2)–(P5) and

$$e_G(A_i, B) = (e_G(A, B) \pm \varepsilon_2 \max\{n, e_G(A, B)\})/K = (1 \pm \varepsilon_2)e_G(A, B)/K. \tag{5.8}$$

Further, Lemma 5.1(vi) implies that

$$d_F(v, A_i) = (d_F(v, A) \pm \varepsilon_1 n)/K$$

for all $1 \leq i \leq K$ and $v \in V(G)$.

For the second application of Lemma 5.1 we let $U := B$, $R_1 := B_0$, $R_2 := A_0$ and $R_j := A_{j-2}$ for all $3 \leq j \leq K+2$. As before, $\Delta(G[U]) \leq \varepsilon' n$ by (FR5) and $d_G(u, R_j) \leq \varepsilon n \leq \varepsilon' n$ for all $u \in U$ and $j = 1, 2$ by (FR4). Moreover, (FR4) and (FR7) together imply that $d_G(x, U) \geq D/3 \geq \varepsilon' n$ for all $3 \leq j \leq K+2$ and each $x \in R_j = A_{j-2}$. Thus we can apply Lemma 5.1 with ε' playing the role of ε to obtain a partition $U_1, \dots, U_K$ of U. Let $B_i := U_i$ for all $i \leq K$. Then the B_i satisfy (P2)–(P5) with A replaced by B, A_i replaced by B_i, and so on. Moreover, for all $1 \leq i, j \leq K$,

$$\begin{aligned} e_G(A_i, B_j) &= (e_G(A_i, B) \pm \varepsilon_2 \max\{n, e_G(A_i, B)\})/K \\ &\overset{(5.8)}{=} ((1 \pm \varepsilon_2)e_G(A, B) \pm \varepsilon_2(1 + \varepsilon_2)e_G(A, B))/K^2 \\ &= (e_G(A, B) \pm 3\varepsilon_2 e_G(A, B))/K^2, \end{aligned}$$

i.e. (P6) holds. Since clearly (P1) holds as well, $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ together form a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Further, Lemma 5.1(vi) implies that

$$d_F(v, B_i) = (d_F(v, B) \pm \varepsilon_1 n)/K$$

for all $1 \leq i \leq K$ and $v \in V(G)$. □

The next lemma gives a decomposition of $G[A']$ and $G[B']$ into suitable smaller edge-disjoint subgraphs H^A_{ij} and H^B_{ij}. We say that the graphs H^A_{ij} and H^B_{ij} guaranteed by Lemma 5.3 are *localized slices* of G. Note that the order of the indices i and j matters here, i.e. $H^A_{ij} \neq H^A_{ji}$. Also, we allow $i = j$.

Lemma 5.3. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and $D \geq n/200$. Let $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Then for all $1 \leq i, j \leq K$ there are graphs H^A_{ij} and H^B_{ij} with the following properties:*

- (i) *H^A_{ij} is a spanning subgraph of $G[A_0, A_i \cup A_j] \cup G[A_i, A_j] \cup G[A_0]$;*
- (ii) *The sets $E(H^A_{ij})$ over all $1 \leq i, j \leq K$ form a partition of the edges of $G[A']$;*
- (iii) *$e(H^A_{ij}) = (e(A') \pm 9\varepsilon_2 \max\{n, e(A')\})/K^2$ for all $1 \leq i, j \leq K$;*
- (iv) *$e_{H^A_{ij}}(A_0, A_i \cup A_j) = (e(A_0, A) \pm 2\varepsilon_2 \max\{n, e(A_0, A)\})/K^2$ for all $1 \leq i, j \leq K$;*
- (v) *$e_{H^A_{ij}}(A_i, A_j) = (e(A) \pm 2\varepsilon_2 \max\{n, e(A)\})/K^2$ for all $1 \leq i, j \leq K$;*
- (vi) *For all $1 \leq i, j \leq K$ and all $v \in A_0$ we have $d_{H^A_{ij}}(v) = d_{H^A_{ij}}(v, A_i \cup A_j) + d_{H^A_{ij}}(v, A_0) = (d(v, A) \pm 4\varepsilon_1 n)/K^2$.*

The analogous assertions hold if we replace A by B, A_i by B_i, and so on.

Proof. In order to construct the graphs H^A_{ij} we perform the following procedure:

- Initially each H^A_{ij} is an empty graph with vertex set $A_0 \cup A_i \cup A_j$.
- For all $1 \leq i \leq K$ choose a random partition $E(A_0, A_i)$ into K sets U_j of equal size and let $E(H^A_{ij}) := U_j$. (If $E(A_0, A_i)$ is not divisible by K, first distribute up to $K-1$ edges arbitrarily among the U_j to achieve divisibility.)
- For all $i \leq K$, we add all the edges in $E(A_i)$ to H^A_{ii}.
- For all $i, j \leq K$ with $i \neq j$, half of the edges in $E(A_i, A_j)$ are added to H^A_{ij} and the other half is added to H^A_{ji} (the choice of the edges is arbitrary).
- The edges in $G[A_0]$ are distributed equally amongst the H^A_{ij}. (So $e_{H^A_{ij}}(A_0) = e(A_0)/K^2 \pm 1$.)

Clearly, the above procedure ensures that properties (i) and (ii) hold. (P5) implies (iv) and (P3) and (P4) imply (v).

Consider any $v \in A_0$. To prove (vi), note that we may assume that $d(v, A) \geq \varepsilon_1 n/K^2$. Let $X := d_{H^A_{ij}}(v, A_i \cup A_j)$. Note that (P2) implies that $\mathbb{E}(X) = (d(v, A) \pm 2\varepsilon_1 n)/K^2$ and note that $\mathbb{E}(X) \leq n$. So the Chernoff-Hoeffding bound for the hypergeometric distribution in Proposition 2.3 implies that

$$\mathbb{P}(|X - \mathbb{E}(X)| > \varepsilon_1 n/K^2) \leq \mathbb{P}(|X - \mathbb{E}(X)| > \varepsilon_1 \mathbb{E}(X)/K^2) \leq 2e^{-\varepsilon_1^2 \mathbb{E}(X)/3K^4} \leq 1/n^2.$$

Since $d_{H^A_{ij}}(v, A_0) \leq |A_0| \leq \varepsilon_1 n/K^2$, a union bound implies the desired result. Finally, observe that for any $a, b_1, \dots, b_4 > 0$, we have

$$\sum_{i=1}^{4} \max\{a, b_i\} \leq 4 \max\{a, b_1, \dots, b_4\} \leq 4 \max\{a, b_1 + \dots + b_4\}.$$

So (iii) follows from (iv), (v) and the fact that $e_{H^A_{ij}}(A_0) = e(A_0)/K^2 \pm 1$. □

Note that the construction implies that if $i \neq j$, then H^A_{ij} will contain edges between A_0 and A_i but not between A_0 and A_j. However, this additional information is not needed in the subsequent argument.

5.2. **Decomposing the localized slices.** Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework. Recall that $a = |A_0|$, $b = |B_0|$ and $a \geq b$. Since G is D-balanced by (FR2), we have $e(A') - e(B') = (a - b)D/2$. So there are an integer $q \geq -b$ and a constant $0 \leq c < 1$ such that

$$e(A') = (a + q + c)D/2 \quad \text{and} \quad e(B') = (b + q + c)D/2. \tag{5.9}$$

The aim of this subsection is to prove Lemma 5.6, which guarantees a decomposition of each localized slice H^A_{ij} into path systems (which will be extended into A_0B_0-path systems in Section 5.4) and a sparse (but not too sparse) leftover graph G^A_{ij}.

The following two results will be used in the proof of Lemma 5.6.

Lemma 5.4. *Let $0 < 1/n \ll \alpha, \beta, \gamma$ so that $\gamma < 1/2$. Suppose that G is a graph on n vertices such that $\Delta(G) \leq \alpha n$ and $e(G) \geq \beta n$. Then G contains a spanning subgraph H such that $e(H) = \lceil (1 - \gamma)e(G) \rceil$ and $\Delta(G - H) \leq 6\gamma\alpha n/5$.*

Proof. Let H' be a spanning subgraph of G such that

- $\Delta(H') \leq 6\gamma\alpha n/5$;
- $e(H') \geq \gamma e(G)$.

To see that such a graph H' exists, consider a random subgraph of G obtained by including each edge of G with probability $11\gamma/10$. Then $\mathbb{E}(\Delta(H')) \leq 11\gamma\alpha n/10$ and $\mathbb{E}(e(H')) = 11\gamma e(G)/10$. Thus applying Proposition 2.3 we have that, with high probability, H' is as desired.

Define H to be a spanning subgraph of G such that $H \supseteq G - H'$ and $e(H) = \lceil (1-\gamma)e(G) \rceil$. Then $\Delta(G-H) \leq \Delta(H') \leq 6\gamma\alpha n/5$, as required. □

Lemma 5.5. *Suppose that G is a graph such that $\Delta(G) \leq D-2$ where $D \in \mathbb{N}$ is even. Suppose A_0, A is a partition of $V(G)$ such that $d_G(x) \leq D/2-1$ for all $x \in A$ and $\Delta(G[A_0]) \leq D/2-1$. Then G has a decomposition into $D/2$ edge-disjoint path systems $P_1, \dots, P_{D/2}$ such that the following conditions hold:*

(i) *For each $i \leq D/2$, any internal vertex on a path in P_i lies in A_0;*
(ii) *$|e(P_i) - e(P_j)| \leq 1$ for all $i, j \leq D/2$.*

Proof. Let G_1 be a maximal spanning subgraph of G under the constraints that $G[A_0] \subseteq G_1$ and $\Delta(G_1) \leq D/2-1$. Note that $G[A_0] \cup G[A] \subseteq G_1$. Set $G_2 := G - G_1$. So G_2 only contains A_0A-edges. Further, since $\Delta(G) \leq D-2$, the maximality of G_1 implies that $\Delta(G_2) \leq D/2-1$.

Define an auxiliary graph G', obtained from G_1 as follows: write $A_0 = \{a_1, \dots, a_m\}$. Add a new vertex set $A_0' = \{a_1', \dots, a_m'\}$ to G_1. For each $i \leq m$ and $x \in A$, we add an edge between a_i' and x if and only if $a_i x$ is an edge in G_2.

Thus $G'[A_0 \cup A]$ is isomorphic to G_1 and $G'[A_0', A]$ is isomorphic to G_2. By construction and since $d_G(x) \leq D/2-1$ for all $x \in A$, we have that $\Delta(G') \leq D/2-1$. Hence, Proposition 4.5 implies that $E(G')$ can be decomposed into $D/2$ edge-disjoint matchings $M_1, \dots, M_{D/2}$ such that $||M_i| - |M_j|| \leq 1$ for all $i, j \leq D/2$.

By identifying each vertex $a_i' \in A_0'$ with the corresponding vertex $a_i \in A_0$, $M_1, \dots, M_{D/2}$ correspond to edge-disjoint subgraphs $P_1, \dots, P_{D/2}$ of G such that

- $P_1, \dots, P_{D/2}$ together cover all the edges in G;
- $|e(P_i) - e(P_j)| \leq 1$ for all $i, j \leq D/2$.

Note that $d_{M_i}(x) \leq 1$ for each $x \in V(G')$. Thus $d_{P_i}(x) \leq 1$ for each $x \in A$ and $d_{P_i}(x) \leq 2$ for each $x \in A_0$. This implies that any cycle in P_i must lie in $G[A_0]$. However, M_i is a matching and $G'[A_0'] \cup G'[A_0, A_0']$ contains no edges. Therefore, P_i contains no cycle, and so P_i is a path system such that any internal vertex on a path in P_i lies in A_0. Hence $P_1, \dots, P_{D/2}$ satisfy (i) and (ii). □

Lemma 5.6. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and $D \geq n/200$. Let $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Let H^A_{ij} be a localized slice of G as guaranteed by Lemma 5.3. Define c and q as in (5.9).*

Suppose that $t := (1-20\varepsilon_4)D/2K^2 \in \mathbb{N}$. *If* $e(B') \geq \varepsilon_3 n$, *set* t^* *to be the largest integer which is at most* ct *and is divisible by* K^2. *Otherwise, set* $t^* := 0$. *Define*

$$\ell_a := \begin{cases} 0 & \text{if } e(A') < \varepsilon_3 n; \\ a-b & \text{if } e(A') \geq \varepsilon_3 n \text{ but } e(B') < \varepsilon_3 n; \\ a+q+c & \text{otherwise} \end{cases}$$

and

$$\ell_b := \begin{cases} 0 & \text{if } e(B') < \varepsilon_3 n; \\ b+q+c & \text{otherwise.} \end{cases}$$

Then H^A_{ij} *has a decomposition into* t *edge-disjoint path systems* $P_1, \dots, P_t$ *and a spanning subgraph* G^A_{ij} *with the following properties:*

(i) *For each* $s \leq t$, *any internal vertex on a path in* P_s *lies in* A_0;
(ii) $e(P_1) = \cdots = e(P_{t^*}) = \lceil \ell_a \rceil$ *and* $e(P_{t^*+1}) = \cdots = e(P_t) = \lfloor \ell_a \rfloor$;
(iii) $e(P_s) \leq \sqrt{\varepsilon} n$ *for every* $s \leq t$;
(iv) $\Delta(G^A_{ij}) \leq 13\varepsilon_4 D/K^2$.

The analogous assertion (with ℓ_a *replaced by* ℓ_b *and* A_0 *replaced by* B_0*) holds for each localized slice* H^B_{ij} *of* G. *Furthermore,* $\lceil \ell_a \rceil - \lceil \ell_b \rceil = \lfloor \ell_a \rfloor - \lfloor \ell_b \rfloor = a-b$.

Proof. Note that (5.9) and (FR3) together imply that $\ell_a D/2 \leq (a+q+c)D/2 = e(A') \leq \varepsilon n^2$ and so $\lceil \ell_a \rceil \leq \sqrt{\varepsilon} n$. Thus (iii) will follow from (ii). So it remains to prove (i), (ii) and (iv). We split the proof into three cases.

Case 1. $e(A') < \varepsilon_3 n$

(FR2) and (FR4) imply that $e(A') - e(B') = (a-b)D/2 \geq 0$. So $e(B') \leq e(A') < \varepsilon_3 n$. Thus $\ell_a = \ell_b = 0$. Set $G^A_{ij} := H^A_{ij}$ and $G^B_{ij} := H^B_{ij}$. Therefore, (iv) is satisfied as $\Delta(H^A_{ij}) \leq e(A') < \varepsilon_3 n \leq 13\varepsilon_4 D/K^2$. Further, (i) and (ii) are vacuous (i.e. we set each P_s to be the empty graph on $V(G)$).

Note that $a = b$ since otherwise $a > b$ and therefore (FR2) implies that $e(A') \geq (a-b)D/2 \geq D/2 > \varepsilon_3 n$, a contradiction. Hence, $\lceil \ell_a \rceil - \lceil \ell_b \rceil = \lfloor \ell_a \rfloor - \lfloor \ell_b \rfloor = 0 = a-b$.

Case 2. $e(A') \geq \varepsilon_3 n$ and $e(B') < \varepsilon_3 n$

Since $\ell_b = 0$ in this case, we set $G^B_{ij} := H^B_{ij}$ and each P_s to be the empty graph on $V(G)$. Then as in Case 1, (i), (ii) and (iv) are satisfied with respect to H^B_{ij}. Further, clearly $\lceil \ell_a \rceil - \lceil \ell_b \rceil = \lfloor \ell_a \rfloor - \lfloor \ell_b \rfloor = a-b$.

Note that $a > b$ since otherwise $a = b$ and thus $e(A') = e(B')$ by (FR2), a contradiction to the case assumptions. Since $e(A') - e(B') = (a-b)D/2$ by (FR2), Lemma 5.3(iii) implies that

$$e(H^A_{ij}) \geq (1-9\varepsilon_2)e(A')/K^2 - 9\varepsilon_2 n/K^2 \geq (1-9\varepsilon_2)(a-b)D/(2K^2) - 9\varepsilon_2 n/K^2$$
$$\geq (1-\varepsilon_3)(a-b)D/(2K^2) > (a-b)t. \tag{5.10}$$

Similarly, Lemma 5.3(iii) implies that

$$e(H^A_{ij}) \leq (1+\varepsilon_4)(a-b)D/(2K^2). \tag{5.11}$$

Therefore, (5.10) implies that there exists a constant $\gamma > 0$ such that

$$(1-\gamma)e(H^A_{ij}) = (a-b)t.$$

Since $(1-19\varepsilon_4)(1-\varepsilon_3) > (1-20\varepsilon_4)$, (5.10) implies that $\gamma > 19\varepsilon_4 \gg 1/n$. Further, since $(1+\varepsilon_4)(1-21\varepsilon_4) < (1-20\varepsilon_4)$, (5.11) implies that $\gamma < 21\varepsilon_4$.

Note that (FR5), (FR7) and Lemma 5.3(vi) imply that

$$\Delta(H^A_{ij}) \leq (D/2 + 5\varepsilon_1 n)/K^2. \tag{5.12}$$

Thus Lemma 5.4 implies that H^A_{ij} contains a spanning subgraph H such that $e(H) = (1-\gamma)e(H^A_{ij}) = (a-b)t$ and

$$\Delta(H^A_{ij} - H) \leq 6\gamma(D/2 + 5\varepsilon_1 n)/(5K^2) \leq 13\varepsilon_4 D/K^2,$$

where the last inequality follows since $\gamma < 21\varepsilon_4$ and $\varepsilon_1 \ll 1$. Setting $G^A_{ij} := H^A_{ij} - H$ implies that (iv) is satisfied.

Our next task is to decompose H into t edge-disjoint path systems so that (i) and (ii) are satisfied. Note that (5.12) implies that

$$\Delta(H) \leq \Delta(H^A_{ij}) \leq (D/2 + 5\varepsilon_1 n)/K^2 < 2t - 2.$$

Further, (FR4) implies that $\Delta(H[A_0]) \leq |A_0| \leq \varepsilon n < t-1$ and (FR5) implies that $d_H(x) \leq \varepsilon' n < t-1$ for all $x \in A$. Since $e(H) = (a-b)t$, Lemma 5.5 implies that H has a decomposition into t edge-disjoint path systems $P_1, \dots, P_t$ satisfying (i) and so that $e(P_s) = a - b = \ell_a$ for all $s \leq t$. In particular, (ii) is satisfied.

Case 3. $e(A'), e(B') \geq \varepsilon_3 n$

By definition of ℓ_a and ℓ_b, we have that $\lceil \ell_a \rceil - \lceil \ell_b \rceil = \lfloor \ell_a \rfloor - \lfloor \ell_b \rfloor = a - b$. Notice that since $e(A') \geq \varepsilon_3 n$ and $\varepsilon_2 \ll \varepsilon_3$, certainly $\varepsilon_3 e(A')/(2K^2) > 9\varepsilon_2 n/K^2$. Therefore, Lemma 5.3(iii) implies that

$$\begin{aligned} e(H^A_{ij}) &\geq (1-9\varepsilon_2)e(A')/K^2 - 9\varepsilon_2 n/K^2 \\ &\geq (1-\varepsilon_3)e(A')/K^2 \\ &\geq \varepsilon_3 n/(2K^2). \end{aligned} \tag{5.13}$$

Note that $1/n \ll \varepsilon_3/(2K^2)$. Further, (5.9) and (5.13) imply that

$$\begin{aligned} e(H^A_{ij}) &\geq (1-\varepsilon_3)e(A')/K^2 \\ &= (1-\varepsilon_3)(a+q+c)D/(2K^2) > (a+q)t + t^*. \end{aligned} \tag{5.14}$$

Similarly, Lemma 5.3(iii) implies that

$$e(H^A_{ij}) \leq (1+\varepsilon_3)(a+q+c)D/(2K^2). \tag{5.15}$$

By (5.14) there exists a constant $\gamma > 0$ such that

$$(1-\gamma)e(H^A_{ij}) = (a+q)t + t^*.$$

Note that (5.14) implies that $1/n \ll 19\varepsilon_4 < \gamma$ and (5.15) implies that $\gamma < 21\varepsilon_4$. Moreover, as in Case 2, (FR5), (FR7) and Lemma 5.3(vi) together show that

$$\Delta(H^A_{ij}) \leq (D/2 + 5\varepsilon_1 n)/K^2. \tag{5.16}$$

Thus (as in Case 2 again), Lemma 5.4 implies that H^A_{ij} contains a spanning subgraph H such that $e(H) = (1-\gamma)e(H^A_{ij}) = (a+q)t + t^*$ and

$$\Delta(H^A_{ij} - H) \leq 6\gamma(D/2 + 5\varepsilon_1 n)/(5K^2) \leq 13\varepsilon_4 D/K^2.$$

Setting $G^A_{ij} := H^A_{ij} - H$ implies that (iv) is satisfied. Next we decompose H into t edge-disjoint path systems so that (i) and (ii) are satisfied. Note that (5.16) implies that

$$\Delta(H) \leq \Delta(H^A_{ij}) \leq (D/2 + 5\varepsilon_1 n)/K^2 < 2t - 2.$$

Further, (FR4) implies that $\Delta(H[A_0]) \leq |A_0| \leq \varepsilon n < t-1$ and (FR5) implies that $d_H(x) \leq \varepsilon' n < t-1$ for all $x \in A$. Since $e(H) = (a+q)t + t^*$, Lemma 5.5 implies that H has a decomposition into t edge-disjoint path systems $P_1, \dots, P_t$ satisfying (i) and (ii). An identical argument implies that (i), (ii) and (iv) are satisfied with respect to H^B_{ij} also. □

5.3. **Decomposing the global graph.** Let G^A_{glob} be the union of the graphs G^A_{ij} guaranteed by Lemma 5.6 over all $1 \leq i, j \leq K$. Define G^B_{glob} similarly. The next lemma gives a decomposition of both G^A_{glob} and G^B_{glob} into suitable path systems. Properties (iii) and (iv) of the lemma guarantee that one can pair up each such path system $Q_A \subseteq G^A_{glob}$ with a different path system $Q_B \subseteq G^B_{glob}$ such that $Q_A \cup Q_B$ is 2-balanced (in particular $e(Q_A) - e(Q_B) = a - b$). This property will then enable us to apply Lemma 4.10 to extend $Q_A \cup Q_B$ into a Hamilton cycle using only edges between A' and B'.

