# YONEDA LEMMA FOR COMPLETE SEGAL SPACES

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ABSTRACT. In this note we formulate and give a self-contained proof of the Yoneda lemma for  $\infty$ -categories in the language of complete Segal spaces.

To the memory of I.M. Gelfand

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### INTRODUCTION

In recent years  $\infty$ -categories or, more formally,  $(\infty, 1)$ -categories appear in various areas of mathematics. For example, they became a necessary ingredient in the geometric Langlands problem. In his books [Lu1, Lu2] Lurie developed a theory of  $\infty$ -categories in the language of quasi-categories and extended many results of the ordinary category theory to this setting.

In his work [Re1] Rezk introduced another model of  $\infty$ -categories, which he called complete Segal spaces. This model has certain advantages. For example, it has a generalization to  $(\infty, n)$ -categories (see [Re2]).

It is natural to extend results of the ordinary category theory to the setting of complete Segal spaces. In this note we do this for the Yoneda lemma.

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To formulate it, we need to construct a convenient model of the " $\infty$ -category of spaces", which an  $\infty$ -analog of the category of sets. Motivated by Lurie's results, we define this  $\infty$ -category to be the simplicial space "classifying left fibrations". After this is done, the construction of the Yoneda embedding and the proof of the Yoneda lemma goes almost like in the case of ordinary categories.

In our next works [KV1, KV2] we study adjoint functors, limits and colimits, show a stronger version of the Yoneda lemma, and generalize results of this paper to the setting of  $(\infty, n)$ -categories.

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This paper is organized as follows. To make the work self-contained, in the first section we introduce basic definitions and discuss properties of model categories, simplicial sets, simplicial spaces and Segal spaces, assuming only basic category theory. In the second section we introduce left fibrations, construct the  $\infty$ -category of spaces  $\mathfrak{S}$ , and formulate and prove the Yoneda lemma. Next, in the third section, we study quasifibrations of simplicial spaces, which are needed for our argument and are also very interesting objects for their own. Finally, the last section is devoted to the proof of the properties of  $\mathfrak{S}$ , formulated in the second section.

#### 1. Preliminaries

#### 1.1. Model categories.

**1.1.1. Notation.** Let C be a category. (a) For an element  $Z \in C$ , we denote by C/Z the overcategory over Z.

(b) For a pair of morphisms  $i : A \to B$  and  $p : X \to Y$  in  $\mathcal{C}$ , we denote by  $\operatorname{Hom}_{\mathcal{C}}(i,p)$  the set of commutative diagrams in  $\mathcal{C}$ 

$$\begin{array}{ccc}
A & \stackrel{a}{\longrightarrow} X \\
\downarrow & \downarrow & p \\
B & \stackrel{b}{\longrightarrow} Y. \end{array}$$

We say that *i* is a *retract* of *p*, if there exist  $\alpha \in \text{Hom}_{\mathcal{C}}(i, p)$  and  $\beta \in \text{Hom}_{\mathcal{C}}(p, i)$ such that  $\beta \circ \alpha = \text{Id}_i$ .

(c) We say that p has the right lifting property (RLP) with respect to i (and that i has the left lifting property (LLP) with respect to p), if for every commutative diagram (1.1) there exists a morphism  $c: B \to X$  such that  $p \circ c = b$  and  $c \circ i = a$ .

Equivalently, this happens if and only if the natural map of sets

$$(i^*, p_*)$$
: Hom $(B, X) \to$  Hom $(A, X) \times_{Hom(A, Y)}$  Hom $(B, Y)$ 

is surjective.

(d) Assume that  $\mathcal{C}$  has fiber products. Then for every morphism  $f: X \to Y$  in  $\mathcal{C}/Z$  and morphism  $g: Z' \to Z$  in  $\mathcal{C}$ , we write  $g^*(f): g^*(X) \to g^*(Y)$  instead of  $f \times_Z Z': X \times_Z Z' \to Y \times_Z Z'$  and call it the pullback of f.

(e) We say that a category  $\mathcal{C}$  is *Cartesian*, if  $\mathcal{C}$  has finite products, and for every  $X, Y \in \mathcal{C}$  there exists an element  $X^Y \in \mathcal{C}$ , representing a functor  $Z \mapsto \text{Hom}(Z \times Y, X)$ , which is called *the internal hom* of X and Y.

**1.1.2. Example.** Let C be the category of functors  $C = \operatorname{Fun}(\Gamma^{op}, Set)$ , where  $\Gamma$  is a small category, and *Set* is the category of sets.

(a) Since category Set has all limits and colimits, category  $\mathcal{C}$  also has these properties. Explicitly, every functor  $\alpha : I \to \mathcal{C}$  defines a functor  $\alpha(\gamma) : I \to Set$  for each  $\gamma \in \Gamma$ , and we have  $\lim_{I}(\alpha)(\gamma) = \lim_{I}(\alpha(\gamma))$  and similarly for colimits. In particular, category  $\mathcal{C}$  has products.

(b) For every  $\gamma \in \Gamma$ , denote by  $F_{\gamma} \in C$  the representable functor  $\operatorname{Hom}_{\Gamma}(\cdot, \gamma)$ . Then for every  $X, Y \in C$  there exists their internal hom  $X^Y \in C$ , defined by the rule  $X^Y(\gamma) = \operatorname{Hom}(Y \times F_{\gamma}, X)$  with obvious transition maps. In other words, category C is Cartesian.

The following lemma is straightforward.

**Lemma 1.1.3.** Let C be a Cartesian category, and let  $i : A \to B$ ,  $j : A' \to B'$  and  $p : X \to Y$  be morphisms in C. Then j has the LLP with respect to  $(i^*, p_*) : X^B \to X^A \times_{Y^A} Y^B$  if and only if  $(i_*, j_*) : (A \times B') \sqcup_{(A \times A')} (B \times A') \to B \times B'$  has the LLP with respect to p.