Lemma 5.7. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and such that $D \geq n/200$ and D is even. Let $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Let G^A_{glob} be the union of the graphs G^A_{ij} guaranteed by Lemma 5.6 over all $1 \leq i, j \leq K$. Define G^B_{glob} similarly. Suppose that $k := 10\varepsilon_4 D \in \mathbb{N}$. Then the following properties hold:*

(i) *There is an integer q' and a real number $0 \leq c' < 1$ so that $e(G^A_{glob}) = (a + q' + c')k$ and $e(G^B_{glob}) = (b + q' + c')k$.*
(ii) $\Delta(G^A_{glob}), \Delta(G^B_{glob}) < 3k/2$.
(iii) *Let $k^* := c'k$. Then G^A_{glob} has a decomposition into k^* path systems, each containing $a + q' + 1$ edges, and $k - k^*$ path systems, each containing $a + q'$ edges. Moreover, each of these k path systems Q satisfies $d_Q(x) \leq 1$ for all $x \in A$.*
(iv) *G^B_{glob} has a decomposition into k^* path systems, each containing $b + q' + 1$ edges, and $k - k^*$ path systems, each containing $b + q'$ edges. Moreover, each of these k path systems Q satisfies $d_Q(x) \leq 1$ for all $x \in B$.*
(v) *Each of the path systems guaranteed in (iii) and (iv) contains at most $\sqrt{\varepsilon}n$ edges.*

Note that in Lemma 5.7 and several later statements the parameter ε_3 is implicitly defined by the application of Lemma 5.6 which constructs the graphs G^A_{glob} and G^B_{glob}.

Proof. Let t^* and t be as defined in Lemma 5.6. Our first task is to show that (i) is satisfied. If $e(A'), e(B') < \varepsilon_3 n$ then $G^A_{glob} = G[A']$ and $G^B_{glob} = G[B']$. Further, $a = b$ in this case since otherwise (FR4) implies that $a > b$ and so (FR2) yields that $e(A') \geq (a-b)D/2 \geq D/2 > \varepsilon_3 n$, a contradiction. Therefore, (FR2) implies that

$$e(G^A_{glob}) - e(G^B_{glob}) = e(A') - e(B') = (a-b)D/2 = 0 = (a-b)k.$$

If $e(A') \geq \varepsilon_3 n$ and $e(B') < \varepsilon_3 n$ then $G^B_{glob} = G[B']$. Further, G^A_{glob} is obtained from $G[A']$ by removing tK^2 edge-disjoint path systems, each of which contains precisely $a-b$ edges. Thus (FR2) implies that

$$e(G^A_{glob}) - e(G^B_{glob}) = e(A') - e(B') - tK^2(a-b) = (a-b)(D/2 - tK^2) = (a-b)k.$$

Finally, consider the case when $e(A'), e(B') > \varepsilon_3 n$. Then G^A_{glob} is obtained from $G[A']$ by removing t^*K^2 edge-disjoint path systems, each of which contain exactly $a+q+1$ edges, and by removing $(t-t^*)K^2$ edge-disjoint path systems, each of which contain exactly $a+q$ edges. Similarly, G^B_{glob} is obtained from $G[B']$ by removing t^*K^2 edge-disjoint path systems, each of which contain exactly $b+q+1$ edges, and by removing $(t-t^*)K^2$ edge-disjoint path systems, each of which contain exactly $b+q$ edges. So (FR2) implies that

$$e(G^A_{glob}) - e(G^B_{glob}) = e(A') - e(B') - (a-b)tK^2 = (a-b)k.$$

Therefore, in every case,

$$e(G^A_{glob}) - e(G^B_{glob}) = (a-b)k. \tag{5.17}$$

Define the integer q' and $0 \leq c' < 1$ by $e(G^A_{glob}) = (a+q'+c')k$. Then (5.17) implies that $e(G^B_{glob}) = (b+q'+c')k$. This proves (i). To prove (ii), note that Lemma 5.6(iv) implies that $\Delta(G^A_{glob}) \leq 13\varepsilon_4 D < 3k/2$ and similarly $\Delta(G^B_{glob}) < 3k/2$.

Note that (FR5) implies that $d_{G^A_{glob}}(x) \leq \varepsilon' n < k-1$ for all $x \in A$ and $\Delta(G^A_{glob}[A_0]) \leq |A_0| \leq \varepsilon n < k-1$. Thus Lemma 5.5 together with (i) implies that (iii) is satisfied. (iv) follows from Lemma 5.5 analogously.

(FR3) implies that $e(G^A_{glob}) \leq e_G(A') \leq \varepsilon n^2$ and $e(G^B_{glob}) \leq e_G(B') \leq \varepsilon n^2$. Therefore, each path system from (iii) and (iv) contains at most $\lceil \varepsilon n^2/k \rceil \leq \sqrt{\varepsilon} n$ edges. So (v) is satisfied. □

We say that a path system $P \subseteq G[A']$ is (i, j, A)-*localized* if

(i) $E(P) \subseteq E(G[A_0, A_i \cup A_j]) \cup E(G[A_i, A_j]) \cup E(G[A_0])$;
(ii) Any internal vertex on a path in P lies in A_0.

We introduce an analogous notion of (i, j, B)-*localized* for path systems $P \subseteq G[B']$.

The following result is a straightforward consequence of Lemmas 5.3, 5.6 and 5.7. It gives a decomposition of $G[A'] \cup G[B']$ into pairs of paths systems so that most of these are localized and so that each pair can be extended into a Hamilton cycle by adding $A'B'$-edges.

Corollary 5.8. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and such that $D \geq n/200$ and D is even. Let $A_0, A_1, \dots, A_K$ and $B_0, B_1, \dots, B_K$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Let $t_K := (1 - 20\varepsilon_4)D/2K^4$ and $k := 10\varepsilon_4 D$. Suppose that $t_K \in \mathbb{N}$. Then there are K^4 sets $\mathcal{M}_{i_1 i_2 i_3 i_4}$, one for each $1 \leq i_1, i_2, i_3, i_4 \leq K$, such that each $\mathcal{M}_{i_1 i_2 i_3 i_4}$ consists of t_K pairs of path systems and satisfies the following properties:*

- (a) *Let (P, P') be a pair of path systems which forms an element of $\mathcal{M}_{i_1 i_2 i_3 i_4}$. Then*
  - (i) *P is an (i_1, i_2, A)-localized path system and P' is an (i_3, i_4, B)-localized path system;*
  - (ii) *$e(P) - e(P') = a - b$;*
  - (iii) *$e(P), e(P') \leq \sqrt{\varepsilon} n$.*
- (b) *The $2t_K$ path systems in the pairs belonging to $\mathcal{M}_{i_1 i_2 i_3 i_4}$ are all pairwise edge-disjoint.*
- (c) *Let $G(\mathcal{M}_{i_1 i_2 i_3 i_4})$ denote the spanning subgraph of G whose edge set is the union of all the path systems in the pairs belonging to $\mathcal{M}_{i_1 i_2 i_3 i_4}$. Then the K^4 graphs $G(\mathcal{M}_{i_1 i_2 i_3 i_4})$ are edge-disjoint. Further, each $x \in A_0$ satisfies $d_{G(\mathcal{M}_{i_1 i_2 i_3 i_4})}(x) \geq (d_G(x, A) - 15\varepsilon_4 D)/K^4$ while each $y \in B_0$ satisfies $d_{G(\mathcal{M}_{i_1 i_2 i_3 i_4})}(y) \geq (d_G(y, B) - 15\varepsilon_4 D)/K^4$.*
- (d) *Let G_{glob} be the subgraph of $G[A'] \cup G[B']$ obtained by removing all edges contained in $G(\mathcal{M}_{i_1 i_2 i_3 i_4})$ for all $1 \leq i_1, i_2, i_3, i_4 \leq K$. Then $\Delta(G_{glob}) \leq 3k/2$. Moreover, G_{glob} has a decomposition into k pairs of path systems $(Q_{1,A}, Q_{1,B}), \dots, (Q_{k,A}, Q_{k,B})$ so that*
  - (i′) *$Q_{i,A} \subseteq G_{glob}[A']$ and $Q_{i,B} \subseteq G_{glob}[B']$ for all $i \leq k$;*
  - (ii′) *$d_{Q_{i,A}}(x) \leq 1$ for all $x \in A$ and $d_{Q_{i,B}}(x) \leq 1$ for all $x \in B$;*
  - (iii′) *$e(Q_{i,A}) - e(Q_{i,B}) = a - b$ for all $i \leq k$;*
  - (iv′) *$e(Q_{i,A}), e(Q_{i,B}) \leq \sqrt{\varepsilon} n$ for all $i \leq k$.*

Proof. Apply Lemma 5.3 to obtain localized slices H^A_{ij} and H^B_{ij} (for all $i, j \leq K$). Let $t := K^2 t_K$ and let t^* be as defined in Lemma 5.6. Since $t/K^2, t^*/K^2 \in \mathbb{N}$ we have $(t - t^*)/K^2 \in \mathbb{N}$. For all $i_1, i_2 \leq K$, let $\mathcal{M}^A_{i_1 i_2}$ be the set of t path systems in $H^A_{i_1 i_2}$ guaranteed by Lemma 5.6. We call the t^* path systems in $\mathcal{M}^A_{i_1 i_2}$ of size $\lceil \ell_a \rceil$ *large* and the others *small*. We define $\mathcal{M}^B_{i_3 i_4}$ as well as large and small path systems in $\mathcal{M}^B_{i_3 i_4}$ analogously (for all $i_3, i_4 \leq K$).

We now construct the sets $\mathcal{M}_{i_1 i_2 i_3 i_4}$ as follows: For all $i_1, i_2 \leq K$, consider a random partition of the set of all large path systems in $\mathcal{M}^A_{i_1 i_2}$ into K^2 sets of equal size t^*/K^2 and assign (all the path systems in) each of these sets to one of the $\mathcal{M}_{i_1 i_2 i_3 i_4}$ with $i_3, i_4 \leq K$. Similarly, randomly partition the set of small path systems in $\mathcal{M}^A_{i_1 i_2}$ into K^2 sets, each containing $(t - t^*)/K^2$ path systems. Assign each of these K^2 sets to one of the $\mathcal{M}_{i_1 i_2 i_3 i_4}$ with $i_3, i_4 \leq K$. Proceed similarly for each $\mathcal{M}^B_{i_3 i_4}$ in order to assign each of its path systems randomly to some $\mathcal{M}_{i_1 i_2 i_3 i_4}$. Then to each $\mathcal{M}_{i_1 i_2 i_3 i_4}$ we have assigned exactly t^*/K^2 large path systems from both $\mathcal{M}^A_{i_1 i_2}$ and $\mathcal{M}^B_{i_3 i_4}$. Pair these off arbitrarily. Similarly, pair off the small path systems assigned to $\mathcal{M}_{i_1i_2i_3i_4}$ arbitrarily. Clearly, the sets $\mathcal{M}_{i_1i_2i_3i_4}$ obtained in this way satisfy (a) and (b).

We now verify (c). By construction, the K^4 graphs $G(\mathcal{M}_{i_1i_2i_3i_4})$ are edge-disjoint. So consider any vertex $x \in A_0$ and write $d := d_G(x, A)$. Note that $d_{H^A_{i_1i_2}}(x) \geq (d - 4\varepsilon_1 n)/K^2$ by Lemma 5.3(vi). Let $G(\mathcal{M}^A_{i_1i_2})$ be the spanning subgraph of G whose edge set is the union of all the path systems in $\mathcal{M}^A_{i_1i_2}$. Then Lemma 5.6(iv) implies that

$$d_{G(\mathcal{M}^A_{i_1i_2})}(x) \geq d_{H^A_{i_1i_2}}(x) - \Delta(G^A_{i_1i_2}) \geq \frac{d - 4\varepsilon_1 n}{K^2} - \frac{13\varepsilon_4 D}{K^2} \geq \frac{d - 14\varepsilon_4 D}{K^2}.$$

So a Chernoff-Hoeffding estimate for the hypergeometric distribution (Proposition 2.3) implies that

$$d_{G(\mathcal{M}_{i_1i_2i_3i_4})}(x) \geq \frac{1}{K^2}\left(\frac{d - 14\varepsilon_4 D}{K^2}\right) - \varepsilon n \geq \frac{d - 15\varepsilon_4 D}{K^4}.$$

(Note that we only need to apply the Chernoff-Hoeffding bound if $d \geq \varepsilon n$ say, as (c) is vacuous otherwise.)

It remains to check condition (d). First note that $k \in \mathbb{N}$ since $t_K, D/2 \in \mathbb{N}$. Thus we can apply Lemma 5.7 to obtain a decomposition of both G^A_{glob} and G^B_{glob} into path systems. Since $G_{glob} = G^A_{glob} \cup G^B_{glob}$, (d) is an immediate consequence of Lemma 5.7(ii)–(v). □

5.4. **Constructing the localized balanced exceptional systems.** The localized path systems obtained from Corollary 5.8 do not yet cover all of the exceptional vertices. This is achieved via the following lemma: we extend the path systems to achieve this additional property, while maintaining the property of being balanced. More precisely, let

$$\mathcal{P} := \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$$

be a (K, m, ε)-partition of a set V of n vertices. Given $1 \leq i_1, i_2, i_3, i_4 \leq K$ and $\varepsilon_0 > 0$, an (i_1, i_2, i_3, i_4)-*balanced exceptional system with respect to* $\mathcal{P}$ *and parameter* ε_0 is a path system J with $V(J) \subseteq A_0 \cup B_0 \cup A_{i_1} \cup A_{i_2} \cup B_{i_3} \cup B_{i_4}$ such that the following conditions hold:

(BES1) Every vertex in $A_0 \cup B_0$ is an internal vertex of a path in J. Every vertex $v \in A_{i_1} \cup A_{i_2} \cup B_{i_3} \cup B_{i_4}$ satisfies $d_J(v) \leq 1$.
(BES2) Every edge of $J[A \cup B]$ is either an $A_{i_1}A_{i_2}$-edge or a $B_{i_3}B_{i_4}$-edge.
(BES3) The edges in J cover precisely the same number of vertices in A as in B.
(BES4) $e(J) \leq \varepsilon_0 n$.

To shorten the notation, we will often refer to J as an (i_1, i_2, i_3, i_4)-*BES*. If V is the vertex set of a graph G and $J \subseteq G$, we also say that J is an (i_1, i_2, i_3, i_4)-*BES in* G. Note that (BES2) implies that an (i_1, i_2, i_3, i_4)-BES does not contain edges between A and B. Furthermore, an (i_1, i_2, i_3, i_4)-BES is also, for example, an (i_2, i_1, i_4, i_3)-BES. We will sometimes omit the indices i_1, i_2, i_3, i_4 and just refer to a balanced exceptional system (or a BES for short). We will sometimes also omit the partition $\mathcal{P}$, if it is clear from the context.

(BES1) implies that each balanced exceptional system is an A_0B_0-path system as defined before Proposition 4.6. (However, the converse is not true since, for example, a 2-balanced A_0B_0-path system need not satisfy (BES4).) So (BES3) and Proposition 4.6 imply that each balanced exceptional system is also 2-balanced.

We now extend each set $\mathcal{M}_{i_1i_2i_3i_4}$ obtained from Corollary 5.8 into a set $\mathcal{J}_{i_1i_2i_3i_4}$ of (i_1, i_2, i_3, i_4)-BES.

Lemma 5.9. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon_0 \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and such that $D \geq n/200$ and D is even. Let $\mathcal{P} := \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Suppose that $t_K := (1 - 20\varepsilon_4)D/2K^4 \in \mathbb{N}$. Let $\mathcal{M}_{i_1i_2i_3i_4}$ be the sets returned by Corollary 5.8. Then for all $1 \leq i_1, i_2, i_3, i_4 \leq K$ there is a set $\mathcal{J}_{i_1i_2i_3i_4}$ which satisfies the following properties:*

- (i) *$\mathcal{J}_{i_1i_2i_3i_4}$ consists of t_K edge-disjoint (i_1, i_2, i_3, i_4)-BES in G with respect to $\mathcal{P}$ and with parameter ε_0.*
- (ii) *For each of the t_K pairs of path systems $(P, P') \in \mathcal{M}_{i_1i_2i_3i_4}$, there is a unique $J \in \mathcal{J}_{i_1i_2i_3i_4}$ which contains all the edges in $P \cup P'$. Moreover, all edges in $E(J) \setminus E(P \cup P')$ lie in $G[A_0, B_{i_3}] \cup G[B_0, A_{i_1}]$.*
- (iii) *Whenever $(i_1, i_2, i_3, i_4) \neq (i'_1, i'_2, i'_3, i'_4)$, $J \in \mathcal{J}_{i_1i_2i_3i_4}$ and $J' \in \mathcal{J}_{i'_1i'_2i'_3i'_4}$, then J and J' are edge-disjoint.*

We let $\mathfrak{J}$ denote the union of the sets $\mathcal{J}_{i_1i_2i_3i_4}$ over all $1 \leq i_1, i_2, i_3, i_4 \leq K$.

Proof. We will construct the sets $\mathcal{J}_{i_1i_2i_3i_4}$ greedily by extending each pair of path systems $(P, P') \in \mathcal{M}_{i_1i_2i_3i_4}$ in turn into an (i_1, i_2, i_3, i_4)-BES containing $P \cup P'$. For this, consider some arbitrary ordering of the K^4 4-tuples (i_1, i_2, i_3, i_4). Suppose that we have already constructed the sets $\mathcal{J}_{i'_1i'_2i'_3i'_4}$ for all (i'_1, i'_2, i'_3, i'_4) preceding (i_1, i_2, i_3, i_4) so that (i)–(iii) are satisfied. So our aim now is to construct $\mathcal{J}_{i_1i_2i_3i_4}$. Consider an enumeration $(P_1, P'_1), \dots, (P_{t_K}, P'_{t_K})$ of the pairs of path systems in $\mathcal{M}_{i_1i_2i_3i_4}$. Suppose that for some $i \leq t_K$ we have already constructed edge-disjoint (i_1, i_2, i_3, i_4)-BES $J_1, \dots, J_{i-1}$, so that for each $i' < i$ the following conditions hold:

- $J_{i'}$ contains the edges in $P_{i'} \cup P'_{i'}$;
- all edges in $E(J_{i'}) \setminus E(P_{i'} \cup P'_{i'})$ lie in $G[A_0, B_{i_3}] \cup G[B_0, A_{i_1}]$;
- $J_{i'}$ is edge-disjoint from all the balanced exceptional systems in $\bigcup_{(i'_1,i'_2,i'_3,i'_4)} \mathcal{J}_{i'_1i'_2i'_3i'_4}$, where the union is over all (i'_1, i'_2, i'_3, i'_4) preceding (i_1, i_2, i_3, i_4).

We will now construct $J := J_i$. For this, we need to add suitable edges to $P_i \cup P'_i$ to ensure that all vertices of $A_0 \cup B_0$ have degree two. We start with A_0. Recall that $a = |A_0|$ and write $A_0 = \{x_1, \dots, x_a\}$. Let G' denote the subgraph of $G[A', B']$ obtained by removing all the edges lying in $J_1, \dots, J_{i-1}$ as well as all those edges lying in the balanced exceptional systems belonging to $\bigcup_{(i'_1,i'_2,i'_3,i'_4)} \mathcal{J}_{i'_1i'_2i'_3i'_4}$ (where as before the union is over all (i'_1, i'_2, i'_3, i'_4) preceding (i_1, i_2, i_3, i_4)). We will choose the new edges incident to A_0 in J inside $G'[A_0, B_{i_3}]$.

Suppose we have already found suitable edges for $x_1, \dots, x_{j-1}$ and let $J(j)$ be the set of all these edges. We will first show that the degree of x_j inside $G'[A_0, B_{i_3}]$ is still large. Let $d_j := d_G(x_j, A')$. Consider any (i'_1, i'_2, i'_3, i'_4) preceding (i_1, i_2, i_3, i_4).