**Definition 1.1.4.** (compare [GJ, II,1]). A model category is a category C, equipped with three collections of morphisms, called cofibrations, fibrations and weak equivalences, which satisfy the following axioms:

CM1: The category  $\mathcal{C}$  has all finite limits and colimits.

CM2 (2-out-of-3): In a diagram  $X \xrightarrow{f} Y \xrightarrow{g} Z$  if any two of the morphisms f, g and  $g \circ f$  are weak equivalences, then so is the third.

CM3 (retract): If f is a retract of g, and g is a weak equivalence/fibration/cofibration then so is f.

CM4 (lifting property): Let i be a cofibration and p be a fibration. Then p has the RLP with respect to i, if either i or p is a weak equivalence.

CM5 (decomposition property): any morphism f has a decomposition

(a)  $f = p \circ i$ , where p is fibration, and i is a cofibration and weak equivalence;

(b)  $f = q \circ j$ , where q is fibration and weak equivalence, and j is a cofibration.

**1.1.5.** Notation. (a) A map in a model category C is called a *trivial cofibration* (resp. *trivial fibration*) if it is cofibration (resp. fibration) and a weak equivalence.

(b) By CM5, every morphism  $f : X \to Y$  can be written as a composition  $X \xrightarrow{i} X' \xrightarrow{p} Y$ , where *i* is a trivial cofibration, and *p* a fibration. In such a case, we say that *p* is a *fibrant replacement* of *f*.

(c) An element  $X \in \mathcal{C}$  is called *fibrant* (resp. *cofibrant*), if the canonical map  $X \to \text{pt}$  (resp.  $\emptyset \to X$ ), where pt (resp  $\emptyset$ ) is the final (resp. initial) object of  $\mathcal{C}$ , is a fibration (resp. cofibration).

For the following basic fact see, for example, [GJ, II, Lem. 1.1].

**Lemma 1.1.6.** A map  $f : X \to Y$  in a model category C is a cofibration (resp. trivial cofibration) if and only if it has the LLP with respect to all trivial fibrations (resp. fibrations).

(b) A map  $f : X \to Y$  in a model category C is a fibration (resp. trivial fibration) if and only if it has the RLP with respect to all trivial cofibrations (resp. cofibrations).

**1.1.7. Remarks.** (a) Lemma 1.1.6 implies in particular that (trivial) cofibrations and (trivial) fibrations are closed under compositions, and that all isomorphisms are trivial cofibrations and trivial fibrations.

(b) It also follows immediately from Lemma 1.1.6 that (trivial) fibrations are preserved by all pullbacks, and that (trivial) cofibrations are preserved by all pushouts.

(c) It follows from CM5 (a) and CM2 that every weak equivalence f has a decomposition  $f = p \circ i$ , where p is trivial fibration, and i is a trivial cofibration.

**Definition 1.1.8.** We call a model category  $\mathcal{C}$  Cartesian, if  $\mathcal{C}$  is a Cartesian category, the final object of  $\mathcal{C}$  is cofibrant, and for every cofibration  $i : A \to B$  and fibration  $p : X \to Y$ , the induced map  $q : X^B \to X^A \times_{Y^A} Y^B$  is a fibration and, additionally, q is a weak equivalence if either i or p is.

**1.1.9. Remark.** Taking  $A = \emptyset$  or Y = pt in the definition of Cartesian model category, we get the following particular cases.

(a) If B is cofibrant, then for every (trivial) fibration  $X \to Y$ , the induced map  $X^B \to Y^B$  is a (trivial) fibration.

(b) If X is fibrant, then for every (trivial) cofibration  $A \to B$ , the induced map  $X^B \to X^A$  is a (trivial) fibration.

**Lemma 1.1.10.** Let C be a model category, which is Cartesian as a category, and such that the final object of C is cofibrant. Then C is a Cartesian model category if and only if for every two cofibrations  $i : A \to B$  and  $i' : A' \to B'$ , the induced morphism  $j : (A \times B') \sqcup_{(A \times A')} (B \times A') \to B \times B'$  is a cofibration and, additionally, j is a weak equivalence, if either i or i' is.

*Proof.* This follows from a combination of Lemma 1.1.6 and Lemma 1.1.3.  $\Box$ 

#### **Definition 1.1.11.** A model category C is called:

- (a) *right proper*, if weak equivalences are preserved by pullbacks along fibrations;
- (b) *left proper*, if weak equivalence are preserved by pushouts along cofibrations;
- (c) *proper*, if it is both left and right proper.

#### 1.2. Simplicial sets.

**1.2.1.** Category  $\Delta$ . (a) For  $n \geq 0$ , we denote by [n] the category, corresponding to a partially ordered set  $\{0 < 1 < \ldots < n\}$ . Let  $\Delta$  be the full subcategory of the category of small categories *Cat*, consisting of objects  $[n], n \geq 0$ .

(b) For each (m + 1)-tuple of integers  $0 \le k_0 \le k_1 \le \ldots \le k_m \le n$ , we denote by  $\delta^{k_0,\ldots,k_m}$  the map  $\delta: [m] \to [n]$  such that  $\delta(i) = k_i$  for all i.

(c) For  $0 \leq i \leq n$  we define an inclusion  $d^i : [n-1] \hookrightarrow [n]$  such that  $i \notin \operatorname{Im} d^i$ ; for  $0 \leq i < j \leq n$ , we define an inclusion  $d^{i,j} : [n-2] \hookrightarrow [n]$  such that  $i, j \notin \operatorname{Im} d^{i,j}$ ; for  $0 \leq i \leq n-m$  we define an inclusion  $e^i : [m] \to [n]$  defined by  $e^i(k) := k+i$ .

**1.2.2. Spaces.** (a) By the category of spaces or, what is the same, the category simplicial sets we mean the category of functors  $Sp := \operatorname{Fun}(\Delta^{op}, Set)$ .

(b) For  $X \in Sp$ , we set  $X_n := X([n])$ . For every  $\tau : [n] \to [m]$  in  $\Delta$ , we denote by  $\tau^* : X_m \to X_n$  the induced map of sets. For every morphism  $f : X \to Y$  in Spwe denote by  $f_n$  the corresponding map  $X_n \to Y_n$ .

(c) By 1.1.2, category Sp is Cartesian and has all limits and colimits.

**1.2.3. The standard** *n*-simplex. (a) For every  $n \ge 0$ , we denote by  $\Delta[n] \in Sp$  the functor  $\operatorname{Hom}_{\Delta}(\cdot, [n]) : \Delta^{op} \to Set$ . Then  $\operatorname{pt} := \Delta[0]$  is a final object of Sp.

(b) The Yoneda lemma defines identifications  $\operatorname{Hom}_{Sp}(\Delta[n], X) = X_n$  and  $\operatorname{Hom}_{Sp}(\Delta[n], \Delta[m]) = \operatorname{Hom}_{\Delta}([n], [m])$  for all  $X \in Sp$  and all  $n, m \ge 0$ .

(c) We denote by  $\Delta^{i}[n]$  the image of the inclusion  $d^{i}: \Delta[n-1] \to \Delta[n]$ , and set  $\partial \Delta[n] := \bigcup_{i=0}^{n} \Delta^{i}[n] \subset \Delta[n]$  and  $\Lambda^{k}[n] := \bigcup_{i \neq k} \Delta^{k}[n] \subset \Delta[n]$  for all  $k = 0, \ldots, n$ .

**1.2.4. Fibers.** (a) For  $X \in Sp$ , we say  $x \in X$  instead of  $x \in X_0$ . By 1.2.3 (b), each  $x \in X$  corresponds to a map  $x : \text{pt} \to X$ .

(b) For every morphism  $f: Y \to X$ , we denote by  $f^{-1}(x)$  or  $Y_x$  the fiber product  $\{x\} \times_X Y := \operatorname{pt} \times_{x,X} Y$  and call it the fiber of f at x.

(c) For every  $Z \in Sp$  and  $X, Y \in Sp/Z$ , we denote by  $\operatorname{Map}_Z(X, Y)$  the fiber of  $Y^X \to Z^X$  over the projection  $(X \to Z) \in Z^X$ .

**Definition 1.2.5.** (a) A map  $f : X \to Y$  in Sp is called a *(Kan) fibration*, if it has the RLP with respect to inclusions  $\Lambda^k[n] \hookrightarrow \Delta[n]$  for all  $n > 0, k = 0, \ldots, n$ .

(b) A map  $f : X \to Y$  in Sp is called a *weak equivalence*, if it induces a weak equivalence  $|f| : |X| \to |Y|$  between geometric realisations (see [GJ, p. 60]).

(c) A map  $f: X \to Y$  in Sp is called a *cofibration*, if  $f_n: X_n \to Y_n$  is an inclusion for all n.

**Theorem 1.2.6.** Category Sp has a structure of a proper Cartesian model category such that cofibrations, fibrations and weak equivalences are defined in Definition 1.2.5. In particular, all  $X \in Sp$  are cofibrant, and trivial fibrations are precisely the maps which have the RLP with respect to inclusions  $\partial \Delta[n] \hookrightarrow \Delta[n], n \ge 0$ .

*Proof.* See [GJ, I, Thm 11.3, Prop 11.5 and II, Cor 8.6] and note that in the case of model category Sp, "Cartesian" means the same as "simplicial".

**Definition 1.2.7.** We say that  $X \in Sp$  is a *(contractible) Kan complex*, if the projection  $X \to pt$  is a (trivial) fibration.

**1.2.8.** Connected components. (a) We say that  $X \in Sp$  is *connected*, if it can not be written as  $X = X' \sqcup X''$ , where  $X', X'' \neq \emptyset$ . We say that  $Y \subset X$  is a *connected component of* X, if it is a maximal connected subspace of X. Notice that X is a disjoint union of its connected components.

(b) We denote the set of connected components of X by  $\pi_0(X)$ . Then every map  $f: X \to Y$  in Sp induces a map  $\pi_0(f): \pi_0(X) \to \pi_0(Y)$ . Note that X is connected if and only if its geometric realization |X| is connected. In particular, we have an equality  $\pi_0(X) = \pi_0(|X|)$ . Therefore for every weak equivalence  $f: X \to Y$  in Sp, the map  $\pi_0(f)$  is a bijection.

(c) For  $x, y \in X$ , we say that  $x \sim y$ , if x and y belong to the same connected component of X. If X is a Kan complex, then  $x \sim y$  if and only if there exists a map  $\alpha : \Delta[1] \to X$  such that  $\alpha(0) = x$  and  $\alpha(1) = y$  (see [GJ, Lem 6.1]).

**Lemma 1.2.9.** (a) Let  $f : X \to Y$  be a trivial fibration. Then the space of sections  $Map_Y(Y, X)$  of f is non-empty and connected.

(b) Let  $f: X \to Y$  be a fibration in Sp. Then f is trivial if and only if the Kan complex  $f^{-1}(y)$  is contractible for every  $y \in Y$ .

(c) Let  $f : X \to Y$  is a map of fibrations over Z in Sp. Then f is a weak equivalence if and only if the map of fibers  $f_z : X_z \to Y_z$  is a weak equivalence for every  $z \in Z$ .

*Proof.* (a) Since Y is cofibrant, the projection  $X^Y \to Y^Y$  is a trivial fibration (by 1.1.9 (a)). Hence its fiber  $\operatorname{Map}_Y(Y, X)$  is a contractible Kan complex (by 1.1.7 (b)), therefore it is non-empty and connected by 1.2.8 (b).

(b) By the last assertion of Theorem 1.2.6, the fibration f is trivial if and only if its pullback  $\tau^*(f)$  is a trivial fibration for all  $\tau : \Delta[n] \to Y$ . Thus we may assume that  $Y = \Delta[n]$ . Then for each  $y \in \Delta[n]$ , the inclusion  $y : \Delta[0] \to \Delta[n]$  is a weak equivalence. Thus  $X_y \to X$  is a weak equivalence, because Sp is right proper. Hence, by 2-out-of-3, f is a weak equivalence if and only if  $X_y \to \Delta[0]$  is.