Let $G(\mathcal{J}_{i'_1 i'_2 i'_3 i'_4})$ denote the union of the t_K balanced exceptional systems belonging to $\mathcal{J}_{i'_1 i'_2 i'_3 i'_4}$. Thus $d_{G(\mathcal{J}_{i'_1 i'_2 i'_3 i'_4})}(x_j) = 2t_K$. However, Corollary 5.8(c) implies that $d_{G(\mathcal{M}_{i'_1 i'_2 i'_3 i'_4})}(x_j) \geq (d_j - 15\varepsilon_4 D)/K^4$. So altogether, when constructing (the balanced exceptional systems in) $\mathcal{J}_{i'_1 i'_2 i'_3 i'_4}$, we have added at most $2t_K - (d_j - 15\varepsilon_4 D)/K^4$ new edges at x_j, and all these edges join x_j to vertices in $B_{i'_3}$. Similarly, when constructing $J_1, \dots, J_{i-1}$, we have added at most $2t_K - (d_j - 15\varepsilon_4 D)/K^4$ new edges at x_j. Since the number of 4-tuples (i'_1, i'_2, i'_3, i'_4) with $i'_3 = i_3$ is K^3, it follows that

$$\begin{aligned} d_G(x_j, B_{i_3}) - d_{G'}(x_j, B_{i_3}) &\leq K^3 \left(2t_K - \frac{d_j - 15\varepsilon_4 D}{K^4}\right) \\ &= \frac{1}{K}\left((1 - 20\varepsilon_4)D - d_j + 15\varepsilon_4 D\right) \\ &= \frac{1}{K}\left(D - d_j - 5\varepsilon_4 D\right). \end{aligned}$$

Also, (P2) with A replaced by B implies that

$$d_G(x_j, B_{i_3}) \geq \frac{d_G(x_j, B) - \varepsilon_1 n}{K} \geq \frac{d_G(x_j) - d_G(x_j, A') - \varepsilon_1 n}{K} = \frac{D - d_j - \varepsilon_1 n}{K},$$

where here we use (FR2) and (FR6). So altogether, we have

$$d_{G'}(x_j, B_{i_3}) \geq (5\varepsilon_4 D - \varepsilon_1 n)/K \geq \varepsilon_4 n/50K.$$

Let B'_{i_3} be the set of vertices in B_{i_3} not covered by the edges of $J(j) \cup P'_i$. Note that $|B'_{i_3}| \geq |B_{i_3}| - 2|A_0| - 2e(P'_i) \geq |B_{i_3}| - 3\sqrt{\varepsilon}n$ since $a = |A_0| \leq \varepsilon n$ by (FR4) and $e(P'_i) \leq \sqrt{\varepsilon}n$ by Corollary 5.8(a)(iii). So $d_{G'}(x_j, B'_{i_3}) \geq \varepsilon_4 n/51K$. We can add up to two of these edges to J in order to ensure that x_j has degree two in J. This completes the construction of the edges of J incident to A_0. The edges incident to B_0 are found similarly.

Let J be the graph on $A_0 \cup B_0 \cup A_{i_1} \cup A_{i_2} \cup B_{i_3} \cup B_{i_4}$ whose edge set is constructed in this way. By construction, J satisfies (BES1) and (BES2) since P_j and P'_j are (i_1, i_2, A)-localized and (i_3, i_4, B)-localized respectively. We now verify (BES3). As mentioned before the statement of the lemma, (BES1) implies that J is an A_0B_0-path system (as defined before Proposition 4.6). Moreover, Corollary 5.8(a)(ii) implies that $P_i \cup P'_i$ is a path system which satisfies (B1) in the definition of 2-balanced. Since J was obtained by adding only $A'B'$-edges, (B1) is preserved in J. Since by construction J satisfies (B2), it follows that J is 2-balanced. So Proposition 4.6 implies (BES3).

Finally, we verify (BES4). For this, note that Corollary 5.8(a)(iii) implies that $e(P_i), e(P'_i) \leq \sqrt{\varepsilon}n$. Moreover, the number of edges added to $P_i \cup P'_i$ when constructing J is at most $2(|A_0| + |B_0|)$, which is at most $2\varepsilon n$ by (FR4). Thus $e(J) \leq 2\sqrt{\varepsilon}n + 2\varepsilon n \leq \varepsilon_0 n$. □

5.5. **Covering G_{glob} by edge-disjoint Hamilton cycles.** We now find a set of edge-disjoint Hamilton cycles covering the edges of the 'leftover' graph obtained from $G - G[A,B]$ by deleting all those edges lying in balanced exceptional systems belonging to $\mathfrak{J}$.

Lemma 5.10. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon_0 \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and such that $D \geq n/200$ and D is even. Let $\mathcal{P} := \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Suppose that $t_K := (1 - 20\varepsilon_4)D/2K^4 \in \mathbb{N}$. Let $\mathfrak{J}$ be as defined after Lemma 5.9 and let $G(\mathfrak{J}) \subseteq G$ be the union of all the balanced exceptional systems lying in $\mathfrak{J}$. Let $G^* := G - G(\mathfrak{J})$, let $k := 10\varepsilon_4 D$ and let $(Q_{1,A}, Q_{1,B}), \dots, (Q_{k,A}, Q_{k,B})$ be as in Corollary 5.8(d).*

- (a) *The graph $G^* - G^*[A,B]$ can be decomposed into k A_0B_0-path systems $Q_1, \dots, Q_k$ which are 2-balanced and satisfy the following properties:*
  - (i) *Q_i contains all edges of $Q_{i,A} \cup Q_{i,B}$;*
  - (ii) *$Q_1, \dots, Q_k$ are pairwise edge-disjoint;*
  - (iii) *$e(Q_i) \leq 3\sqrt{\varepsilon} n$.*
- (b) *Let $Q_1, \dots, Q_k$ be as in (a). Suppose that F is a graph on $V(G)$ such that $G \subseteq F$, $\delta(F) \geq 2n/5$ and such that F satisfies (WF5) with respect to ε'. Then there are edge-disjoint Hamilton cycles $C_1, \dots, C_k$ in $F - G(\mathfrak{J})$ such that $Q_i \subseteq C_i$ and $C_i \cap G$ is 2-balanced for each $i \leq k$.*

Proof. We first prove (a). The argument is similar to that of Lemma 5.7. Roughly speaking, we will extend each $Q_{i,A}$ into a path system $Q'_{i,A}$ by adding suitable A_0B-edges which ensure that every vertex in A_0 has degree exactly two in $Q'_{i,A}$. Similarly, we will extend each $Q_{i,B}$ into $Q'_{i,B}$ by adding suitable AB_0-edges. We will ensure that no vertex is an endvertex of both an edge in $Q'_{i,A}$ and an edge in $Q'_{i,B}$ and take Q_i to be the union of these two path systems. We first construct all the $Q'_{i,A}$.

Claim 1. *$G^*[A'] \cup G^*[A_0, B]$ has a decomposition into edge-disjoint path systems $Q'_{1,A}, \dots, Q'_{k,A}$ such that*

- *$Q_{i,A} \subseteq Q'_{i,A}$ and $E(Q'_{i,A}) \setminus E(Q_{i,A})$ consists of A_0B-edges in G^* (for each $i \leq k$);*
- *$d_{Q'_{i,A}}(x) = 2$ for every $x \in A_0$ and $d_{Q'_{i,A}}(x) \leq 1$ for every $x \notin A_0$;*
- *no vertex is an endvertex of both an edge in $Q'_{i,A}$ and an edge in $Q_{i,B}$ (for each $i \leq k$).*

To prove Claim 1, let G_{glob} be as defined in Corollary 5.8(d). Thus $G_{glob}[A'] = Q_{1,A} \cup \dots \cup Q_{k,A}$. On the other hand, Lemma 5.9(ii) implies that $G^*[A'] = G_{glob}[A']$. Hence,

$$G^*[A'] = G_{glob}[A'] = Q_{1,A} \cup \dots \cup Q_{k,A}. \tag{5.18}$$

Similarly, $G^*[B'] = G_{glob}[B'] = Q_{1,B} \cup \dots \cup Q_{k,B}$. Moreover, $G_{glob} = G^*[A'] \cup G^*[B']$. Consider any vertex $x \in A_0$. Let $d_{glob}(x)$ denote the degree of x in $Q_{1,A} \cup \dots \cup Q_{k,A}$.

So $d_{glob}(x) = d_{G^*}(x, A')$ by (5.18). Let

$$d_{loc}(x) := d_G(x, A') - d_{glob}(x) \tag{5.19}$$
$$= d_G(x, A') - d_{G^*}(x, A') = d_{G(\mathfrak{J})}(x, A'). \tag{5.20}$$

Then

$$d_{loc}(x) + d_G(x, B') + d_{glob}(x) \overset{(5.19)}{=} d_G(x) = D, \tag{5.21}$$

where the final equality follows from (FR2). Recall that $\mathfrak{J}$ consists of $K^4 t_K$ edge-disjoint balanced exceptional systems. Since x has two neighbours in each of these balanced exceptional systems, the degree of x in $G(\mathfrak{J})$ is $2K^4 t_K = D - 2k$. Altogether this implies that

$$\begin{aligned} d_{G^*}(x, B') &= d_G(x, B') - d_{G(\mathfrak{J})}(x, B') = d_G(x, B') - (d_{G(\mathfrak{J})}(x) - d_{G(\mathfrak{J})}(x, A')) \\ &\overset{(5.20)}{=} d_G(x, B') - (D - 2k - d_{loc}(x)) \overset{(5.21)}{=} 2k - d_{glob}(x). \end{aligned} \tag{5.22}$$

Note that this is precisely the total number of edges at x which we need to add to $Q_{1,A}, \dots, Q_{k,A}$ in order to obtain $Q'_{1,A}, \dots, Q'_{k,A}$ as in Claim 1.

We can now construct the path systems $Q'_{i,A}$. For each $x \in A_0$, let $n_i(x) = 2 - d_{Q_{i,A}}(x)$. So $0 \le n_i(x) \le 2$ for all $i \le k$. Recall that $a := |A_0|$ and consider an ordering $x_1, \dots, x_a$ of the vertices in A_0. Let $G^*_j := G^*[\{x_1, \dots, x_j\}, B]$. Assume that for some $0 \le j < a$, we have already found a decomposition of G^*_j into edge-disjoint path systems $Q_{1,j}, \dots, Q_{k,j}$ satisfying the following properties (for all $i \le k$):

- (i′) no vertex is an endvertex of both an edge in $Q_{i,j}$ and an edge in $Q_{i,B}$;
- (ii′) $x_{j'}$ has degree $n_i(x_{j'})$ in $Q_{i,j}$ for all $j' \le j$ and all other vertices have degree at most one in $Q_{i,j}$.

We call this assertion $\mathcal{A}_j$. We will show that $\mathcal{A}_{j+1}$ holds (i.e. the above assertion also holds with j replaced by $j+1$). This in turn implies Claim 1 if we let $Q'_{i,A} := Q_{i,a} \cup Q_{i,A}$ for all $i \le k$.

To prove $\mathcal{A}_{j+1}$, consider the following bipartite auxiliary graph H_{j+1}. The vertex classes of H_{j+1} are $N_{j+1} := N_{G^*}(x_{j+1}) \cap B$ and Z_{j+1}, where Z_{j+1} is a multiset whose elements are chosen from $Q_{1,B}, \dots, Q_{k,B}$. Each $Q_{i,B}$ is included exactly $n_i(x_{j+1})$ times in Z_{j+1}. Note that $N_{j+1} = N_{G^*}(x_{j+1}) \cap B'$ since $e(G[A_0, B_0]) = 0$ by (FR6). Altogether this implies that

$$\begin{aligned} |Z_{j+1}| &= \sum_{i=1}^{k} n_i(x_{j+1}) = 2k - \sum_{i=1}^{k} d_{Q_{i,A}}(x_{j+1}) = 2k - d_{glob}(x_{j+1}) \\ &\overset{(5.22)}{=} d_{G^*}(x_{j+1}, B') = |N_{j+1}| \ge k/2. \end{aligned} \tag{5.23}$$

The final inequality follows from (5.22) since

$$d_{glob}(x_{j+1}) \overset{(5.18)}{\le} \Delta(G_{glob}[A']) \le 3k/2$$

by Corollary 5.8(d). We include an edge in H_{j+1} between $v \in N_{j+1}$ and $Q_{i,B} \in Z_{j+1}$ if v is not an endvertex of an edge in $Q_{i,B} \cup Q_{i,j}$.

Claim 2. *H_{j+1} has a perfect matching M'_{j+1}.*

Given the perfect matching guaranteed by the claim, we construct $Q_{i,j+1}$ from $Q_{i,j}$ as follows: the edges of $Q_{i,j+1}$ incident to x_{j+1} are precisely the edges $x_{j+1}v$ where $vQ_{i,B}$ is an edge of M'_{j+1} (note that there are up to two of these). Thus Claim 2 implies that $\mathcal{A}_{j+1}$ holds. (Indeed, (i′)–(ii′) are immediate from the definition of H_{j+1}.)

To prove Claim 2, consider any vertex $v \in N_{j+1}$. Since $v \in B$, the number of path systems $Q_{i,B}$ containing an edge at v is at most $d_G(v, B')$. The number of indices i for which $Q_{i,j}$ contains an edge at v is at most $d_G(v, A_0) \leq |A_0|$. Since each path system $Q_{i,B}$ occurs at most twice in the multiset Z_{j+1}, it follows that the degree of v in H_{j+1} is at least $|Z_{j+1}| - 2d_G(v, B') - 2|A_0|$. Moreover, $d_G(v, B') \leq \varepsilon' n \leq k/16$ (say) by (FR5). Also, $|A_0| \leq \varepsilon n \leq k/16$ by (FR4). So v has degree at least $|Z_{j+1}| - k/4 \geq |Z_{j+1}|/2$ in H_{j+1}.

Now consider any path system $Q_{i,B} \in Z_{j+1}$. Recall that $e(Q_{i,B}) \leq \sqrt{\varepsilon} n \leq k/16$ (say), where the first inequality follows from Corollary 5.8(d)(iv′). Moreover, $e(Q_{i,j}) \leq 2|A_0| \leq 2\varepsilon n \leq k/8$, where the second inequality follows from (FR4). Thus the degree of $Q_{i,B}$ in H_{j+1} is at least

$$|N_{j+1}| - 2e(Q_{i,B}) - e(Q_{i,j}) \geq |N_{j+1}| - k/4 \geq |N_{j+1}|/2.$$

Altogether this implies that H_{j+1} has a perfect matching M'_{j+1}, as required.

This completes the construction of $Q'_{1,A}, \dots, Q'_{k,A}$. Next we construct $Q'_{1,B}, \dots, Q'_{k,B}$ using the same approach.

Claim 3. *$G^*[B'] \cup G^*[B_0, A]$ has a decomposition into edge-disjoint path systems $Q'_{1,B}, \dots, Q'_{k,B}$ such that*

- *$Q_{i,B} \subseteq Q'_{i,B}$ and $E(Q'_{i,B}) \setminus E(Q_{i,B})$ consists of B_0A-edges in G^* (for each $i \leq k$);*
- *$d_{Q'_{i,B}}(x) = 2$ for every $x \in B_0$ and $d_{Q'_{i,B}}(x) \leq 1$ for every $x \notin B_0$;*
- *no vertex is an endvertex of both an edge in $Q'_{i,A}$ and an edge in $Q'_{i,B}$ (for each $i \leq k$).*

The proof of Claim 3 is similar to that of Claim 1. The only difference is that when constructing $Q'_{i,B}$, we need to avoid the endvertices of all the edges in $Q'_{i,A}$ (not just the edges in $Q_{i,A}$). However, $e(Q'_{i,A} - Q_{i,A}) \leq 2|A_0|$, so this does not affect the calculations significantly.

We now take $Q_i := Q'_{i,A} \cup Q'_{i,B}$ for all $i \leq k$. Then the Q_i are pairwise edge-disjoint and

$$e(Q_i) \leq e(Q_{i,A}) + e(Q_{i,B}) + 2|A_0 \cup B_0| \leq 2\sqrt{\varepsilon} n + 2\varepsilon n \leq 3\sqrt{\varepsilon} n$$

by Corollary 5.8(d)(iv′) and (FR4). Moreover, Corollary 5.8(d)(iii′) implies that

$$e_{Q_i}(A') - e_{Q_i}(B') = e(Q_{i,A}) - e(Q_{i,B}) = a - b. \tag{5.24}$$

Thus each Q_i is a 2-balanced A_0B_0-path system. Further, $Q_1, \dots, Q_k$ form a decomposition of

$$G^*[A'] \cup G^*[A_0, B] \cup G^*[B'] \cup G^*[B_0, A] = G^* - G^*[A, B].$$

(The last equality follows since $e(G[A_0, B_0]) = 0$ by (FR6).) This completes the proof of (a).

To prove (b), note that (F, G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', D)$-pre-framework, i.e. it satisfies (WF1)–(WF5). Indeed, recall that (FR1)–(FR4) imply (WF1)–(WF4) and that (WF5) holds by assumption. So we can apply Lemma 4.10 (with Q_1 playing the role of Q) to extend Q_1 into a Hamilton cycle C_1. Moreover, Lemma 4.10(iii) implies that $C_1 \cap G$ is 2-balanced, as required. (Lemma 4.10(ii) guarantees that C_1 is edge-disjoint from $Q_2, \dots, Q_k$ and $G(\mathfrak{J})$.)

Let $G_1 := G - C_1$ and $F_1 := F - C_1$. Proposition 4.3 (with C_1 playing the role of H) implies that $(F_1, G_1, A, A_0, B, B_0)$ is an $(\varepsilon, \varepsilon', D-2)$-pre-framework. So we can now apply Lemma 4.10 to $(F_1, G_1, A, A_0, B, B_0)$ to extend Q_2 into a Hamilton cycle C_2, where $C_2 \cap G$ is also 2-balanced.

We can continue this way to find $C_3, \dots, C_k$. Indeed, suppose that we have found $C_1, \dots, C_i$ for $i < k$. Then we can still apply Lemma 4.10 since $\delta(F) - 2i \geq \delta(F) - 2k \geq n/3$. Moreover, $C_j \cap G$ is 2-balanced for all $j \leq i$, so $(C_1 \cup \dots \cup C_i) \cap G$ is $2i$-balanced. This in turn means that Proposition 4.3 (applied with $C_1 \cup \dots \cup C_i$ playing the role of H) implies that after removing $C_1, \dots, C_i$, we still have an $(\varepsilon, \varepsilon', D-2i)$-pre-framework and can find C_{i+1}. □

We can now put everything together to find a set of localized balanced exceptional systems and a set of Hamilton cycles which altogether cover all edges of G outside $G[A, B]$. The localized balanced exceptional systems will be extended to Hamilton cycles later on.

Corollary 5.11. *Let $0 < 1/n \ll \varepsilon \ll \varepsilon_0 \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll 1$. Suppose that (G, A, A_0, B, B_0) is an $(\varepsilon, \varepsilon', K, D)$-framework with $|G| = n$ and such that $D \geq n/200$ and D is even. Let $\mathcal{P} := \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a $(K, m, \varepsilon, \varepsilon_1, \varepsilon_2)$-partition for G. Suppose that $t_K := (1 - 20\varepsilon_4)D/2K^4 \in \mathbb{N}$ and let $k := 10\varepsilon_4 D$. Suppose that F is a graph on $V(G)$ such that $G \subseteq F$, $\delta(F) \geq 2n/5$ and such that F satisfies (WF5) with respect to ε'. Then there are k edge-disjoint Hamilton cycles $C_1, \dots, C_k$ in F and for all $1 \leq i_1, i_2, i_3, i_4 \leq K$ there is a set $\mathcal{J}_{i_1 i_2 i_3 i_4}$ such that the following properties are satisfied:*

(i) *$\mathcal{J}_{i_1 i_2 i_3 i_4}$ consists of t_K (i_1, i_2, i_3, i_4)-BES in G with respect to $\mathcal{P}$ and with parameter ε_0 which are edge-disjoint from each other and from $C_1 \cup \dots \cup C_k$.*
(ii) *Whenever $(i_1, i_2, i_3, i_4) \neq (i'_1, i'_2, i'_3, i'_4)$, $J \in \mathcal{J}_{i_1 i_2 i_3 i_4}$ and $J' \in \mathcal{J}_{i'_1 i'_2 i'_3 i'_4}$, then J and J' are edge-disjoint.*
(iii) *Given any $i \leq k$ and $v \in A_0 \cup B_0$, the two edges incident to v in C_i lie in G.*
(iv) *Let $G^\diamond$ be the subgraph of G obtained by deleting the edges of all the C_i and all the balanced exceptional systems in $\mathcal{J}_{i_1 i_2 i_3 i_4}$ (for all $1 \leq i_1, i_2, i_3, i_4 \leq K$). Then $G^\diamond$ is bipartite with vertex classes A', B' and $V_0 = A_0 \cup B_0$ is an isolated set in $G^\diamond$.*

Proof. This follows immediately from Lemmas 5.9 and 5.10(b). Indeed, clearly (i)–(iii) are satisfied. To check (iv), note that $G^\diamond$ is obtained from the graph G^* defined in Lemma 5.10 by deleting all the edges of the Hamilton cycles C_i. But Lemma 5.10

implies that the C_i together cover all the edges in $G^* - G^*[A,B]$. Thus this implies that $G^\diamond$ is bipartite with vertex classes A', B' and V_0 is an isolated set in $G^\diamond$. □

6. Special factors and balanced exceptional factors

As discussed in the proof sketch, the proof of Theorem 1.5 proceeds as follows. First we find an approximate decomposition of the given graph G and finally we find a decomposition of the (sparse) leftover from the approximate decomposition (with the aid of a 'robustly decomposable' graph we removed earlier). Both the approximate decomposition as well as the actual decomposition steps assume that we work with a bipartite graph on $A \cup B$ (with $|A| = |B|$). So in both steps, we would need $A_0 \cup B_0$ to be empty, which we clearly cannot assume. On the other hand, in both steps, one can specify 'balanced exceptional path systems' (BEPS) in G with the following crucial property: one can replace each BEPS with a path system BEPS* so that

- (α_1) BEPS* is bipartite with vertex classes A and B;
- (α_2) a Hamilton cycle C^* in $G^* := G[A,B] + \text{BEPS}^*$ which contains BEPS* corresponds to a Hamilton cycle C in G which contains BEPS (see Section 6.1).