(c) will be proven in 1.3.13.

# 1.3. Simplicial Spaces.

**1.3.1. Notation.** (a) By the category of *simplicial spaces*, we mean the category of functors  $sSp = \operatorname{Fun}(\Delta^{op}, Sp) = \operatorname{Fun}(\Delta^{op} \times \Delta^{op}, Set)$ .

(b) For  $X \in sSp$  and  $n, m \geq 0$ , we set  $X_n := X([n]) \in Sp$  and  $X_{n,m} := (X_n)_m \in Set$ . For every morphism  $f : X \to Y$  in sSp, we denote by  $f_n : X_n \to Y_n$  the corresponding morphism in Sp.

(c) For every  $\tau : [n] \to [m]$  in  $\Delta$ , we denote by  $\tau^* : X_m \to X_n$  the induced map of spaces. We also set  $\delta_{k_0,\ldots,k_m} := (\delta^{k_0,\ldots,k_m})^* : X_n \to X_m, d_i := (d^i)^* : X_n \to X_{n-1},$  and  $e_i := (e^i)^* : X_n \to X_m$ .

(d) By 1.1.2, category sSp is Cartesian and has all limits and colimits. For  $X, Y \in sSp$ , we define the mapping space  $Map(Y, X) := (X^Y)_0 \in Sp$ .

**1.3.2. Two embeddings**  $Sp \hookrightarrow sSp$ . (a) Denote by  $diag : Sp \to sSp$  (resp.  $diag : Set \to Sp$ ) the map which associates to each X the constant simplicial space (resp. set)  $[n] \mapsto X, \tau \mapsto \operatorname{Id}_X$ . For each  $X \in Sp$ , we denote the constant simplicial space  $diag(X) \in sSp$  simply by X.

(b) The embedding  $diag : Set \to Sp$  gives rise to an embedding  $disc : Sp = Fun(\Delta^{op}, Set) \to sSp = Fun(\Delta^{op}, Sp)$ . Then the image of disc, consists of discrete simplicial spaces, that is,  $X \in sSp$  such that  $X_n \in Sp$  is discrete (that is, each map  $\Delta[1] \to X_n$  is constant) for all n.

(c) We set  $F[n] := disc(\Delta[n]), \ \partial F[n] := disc(\partial \Delta[n]) \text{ and } F^i[n] := disc(\Lambda^i[n]).$ 

**1.3.3. Standard bisimplex.** (a) For  $n, m \ge 0$ , we set  $[n, m] := ([n], [m]) \in \Delta^2$ and  $\Box[n, m] := F[n] \times \Delta[m] \in sSp$ . In particular, we have equalities  $F[n] = \Box[n, 0]$ ,  $\Delta[m] = \Box[0, m]$  and  $\operatorname{pt} = F[0] = \Delta[0]$ .

(b) Note that  $\Box[n,m]$  is the functor  $\operatorname{Hom}_{\Delta \times \Delta}(\cdot, [n,m])$ . Then, by the Yoneda lemma, we get identifications  $\operatorname{Hom}(\Box[n,m],\Box[n',m']) = \operatorname{Hom}([n,m],[n',m'])$  and  $\operatorname{Hom}(\Box[n,m],X) = X_{n,m}$ . In particular, we have identifications

 $\operatorname{Map}(F[n], X) = X_n \text{ and } \operatorname{Hom}(F[n], F[m]) = \operatorname{Hom}([n], [m]).$ 

(c) We also set  $\partial \Box[n,m] := (\partial F[n] \times \Delta[m]) \sqcup_{(\partial F[n] \times \partial \Delta[m])} (F[n] \times \partial \Delta[m])$  and  $X_{\partial n} := \operatorname{Map}(\partial F[n], X).$ 

**1.3.4. Fibers.** (a) For  $X \in sSp$ , we say  $x \in X$  instead of  $x \in X_{0,0}$ , and  $x \sim y \in X$  instead of  $x \sim y \in X_0$ . By 1.3.3 (b), each  $x \in X$  corresponds to a map  $x : \text{pt} \to X$ .

(b) As in 1.2.4, for every morphism  $f: Y \to X$  in sSp, we denote by  $f^{-1}(x)$  or  $Y_x$  the fiber product  $\{x\} \times_X Y := \operatorname{pt} \times_{x,X} Y$  and call it the fiber of f at x.

(c) For every  $Z \in sSp$  and  $X, Y \in sSp/Z$  we denote by  $\mathcal{M}ap_Z(X, Y) \in sSp$  the fiber of  $Y^X \to Z^X$  over the projection  $(X \to Z) \in Z^X$ . We also set

 $\operatorname{Map}_{Z}(X,Y) := \mathcal{M}ap_{Z}(X,Y)_{0} \in Sp.$ 

**Definition 1.3.5.** We say that a map  $f: X \to Y$  in sSp is a *(Reedy) fibration* if for every  $n \ge 0$  the induced map  $\overline{f}_n: X_n \to Y_n \times_{Y_{\partial n}} X_{\partial n}$  is a Kan fibration in Sp.

**Theorem 1.3.6.** Category sSp has Cartesian proper model category such that cofibrations and weak equivalences are degree-wise and fibrations are Reedy fibrations.

*Proof.* All the assertions, except that the model category is Cartesian, are proven in [GJ, IV,Thm 3.9]. For the remaining assertion, we use Lemma 1.1.10. Now the assertion follows from the fact that pushouts, products, cofibrations and weak equivalences are defined degree-wise, and the model category Sp is Cartesian.  $\Box$ 

**1.3.7. Remarks.** (a) It follows from Lemma 1.1.3 that a map  $f: X \to Y$  in sSp is a fibration if and only if it has the RLP with respect to all inclusions  $(\partial F[n] \times \Delta[m]) \sqcup_{(\partial F[n] \times \Lambda^i[m])} (F[n] \times \Lambda^i[m]) \hookrightarrow \Box[n,m].$ 

(b) If  $f: X \to Y$  is a Reedy fibration, then the map  $f_n: X_n \to Y_n$  is a fibration for all n. Indeed,  $f_0 = \bar{f}_0$  is a fibration by definition,  $f^{F[n]}: X^{F[n]} \to Y^{F[n]}$  is a fibration by 1.1.9 (a), hence  $f_n = (f^{F[n]})_0$  is a fibration.

(c) Let  $X \to Y$  be a fibration in sSp, and let  $i : A \to B$  be a cofibration over Y. Then the map  $X^B \to Y^B \times_{Y^A} X^A$  is fibration by Theorem 1.3.6. Hence taking fibers at  $(B \to Y) \in Y^B$  and passing to zero spaces, we get that the map  $i^* : \operatorname{Map}_Y(B, X) \to \operatorname{Map}_Y(A, X)$  is a fibration, thus  $\operatorname{Map}_Y(B, X)$  is a Kan complex.

### **1.3.8. Homotopy equivalence.** Let $Z \in sSp$ (resp. $Z \in Sp$ ).

(a) We say that maps  $f: X \to Y$  and  $g: X \to Y$  in sSp/Z (resp. Sp/Z) are homotopic over Z and write  $f \sim_Z g$ , if  $f \sim g$  as elements of  $Map_Z(X,Y)$ .

Notice that if  $Y \to Z$  is a fibration, then  $\operatorname{Map}_Z(X, Y) \in Sp$  is a Kan complex (by 1.3.7 (c)), thus by 1.2.8 (c)  $f \sim_Z g$  means that there exists a map  $h: X \times \Delta[1] \to Y$  over Z such that  $h|_0 = f$  and  $h|_1 = f$ .

(b) We say that a map  $f: X \to Y$  is a homotopy equivalence over Z, if there exists a map  $g: Y \to X$  over Z, called a homotopy inverse of f, such that  $f \circ g \sim_Z \operatorname{Id}_Y$ and  $g \circ f \sim_Z \operatorname{Id}_X$ .

**1.3.9. Remarks.** (a) Let  $f: X \to Y$  be a homotopy equivalence over Z with homotopy inverse g. Then for every  $\tau: Z' \to Z$ , the pullback  $\tau^*(f)$  is a homotopy equivalence over Z' with homotopy inverse  $\tau^*(g)$ . Similarly, for every  $K \in sSp$ , the map  $f^K: X^K \to Y^K$  is a homotopy equivalence with homotopy inverse  $g^K: Y^K \to X^K$ . Also, a composition of homotopy equivalences is a homotopy equivalence.

(b) Any homotopy equivalence is a weak equivalence. Indeed, the assertion for Sp follows from the fact that if  $f \sim_Z g$ , then the geometric realizations satisfy  $|f| \sim_{|Z|} |g|$ , and the assertion for sSp follows from that for Sp.

# **1.3.10. Strong deformation retract.** Let $Z \in sSp$ (resp. $Z \in Sp$ ).

(a) We say that an inclusion  $i: Y \hookrightarrow X$  over Z is a strong deformation retract over Z, if there exists a map  $h: X \times \Delta[1] \to X$  over Z such that  $h|_0 = \mathrm{Id}_X$ , and  $h|_1(X) \subset Y$ ,  $h|_{Y \times \Delta[1]}$  is  $Y \times \Delta[1] \xrightarrow{\mathrm{pr}_2} Y \hookrightarrow X$ .

(b) If  $i: Y \hookrightarrow X$  is a strong deformation retract over Z, then i is a homotopy equivalence over Z, and  $h|_1: X \to Y$  is its homotopy inverse. Also in this case,  $Y \to Z$  is a retract of  $X \to Z$ . In particular, if  $X \to Z$  is a fibration, then  $Y \to Z$  is a fibration as well (by CM3).

(c) Conversely, a trivial cofibration  $i: Y \to X$  between fibrations over Z is a strong deformation retract over Z.

*Proof.* Since  $i: Y \to X$  is trivial cofibration, while  $Y \to Z$  is a fibration, there exists a map  $p: X \to Y$  over Y such that  $p \circ i = \operatorname{Id}_Y$ . Next, the induced map  $(\partial \Delta[1] \times X) \sqcup_{(\partial \Delta[1] \times Y)} (\Delta[1] \times Y) \hookrightarrow \Delta[1] \times X$  is a trivial cofibration (see Lemma 1.1.10). Since  $X \to Z$  is a fibration, there exists a map  $h: \Delta[1] \times X \to X$  over Z such that  $h|_0 = \operatorname{Id}_X$ ,  $h|_1 = p$  and  $h|_{\Delta[1] \times Y} = \operatorname{pr}_2$ .

**Lemma 1.3.11.** (a) A morphism  $f : X \to Y$  over Z is a homotopy equivalence over Z if and only if for every map  $\alpha : K \to Z$  in sSp, the induced map  $\pi_0(\operatorname{Map}_Z(K, X)) \to \pi_0(\operatorname{Map}_Z(K, Y))$  is a bijection.

(b) Every weak equivalence between fibrations is a homotopy equivalence. In particular, a pullback of a weak equivalence between fibrations is a weak equivalence.

(c) For each fibration  $X \to Y \times \Delta[1]$ , there exists a weak equivalence  $X|_0 \to X|_1$  of fibrations over Y.

(d) Let  $f: Y_A \to A$  be a fibration and let  $A \hookrightarrow B$  be trivial cofibration. Then there exists a fibration  $g: Y_B \to B$ , whose restriction to A is f.