Each BEPS will contain one of the balanced exceptional sequences BES constructed in Section 5. BEPS* will then be obtained by replacing the edges in BES by suitable 'fictive' edges (i.e. which are not necessarily contained in G).

So, roughly speaking, this allows us to work with G^* rather than G in the two steps. A convenient way of specifying and handling these balanced exceptional path systems is to combine them into 'balanced exceptional factors' BF (see Section 6.3 for the definition).

One complication is that the 'robust decomposition lemma' (Lemma 7.4) we use from [15] deals with digraphs rather than undirected graphs. So to be able to apply it, we need a suitable orientation of the edges of G and so we will actually consider directed path systems $\text{BEPS}^*_{\text{dir}}$ instead of BEPS* above (whereas the path systems BEPS are undirected).

The formulation of the robust decomposition lemma is quite general and rather than guaranteeing (α_2) directly, it assumes the existence of certain directed 'special paths systems' SPS which are combined into 'special factors' SF. These are introduced in Section 6.2. Each of the Hamilton cycles produced by the lemma then contains exactly one of these special path systems. So to apply the lemma, it suffices to check separately that each $\text{BEPS}^*_{\text{dir}}$ satisfies the conditions required of a special path system and that it also satisfies (α_2).

6.1. Constructing the graphs J^* from the balanced exceptional systems J.

Suppose that J is a balanced exceptional system in a graph G with respect to a (K, m, ε_0)-partition $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ of $V(G)$. We will now use J to define an auxiliary matching J^*. Every edge of J^* will have one endvertex in A and its other endvertex in B. We will regard J^* as being edge-disjoint from the original graph G. So even if both J^* and G have an edge between the same pair of endvertices, we will regard these as different edges. The edges of such a J^*

will be called *fictive edges*. Proposition 6.1(ii) below shows that a Hamilton cycle in $G[A \cup B] + J^*$ containing all edges of J^* in a suitable order will correspond to a Hamilton cycle in G which contains J. So when finding our Hamilton cycles, this property will enable us to ignore all the vertices in $V_0 = A_0 \cup B_0$ and to consider a bipartite (multi-)graph between A and B instead.

We construct J^* in two steps. First we will construct a matching J^*_{AB} on $A \cup B$ and then J^*. Since each maximal path in J has endpoints in $A \cup B$ and internal vertices in V_0 by (BES1), a balanced exceptional system J naturally induces a matching J^*_{AB} on $A \cup B$. More precisely, if $P_1, \dots, P_{\ell'}$ are the non-trivial paths in J and x_i, y_i are the endpoints of P_i, then we define $J^*_{AB} := \{x_i y_i : i \leq \ell'\}$. Thus J^*_{AB} is a matching by (BES1) and $e(J^*_{AB}) \leq e(J)$. Moreover, J^*_{AB} and $E(J)$ cover exactly the same vertices in A. Similarly, they cover exactly the same vertices in B. So (BES3) implies that $e(J^*_{AB}[A]) = e(J^*_{AB}[B])$. We can write $E(J^*_{AB}[A]) = \{x_1x_2, \dots, x_{2s-1}x_{2s}\}$, $E(J^*_{AB}[B]) = \{y_1y_2, \dots, y_{2s-1}y_{2s}\}$ and $E(J^*_{AB}[A, B]) = \{x_{2s+1}y_{2s+1}, \dots, x_{s'}y_{s'}\}$, where $x_i \in A$ and $y_i \in B$. Define $J^* := \{x_i y_i : 1 \leq i \leq s'\}$. Note that $e(J^*) = e(J^*_{AB}) \leq e(J)$. All edges of J^* are called *fictive edges*.

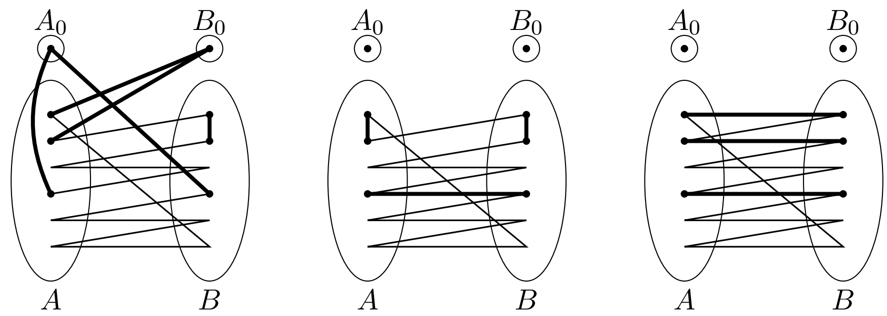
As mentioned before, we regard J^* as being edge-disjoint from the original graph G. Suppose that P is an orientation of a subpath of (the multigraph) $G[A \cup B] + J^*$. We say that P is *consistent with* J^* if P contains all the edges of J^* and P traverses the vertices $x_1, y_1, x_2, \dots, y_{s'-1}, x_{s'}, y_{s'}$ in this order. (This ordering will be crucial for the vertices $x_1, y_1, \dots, x_{2s}, y_{2s}$, but it is also convenient to have an ordering involving all vertices of J^*.) Similarly, we say that a cycle D in $G[A \cup B] + J^*$ is *consistent with* J^* if D contains all the edges of J^* and there exists some orientation of D which traverses the vertices $x_1, y_1, x_2, \dots, y_{s'-1}, x_{s'}, y_{s'}$ in this order.

The next result shows that if J is a balanced exceptional system and C is a Hamilton cycle on $A \cup B$ which is consistent with J^*, then the graph obtained from C by replacing J^* with J is a Hamilton cycle on $V(G)$ which contains J, see Figure 2. When choosing our Hamilton cycles, this property will enable us ignore all the vertices in V_0 and edges in A and B and to consider the (almost complete) bipartite graph with vertex classes A and B instead.

Proposition 6.1. *Let $\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$ be a (K, m, ε)-partition of a vertex set V. Let G be a graph on V and let J be a balanced exceptional system with respect to $\mathcal{P}$.*

- (i) *Assume that P is an orientation of a subpath of $G[A \cup B] + J^*$ such that P is consistent with J^*. Then the graph obtained from $P - J^* + J$ by ignoring the orientations of the edges is a path on $V(P) \cup V_0$ whose endvertices are the same as those of P.*
- (ii) *If $J \subseteq G$ and D is a Hamilton cycle of $G[A \cup B] + J^*$ which is consistent with J^*, then $D - J^* + J$ is a Hamilton cycle of G.*

Proof. We first prove (i). Let $s := e(J^*_{AB}[A]) = e(J^*_{AB}[B])$ and $J^\diamond := \{x_1y_1, \dots, x_{2s}y_{2s}\}$ (where the x_i and y_i are as in the definition of J^*). So $J^* := J^\diamond \cup \{x_{2s+1}y_{2s+1}, \dots, x_{s'}y_{s'}\}$, where $s' := e(J^*)$. Let P^c denote the path obtained from $P = z_1 \dots z_2$ by reversing its direction. (So $P^c = z_2 \dots z_1$ traverses the vertices $y_{s'}, x_{s'}, y_{2s'-1}, \dots, x_2, y_1, x_1$ in



(A) J (B) J^*_{AB} (C) J^*

FIGURE 2. The thick lines illustrate the edges of J, J^*_{AB} and J^* respectively.

this order.) First note

$$P' := z_1 P x_1 x_2 P^c y_1 y_2 P x_3 x_4 P^c y_3 y_4 \dots x_{2s-1} x_{2s} P^c y_{2s-1} y_{2s} P z_2$$

is a path on $V(P)$. Moreover, the underlying undirected graph of P' is precisely

$$P - J^\diamond + (J^*_{AB}[A] \cup J^*_{AB}[B]) = P - J^* + J^*_{AB}.$$

In particular, P' contains J^*_{AB}. Now recall that if $w_1 w_2$ is an edge in J^*_{AB}, then the vertices w_1 and w_2 are the endpoints of some path P^* in J (where the internal vertices on P^* lie in V_0). Clearly, $P' - w_1 w_2 + P^*$ is also a path. Repeating this step for every edge $w_1 w_2$ of J^*_{AB} gives a path P'' on $V(P) \cup V_0$. Moreover, $P'' = P - J^* + J$. This completes the proof of (i).

(ii) now follows immediately from (i). □

6.2. **Special path systems and special factors.** As mentioned earlier, in order to apply Lemma 7.4, we first need to prove the existence of certain 'special path systems'. These are defined below.

Suppose that

$$\mathcal{P} = \{A_0, A_1, \dots, A_K, B_0, B_1, \dots, B_K\}$$

is a (K, m, ε_0)-partition of a vertex set V and $L, m/L \in \mathbb{N}$. We say that $(\mathcal{P}, \mathcal{P}')$ is a (K, L, m, ε_0)*-partition of* V if $\mathcal{P}'$ is obtained from $\mathcal{P}$ by partitioning A_i into L sets $A_{i,1}, \dots, A_{i,L}$ of size m/L for all $1 \le i \le K$ and partitioning B_i into L sets $B_{i,1}, \dots, B_{i,L}$ of size m/L for all $1 \le i \le K$. (So $\mathcal{P}'$ consists of the exceptional sets A_0, B_0, the KL clusters $A_{i,j}$ and the KL clusters $B_{i,j}$.) Unless stated otherwise, whenever considering a (K, L, m, ε_0)-partition $(\mathcal{P}, \mathcal{P}')$ of a vertex set V we use the above notation to denote the elements of $\mathcal{P}$ and $\mathcal{P}'$.

Let $(\mathcal{P}, \mathcal{P}')$ be a (K, L, m, ε_0)-partition of V. Consider a spanning cycle $C = A_1 B_1 \dots A_K B_K$ on the clusters of $\mathcal{P}$. Given an integer f dividing K, the *canonical interval partition* $\mathcal{I}$ of C into f intervals consists of the intervals

$$A_{(i-1)K/f+1} B_{(i-1)K/f+1} A_{(i-1)K/f+2} \dots B_{iK/f} A_{iK/f+1}$$

for all $i \le f$. (Here $A_{K+1} := A_1$.)

Suppose that G is a digraph on $V \setminus V_0$ and $h \le L$. Let $I = A_j B_j A_{j+1} \dots A_{j'}$ be an interval in $\mathcal{I}$. A *special path system SPS of style h in G spanning the interval I* consists of precisely m/L (non-trivial) vertex-disjoint directed paths $P_1, \dots, P_{m/L}$ such that the following conditions hold:

(SPS1) Every P_s has its initial vertex in $A_{j,h}$ and its final vertex in $A_{j',h}$.

(SPS2) SPS contains a matching $\mathrm{Fict}(SPS)$ such that all the edges in $\mathrm{Fict}(SPS)$ avoid the endclusters A_j and $A_{j'}$ of I and such that $E(P_s) \setminus \mathrm{Fict}(SPS) \subseteq E(G)$.

(SPS3) The vertex set of SPS is $A_{j,h} \cup B_{j,h} \cup A_{j+1,h} \cup \dots \cup B_{j'-1,h} \cup A_{j',h}$.

The edges in $\mathrm{Fict}(SPS)$ are called *fictive edges of SPS*.

Let $\mathcal{I} = \{I_1, \dots, I_f\}$ be the canonical interval partition of C into f intervals. A *special factor SF with parameters (L, f) in G (with respect to C, $\mathcal{P}'$)* is a 1-regular digraph on $V \setminus V_0$ which is the union of Lf digraphs $SPS_{j,h}$ (one for all $j \le f$ and $h \le L$) such that each $SPS_{j,h}$ is a special path system of style h in G which spans I_j. We write $\mathrm{Fict}(SF)$ for the union of the sets $\mathrm{Fict}(SPS_{j,h})$ over all $j \le f$ and $h \le L$ and call the edges in $\mathrm{Fict}(SF)$ *fictive edges of SF*.

We will always view fictive edges as being distinct from each other and from the edges in other digraphs. So if we say that special factors $SF_1, \dots, SF_r$ are pairwise edge-disjoint from each other and from some digraph Q on $V \setminus V_0$, then this means that Q and all the $SF_i - \mathrm{Fict}(SF_i)$ are pairwise edge-disjoint, but for example there could be an edge from x to y in Q as well as in $\mathrm{Fict}(SF_i)$ for several indices $i \le r$. But these are the only instances of multiedges that we allow, i.e. if there is more than one edge from x to y, then all but at most one of these edges are fictive edges.

6.3. **Balanced exceptional path systems and balanced exceptional factors.** We now define balanced exceptional path systems BEPS. It will turn out that they (or rather their bipartite directed versions $\mathrm{BEPS}^*_{\mathrm{dir}}$ involving fictive edges) will satisfy the conditions of the special path systems defined above. Moreover, (bipartite) Hamilton cycles containing $\mathrm{BEPS}^*_{\mathrm{dir}}$ correspond to Hamilton cycles in the 'original' graph G (see Proposition 6.2).

Let $(\mathcal{P}, \mathcal{P}')$ be a (K, L, m, ε_0)-partition of a vertex set V. Suppose that $K/f \in \mathbb{N}$ and $h \le L$. Consider a spanning cycle $C = A_1 B_1 \dots A_K B_K$ on the clusters of $\mathcal{P}$. Let $\mathcal{I}$ be the canonical interval partition of C into f intervals of equal size. Suppose that G is an oriented bipartite graph with vertex classes A and B. Suppose that $I = A_j B_j \dots A_{j'}$ is an interval in $\mathcal{I}$. A *balanced exceptional path system $BEPS$ of style h for G spanning I* consists of precisely m/L (non-trivial) vertex-disjoint undirected paths $P_1, \dots, P_{m/L}$ such that the following conditions hold:

(BEPS1) Every P_s has one endvertex in $A_{j,h}$ and its other endvertex in $A_{j',h}$.

(BEPS2) $J := BEPS - BEPS[A, B]$ is a balanced exceptional system with respect to $\mathcal{P}$ such that P_1 contains all edges of J and so that the edge set of J is disjoint from $A_{j,h}$ and $A_{j',h}$. Let $P_{1,\mathrm{dir}}$ be the path obtained by orienting P_1 towards its endvertex in $A_{j',h}$ and let J_{dir} be the orientation of J obtained in this way. Moreover, let J^*_{dir} be obtained from J^* by orienting every edge in

J^* towards its endvertex in B. Then $P^*_{1,\rm dir} := P_{1,\rm dir} - J_{\rm dir} + J^*_{\rm dir}$ is a directed path from $A_{j,h}$ to $A_{j',h}$ which is consistent with J^*.

(BEPS3) The vertex set of $BEPS$ is $V_0 \cup A_{j,h} \cup B_{j,h} \cup A_{j+1,h} \cup \dots \cup B_{j'-1,h} \cup A_{j',h}$.

(BEPS4) For each $2 \le s \le m/L$, define $P_{s,\rm dir}$ similarly as $P_{1,\rm dir}$. Then $E(P_{s,\rm dir}) \setminus E(J_{\rm dir}) \subseteq E(G)$ for every $1 \le s \le m/L$.

Let $BEPS^*_{\rm dir}$ be the path system consisting of $P^*_{1,\rm dir}, P_{2,\rm dir}, \dots, P_{m/L,\rm dir}$. Then $BEPS^*_{\rm dir}$ is a special path system of style h in G which spans the interval I and such that $\mathrm{Fict}(BEPS^*_{\rm dir}) = J^*_{\rm dir}$.

Let $\mathcal{I} = \{I_1, \dots, I_f\}$ be the canonical interval partition of C into f intervals. A *balanced exceptional factor BF with parameters (L, f) for G (with respect to C, $\mathcal{P}'$)* is the union of Lf undirected graphs $BEPS_{j,h}$ (one for all $j \le f$ and $h \le L$) such that each $BEPS_{j,h}$ is a balanced exceptional path system of style h for G which spans I_j. We write $BF^*_{\rm dir}$ for the union of $BEPS^*_{j,h,\rm dir}$ over all $j \le f$ and $h \le L$. Note that $BF^*_{\rm dir}$ is a special factor with parameters (L, f) in G (with respect to C, $\mathcal{P}'$) such that $\mathrm{Fict}(BF^*_{\rm dir})$ is the union of $J^*_{j,h,\rm dir}$ over all $j \le f$ and $h \le L$, where $J_{j,h} = BEPS_{j,h} - BEPS_{j,h}[A, B]$ is the balanced exceptional system contained in $BEPS_{j,h}$ (see condition (BEPS2)). In particular, $BF^*_{\rm dir}$ is a 1-regular digraph on $V \setminus V_0$ while BF is an undirected graph on V with

$$d_{BF}(v) = 2 \ \text{ for all } v \in V \setminus V_0 \quad \text{and} \quad d_{BF}(v) = 2Lf \ \text{ for all } v \in V_0. \tag{6.1}$$

Given a balanced exceptional path system $BEPS$, let J be as in (BEPS2) and let $BEPS^* := BEPS - J + J^*$. So $BEPS^*$ consists of $P_1^* := P_1 - J + J^*$ as well as $P_2, \dots, P_{m/L}$. The following is an immediate consequence of (BEPS2) and Proposition 6.1.

Proposition 6.2. *Let $(\mathcal{P}, \mathcal{P}')$ be a (K, L, m, ε_0)-partition of a vertex set V. Suppose that G is a graph on $V \setminus V_0$, that $G_{\rm dir}$ is an orientation of $G[A, B]$ and that $BEPS$ is a balanced exceptional path system for $G_{\rm dir}$. Let J be as in (BEPS2). Let C be a Hamilton cycle of $G + J^*$ which contains $BEPS^*$. Then $C - BEPS^* + BEPS$ is a Hamilton cycle of $G \cup J$.*

Proof. Note that $C - BEPS^* + BEPS = C - J^* + J$. Moreover, (BEPS2) implies that C contains all edges of J^* and is consistent with J^*. So the proposition follows from Proposition 6.1(ii) applied with $G \cup J$ playing the role of G. □

6.4. **Finding balanced exceptional factors in a scheme.** The following definition of a 'scheme' captures the 'non-exceptional' part of the graphs we are working with. For example, this will be the structure within which we find the edges needed to extend a balanced exceptional system into a balanced exceptional path system.

Given an oriented graph G and partitions $\mathcal{P}$ and $\mathcal{P}'$ of a vertex set V, we call $(G, \mathcal{P}, \mathcal{P}')$ a $[K, L, m, \varepsilon_0, \varepsilon]$-*scheme* if the following properties hold:

(Sch1′) $(\mathcal{P}, \mathcal{P}')$ is a (K, L, m, ε_0)-partition of V. Moreover, $V(G) = A \cup B$.

(Sch2′) Every edge of G has one endvertex in A and its other endvertex in B.

(Sch3′) $G[A_{i,j}, B_{i',j'}]$ and $G[B_{i',j'}, A_{i,j}]$ are $[\varepsilon, 1/2]$-superregular for all $i, i' \leq K$ and all $j, j' \leq L$. Further, $G[A_i, B_j]$ and $G[B_j, A_i]$ are $[\varepsilon, 1/2]$-superregular for all $i, j \leq K$.

(Sch4′) $|N^+_G(x) \cap N^-_G(y) \cap B_{i,j}| \geq (1-\varepsilon)m/5L$ for all distinct $x, y \in A$, all $i \leq K$ and all $j \leq L$. Similarly, $|N^+_G(x) \cap N^-_G(y) \cap A_{i,j}| \geq (1-\varepsilon)m/5L$ for all distinct $x, y \in B$, all $i \leq K$ and all $j \leq L$.

If $L = 1$ (and so $\mathcal{P} = \mathcal{P}'$), then (Sch1′) just says that $\mathcal{P}$ is a (K, m, ε_0)-partition of $V(G)$.