*Proof.* (a) If f is a homotopy equivalence, then the induced map  $\operatorname{Map}_Z(K, X) \to \operatorname{Map}_Z(K, Y)$  is a homotopy equivalence (by 1.3.9 (a)), thus the assertion follows from 1.3.9 (b) and 1.2.8 (b). Conversely, applying the assumption for the projection  $Y \to Z$ , we find a morphism  $g: Y \to X$  over Z such that  $f \circ g \sim_Z \operatorname{Id}_Y$ . Next applying it to the projection  $X \to Z$ , we find that  $g \circ f \sim_Z \operatorname{Id}_X$ .

(b) By 1.1.7 (c) and 1.3.9 (a), it is enough to consider separately cases of a trivial cofibration and a trivial fibration. When f is a trivial cofibration, the assertion follows from 1.3.10 (c) and (b). When f is a trivial fibration, the assertion follows from (a). Indeed, each map  $\operatorname{Map}_Z(K, X) \to \operatorname{Map}_Z(K, Y)$  is a trivial fibration, thus the map on  $\pi_0$  is a bijection by 1.2.8 (b). The last assertion follows from 1.3.9 (a).

(c) Since each map  $\delta^i : \Delta[0] \hookrightarrow \Delta[1]$  is a trivial cofibration, the induced map  $(\delta^i)^* : X^{\Delta[1]} \to X \times_{\Delta[1] \times Y} (Y \times \Delta[1])^{\Delta[1]}$  is a trivial fibration. Taking the pullback with respect to the inclusion  $Y \hookrightarrow (Y \times \Delta[1])^{\Delta[1]}$ , corresponding to  $\mathrm{Id}_{Y \times \Delta[1]}$ , we get a trivial fibration  $\widetilde{X} := X^{\Delta[1]} \times_{(\Delta[1] \times Y)^{\Delta[1]}} Y \to X|_i$  over Y. Thus both  $X|_0$  and  $X|_1$  are homotopy equivalent to  $\widetilde{X}$  over Y (by (b)), hence they are homotopy equivalent.

(d) will be proven in 3.2.3.

**1.3.12. Remark.** It follows from Lemma 1.3.11 (b) and 1.2.8 (b) that a Kan complex  $X \in Sp$  is contractible if and only if the projection  $X \to pt$  is a homotopy equivalence. Thus by definition this happens if and only if X is non-empty and  $\mathrm{Id}_X$  is homotopic to a constant map  $X \to \{x\} \subset X$ .

**1.3.13.** Proof of Lemma 1.2.9 (c). If f is a weak equivalence, then each  $f_z : X_z \to Y_z$  is a weak equivalence by Lemma 1.3.11 (b).

Conversely, write f as  $p \circ i$ , where  $i : X \to X'$  is a trivial cofibration, and  $p: X' \to Y$  is fibration. By the "only if" assertion, each  $i_z$  is a weak equivalence. Since each  $f_z$  is a trivial fibration by assumption, each  $p_z$  is a weak equivalence by 2-out-of-3. Since  $p_z: X'_z \to Y_z$  is a fibration, it is a trivial fibration. Hence all fibers of each  $p_z$  are contractible. Thus all fibers of p are contractible, hence p is a trivial fibration by Lemma 1.2.9 (b). Therefore f is a weak equivalence.

1.4. Segal spaces. We follow closely [Re1].

**1.4.1.** Notation. (a) We say that  $X \in sSp$  is a *Sequence*, if X is fibrant, and

 $\varphi_n =: \delta_{01} \times_{\delta_1} \ldots \times_{\delta_{n-1}} \delta_{n-1,n} : X_n \to X_1 \times_{X_0} \ldots \times_{X_0} X_1$ 

is a weak equivalence for each  $n \geq 2$ .

(b) Notice that since Reedy model category is Cartesian, when X is fibrant, the map  $\varphi_n$  is a fibration. Thus a fibrant object  $X \in sSp$  is a Segal space if and only if each map  $\varphi_n$  is a trivial fibration.

#### **1.4.2.** "Objects" and "Mapping spaces". Let X be a Segal space.

(a) By a space of objects of X we mean space  $X_0$ . We set  $Ob X := X_{0,0}$  and call it the set of objects of X. As in 1.3.4, we say  $x \in X$  instead of  $x \in Ob X$ .

(b) For each  $x, y \in X$ , we denote by  $map(x, y) = map_X(x, y) \in Sp$  the fiber of  $(\delta_0, \delta_1): X_1 \to X_0 \times X_0$  over (x, y). Notice that since X is fibrant, the map  $(\delta_0, \delta_1)$ is a fibration, thus each space map(x, y) is a Kan complex. For each  $x, y, z \in X$ , we denote by map(x, y, z) the fiber of  $(\delta_0, \delta_1, \delta_2) : X_3 \to (X_0)^3$  over (x, y, z).

(c) For each  $x \in X$ , we set  $\operatorname{id}_x := \delta_{0,0}(x) \in \operatorname{map}_X(x, x)$ .

(d) We call a map between Segal spaces  $f: X \to Y$  is fully faithful, if for every  $x, y \in X$  the induced map  $\operatorname{map}_X(x, y) \to \operatorname{map}_Y(f(x), f(y))$  is a weak equivalence.

**1.4.3.** The homotopy category. (a) Let X be a Segal space, and  $x, y, z \in$ X. Then the trivial fibration  $\varphi_2: X_2 \to X_1 \times_{X_0} X_1$  induces a trivial fibration  $\operatorname{map}(x, y, z) \to \operatorname{map}(x, y) \times \operatorname{map}(y, z)$  (by 1.1.7 (b)), which by Lemma 1.2.9 (a) has a section s, unique up to homotopy. Thus we have a well-defined map

$$[s] := \pi_0(s) : \pi_0(\max(x, y)) \times \pi_0(\max(y, z)) \to \pi_0(\max(x, y, z)).$$

(b) The map  $\delta_{02} : X_2 \to X_1$  induces a map  $\delta_{02} : \operatorname{map}(x, y, z) \to \operatorname{map}(x, z)$ . Therefore for every  $[\alpha] \in \pi_0(\max(x, y))$  and  $[\beta] \in \pi_0(\max(y, z))$  we can define

(1.2) 
$$[\beta] \circ [\alpha] := \pi_0(\delta_{02})([s]([\alpha], [\beta])) \in \pi_0(\max(x, z)).$$

It is not difficult to prove (see [Re1, Prop. 5.4]) that this composition is associative and satisfies  $[\alpha] \circ [\operatorname{id}_x] = [\alpha] = [\operatorname{id}_y] \circ [\alpha]$  for all  $\alpha \in \operatorname{map}(x, y)$ .

(c) Using (b), one can associate to X its homotopy category Ho X, whose objects are Ob X, morphisms defined by  $\operatorname{Hom}_{\operatorname{Ho} X}(x,y) := \pi_0(\operatorname{map}_X(x,y))$ , the composition is defined by (1.2), and the identity map is  $[id_x] \in Hom_{Ho X}(x, x)$ .

### **1.4.4.** Complete Segal spaces. Let X be a Segal space.

(a) We say that  $\alpha \in \max_X(x,y) \subset X_1$  is a homotopy equivalence, if the corresponding morphism  $[\alpha] \in Mor Ho X$  is an isomorphism. Explicitly, this means that there exist  $\beta \in \max(y, x)$  such that  $[\beta] \circ [\alpha] = [\operatorname{id}_x]$  and  $[\alpha] \circ [\beta] = [\operatorname{id}_y]$ .

(b) Let  $X_{heq} \subset X_1$  be the maximal subspace such that each  $\alpha \in X_{heq}$  is a homotopy equivalence. It is not difficult to prove (see [Re1, Lem. 5.8]) that  $X_{heq} \subset$  $X_1$  is a union of connected components.

(c) Notice that since each  $[id_x]$  is an isomorphism, we have  $id_x \in X_{heq}$  for every  $x \in X$ . Therefore the map  $s_0 := \delta_{0,0} : X_0 \to X_1$  factors through  $X_{heq}$ . We say that X is called a *complete Segal space*, if the map  $s_0 : X_0 \to X_{heq}$  is a weak equivalence.

# **Lemma 1.4.5.** Let $X \in Sp$ be a Segal space.

(a) Let  $X'_1 \subset X_1$  be the union of connected components, intersecting  $s_0(X_0)$ , and set  $X'_3 := \delta_{02}^{-1}(X'_1) \cap \delta_{13}^{-1}(X'_1) \subset X_3$ . Then  $X'_1 \subset X_{heq}$  and  $\delta_{12}(X'_3) = X_{heq}$ . (b) X is complete if and only if  $\delta_0 : X_{heq} \hookrightarrow X_1 \to X_0$  is a trivial fibration.

*Proof.* (a) Since  $X_{heq} \subset X_1$  is a union of connected components, inclusion  $s_0(X_0) \subset X_1$  $X_{heq}$  implies that  $X'_1 \subset X_{heq}$ . Next, for each  $\alpha \in X'_3$  we have  $\delta_{02}(\alpha), \delta_{13}(\alpha) \in$  $X'_1 \subset X_{heq}$ , hence  $[\delta_{02}(\alpha)] = [\delta_{12}(\alpha)] \circ [\delta_{01}(\alpha)]$  and  $[\delta_{13}(\alpha)] = [\delta_{23}(\alpha)] \circ [\delta_{12}(\alpha)]$  are isomorphisms in Ho X. Therefore  $[\delta_{12}(\alpha)]$  is isomorphism, thus  $\delta_{12}(\alpha) \in X_{heq}$ .

Conversely, let  $\alpha \in \max(x, y) \subset X_{heq}$  and let  $\beta \in \max(y, x)$  such that  $[\beta] \circ [\alpha] =$  $[\mathrm{id}_x]$  and  $[\alpha] \circ [\beta] = [\mathrm{id}_y]$ . Since  $\varphi_3$  is a trivial fibration, it is surjective. Thus there exists  $\gamma \in X_3$  such that  $\delta_{01}(\gamma) = \delta_{23}(\gamma) = \beta$  and  $\delta_{12}(\gamma) = \alpha$ . Then, by assumption,  $\delta_{02}(\gamma) \sim \mathrm{id}_y$  and  $\delta_{13}(\gamma) \sim \mathrm{id}_x$ , thus  $\gamma \in X'_3$ .

(c) By 1.4.4 (b), the composition  $X_{heq} \hookrightarrow X_1 \to X_0 \times X_0$  is a fibration, thus a projection  $\delta_0 : X_{heq} \to X_0$  is a fibration. Since  $\delta_0 \circ s_0 = \mathrm{Id}_{X_0}$ , we conclude that  $s_0 : X_0 \to X_{heq}$  is a weak equivalence if and only if  $\delta_0 : X_{heq} \to X_0$  is a trivial fibration.

**1.4.6. Cartesian structure.** Rezk showed (see [Re1, Cor 7.3]) that if X is a (complete) Segal space, then  $X^K$  is a (complete) Segal space for every  $K \in sSp$ .

#### 2. The Yoneda Lemma

### 2.1. Left fibrations.

**Definition 2.1.1.** We call a fibration  $f : X \to Y$  in sSp a *left fibration*, if the map  $(f_*, (\delta^0)^*) : X^{F[1]} \to X \times_Y Y^{F[1]}$ , induced by  $\delta^0 : F[0] \hookrightarrow F[1]$ , is a trivial fibration.

Lemma 2.1.2. (a) A pullback of a left fibration is a left fibration.

(b) If  $f: X \to Y$  is a left fibration, then  $f^Z: X^Z \to Y^Z$  is a left fibration for every  $Z \in sSp$ .

*Proof.* (a) follows from the fact that a pullback of a (trivial) fibration is a (trivial) fibration (see 1.1.7 (b)).