The next lemma allows us to extend a suitable balanced exceptional system into a balanced exceptional path system. Given $h \leq L$, we say that an (i_1, i_2, i_3, i_4)-BES J has *style* h *(with respect to the* (K, L, m, ε_0)*-partition* $(\mathcal{P}, \mathcal{P}')$*)* if all the edges of J have their endvertices in $V_0 \cup A_{i_1,h} \cup A_{i_2,h} \cup B_{i_3,h} \cup B_{i_4,h}$.

Lemma 6.3. *Suppose that* $K, L, n, m/L \in \mathbb{N}$*, that* $0 < 1/n \ll \varepsilon, \varepsilon_0 \ll 1$ *and* $\varepsilon_0 \ll 1/K, 1/L$*. Let* $(G, \mathcal{P}, \mathcal{P}')$ *be a* $[K, L, m, \varepsilon_0, \varepsilon]$*-scheme with* $|V(G) \cup V_0| = n$*. Consider a spanning cycle* $C = A_1B_1 \dots A_KB_K$ *on the clusters of* $\mathcal{P}$ *and let* $I = A_jB_jA_{j+1} \dots A_{j'}$ *be an interval on* C *of length at least* 10*. Let* J *be an* (i_1, i_2, i_3, i_4)*-BES of style* $h \leq L$ *with parameter* ε_0 *(with respect to* $(\mathcal{P}, \mathcal{P}')$*), for some* $i_1, i_2, i_3, i_4 \in \{j+1, \dots, j'-1\}$*. Then there exists a balanced exceptional path system of style* h *for* G *which spans the interval* I *and contains all edges in* J*.*

Proof. For each $k \leq 4$, let m_k denote the number of vertices in $A_{i_k,h} \cup B_{i_k,h}$ which are incident to edges of J. We only consider the case when i_1, i_2, i_3 and i_4 are distinct and $m_k > 0$ for each $k \leq 4$, as the other cases can be proved by similar arguments. Clearly $m_1 + \dots + m_4 \leq 2\varepsilon_0 n$ by (BES4). For every vertex $x \in A$, we define $B(x)$ to be the cluster $B_{i,h} \in \mathcal{P}'$ such that A_i contains x. Similarly, for every $y \in B$, we define $A(y)$ to be the cluster $A_{i,h} \in \mathcal{P}'$ such that B_i contains y.

Let $x_1y_1, \dots, x_{s'}y_{s'}$ be the edges of J^*, with $x_i \in A$ and $y_i \in B$ for all $i \leq s'$. (Recall that the ordering of these edges is fixed in the definition of J^*.) Thus $s' = (m_1 + \dots + m_4)/2 \leq \varepsilon_0 n$. Moreover, our assumption that $\varepsilon_0 \ll 1/K, 1/L$ implies that $\varepsilon_0 n \leq m/100L$ (say). Together with (Sch4′) this in turn ensures that for every $r \leq s'$, we can pick vertices $w_r \in B(x_r)$ and $z_r \in A(y_r)$ such that w_rx_r, y_rz_r and z_rw_{r+1} are (directed) edges in G and such that all the $4s'$ vertices x_r, y_r, w_r, z_r (for $r \leq s'$) are distinct from each other. Let P'_1 be the path $w_1x_1y_1z_1w_2x_2y_2z_2w_3 \dots y_{s'}z_{s'}$. Thus P'_1 is a directed path from B to A in $G + J^*_{\rm dir}$ which is consistent with J^*. (Here $J^*_{\rm dir}$ is obtained from J^* by orienting every edge towards B.) Note that $|V(P'_1) \cap A_{i_k,h}| = m_k = |V(P'_1) \cap B_{i_k,h}|$ for all $k \leq 4$. (This follows from our assumption that i_1, i_2, i_3 and i_4 are distinct.) Moreover, $V(P'_1) \cap (A_i \cup B_i) = \emptyset$ for all $i \notin \{i_1, i_2, i_3, i_4\}$.

Pick a vertex z' in $A_{j,h}$ so that $z'w_1$ is an edge of G. Find a path P''_1 from $z_{s'}$ to $A_{j',h}$ in G such that the vertex set of P''_1 consists of $z_{s'}$ and precisely one vertex in each $A_{i,h}$ for all $i \in \{j+1, \dots, j'\} \setminus \{i_1, i_2, i_3, i_4\}$ and one vertex in each $B_{i,h}$ for all $i \in \{j, \dots, j'-1\} \setminus \{i_1, i_2, i_3, i_4\}$ and no other vertices. (Sch4′) ensures that this can be done greedily. Define $P^*_{1,\rm dir}$ to be the concatenation of $z'w_1$, P'_1 and P''_1. Note that $P^*_{1,\rm dir}$ is a directed path from $A_{j,h}$ to $A_{j',h}$ in $G + J^*_{\rm dir}$ which is consistent with

J^*. Moreover, $V(P^*_{1,\mathrm{dir}}) \subseteq \bigcup_{i \le K} A_{i,h} \cup B_{i,h}$,

$$|V(P^*_{1,\mathrm{dir}}) \cap A_{i,h}| = \begin{cases} 1 & \text{for } i \in \{j, \dots, j'\} \setminus \{i_1, i_2, i_3, i_4\}, \\ m_k & \text{for } i = i_k \text{ and } k \le 4, \\ 0 & \text{otherwise,} \end{cases}$$

while

$$|V(P^*_{1,\mathrm{dir}}) \cap B_{i,h}| = \begin{cases} 1 & \text{for } i \in \{j, \dots, j'-1\} \setminus \{i_1, i_2, i_3, i_4\}, \\ m_k & \text{for } i = i_k \text{ and } k \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

(Sch4′) ensures that for each $k \le 4$, there exist $m_k - 1$ (directed) paths $P^k_1, \dots, P^k_{m_k-1}$ in G such that

- P^k_r is a path from $A_{j,h}$ to $A_{j',h}$ for each $r \le m_k - 1$ and $k \le 4$;
- each P^k_r contains precisely one vertex in $A_{i,h}$ for each $i \in \{j, \dots, j'\} \setminus \{i_k\}$, one vertex in $B_{i,h}$ for each $i \in \{j, \dots, j'-1\} \setminus \{i_k\}$ and no other vertices;
- $P^*_{1,\mathrm{dir}}, P^1_1, \dots, P^1_{m_1-1}, P^2_1, \dots, P^4_{m_4-1}$ are vertex-disjoint.

Let Q be the union of $P^*_{1,\mathrm{dir}}$ and all the P^k_r over all $k \le 4$ and $r \le m_k - 1$. Thus Q is a path system consisting of $m_1 + \dots + m_4 - 3$ vertex-disjoint directed paths from $A_{j,h}$ to $A_{j',h}$. Moreover, $V(Q)$ consists of precisely $m_1 + \dots + m_4 - 3 \le 2\varepsilon_0 n$ vertices in $A_{i,h}$ for every $j \le i \le j'$ and precisely $m_1 + \dots + m_4 - 3$ vertices in $B_{i,h}$ for every $j \le i < j'$. Set $A'_{i,h} := A_{i,h} \setminus V(Q)$ and $B'_{i,h} := B_{i,h} \setminus V(Q)$ for all $i \le K$. Note that, for all $j \le i \le j'$,

$$(6.2) \quad |A'_{i,h}| = \frac{m}{L} - (m_1 + \dots + m_4 - 3) \ge \frac{m}{L} - 2\varepsilon_0 n \ge \frac{m}{L} - 5\varepsilon_0 mK \ge (1 - \sqrt{\varepsilon_0})\frac{m}{L}$$

since $\varepsilon_0 \ll 1/K, 1/L$. Similarly, $|B'_{i,h}| \ge (1 - \sqrt{\varepsilon_0})m/L$ for all $j \le i < j'$. Pick a new constant ε' such that $\varepsilon, \varepsilon_0 \ll \varepsilon' \ll 1$. Then (Sch3′) and (6.2) together with Proposition 2.1 imply that $G[A'_{i,h}, B'_{i,h}]$ is still $[\varepsilon', 1/2]$-superregular and so we can find a perfect matching in $G[A'_{i,h}, B'_{i,h}]$ for all $j \le i < j'$. Similarly, we can find a perfect matching in $G[B'_{i,h}, A'_{i+1,h}]$ for all $j \le i < j'$. The union Q' of all these matchings forms $m/L - (m_1 + \dots + m_4) + 3$ vertex-disjoint directed paths.

Let P_1 be the undirected graph obtained from $P^*_{1,\mathrm{dir}} - J^*_{\mathrm{dir}} + J$ by ignoring the directions of all the edges. Proposition 6.1(i) implies that P_1 is a path on $V(P^*_{1,\mathrm{dir}}) \cup V_0$ with the same endvertices as $P^*_{1,\mathrm{dir}}$. Consider the path system obtained from $(Q \cup Q') \setminus \{P^*_{1,\mathrm{dir}}\}$ by ignoring the directions of the edges on all the paths. Let $BEPS$ be the union of this path system and P_1. Then $BEPS$ is a balanced exceptional path system for G, as required. □

The next lemma shows that we can obtain many edge-disjoint balanced exceptional factors by extending balanced exceptional systems with suitable properties.

Lemma 6.4. *Suppose that $L, f, q, n, m/L, K/f \in \mathbb{N}$, that $K/f \ge 10$, that $0 < 1/n \ll \varepsilon, \varepsilon_0 \ll 1$, that $\varepsilon_0 \ll 1/K, 1/L$ and $Lq/m \ll 1$. Let $(G, \mathcal{P}, \mathcal{P}')$ be a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme with $|V(G) \cup V_0| = n$. Consider a spanning cycle $C = A_1B_1 \dots A_KB_K$ on* *the clusters of $\mathcal{P}$. Suppose that there exists a set $\mathcal{J}$ of Lfq edge-disjoint balanced exceptional systems with parameter ε_0 such that*

- *for all $i \leq f$ and all $h \leq L$, $\mathcal{J}$ contains precisely q (i_1, i_2, i_3, i_4)-BES of style h (with respect to $(\mathcal{P}, \mathcal{P}')$) for which $i_1, i_2, i_3, i_4 \in \{(i-1)K/f+2, \dots, iK/f\}$.*

Then there exist q edge-disjoint balanced exceptional factors with parameters (L, f) for G (with respect to C, $\mathcal{P}'$) covering all edges in $\bigcup \mathcal{J}$.

Recall that the canonical interval partition $\mathcal{I}$ of C into f intervals consists of the intervals

$$A_{(i-1)K/f+1} B_{(i-1)K/f+1} A_{(i-1)K/f+2} \dots A_{iK/f+1}$$

for all $i \leq f$. So the condition on $\mathcal{J}$ ensures that for each interval $I \in \mathcal{I}$ and each $h \leq L$, the set $\mathcal{J}$ contains precisely q balanced exceptional systems of style h whose edges are only incident to vertices in V_0 and vertices belonging to clusters in the interior of I. We will use Lemma 6.3 to extend each such balanced exceptional system into a balanced exceptional path system of style h spanning I.

Proof of Lemma 6.4. Choose a new constant ε' with $\varepsilon, Lq/m \ll \varepsilon' \ll 1$. Let $\mathcal{J}_1, \dots, \mathcal{J}_q$ be a partition of $\mathcal{J}$ such that for all $j \leq q$, $h \leq L$ and $i \leq f$, the set $\mathcal{J}_j$ contains precisely one (i_1, i_2, i_3, i_4)-BES of style h with $i_1, i_2, i_3, i_4 \in \{(i-1)K/f + 2, \dots, iK/f\}$. Thus each $\mathcal{J}_j$ consists of Lf balanced exceptional systems. For each $j \leq q$ in turn, we will choose a balanced exceptional factor EF_j with parameters (L, f) for G such that BF_j and $BF_{j'}$ are edge-disjoint for all $j' < j$ and BF_j contains all edges of the balanced exceptional systems in $\mathcal{J}_j$. Assume that we have already constructed $BF_1, \dots, BF_{j-1}$. In order to construct BF_j, we will choose the Lf balanced exceptional path systems forming BF_j one by one, such that each of these balanced exceptional path systems is edge-disjoint from $BF_1, \dots, BF_{j-1}$ and contains precisely one of the balanced exceptional systems in $\mathcal{J}_j$. Suppose that we have already chosen some of these balanced exceptional path systems and that next we wish to choose a balanced exceptional path system of style h which spans the interval $I \in \mathcal{I}$ of C and contains $J \in \mathcal{J}_j$. Let G' be the oriented graph obtained from G by deleting all the edges in the balanced path systems already chosen for BF_j as well as deleting all the edges in $BF_1, \dots, BF_{j-1}$. Recall from (Sch1$'$) that $V(G) = A \cup B$. Thus $\Delta(G - G') \leq 2j < 3q$ by (6.1). Together with Proposition 2.1 this implies that $(G', \mathcal{P}, \mathcal{P}')$ is still a $[K, L, m, \varepsilon_0, \varepsilon']$-scheme. (Here we use that $\Delta(G - G') < 3q = 3Lq/m \cdot m/L$ and $\varepsilon, Lq/m \ll \varepsilon' \ll 1$.) So we can apply Lemma 6.3 with ε' playing the role of ε to obtain a balanced exceptional path system of style h for G' (and thus for G) which spans I and contains all edges of J. This completes the proof of the lemma. □

7. The robust decomposition lemma

The robust decomposition lemma (Corollary 7.5) allows us to transform an approximate Hamilton decomposition into an exact one. As discussed in Section 3, it will only be used in the proof of Theorem 1.5 (and not in the proof of Theorem 1.6).

In the next subsection, we introduce the necessary concepts. In particular, Corollary 7.5 relies on the existence of a so-called bi-universal walk. The (proof of the) robust decomposition lemma then uses edges guaranteed by this universal walk to 'balance out' edges of the graph H when constructing the Hamilton decomposition of $G^{\rm rob} + H$.

7.1. **Chord sequences and bi-universal walks.** Let R be a digraph whose vertices are $V_1, \dots, V_k$ and suppose that $C = V_1 \dots V_k$ is a Hamilton cycle of R. (Later on the vertices of R will be clusters. So we denote them by capital letters.)

A *chord sequence* $CS(V_i, V_j)$ from V_i to V_j in R is an ordered sequence of edges of the form

$$CS(V_i, V_j) = (V_{i_1-1}V_{i_2}, V_{i_2-1}V_{i_3}, \dots, V_{i_t-1}V_{i_{t+1}}),$$

where $V_{i_1} = V_i$, $V_{i_{t+1}} = V_j$ and the edge $V_{i_s-1}V_{i_{s+1}}$ belongs to R for each $s \le t$.

If $i = j$ then we consider the empty set to be a chord sequence from V_i to V_j. Without loss of generality, we may assume that $CS(V_i, V_j)$ does not contain any edges of C. (Indeed, suppose that $V_{i_s-1}V_{i_{s+1}}$ is an edge of C. Then $i_s = i_{s+1}$ and so we can obtain a chord sequence from V_i to V_j with fewer edges.) For example, if $V_{i-1}V_{i+2} \in E(R)$, then the edge $V_{i-1}V_{i+2}$ is a chord sequence from V_i to V_{i+2}.

The crucial property of chord sequences is that they satisfy a 'local balance' condition. Suppose that CS is obtained by concatenating several chord sequences

$$CS(V_{i_1}, V_{i_2}), CS(V_{i_2}, V_{i_3}), \dots, CS(V_{i_{\ell-1}}, V_{i_\ell}), CS(V_{i_\ell}, V_{i_{\ell+1}})$$

where $V_{i_1} = V_{i_{\ell+1}}$. Then for every V_i, the number of edges of CS leaving V_{i-1} equals the number of edges entering V_i. We will not use this property explicitly, but it underlies the proofs of e.g. Lemma 7.4 and appears implicitly e.g. in (BU3) below.

A closed walk U in R is a *bi-universal walk for C with parameter ℓ'* if the following conditions hold:

- (BU1) The edge set of U has a partition into $U_{\rm odd}$ and $U_{\rm even}$. For every $1 \le i \le k$ there is a chord sequence $ECS^{\rm bi}(V_i, V_{i+2})$ from V_i to V_{i+2} such that $U_{\rm even}$ contains all edges of all these chord sequences for even i (counted with multiplicities) and $U_{\rm odd}$ contains all edges of these chord sequences for odd i. All remaining edges of U lie on C.
- (BU2) Each $ECS^{\rm bi}(V_i, V_{i+2})$ consists of at most $\sqrt{\ell'}/2$ edges.
- (BU3) $U_{\rm even}$ enters every cluster V_i exactly $\ell'/2$ times and it leaves every cluster V_i exactly $\ell'/2$ times. The same assertion holds for $U_{\rm odd}$.

Note that condition (BU1) means that if an edge $V_iV_j \in E(R) \setminus E(C)$ occurs in total 5 times (say) in $ECS^{\rm bi}(V_1, V_3), \dots, ECS^{\rm bi}(V_k, V_2)$ then it occurs precisely 5 times in U. We will identify each occurrence of V_iV_j in $ECS^{\rm bi}(V_1, V_3), \dots, ECS^{\rm bi}(V_k, V_2)$ with a (different) occurrence of V_iV_j in U. Note that the edges of $ECS^{\rm bi}(V_i, V_{i+2})$ are allowed to appear in a different order within U.

Lemma 7.1. *Let R be a digraph with vertices $V_1, \dots, V_k$ where $k \ge 4$ is even. Suppose that $C = V_1 \dots V_k$ is a Hamilton cycle of R and that $V_{i-1}V_{i+2} \in E(R)$ for every $1 \le i \le k$. Let $\ell' \ge 4$ be an even integer. Let $U_{{\rm bi},\ell'}$ denote the multiset obtained from $\ell' - 1$ copies of $E(C)$ by adding $V_{i-1}V_{i+2} \in E(R)$ for every $1 \le i \le k$. Then* *the edges in $U_{\mathrm{bi},\ell'}$ can be ordered so that the resulting sequence forms a bi-universal walk for C with parameter ℓ'.*

In the remainder of the paper, we will also write $U_{\mathrm{bi},\ell'}$ for the bi-universal walk guaranteed by Lemma 7.1.

Proof. Let us first show that the edges in $U_{\mathrm{bi},\ell'}$ can be ordered so that the resulting sequence forms a closed walk in R. To see this, consider the multidigraph U obtained from $U_{\mathrm{bi},\ell'}$ by deleting one copy of $E(C)$. Then U is $(\ell'-1)$-regular and thus has a decomposition into 1-factors. We order the edges of $U_{\mathrm{bi},\ell'}$ as follows: We first traverse all cycles of the 1-factor decomposition of U which contain the cluster V_1. Next, we traverse the edge V_1V_2 of C. Next we traverse all those cycles of the 1-factor decomposition which contain V_2 and which have not been traversed so far. Next we traverse the edge V_2V_3 of C and so on until we reach V_1 again.

Recall that, for each $1 \le i \le k$, the edge $V_{i-1}V_{i+2}$ is a chord sequence from V_i to V_{i+2}. Thus we can take $ECS^{\mathrm{bi}}(V_i, V_{i+2}) := V_{i-1}V_{i+2}$. Then $U_{\mathrm{bi},\ell'}$ satisfies (BU1)–(BU3). Indeed, (BU2) is clearly satisfied. Partition one of the copies of $E(C)$ in $U_{\mathrm{bi},\ell'}$ into E_{even} and E_{odd} where $E_{\mathrm{even}} = \{V_iV_{i+1} |\ i \text{ even}\}$ and $E_{\mathrm{odd}} = \{V_iV_{i+1} |\ i \text{ odd}\}$. Note that the union of E_{even} together with all $ECS^{\mathrm{bi}}(V_i, V_{i+2})$ for even i is a 1-factor in R. Add $\ell'/2 - 1$ of the remaining copies of $E(C)$ to this 1-factor to obtain U_{even}. Define U_{odd} to be $E(U_{\mathrm{bi},\ell'}) \setminus U_{\mathrm{even}}$. By construction of U_{even} and U_{odd}, (BU1) and (BU3) are satisfied. □

7.2. **Bi-setups and the robust decomposition lemma.** The aim of this subsection is to state the robust decomposition lemma (Lemma 7.4, proved in [15]) and derive Corollary 7.5, which we shall use later on. The robust decomposition lemma guarantees the existence of a 'robustly decomposable' digraph $G^{\mathrm{rob}}_{\mathrm{dir}}$ within a 'bi-setup'. Roughly speaking, a bi-setup is a digraph G together with its 'reduced digraph' R, which contains a Hamilton cycle C and a universal walk U. In our application, $G[A, B]$ will play the role of G and R will be the complete bipartite digraph. To define a bi-setup formally, we first need to define certain 'refinements' of partitions.

Given a digraph G and a partition $\mathcal{P}$ of $V(G)$ into k clusters $V_1, \dots, V_k$ of equal size, we say that a partition $\mathcal{P}'$ of $V(G)$ is an *ℓ'-refinement of $\mathcal{P}$* if $\mathcal{P}'$ is obtained by splitting each V_i into ℓ' subclusters of equal size. (So $\mathcal{P}'$ consists of $\ell'k$ clusters.) $\mathcal{P}'$ is an *ε-uniform ℓ'-refinement* of $\mathcal{P}$ if it is an ℓ'-refinement of $\mathcal{P}$ which satisfies the following condition: Whenever x is a vertex of G, V is a cluster in $\mathcal{P}$ and $|N^+_G(x) \cap V| \ge \varepsilon|V|$ then $|N^+_G(x) \cap V'| = (1 \pm \varepsilon)|N^+_G(x) \cap V|/\ell'$ for each cluster $V' \in \mathcal{P}'$ with $V' \subseteq V$. The inneighbourhoods of the vertices of G satisfy an analogous condition. We will use the following lemma from [15].

Lemma 7.2. *Suppose that $0 < 1/m \ll 1/k, \varepsilon \ll \varepsilon', d, 1/\ell \le 1$ and that $k, \ell, m/\ell \in \mathbb{N}$. Suppose that G is a digraph and that $\mathcal{P}$ is a partition of $V(G)$ into k clusters of size m. Then there exists an ε-uniform ℓ-refinement of $\mathcal{P}$. Moreover, any ε-uniform ℓ-refinement $\mathcal{P}'$ of $\mathcal{P}$ automatically satisfies the following condition:*

- *Suppose that V, W are clusters in $\mathcal{P}$ and V', W' are clusters in $\mathcal{P}'$ with $V' \subseteq V$ and $W' \subseteq W$. If $G[V,W]$ is $[\varepsilon, d']$-superregular for some $d' \geq d$ then $G[V', W']$ is $[\varepsilon', d']$-superregular.*

We will also need the following definition from [15], which describes the structure within which the robust decomposition lemma finds the robustly decomposable graph. $(G, \mathcal{P}, \mathcal{P}', R, C, U, U')$ is called an $(\ell', k, m, \varepsilon, d)$-*bi-setup* if the following properties are satisfied:

(ST1) G and R are digraphs. $\mathcal{P}$ is a partition of $V(G)$ into k clusters of size m where k is even. The vertex set of R consists of these clusters.
(ST2) For every edge VW of R, the corresponding pair $G[V,W]$ is $(\varepsilon, \geq d)$-regular.
(ST3) $C = V_1 \dots V_k$ is a Hamilton cycle of R and for every edge V_iV_{i+1} of C the corresponding pair $G[V_i, V_{i+1}]$ is $[\varepsilon, \geq d]$-superregular.
(ST4) U is a bi-universal walk for C in R with parameter ℓ' and $\mathcal{P}'$ is an ε-uniform ℓ'-refinement of $\mathcal{P}$.
(ST5) Let $V_j^1, \dots, V_j^{\ell'}$ denote the clusters in $\mathcal{P}'$ which are contained in V_j (for each $1 \leq j \leq k$). Then U' is a closed walk on the clusters in $\mathcal{P}'$ which is obtained from U as follows: When U visits V_j for the ath time, we let U' visit the subcluster V_j^a (for all $1 \leq a \leq \ell'$).
(ST6) For every edge $V_i^j V_{i'}^{j'}$ of U' the corresponding pair $G[V_i^j, V_{i'}^{j'}]$ is $[\varepsilon, \geq d]$-superregular.

In [15], in a bi-setup, the digraph G could also contain an exceptional set, but since we are only using the definition in the case when there is no such exceptional set, we have only stated it in this special case.

Suppose that $(G, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme and that $C = A_1B_1 \dots A_KB_K$ is a spanning cycle on the clusters of $\mathcal{P}$. Let $\mathcal{P}_{\rm bi} := \{A_1, \dots, A_K, B_1, \dots, B_K\}$. Suppose that $\ell', m/\ell' \in \mathbb{N}$ with $\ell' \geq 4$. Let $\mathcal{P}''_{\rm bi}$ be an ε-uniform ℓ'-refinement of $\mathcal{P}_{\rm bi}$ (which exists by Lemma 7.2). Let $C_{\rm bi}$ be the directed cycle obtained from C in which the edge A_1B_1 is oriented towards B_1 and so on. Let $R_{\rm bi}$ be the complete bipartite digraph whose vertex classes are $\{A_1, \dots, A_K\}$ and $\{B_1, \dots, B_K\}$. Let $U_{{\rm bi},\ell'}$ be a bi-universal walk for C with parameter ℓ' as defined in Lemma 7.1. Let $U'_{{\rm bi},\ell'}$ be the closed walk obtained from $U_{{\rm bi},\ell'}$ as described in (ST5). We will call

$$(G, \mathcal{P}_{\rm bi}, \mathcal{P}''_{\rm bi}, R_{\rm bi}, C_{\rm bi}, U_{{\rm bi},\ell'}, U'_{{\rm bi},\ell'})$$

the *bi-setup associated to* $(G, \mathcal{P}, \mathcal{P}')$. The following lemma shows that it is indeed a bi-setup.

Lemma 7.3. *Suppose that $K, L, m/L, \ell', m/\ell' \in \mathbb{N}$ with $\ell' \geq 4$, $K \geq 2$ and $0 < 1/m \ll 1/K, \varepsilon \ll \varepsilon', 1/\ell'$. Suppose that $(G, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme and that $C = A_1B_1 \dots A_KB_K$ is a spanning cycle on the clusters of $\mathcal{P}$. Then*

$$(G, \mathcal{P}_{\rm bi}, \mathcal{P}''_{\rm bi}, R_{\rm bi}, C_{\rm bi}, U_{{\rm bi},\ell'}, U'_{{\rm bi},\ell'})$$

is an $(\ell', 2K, m, \varepsilon', 1/2)$-bi-setup.

Proof. Clearly, $(G, \mathcal{P}_{\rm bi}, \mathcal{P}''_{\rm bi}, R_{\rm bi}, C_{\rm bi}, U_{{\rm bi},\ell'}, U'_{{\rm bi},\ell'})$ satisfies (ST1). (Sch3$'$) implies that (ST2) and (ST3) hold. Lemma 7.1 implies (ST4). (ST5) follows from the definition of $U'_{\mathrm{bi},\ell'}$. Finally, (ST6) follows from (Sch3′) and Lemma 7.2 since $\mathcal{P}''_{\mathrm{bi}}$ is an ε-uniform ℓ'-refinement of $\mathcal{P}_{\mathrm{bi}}$. □

We now state the robust decomposition lemma from [15]. This guarantees the existence of a 'robustly decomposable' digraph $G^{\mathrm{rob}}_{\mathrm{dir}}$, whose crucial property is that $H + G^{\mathrm{rob}}_{\mathrm{dir}}$ has a Hamilton decomposition for any sparse bipartite regular digraph H which is edge-disjoint from $G^{\mathrm{rob}}_{\mathrm{dir}}$.

$G^{\mathrm{rob}}_{\mathrm{dir}}$ consists of digraphs $CA_{\mathrm{dir}}(r)$ (the 'chord absorber') and $PCA_{\mathrm{dir}}(r)$ (the 'parity extended cycle switcher') together with some special factors. $G^{\mathrm{rob}}_{\mathrm{dir}}$ is constructed in two steps: given a suitable set $\mathcal{SF}$ of special factors, the lemma first 'constructs' $CA_{\mathrm{dir}}(r)$ and then, given another suitable set $\mathcal{SF}'$ of special factors, the lemma 'constructs' $PCA_{\mathrm{dir}}(r)$. The reason for having two separate steps is that in [15], it is not clear how to construct $CA_{\mathrm{dir}}(r)$ after constructing $\mathcal{SF}'$ (rather than before), as the removal of $\mathcal{SF}'$ from the digraph under consideration affects its properties considerably.

Lemma 7.4. *Suppose that $0 < 1/m \ll 1/k \ll \varepsilon \ll 1/q \ll 1/f \ll r_1/m \ll d \ll 1/\ell', 1/g \ll 1$ where ℓ' is even and that $rk^2 \le m$. Let*

$$r_2 := 96\ell' g^2 kr, \quad r_3 := rfk/q, \quad r^\diamond := r_1 + r_2 + r - (q-1)r_3, \quad s' := rfk + 7r^\diamond$$

and suppose that $k/14, k/f, k/g, q/f, m/4\ell', fm/q, 2fk/3g(g-1) \in \mathbb{N}$. Suppose that $(G, \mathcal{P}, \mathcal{P}', R, C, U, U')$ is an $(\ell', k, m, \varepsilon, d)$-bi-setup and $C = V_1 \dots V_k$. Suppose that $\mathcal{P}^*$ is a (q/f)-refinement of $\mathcal{P}$ and that $SF_1, \dots, SF_{r_3}$ are edge-disjoint special factors with parameters $(q/f, f)$ with respect to C, $\mathcal{P}^*$ in G. Let $\mathcal{SF} := SF_1 + \dots + SF_{r_3}$. Then there exists a digraph $CA_{\mathrm{dir}}(r)$ for which the following holds:

(i) *$CA_{\mathrm{dir}}(r)$ is an $(r_1 + r_2)$-regular spanning subdigraph of G which is edge-disjoint from $\mathcal{SF}$.*
(ii) *Suppose that $SF'_1, \dots, SF'_{r^\diamond}$ are special factors with parameters $(1, 7)$ with respect to C, $\mathcal{P}$ in G which are edge-disjoint from each other and from $CA_{\mathrm{dir}}(r) + \mathcal{SF}$. Let $\mathcal{SF}' := SF'_1 + \dots + SF'_{r^\diamond}$. Then there exists a digraph $PCA_{\mathrm{dir}}(r)$ for which the following holds:*
   (a) *$PCA_{\mathrm{dir}}(r)$ is a $5r^\diamond$-regular spanning subdigraph of G which is edge-disjoint from $CA_{\mathrm{dir}}(r) + \mathcal{SF} + \mathcal{SF}'$.*
   (b) *Let $\mathcal{SPS}$ be the set consisting of all the s' special path systems contained in $\mathcal{SF} + \mathcal{SF}'$. Let V_{even} denote the union of all V_i over all even $1 \le i \le k$ and define V_{odd} similarly. Suppose that H is an r-regular bipartite digraph on $V(G)$ with vertex classes V_{even} and V_{odd} which is edge-disjoint from $G^{\mathrm{rob}}_{\mathrm{dir}} := CA_{\mathrm{dir}}(r) + PCA_{\mathrm{dir}}(r) + \mathcal{SF} + \mathcal{SF}'$. Then $H + G^{\mathrm{rob}}_{\mathrm{dir}}$ has a decomposition into s' edge-disjoint Hamilton cycles $C_1, \dots, C_{s'}$. Moreover, C_i contains one of the special path systems from $\mathcal{SPS}$, for each $i \le s'$.*

Recall from Section 6.2 that we always view fictive edges in special factors as being distinct from each other and from the edges in other graphs. So for example, saying that $CA_{\mathrm{dir}}(r)$ and $\mathcal{SF}$ are edge-disjoint in Lemma 7.4 still allows for a fictive edge

xy in $\mathcal{SF}$ to occur in $CA_{\rm dir}(r)$ as well (but $CA_{\rm dir}(r)$ will avoid all non-fictive edges in $\mathcal{SF}$).

We will use the following 'undirected' consequence of Lemma 7.4.

Corollary 7.5. *Suppose that $0 < 1/m \ll \varepsilon_0, 1/K \ll \varepsilon \ll 1/L \ll 1/f \ll r_1/m \ll 1/\ell', 1/g \ll 1$ where ℓ' is even and that $4rK^2 \le m$. Let*

$$r_2 := 192\ell' g^2 Kr, \quad r_3 := 2rK/L, \quad r^\diamond := r_1 + r_2 + r - (Lf-1)r_3, \quad s' := 2rfK + 7r^\diamond$$

and suppose that $L, K/7, K/f, K/g, m/4\ell', m/L, 4fK/3g(g-1) \in \mathbb{N}$. Suppose that $(G_{\rm dir}, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, \varepsilon]$-scheme and let G' denote the underlying undirected graph of $G_{\rm dir}$. Let $C = A_1B_1 \dots A_KB_K$ be a spanning cycle on the clusters in $\mathcal{P}$. Suppose that $BF_1, \dots, BF_{r_3}$ are edge-disjoint balanced exceptional factors with parameters (L, f) for $G_{\rm dir}$ (with respect to C, $\mathcal{P}'$). Let $\mathcal{BF} := BF_1 + \dots + BF_{r_3}$. Then there exists a graph $CA(r)$ for which the following holds:

- (i) *$CA(r)$ is a $2(r_1 + r_2)$-regular spanning subgraph of G' which is edge-disjoint from $\mathcal{BF}$.*
- (ii) *Suppose that $BF'_1, \dots, BF'_{r^\diamond}$ are balanced exceptional factors with parameters $(1, 7)$ for $G_{\rm dir}$ (with respect to C, $\mathcal{P}$) which are edge-disjoint from each other and from $CA(r) + \mathcal{BF}$. Let $\mathcal{BF}' := BF'_1 + \dots + BF'_{r^\diamond}$. Then there exists a graph $PCA(r)$ for which the following holds:*
  - (a) *$PCA(r)$ is a $10r^\diamond$-regular spanning subgraph of G' which is edge-disjoint from $CA(r) + \mathcal{BF} + \mathcal{BF}'$.*
  - (b) *Let $\mathcal{BEPS}$ be the set consisting of all the s' balanced exceptional path systems contained in $\mathcal{BF} + \mathcal{BF}'$. Suppose that H is a $2r$-regular bipartite graph on $V(G_{\rm dir})$ with vertex classes $\bigcup_{i=1}^K A_i$ and $\bigcup_{i=1}^K B_i$ which is edge-disjoint from $G^{\rm rob} := CA(r) + PCA(r) + \mathcal{BF} + \mathcal{BF}'$. Then $H + G^{\rm rob}$ has a decomposition into s' edge-disjoint Hamilton cycles $C_1, \dots, C_{s'}$. Moreover, C_i contains one of the balanced exceptional path systems from $\mathcal{BEPS}$, for each $i \le s'$.*

We remark that we write $A_1, \dots, A_K, B_1, \dots, B_K$ for the clusters in $\mathcal{P}$. Note that the vertex set of each of $\mathcal{EF}$, $\mathcal{EF}'$, $G^{\rm rob}$ includes V_0 while that of $G_{\rm dir}$, $CA(r)$, $PCA(r)$, H does not. Here $V_0 = A_0 \cup B_0$, where A_0 and B_0 are the exceptional sets of $\mathcal{P}$.

Proof. Choose new constants ε' and d such that $\varepsilon \ll \varepsilon' \ll 1/L$ and $r_1/m \ll d \ll 1/\ell', 1/g$. Consider the bi-setup $(G_{\rm dir}, \mathcal{P}_{\rm bi}, \mathcal{P}''_{\rm bi}, R_{\rm bi}, C_{\rm bi}, U_{{\rm bi},\ell'}, U'_{{\rm bi},\ell'})$ associated to $(G_{\rm dir}, \mathcal{P}, \mathcal{P}')$. By Lemma 7.3, $(G_{\rm dir}, \mathcal{P}_{\rm bi}, \mathcal{P}''_{\rm bi}, R_{\rm bi}, C_{\rm bi}, U_{{\rm bi},\ell'}, U'_{{\rm bi},\ell'})$ is an $(\ell', 2K, m, \varepsilon', 1/2)$-bi-setup and thus also an $(\ell', 2K, m, \varepsilon', d)$-bi-setup. Let $BF^*_{i,{\rm dir}}$ be as defined in Section 6.3. Recall from there that, for each $i \le r_3$, $BF^*_{i,{\rm dir}}$ is a special factor with parameters (L, f) with respect to C, $\mathcal{P}'$ in $G_{\rm dir}$ such that ${\rm Fict}(BF^*_{i,{\rm dir}})$ consists of all the edges in the J^* for all the Lf balanced exceptional systems J contained in BF_i. Thus we can apply Lemma 7.4 to $(G_{\rm dir}, \mathcal{P}_{\rm bi}, \mathcal{P}''_{\rm bi}, R_{\rm bi}, C_{\rm bi}, U_{{\rm bi},\ell'}, U'_{{\rm bi},\ell'})$ with $2K$, Lf, ε' playing the roles of k, q, ε in order to obtain a spanning subdigraph $CA_{\rm dir}(r)$ of $G_{\rm dir}$ which satisfies Lemma 7.4(i). Hence the underlying undirected graph $CA(r)$ of

$CA_{\rm dir}(r)$ satisfies Corollary 7.5(i). Indeed, to check that $CA(r)$ and $\mathcal{BF}$ are edge-disjoint, by Lemma 7.4(i) it suffices to check that $CA(r)$ avoids all edges in all the balanced exceptional systems J contained in BF_i (for all $i \le r_3$). But this follows since $E(G_{\rm dir}) \supseteq E(CA(r))$ consists only of AB-edges by (Sch2′) and since no balanced exceptional system contains an AB-edge by (BES2).

Now let $BF'_1, \dots, BF'_{r^\diamond}$ be balanced exceptional factors as described in Corollary 7.5(ii). Similarly as before, for each $i \le r^\diamond$, $(BF'_i)^*_{\rm dir}$ is a special factor with parameters $(1, 7)$ with respect to C, $\mathcal{P}$ in $G_{\rm dir}$ such that ${\rm Fict}((BF'_i)^*_{\rm dir})$ consists of all the edges in the J^* over all the 7 balanced exceptional systems J contained in BF'_i. Thus we can apply Lemma 7.4 to obtain a spanning subdigraph $PCA_{\rm dir}(r)$ of $G_{\rm dir}$ which satisfies Lemma 7.4(ii)(a) and (ii)(b). Hence the underlying undirected graph $PCA(r)$ of $PCA_{\rm dir}(r)$ satisfies Corollary 7.5(ii)(a).

It remains to check that Corollary 7.5(ii)(b) holds too. Thus let H be as described in Corollary 7.5(ii)(b). Let $H_{\rm dir}$ be an r-regular orientation of H. (To see that such an orientation exists, apply Petersen's theorem to obtain a decomposition of H into 2-factors and then orient each 2-factor to obtain a (directed) 1-factor.) Let $\mathcal{BF}^*_{\rm dir}$ be the union of the $BF^*_{i,{\rm dir}}$ over all $i \le r_3$ and let $(\mathcal{BF}')^*_{\rm dir}$ be the union of the $(BF'_i)^*_{\rm dir}$ over all $i \le r^\diamond$. Then Lemma 7.4(ii)(b) implies that $H_{\rm dir} + CA_{\rm dir}(r) + PCA_{\rm dir}(r) + \mathcal{BF}^*_{\rm dir} + (\mathcal{BF}')^*_{\rm dir}$ has a decomposition into s' edge-disjoint (directed) Hamilton cycles $C'_1, \dots, C'_{s'}$ such that each C'_i contains $BEPS^*_{i,{\rm dir}}$ for some balanced exceptional path system $BEPS_i$ from $\mathcal{BEPS}$. Let C_i be the undirected graph obtained from $C'_i - BEPS^*_{i,{\rm dir}} + BEPS_i$ by ignoring the directions of all the edges. Then Proposition 6.2 (applied with G' playing the role of G) implies that $C_1, \dots, C_{s'}$ is a decomposition of $H + G^{\rm rob} = H + CA(r) + PCA(r) + \mathcal{BF} + \mathcal{BF}'$ into edge-disjoint Hamilton cycles. □

8. Proof of Theorem 1.6

The proof of Theorem 1.6 is similar to that of Theorem 1.5 except that we do not need to apply the robust decomposition lemma in the proof of Theorem 1.6. For both results, we will need an approximate decomposition result (Lemma 8.1), which is stated below and proved as Lemma 3.2 in [4]. The lemma extends a suitable set of balanced exceptional systems into a set of edge-disjoint Hamilton cycles covering most edges of an almost complete and almost balanced bipartite graph.

Lemma 8.1. *Suppose that $0 < 1/n \ll \varepsilon_0 \ll 1/K \ll \rho \ll 1$ and $0 \le \mu \ll 1$, where $n, K \in \mathbb{N}$ and K is even. Suppose that G is a graph on n vertices and $\mathcal{P}$ is a (K, m, ε_0)-partition of $V(G)$. Furthermore, suppose that the following conditions hold:*

(a) *$d(w, B_i) = (1 - 4\mu \pm 4/K)m$ and $d(v, A_i) = (1 - 4\mu \pm 4/K)m$ for all $w \in A$, $v \in B$ and $1 \le i \le K$.*
(b) *There is a set $\mathcal{J}$ which consists of at most $(1/4 - \mu - \rho)n$ edge-disjoint balanced exceptional systems with parameter ε_0 in G.*

(c) *$\mathcal{J}$ has a partition into K^4 sets $\mathcal{J}_{i_1,i_2,i_3,i_4}$ (one for all $1 \le i_1, i_2, i_3, i_4 \le K$) such that each $\mathcal{J}_{i_1,i_2,i_3,i_4}$ consists of precisely $|\mathcal{J}|/K^4$ (i_1, i_2, i_3, i_4)-BES with respect to $\mathcal{P}$.*

(d) *Each $v \in A \cup B$ is incident with an edge in J for at most $2\varepsilon_0 n$ $J \in \mathcal{J}$.*

Then G contains $|\mathcal{J}|$ edge-disjoint Hamilton cycles such that each of these Hamilton cycles contains some $J \in \mathcal{J}$.

To prove Theorem 1.6, we find a framework via Corollary 4.13. Then we choose suitable balanced exceptional systems using Corollary 5.11. Finally, we extend these into Hamilton cycles using Lemma 8.1.

Proof of Theorem 1.6. Step 1: Choosing the constants and a framework. By making α smaller if necessary, we may assume that $\alpha \ll 1$. Define new constants such that

$$0 < 1/n_0 \ll \varepsilon_{\rm ex} \ll \varepsilon_0 \ll \varepsilon_0' \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K \ll \alpha \ll \varepsilon \ll 1,$$

where $K \in \mathbb{N}$ and K is even.

Let G, F and D be as in Theorem 1.6. Apply Corollary 4.13 with $\varepsilon_{\rm ex}$, ε_0 playing the role of ε, ε^* to find a set $\mathcal{C}_1$ of at most $\varepsilon_{\rm ex}^{1/3} n$ edge-disjoint Hamilton cycles in F so that the graph G_1 obtained from G by deleting all the edges in these Hamilton cycles forms part of an $(\varepsilon_0, \varepsilon', K, D_1)$-framework (G_1, A, A_0, B, B_0) with $D_1 \ge D - 2\varepsilon_{\rm ex}^{1/3} n$. Moreover, F satisfies (WF5) with respect to ε' and

$$|\mathcal{C}_1| = (D - D_1)/2. \tag{8.1}$$

In particular, this implies that $\delta(G_1) \ge D_1$ and that D_1 is even (since D is even). Let F_1 be the graph obtained from F by deleting all those edges lying on Hamilton cycles in $\mathcal{C}_1$. Then

$$\delta(F_1) \ge \delta(F) - 2|\mathcal{C}_1| \ge (1/2 - 3\varepsilon_{\rm ex}^{1/3})n. \tag{8.2}$$

Let

$$m := \frac{|A|}{K} = \frac{|B|}{K} \qquad \text{and} \qquad t_K := \frac{(1 - 20\varepsilon_4)D_1}{2K^4}.$$

By changing ε_4 slightly, we may assume that $t_K \in \mathbb{N}$.

Step 2: Choosing a (K, m, ε_0)-partition $\mathcal{P}$. Apply Lemma 5.2 to (G_1, A, A_0, B, B_0) with F_1, ε_0 playing the roles of F, ε in order to obtain partitions $A_1, \dots, A_K$ and $B_1, \dots, B_K$ of A and B into sets of size m such that together with A_0 and B_0 the sets A_i and B_i form a $(K, m, \varepsilon_0, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P}$ for G_1.

Note that by Lemma 5.2(ii) and since F satisfies (WF5), for all $x \in A$ and $1 \le j \le K$, we have

$$\begin{aligned} d_{F_1}(x, B_j) &\ge \frac{d_{F_1}(x, B) - \varepsilon_1 n}{K} \overset{\rm (WF5)}{\ge} \frac{d_{F_1}(x) - \varepsilon' n - |B_0| - \varepsilon_1 n}{K} \\ &\overset{(8.2)}{\ge} \frac{(1/2 - 3\varepsilon_{\rm ex}^{1/3})n - 2\varepsilon_1 n}{K} \ge (1 - 5\varepsilon_1)m. \end{aligned} \tag{8.3}$$

Similarly, $d_{F_1}(y, A_i) \ge (1 - 5\varepsilon_1)m$ for all $y \in B$ and $1 \le i \le K$.

Step 3: Choosing balanced exceptional systems for the almost decomposition. Apply Corollary 5.11 to the $(\varepsilon_0, \varepsilon', K, D_1)$-framework (G_1, A, A_0, B, B_0) with F_1, G_1, ε_0, ε'_0, D_1 playing the roles of F, G, ε, ε_0, D. Let $\mathcal{J}'$ be the union of the sets $\mathcal{J}_{i_1 i_2 i_3 i_4}$ guaranteed by Corollary 5.11. So $\mathcal{J}'$ consists of $K^4 t_K$ edge-disjoint balanced exceptional systems with parameter ε'_0 in G_1 (with respect to $\mathcal{P}$). Let $\mathcal{C}_2$ denote the set of $10\varepsilon_4 D_1$ Hamilton cycles guaranteed by Corollary 5.11. Let F_2 be the subgraph obtained from F_1 by deleting all the Hamilton cycles in $\mathcal{C}_2$. Note that

$$D_2 := D_1 - 2|\mathcal{C}_2| = (1 - 20\varepsilon_4)D_1 = 2K^4 t_K = 2|\mathcal{J}'|. \tag{8.4}$$

Step 4: Finding the remaining Hamilton cycles. Our next aim is to apply Lemma 8.1 with F_2, $\mathcal{J}'$, ε' playing the roles of G, $\mathcal{J}$, ε_0.

Clearly, condition (c) of Lemma 8.1 is satisfied. In order to see that condition (a) is satisfied, let $\mu := 1/K$ and note that for all $w \in A$ we have

$$d_{F_2}(w, B_i) \geq d_{F_1}(w, B_i) - 2|\mathcal{C}_2| \overset{(8.3)}{\geq} (1 - 5\varepsilon_1)m - 20\varepsilon_4 D_1 \geq (1 - 1/K)m.$$

Similarly $d_{F_2}(v, A_i) \geq (1 - 1/K)m$ for all $v \in B$.

To check condition (b), note that

$$|\mathcal{J}'| \overset{(8.4)}{=} \frac{D_2}{2} \leq \frac{D}{2} \leq (1/2 - \alpha)\frac{n}{2} \leq (1/4 - \mu - \alpha/3)n.$$

Thus condition (b) of Lemma 8.1 holds with $\alpha/3$ playing the role of ρ. Since the edges in $\mathcal{J}'$ lie in G_1 and (G_1, A, A_0, B, B_0) is an $(\varepsilon_0, \varepsilon', K, D_1)$-framework, (FR5) implies that each $v \in A \cup B$ is incident with an edge in J for at most $\varepsilon' n + |V_0| \leq 2\varepsilon' n$ $J \in \mathcal{J}'$. (Recall that in a balanced exceptional system there are no edges between A and B.) So condition (d) of Lemma 8.1 holds with ε' playing the role of ε_0.

So we can indeed apply Lemma 8.1 to obtain a collection $\mathcal{C}_3$ of $|\mathcal{J}'|$ edge-disjoint Hamilton cycles in F_2 which cover all edges of $\bigcup \mathcal{J}'$. Then $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is a set of edge-disjoint Hamilton cycles in F of size

$$|\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3| \overset{(8.1),(8.4)}{=} \frac{D - D_1}{2} + \frac{D_1 - D_2}{2} + \frac{D_2}{2} = \frac{D}{2},$$

as required. □

9. Proof of Theorem 1.5

As mentioned earlier, the proof of Theorem 1.5 is similar to that of Theorem 1.6 except that we will also need to apply the robust decomposition lemma. This means Steps 2–4 and Step 8 in the proof of Theorem 1.5 do not appear in the proof of Theorem 1.6. Steps 2–4 prepare the ground for the application of the robust decomposition lemma and in Step 8 we apply it to cover the leftover from the approximate decomposition step with Hamilton cycles. Steps 5–7 contain the approximate decomposition step, using Lemma 8.1.

In our proof of Theorem 1.5 it will be convenient to work with an undirected version of the schemes introduced in Section 6.4. Given a graph G and partitions $\mathcal{P}$ and $\mathcal{P}'$ of a vertex set V, we call $(G, \mathcal{P}, \mathcal{P}')$ a $(K, L, m, \varepsilon_0, \varepsilon)$-*scheme* if the following properties hold:

(Sch1) $(\mathcal{P}, \mathcal{P}')$ is a (K, L, m, ε_0)-partition of V. Moreover, $V(G) = A \cup B$.
(Sch2) Every edge of G joins some vertex in A to some vertex in B.
(Sch3) $d_G(v, A_{i,j}) \geq (1-\varepsilon)m/L$ and $d_G(w, B_{i,j}) \geq (1-\varepsilon)m/L$ for all $v \in B$, $w \in A$, $i \leq K$ and $j \leq L$.

We will also use the following proposition.

Proposition 9.1. *Suppose that $K, L, n, m/L \in \mathbb{N}$ and $0 < 1/n \ll \varepsilon, \varepsilon_0 \ll 1$. Let $(G, \mathcal{P}, \mathcal{P}')$ be a $(K, L, m, \varepsilon_0, \varepsilon)$-scheme with $|G| = n$. Then there exists an orientation $G_{\rm dir}$ of G such that $(G_{\rm dir}, \mathcal{P}, \mathcal{P}')$ is a $[K, L, m, \varepsilon_0, 2\sqrt{\varepsilon}]$-scheme.*

Proof. Randomly orient every edge in G to obtain an oriented graph $G_{\rm dir}$. (So given any edge xy in G with probability 1/2, $xy \in E(G_{\rm dir})$ and with probability 1/2, $yx \in E(G_{\rm dir})$.) (Sch1′) and (Sch2′) follow immediately from (Sch1) and (Sch2).

Note that Fact 2.2 and (Sch3) imply that $G[A_{i,j}, B_{i',j'}]$ is $[1, \sqrt{\varepsilon}]$-superregular with density at least $1 - \varepsilon$, for all $i, i' \leq K$ and $j, j' \leq L$. Using this, (Sch3′) follows easily from the large deviation bound in Proposition 2.3. (Sch4′) follows from Proposition 2.3 in a similar way. □

Proof of Theorem 1.5.
Step 1: Choosing the constants and a framework. Define new constants such that

$$(9.1) \qquad \begin{aligned} 0 < 1/n_0 \ll \varepsilon_{\rm ex} \ll \varepsilon_* \ll \varepsilon_0 \ll \varepsilon_0' \ll \varepsilon' \ll \varepsilon_1 \ll \varepsilon_2 \ll \varepsilon_3 \ll \varepsilon_4 \ll 1/K_2 \\ \ll \gamma \ll 1/K_1 \ll \varepsilon'' \ll 1/L \ll 1/f \ll \gamma_1 \ll 1/g \ll \varepsilon \ll 1, \end{aligned}$$

where $K_1, K_2, L, f, g \in \mathbb{N}$ and both K_2, g are even. Note that we can choose the constants such that

$$\frac{K_1}{28fgL}, \frac{K_2}{4gLK_1}, \frac{4fK_1}{3g(g-1)} \in \mathbb{N}.$$

Let G and D be as in Theorem 1.5. By applying Dirac's theorem to remove a suitable number of edge-disjoint Hamilton cycles if necessary, we may assume that $D \leq n/2$. Apply Corollary 4.13 with G, $\varepsilon_{\rm ex}$, ε_*, ε_0, K_2 playing the roles of F, ε, ε^*, ε', K to find a set $\mathcal{C}_1$ of at most $\varepsilon_{\rm ex}^{1/3} n$ edge-disjoint Hamilton cycles in G so that the graph G_1 obtained from G by deleting all the edges in these Hamilton cycles forms part of an $(\varepsilon_*, \varepsilon_0, K_2, D_1)$-framework (G_1, A, A_0, B, B_0), where

$$(9.2) \qquad |A| + \varepsilon_0 n \geq n/2 \geq D_1 = D - 2|\mathcal{C}_1| \geq D - 2\varepsilon_{\rm ex}^{1/3} n \geq D - \varepsilon_0 n \geq n/2 - 2\varepsilon_0 n \geq |A| - 2\varepsilon_0 n.$$

Note that G_1 is D_1-regular and that D_1 is even since D was even. Moreover, since $K_2/LK_1 \in \mathbb{N}$, (G_1, A, A_0, B, B_0) is also an $(\varepsilon_*, \varepsilon_0, K_1 L, D_1)$-framework and thus an $(\varepsilon_*, \varepsilon', K_1 L, D_1)$-framework.

Let

$$m_1 := \frac{|A|}{K_1} = \frac{|B|}{K_1}, \qquad r := \gamma m_1, \qquad r_1 := \gamma_1 m_1, \qquad r_2 := 192 g^3 K_1 r,$$
$$r_3 := \frac{2rK_1}{L}, \qquad r^\diamond := r_1 + r_2 + r - (Lf-1)r_3,$$
$$D_4 := D_1 - 2(Lfr_3 + 7r^\diamond), \qquad t_{K_1L} := \frac{(1-20\varepsilon_4)D_1}{2(K_1L)^4}.$$

Note that (FR4) implies $m_1/L \in \mathbb{N}$. Moreover,

$$r_2, r_3 \le \gamma^{1/2} m_1 \le \gamma^{1/3} r_1, \qquad r_1/2 \le r^\diamond \le 2r_1. \tag{9.3}$$

Further, by changing $\gamma, \gamma_1, \varepsilon_4$ slightly, we may assume that $r/K_2^2, r_1, t_{K_1L} \in \mathbb{N}$. Since $K_1/L \in \mathbb{N}$ this implies that $r_3 \in \mathbb{N}$. Finally, note that

$$(1+3\varepsilon_*)|A| \ge D \ge D_4 \overset{(9.3)}{\ge} D_1 - \gamma_1 n \overset{(9.2)}{\ge} |A| - 2\gamma_1 n \ge (1-5\gamma_1)|A|. \tag{9.4}$$

Step 2: Choosing a $(K_1, L, m_1, \varepsilon_0)$-partition $(\mathcal{P}_1, \mathcal{P}_1')$. We now prepare the ground for the construction of the robustly decomposable graph G^{rob}, which we will obtain via the robust decomposition lemma (Corollary 7.5) in Step 4.

Recall that (G_1, A, A_0, B, B_0) is an $(\varepsilon_*, \varepsilon', K_1L, D_1)$-framework. Apply Lemma 5.2 with G_1, D_1, K_1L, ε_* playing the roles of G, D, K, ε to obtain partitions $A_1', \dots, A_{K_1L}'$ of A and $B_1', \dots, B_{K_1L}'$ of B into sets of size m_1/L such that together with A_0 and B_0 all these sets A_i' and B_i' form a $(K_1L, m_1/L, \varepsilon_*, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P}_1'$ for G_1. Note that $(1-\varepsilon_0)n \le n - |A_0 \cup B_0| = 2K_1m_1 \le n$ by (FR4). For all $i \le K_1$ and all $h \le L$, let $A_{i,h} := A'_{(i-1)L+h}$. (So this is just a relabeling of the sets A_i'.) Define $B_{i,h}$ similarly and let $A_i := \bigcup_{h \le L} A_{i,h}$ and $B_i := \bigcup_{h \le L} B_{i,h}$. Let $\mathcal{P}_1 := \{A_0, B_0, A_1, \dots, A_{K_1}, B_1, \dots, B_{K_1}\}$ denote the corresponding $(K_1, m_1, \varepsilon_0)$-partition of $V(G)$. Thus $(\mathcal{P}_1, \mathcal{P}_1')$ is a $(K, L, m_1, \varepsilon_0)$-partition of $V(G)$, as defined in Section 6.2.

Let $G_2 := G_1[A, B]$. We claim that $(G_2, \mathcal{P}_1, \mathcal{P}'_1)$ is a $(K_1, L, m_1, \varepsilon_0, \varepsilon')$-scheme. Indeed, clearly (Sch1) and (Sch2) hold. To verify (Sch3), recall that that (G_1, A, A_0, B, B_0) is an $(\varepsilon_*, \varepsilon_0, K_1L, D_1)$-framework and so by (FR5) for all $x \in B$ we have

$$d_{G_2}(x, A) \ge d_{G_1}(x) - d_{G_1}(x, B') - |A_0| \ge D_1 - \varepsilon_0 n - |A_0| \overset{(9.2)}{\ge} |A| - 4\varepsilon_0 n$$

and similarly $d_{G_2}(y, B) \ge |B| - 4\varepsilon_0 n$ for all $y \in A$. Since $\varepsilon_0 \ll \varepsilon'/K_1L$, this implies (Sch3).

Step 3: Balanced exceptional systems for the robustly decomposable graph. In order to apply Corollary 7.5, we first need to construct suitable balanced exceptional systems. Apply Corollary 5.11 to the $(\varepsilon_*, \varepsilon', K_1L, D_1)$-framework (G_1, A, A_0, B, B_0) with G_1, K_1L, $\mathcal{P}_1'$, ε_* playing the roles of F, K, $\mathcal{P}$, ε in order to obtain a set $\mathcal{J}$ of $(K_1L)^4 t_{K_1L}$ edge-disjoint balanced exceptional systems in G_1 with parameter ε_0 such that for all $1 \le i_1', i_2', i_3', i_4' \le K_1L$ the set $\mathcal{J}$ contains precisely t_{K_1L} (i_1', i_2', i_3', i_4')-BES with respect to the partition $\mathcal{P}_1'$. (Note that F in Corollary 5.11 satisfies (WF5) since G_1 satisfies (FR5).) So $\mathcal{J}$ is the union of all the sets $\mathcal{J}_{i_1'i_2'i_3'i_4'}$ returned by Corollary 5.11. (Note that we will not use all the balanced exceptional systems in $\mathcal{J}$ and we do not need to consider the Hamilton cycles guaranteed by this result. So we do not need the full strength of Corollary 5.11 at this point.)

Our next aim is to choose two disjoint subsets $\mathcal{J}_{\rm CA}$ and $\mathcal{J}_{\rm PCA}$ of $\mathcal{J}$ with the following properties:

(a) In total $\mathcal{J}_{\rm CA}$ contains Lfr_3 balanced exceptional systems. For each $i \le f$ and each $h \le L$, $\mathcal{J}_{\rm CA}$ contains precisely r_3 (i_1, i_2, i_3, i_4)-BES of style h (with respect to the $(K, L, m_1, \varepsilon_0)$-partition $(\mathcal{P}_1, \mathcal{P}'_1)$) such that $i_1, i_2, i_3, i_4 \in \{(i-1)K_1/f + 2, \dots, iK_1/f\}$.

(b) In total $\mathcal{J}_{\rm PCA}$ contains $7r^\diamond$ balanced exceptional systems. For each $i \le 7$, $\mathcal{J}_{\rm PCA}$ contains precisely $r^\diamond$ (i_1, i_2, i_3, i_4)-BES (with respect to the partition $\mathcal{P}_1$) with $i_1, i_2, i_3, i_4 \in \{(i-1)K_1/7 + 2, \dots, iK_1/7\}$.

(Recall that we defined in Section 6.4 when an (i_1, i_2, i_3, i_4)-BES has style h with respect to a $(K, L, m_1, \varepsilon_0)$-partition $(\mathcal{P}_1, \mathcal{P}'_1)$.) To see that it is possible to choose $\mathcal{J}_{\rm CA}$ and $\mathcal{J}_{\rm PCA}$, split $\mathcal{J}$ into two sets $\mathcal{J}_1$ and $\mathcal{J}_2$ such that both $\mathcal{J}_1$ and $\mathcal{J}_2$ contain at least $t_{K_1L}/3$ (i'_1, i'_2, i'_3, i'_4)-BES with respect to $\mathcal{P}'_1$, for all $1 \le i'_1, i'_2, i'_3, i'_4 \le K_1L$. Note that there are $(K_1/f - 1)^4$ choices of 4-tuples (i_1, i_2, i_3, i_4) with $i_1, i_2, i_3, i_4 \in \{(i-1)K_1/f + 2, \dots, iK_1/f\}$. Moreover, for each such 4-tuple (i_1, i_2, i_3, i_4) and each $h \le L$ there is one 4-tuple (i'_1, i'_2, i'_3, i'_4) with $1 \le i'_1, i'_2, i'_3, i'_4 \le K_1L$ and such that any (i'_1, i'_2, i'_3, i'_4)-BES with respect to $\mathcal{P}'_1$ is an (i_1, i_2, i_3, i_4)-BES of style h with respect to $(\mathcal{P}_1, \mathcal{P}'_1)$. Together with the fact that

$$\frac{(K_1/f - 1)^4 t_{K_1L}}{3} \ge \frac{D_1}{7(Lf)^4} \ge \gamma^{1/2} n \overset{(9.3)}{\ge} r_3,$$

this implies that we can choose a set $\mathcal{J}_{\rm CA} \subseteq \mathcal{J}_1$ satisfying (a).

Similarly, there are $(K_1/7 - 1)^4$ choices of 4-tuples (i_1, i_2, i_3, i_4) with $i_1, i_2, i_3, i_4 \in \{(i-1)K_1/7 + 2, \dots, iK_1/7\}$. Moreover, for each such 4-tuple (i_1, i_2, i_3, i_4) there are L^4 distinct 4-tuples (i'_1, i'_2, i'_3, i'_4) with $1 \le i'_1, i'_2, i'_3, i'_4 \le K_1L$ and such that any (i'_1, i'_2, i'_3, i'_4)-BES with respect to $\mathcal{P}'_1$ is an (i_1, i_2, i_3, i_4)-BES with respect to $\mathcal{P}_1$. Together with the fact that

$$\frac{(K_1/7 - 1)^4 L^4 t_{K_1L}}{3} \ge \frac{D_1}{7^5} \ge \frac{n}{3 \cdot 7^5} \overset{(9.3)}{\ge} r^\diamond,$$

this implies that we can choose a set $\mathcal{J}_{\rm PCA} \subseteq \mathcal{J}_2$ satisfying (b).

Step 4: Finding the robustly decomposable graph. Recall that $(G_2, \mathcal{P}_1, \mathcal{P}'_1)$ is a $(K_1, L, m_1, \varepsilon_0, \varepsilon')$-scheme. Apply Proposition 9.1 with G_2, $\mathcal{P}_1$, $\mathcal{P}'_1$, K_1, m_1, ε' playing the roles of G, $\mathcal{P}$, $\mathcal{P}'$, K, m, ε to obtain an orientation $G_{2,\rm dir}$ of G_2 such that $(G_{2,\rm dir}, \mathcal{P}_1, \mathcal{P}'_1)$ is a $[K_1, L, m_1, \varepsilon_0, 2\sqrt{\varepsilon'}]$-scheme. Let $C = A_1B_1A_2 \dots A_{K_1}B_{K_1}$ be a spanning cycle on the clusters in $\mathcal{P}_1$.

Our next aim is to use Lemma 6.4 in order to extend the balanced exceptional systems in $\mathcal{J}_{\rm CA}$ into r_3 edge-disjoint balanced exceptional factors with parameters (L, f) for $G_{2,\rm dir}$ (with respect to C, $\mathcal{P}'_1$). For this, note that the condition on $\mathcal{J}_{CA}$ in Lemma 6.4 with r_3 playing the role of q is satisfied by (a). Moreover, $Lr_3/m_1 = 2rK_1/m_1 = 2\gamma K_1 \ll 1$. Thus we can indeed apply Lemma 6.4 to $(G_{2,\rm dir}, \mathcal{P}_1, \mathcal{P}'_1)$

with $\mathcal{J}_{CA}$, $2\sqrt{\varepsilon'}$, K_1, r_3 playing the roles of $\mathcal{J}$, ε, K, q in order to obtain r_3 edge-disjoint balanced exceptional factors $BF_1, \dots, BF_{r_3}$ with parameters (L, f) for $G_{2,\mathrm{dir}}$ (with respect to C, $\mathcal{P}'_1$) such that together these balanced exceptional factors cover all edges in $\bigcup \mathcal{J}_{\mathrm{CA}}$. Let $\mathcal{BF}_{\mathrm{CA}} := BF_1 + \dots + BF_{r_3}$.

Note that $m_1/4g, m_1/L \in \mathbb{N}$ since $m_1 = |A|/K_1$ and $|A|$ is divisible by K_2 and thus m_1 is divisible by $4gL$ (since $K_2/4gLK_1 \in \mathbb{N}$ by our assumption). Furthermore, $4rK_1^2 = 4\gamma m_1 K_1^2 \le \gamma^{1/2} m_1 \le m_1$. Thus we can apply Corollary 7.5 to the $[K_1, L, m_1, \varepsilon_0, \varepsilon'']$-scheme $(G_{2,\mathrm{dir}}, \mathcal{P}_1, \mathcal{P}'_1)$ with K_1, ε'', g playing the roles of K, ε, ℓ' to obtain a spanning subgraph $CA(r)$ of G_2 as described there. (Note that G_2 equals the graph G' defined in Corollary 7.5.) In particular, $CA(r)$ is $2(r_1+r_2)$-regular and edge-disjoint from $\mathcal{BF}_{\mathrm{CA}}$.

Let G_3 be the graph obtained from G_2 by deleting all the edges of $CA(r)+\mathcal{BF}_{\mathrm{CA}}$. Thus G_3 is obtained from G_2 by deleting at most $2(r_1 + r_2 + r_3) \le 6r_1 = 6\gamma_1 m_1$ edges at every vertex in $A \cup B = V(G_3)$. Let $G_{3,\mathrm{dir}}$ be the orientation of G_3 in which every edge is oriented in the same way as in $G_{2,\mathrm{dir}}$. Then Proposition 2.1 implies that $(G_{3,\mathrm{dir}}, \mathcal{P}_1, \mathcal{P}_1)$ is still a $[K_1, 1, m_1, \varepsilon_0, \varepsilon]$-scheme. Moreover,

$$\frac{r^\diamond}{m_1} \overset{(9.3)}{\le} \frac{2r_1}{m_1} = 2\gamma_1 \ll 1.$$

Together with (b) this ensures that we can apply Lemma 6.4 to $(G_{3,\mathrm{dir}}, \mathcal{P}_1)$ with $\mathcal{P}_1$, $\mathcal{J}_{PCA}$, K_1, 1, 7, $r^\diamond$ playing the roles of $\mathcal{P}$, $\mathcal{J}$, K, L, f, q in order to obtain $r^\diamond$ edge-disjoint balanced exceptional factors $BF'_1, \dots, BF'_{r^\diamond}$ with parameters $(1, 7)$ for $G_{3,\mathrm{dir}}$ (with respect to C, $\mathcal{P}_1$) such that together these balanced exceptional factors cover all edges in $\bigcup \mathcal{J}_{\mathrm{PCA}}$. Let $\mathcal{BF}_{\mathrm{PCA}} := BF'_1 + \dots + BF'_{r^\diamond}$.

Apply Corollary 7.5 to obtain a spanning subgraph $PCA(r)$ of G_2 as described there. In particular, $PCA(r)$ is $10r^\diamond$-regular and edge-disjoint from $CA(r)+\mathcal{BF}_{\mathrm{CA}}+\mathcal{BF}_{\mathrm{PCA}}$.

Let $G^{\mathrm{rob}} := CA(r)+PCA(r)+\mathcal{BF}_{\mathrm{CA}}+\mathcal{BF}_{\mathrm{PCA}}$. Note that by (6.1) all the vertices in $V_0 := A_0 \cup B_0$ have the same degree $r_0^{\mathrm{rob}} := 2(Lfr_3 + 7r^\diamond)$ in G^{rob}. So

$$7r_1 \overset{(9.3)}{\le} r_0^{\mathrm{rob}} \overset{(9.3)}{\le} 30r_1. \tag{9.5}$$

Moreover, (6.1) also implies that all the vertices in $A \cup B$ have the same degree r^{rob} in G^{rob}, where $r^{\mathrm{rob}} = 2(r_1 + r_2 + r_3 + 6r^\diamond)$. So

$$r_0^{\mathrm{rob}} - r^{\mathrm{rob}} = 2\left(Lfr_3 + r^\diamond - (r_1 + r_2 + r_3)\right) = 2(Lfr_3 + r - (Lf-1)r_3 - r_3) = 2r.$$

Step 5: Choosing a $(K_2, m_2, \varepsilon_0)$-partition $\mathcal{P}_2$. We now prepare the ground for the approximate decomposition step (i.e. to apply Lemma 8.1). For this, we need to work with a finer partition of $A \cup B$ than the previous one (this will ensure that the leftover from the approximate decomposition step is sufficiently sparse compared to G^{rob}).

Let $G_4 := G_1 - G^{\mathrm{rob}}$ (where G_1 was defined in Step 1) and note that

$$D_4 = D_1 - r_0^{\mathrm{rob}} = D_1 - r^{\mathrm{rob}} - 2r. \tag{9.6}$$

So

$$d_{G_4}(x) = D_4 + 2r \text{ for all } x \in A \cup B \qquad \text{and} \qquad d_{G_4}(x) = D_4 \text{ for all } x \in V_0. \tag{9.7}$$

(Note that D_4 is even since D_1 and $r_0^{\rm rob}$ are even.) So G_4 is D_4-balanced with respect to (A, A_0, B, B_0) by Proposition 4.1. Together with the fact that (G_1, A, A_0, B, B_0) is an $(\varepsilon_*, \varepsilon_0, K_2, D_1)$-framework, this implies that $(G_4, G_4, A, A_0, B, B_0)$ satisfies conditions (WF1)–(WF5) in the definition of an $(\varepsilon_*, \varepsilon_0, K_2, D_4)$-weak framework. However, some vertices in $A_0 \cup B_0$ might violate condition (WF6). (But every vertex in $A \cup B$ will still satisfy (WF6) with room to spare.) So we need to modify the partition of $V_0 = A_0 \cup B_0$ to obtain a new weak framework.

Consider a partition A_0^*, B_0^* of $A_0 \cup B_0$ which maximizes the number of edges in G_4 between $A_0^* \cup A$ and $B_0^* \cup B$. Then $d_{G_4}(v, A_0^* \cup A) \leq d_{G_4}(v)/2$ for all $v \in A_0^*$ since otherwise $A_0^* \setminus \{v\}, B_0^* \cup \{v\}$ would be a better partition of $A_0 \cup B_0$. Similarly $d_{G_4}(v, B_0^* \cup B) \leq d_{G_4}(v)/2$ for all $v \in B_0^*$. Thus (WF6) holds in G_4 (with respect to the partition $A \cup A_0^*$ and $B \cup B_0^*$). Moreover, Proposition 4.2 implies that G_4 is still D_4-balanced with respect to (A, A_0^*, B, B_0^*). Furthermore, with (FR3) and (FR4) applied to G_1, we obtain $e_{G_4}(A \cup A_0^*) \leq e_{G_1}(A \cup A_0) + |A_0^*||A \cup A_0^*| \leq \varepsilon_0 n^2$ and similarly $e_{G_4}(B \cup B_0^*) \leq \varepsilon_0 n^2$. Finally, every vertex in $A \cup B$ has internal degree at most $\varepsilon_0 n + |A_0 \cup B_0| \leq 2\varepsilon_0 n$ in G_4 (with respect to the partition $A \cup A_0^*$ and $B \cup B_0^*$). Altogether this implies that $(G_4, G_4, A, A_0^*, B, B_0^*)$ is an $(\varepsilon_0, 2\varepsilon_0, K_2, D_4)$-weak framework and thus also an $(\varepsilon_0, \varepsilon', K_2, D_4)$-weak framework.

Without loss of generality we may assume that $|A_0^*| \geq |B_0^*|$. Apply Lemma 4.12 to the $(\varepsilon_0, \varepsilon', K_2, D_4)$-weak framework $(G_4, G_4, A, A_0^*, B, B_0^*)$ to find a set $\mathcal{C}_2$ of $|\mathcal{C}_2| \leq \varepsilon_0 n$ edge-disjoint Hamilton cycles in G_4 so that the graph G_5 obtained from G_4 by deleting all the edges of these Hamilton cycles forms part of an $(\varepsilon_0, \varepsilon', K_2, D_5)$-framework $(G_5, A, A_0^*, B, B_0^*)$, where

$$D_5 = D_4 - 2|\mathcal{C}_2| \geq D_4 - 2\varepsilon_0 n. \tag{9.8}$$

Since D_4 is even, D_5 is even. Further,

(9.9)

$$d_{G_5}(x) \overset{(9.7)}{=} D_5 + 2r \text{ for all } x \in A \cup B \qquad \text{and} \qquad d_{G_5}(x) \overset{(9.7)}{=} D_5 \text{ for all } x \in A_0^* \cup B_0^*.$$

Choose an additional constant ε_4' such that $\varepsilon_3 \ll \varepsilon_4' \ll 1/K_2$ and so that

$$t_{K_2} := \frac{(1 - 20\varepsilon_4')D_5}{2K_2^4} \in \mathbb{N}.$$

Now apply Lemma 5.2 to $(G_5, A, A_0^*, B, B_0^*)$ with D_5, K_2, ε_0 playing the roles of D, K, ε in order to obtain partitions $A_1, \dots, A_{K_2}$ and $B_1, \dots, B_{K_2}$ of A and B into sets of size

$$m_2 := |A|/K_2 \tag{9.10}$$

such that together with A_0^* and B_0^* the sets A_i and B_i form a $(K_2, m_2, \varepsilon_0, \varepsilon_1, \varepsilon_2)$-partition $\mathcal{P}_2$ for G_5. (Note that the previous partition of A and B plays no role in the subsequent argument, so denoting the clusters in $\mathcal{P}_2$ by A_i and B_i again will cause no notational conflicts.)

Step 6: Balanced exceptional systems for the approximate decomposition. In order to apply Lemma 8.1, we first need to construct suitable balanced exceptional systems. Apply Corollary 5.11 to the $(\varepsilon_0, \varepsilon', K_2, D_5)$-framework $(G_5, A, A_0^*, B, B_0^*)$ with G_5, ε_0, ε'_0, ε'_4, K_2, D_5, $\mathcal{P}_2$ playing the roles of F, ε, ε_0, ε_4, K, D, $\mathcal{P}$. (Note that since we are letting G_5 play the role of F, condition (WF5) in the corollary immediately follows from (FR5).) Let $\mathcal{J}'$ be the union of the sets $\mathcal{J}_{i_1 i_2 i_3 i_4}$ guaranteed by Corollary 5.11. So $\mathcal{J}'$ consists of $K_2^4 t_{K_2}$ edge-disjoint balanced exceptional systems with parameter ε'_0 in G_5 (with respect to $\mathcal{P}_2$). Let $\mathcal{C}_3$ denote the set of Hamilton cycles guaranteed by Corollary 5.11. So $|\mathcal{C}_3| = 10\varepsilon'_4 D_5$.

Let G_6 be the subgraph obtained from G_5 by deleting all those edges lying in the Hamilton cycles from $\mathcal{C}_3$. Set $D_6 := D_5 - 2|\mathcal{C}_3|$. So

(9.11)
$$d_{G_6}(x) \overset{(9.9)}{=} D_6 + 2r \text{ for all } x \in A \cup B \qquad \text{and} \qquad d_{G_6}(x) \overset{(9.9)}{=} D_6 \text{ for all } x \in V_0.$$

(Note that $V_0 = A_0 \cup B_0 = A_0^* \cup B_0^*$.) Let G'_6 denote the subgraph of G_6 obtained by deleting all those edges lying in the balanced exceptional systems from $\mathcal{J}'$. Thus $G'_6 = G^\diamond$, where $G^\diamond$ is as defined in Corollary 5.11(iv). In particular, V_0 is an isolated set in G'_6 and G'_6 is bipartite with vertex classes $A \cup A_0^*$ and $B \cup B_0^*$ (and thus also bipartite with vertex classes $A' = A \cup A_0$ and $B' = B \cup B_0$).

Consider any vertex $v \in V_0$. Then v has degree D_5 in G_5, degree two in each Hamilton cycle from $\mathcal{C}_3$, degree two in each balanced exceptional system from $\mathcal{J}'$ and degree zero in G'_6. Thus

$$D_6 + 2|\mathcal{C}_3| = D_5 \overset{(9.9)}{=} d_{G_5}(v) = 2|\mathcal{C}_3| + 2|\mathcal{J}'| + d_{G'_6}(v) = 2|\mathcal{C}_3| + 2|\mathcal{J}'|$$

and so

$$D_6 = 2|\mathcal{J}'|. \tag{9.12}$$

Step 7: Approximate Hamilton cycle decomposition. Our next aim is to apply Lemma 8.1 with G_6, $\mathcal{P}_2$, K_2, m_2, $\mathcal{J}'$, ε' playing the roles of G, $\mathcal{P}$, K, m, $\mathcal{J}$, ε_0. Clearly, condition (c) of Lemma 8.1 is satisfied. In order to see that condition (a) is satisfied, let $\mu := (r_0^{\mathrm{rob}} - 2r)/4K_2 m_2$ and note that

$$0 \le \frac{\gamma_1 m_1}{4K_2 m_2} \le \frac{7r_1 - 2r}{4K_2 m_2} \overset{(9.5)}{\le} \mu \overset{(9.5)}{\le} \frac{30 r_1}{4K_2 m_2} \le \frac{30\gamma_1}{K_1} \ll 1.$$

Recall that every vertex $v \in B$ satisfies

$$d_{G_5}(v) \overset{(9.9)}{=} D_5 + 2r \overset{(9.6),(9.8)}{=} D_1 - r_0^{\mathrm{rob}} + 2r \pm 2\varepsilon_0 n \overset{(9.2)}{=} |A| - r_0^{\mathrm{rob}} + 2r \pm 4\varepsilon_0 n.$$

Moreover,

$$d_{G_5}(v, A) = d_{G_5}(v) - d_{G_5}(v, B \cup B_0^*) - |A_0^*| \ge d_{G_5}(v) - 2\varepsilon' n,$$

where the last inequality holds since $(G_5, A, A_0^*, B, B_0^*)$ is an $(\varepsilon_0, \varepsilon', K_2, D_5)$-framework (c.f. conditions (FR4) and (FR5)). Together with the fact that $\mathcal{P}_2$ is a $(K_2, m_2, \varepsilon_0, \varepsilon_1, \varepsilon_2)$-partition for G_5 (c.f. condition (P2)), this implies that

$$\begin{aligned} d_{G_5}(v, A_i) &= \frac{d_{G_5}(v, A) \pm \varepsilon_1 n}{K_2} = \frac{|A| - r_0^{\rm rob} + 2r \pm 2\varepsilon_1 n}{K_2} = \left(1 - \frac{r_0^{\rm rob} - 2r}{K_2 m_2} \pm 5\varepsilon_1\right) m_2 \\ &= (1 - 4\mu \pm 5\varepsilon_1) m_2 = (1 - 4\mu \pm 1/K_2) m_2. \end{aligned}$$

Recall that G_6 is obtained from G_5 by deleting all those edges lying in the Hamilton cycles in $\mathcal{C}_3$ and that

$$|\mathcal{C}_3| = 10\varepsilon_4' D_5 \le 10\varepsilon_4' D_4 \overset{(9.4)}{\le} 11\varepsilon_4' |A| \overset{(9.10)}{\le} m_2/K_2.$$

Altogether this implies that $d_{G_6}(v, A_i) = (1 - 4\mu \pm 4/K_2) m_2$. Similarly one can show that $d_{G_6}(w, B_j) = (1 - 4\mu \pm 4/K_2) m_2$ for all $w \in A$. So condition (a) of Lemma 8.1 holds.

To check condition (b), note that

$$|\mathcal{J}'| \overset{(9.12)}{=} \frac{D_6}{2} \le \frac{D_4}{2} \overset{(9.6)}{\le} \frac{D_1 - r_0^{\rm rob}}{2} \le \frac{n}{4} - \mu \cdot 2K_2 m_2 - r \le \left(\frac{1}{4} - \mu - \frac{\gamma}{3K_1}\right) n.$$

Thus condition (b) of Lemma 8.1 holds with $\gamma/3K_1$ playing the role of ρ.

Since the edges in $\mathcal{J}'$ lie in G_5 and $(G_5, A, A_0^*, B, B_0^*)$ is an $(\varepsilon_0, \varepsilon', K_2, D_5)$-framework, (FR5) implies that each $v \in A \cup B$ is incident with an edge in J for at most $\varepsilon' n + |V_0| \le 2\varepsilon' n$ of the $J \in \mathcal{J}'$. (Recall that in a balanced exceptional system there are no edges between A and B.) So condition (d) of Lemma 8.1 holds with ε' playing the role of ε_0.

So we can indeed apply Lemma 8.1 to obtain a collection $\mathcal{C}_4$ of $|\mathcal{J}'|$ edge-disjoint Hamilton cycles in G_6 which cover all edges of $\bigcup \mathcal{J}'$.

Step 8: Decomposing the leftover and the robustly decomposable graph. Finally, we can apply the 'robust decomposition property' of $G^{\rm rob}$ guaranteed by Corollary 7.5 to obtain a Hamilton decomposition of the leftover from the previous step together with $G^{\rm rob}$.

To achieve this, let H' denote the subgraph of G_6 obtained by deleting all those edges lying in the Hamilton cycles from $\mathcal{C}_4$. Thus (9.11) and (9.12) imply that every vertex in V_0 is isolated in H' while every vertex $v \in A \cup B$ has degree $d_{G_6}(v) - 2|\mathcal{J}'| = D_6 + 2r - 2|\mathcal{J}'| = 2r$ in H' (the last equality follows from (9.12)). Moreover, $H'[A]$ and $H'[B]$ contain no edges. (This holds since H' is a spanning subgraph of $G_6 - \bigcup \mathcal{J}' = G_6'$ and since we have already seen that G_6' is bipartite with vertex classes A' and B'.) Now let $H := H'[A, B]$. Then Corollary 7.5(ii)(b) implies that $H + G^{\rm rob}$ has a Hamilton decomposition. Let $\mathcal{C}_5$ denote the set of Hamilton cycles thus obtained. Note that $H + G^{\rm rob}$ is a spanning subgraph of G which contains all edges of G which were not covered by $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$. So $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_5$ is a Hamilton decomposition of G. □

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Béla Csaba
Bolyai Institute,
University of Szeged,
H-6720 Szeged, Aradi vértanúk tere 1.
Hungary

E-mail address: `bcsaba@math.u-szeged.hu`

Daniela Kühn, Allan Lo, Deryk Osthus
School of Mathematics
University of Birmingham
Edgbaston
Birmingham
B15 2TT
UK

E-mail addresses: {`d.kuhn, s.a.lo, d.osthus`}`@bham.ac.uk`

Andrew Treglown
School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London
E1 4NS
UK

E-mail addresses: `a.treglown@qmul.ac.uk`