(b) By definition, the map  $X^{F[1]} \to X \times_Y Y^{F[1]}$  is a trivial fibration. Since Reedy model structure is Cartesian, we conclude that  $f^Z$  is a fibration, while the map  $(X^Z)^{F[1]} = (X^{F[1]})^Z \to (X \times_Y Y^{F[1]})^Z = X^Z \times_{Y_Z} (Y^Z)^{F[1]}$  is a trivial fibration (use 1.1.9). Thus  $f^Z : X^Z \to Y^Z$  is a left fibration.  $\Box$ 

**Lemma 2.1.3.** Let  $f : X \to Y$  be a fibration in sSp. The following conditions are equivalent:

(a) f is a left fibration.

(b) For every  $n \ge 1$ , the map  $(f_*, (\delta^0)^*) : X^{F[n]} \to X \times_Y Y^{F[n]}$ , induced by  $\delta^0 : [0] \hookrightarrow [n]$ , is a trivial fibration.

(c) For every  $n \ge 1$ , the map  $p_n : X_n \to X_0 \times_{Y_0} Y_n$ , induced by  $\delta^0 : [0] \hookrightarrow [n]$ , is a trivial fibration.

*Proof.* (a)  $\implies$  (b) By (a) and 1.1.9 (a), the map

$$p: X^{F[1] \times F[n]} = (X^{F[1]})^{F[n]} \to (X \times_Y Y^{F[1]})^{F[n]} = X^{F[n]} \times_{Y^{F[n]}} Y^{F[1] \times F[n]}$$

is a trivial fibration. Since trivial fibrations are stable under retracts (axiom CM3), it remains to show that the map  $X^{F[n+1]} \to X \times_Y Y^{F[n+1]}$  is a retract of p. It is enough to show that  $\delta^0 : [0] \hookrightarrow [n+1]$  is a retract of  $\delta^0 : [n] \times [0] \hookrightarrow [n] \times [1]$ . Consider maps  $[n+1] \xrightarrow{\alpha} [n] \times [1] \xrightarrow{\beta} [n+1]$ , where  $\alpha(0) = (0,0), \ \alpha(i) = (i-1,1)$  for  $i \ge 1$  and  $\beta(i,j) = (i+1)j$ . Then  $\beta \circ \alpha = \operatorname{Id}, \ \alpha(0) \in [n] \times \{0\}$  and  $\beta([n] \times \{0\}) = 0$ , thus  $\alpha$  and  $\beta$  realize  $\delta^0 : [0] \hookrightarrow [n+1]$  as a retract of  $\delta^0 : [n] \times [0] \hookrightarrow [n] \times [1]$ .

(b)  $\implies$  (c) Pass to the zero spaces.

(c)  $\Longrightarrow$  (a) First we assume that Y is fibrant. Since f is a fibration, X is fibrant, and the induced map  $X^{F[1]} \to X \times_Y Y^{F[1]}$  is a fibration. It remains to show that each map  $(X^{F[1]})_n \to X_n \times_{Y_n} (Y^{F[1]})_n$  is a weak equivalence. Since the map  $X_n \to X_0 \times_{Y_0} Y_n$  is a trivial fibration, its pullback

$$X_n \times_{Y_n} (Y^{F[1]})_n \to (X_0 \times_{Y_0} Y_n) \times_{Y_n} (Y^{F[1]})_n = X_0 \times_{Y_0} (Y^{F[1]})_n$$

is a trivial fibration. It remains to show that the map  $(X^{F[1]})_n \to X_0 \times_{Y_0} (Y^{F[1]})_n$ or, equivalently,  $q_n : (X^{F[1] \times F[n]})_0 \to X_0 \times_{Y_0} (Y^{F[1] \times F[n]})_0$  is a weak equivalence. We follow the argument of [Re1, Lem 10.3]. Let  $\gamma^i : [n+1] \to [n] \times [1]$  (resp.  $\epsilon^i : [n] \to [n] \times [1]$ ) be the map with sends j to (j, 0), if  $j \leq i$  and to (j - 1, 1) (resp. (j, 1)) otherwise. Then maps  $\gamma^i$  and  $\epsilon^i$  induce decomposition of  $F[n] \times F[1]$  as  $F[n+1] \sqcup_{F[n]} \ldots \sqcup_{F[n]} F[n+1]$ , where all maps  $F[n] \to F[n+1]$  are cofibrations.

Therefore we get decompositions of  $(X^{F[n] \times F[1]})_0$  and  $X_0 \times_{Y_0} (Y^{F[1] \times F[n]})_0$  as

$$(X^{F[n+1]})_0 \times_{(X^{F[n]})_0} \ldots \times_{(X^{F[n]})_0} (X^{F[n+1]})_0$$
 and

 $(X_0 \times_{Y_0} (Y^{F[n+1]})_0) \times_{(X_0 \times_{Y_0} (Y^{F[n]})_0)} \dots \times_{(X_0 \times_{Y_0} (Y^{F[n]})_0)} (X_0 \times_{Y_0} (Y^{F[n+1]})_0).$ 

Since X and Y are fibrant, all maps in both fiber products are fibrations.

Thus  $q_n$  can be written as a fiber products of  $X_{n+1} \to X_0 \times_{Y_0} Y_{n+1}$ 's over  $X_n \to X_0 \times_{Y_0} Y_n$ 's. Since these maps are weak equivalences by (c), we conclude that p is a weak equivalence by Corollary 3.1.5.

For a general Y, we choose a fibrant replacement  $Y \hookrightarrow Y'$ . Then by Lemma 1.3.11 (d) there exists a fibration  $f': X' \to Y'$ , whose restriction to Y is f. We claim that f' satisfies assumption (c). Since f' is a fibration, each  $p'_n: X'_n \to X'_0 \times_{Y'_0} Y'_n$  is a fibration. Thus it remains to show that  $p'_n$  is a weak equivalence.

Consider Cartesian diagram

and note that all horizontal maps are fibrations. Since i is a weak equivalence and Sp is right proper, we conclude that i' and i'' are weak equivalences. Since  $p_n$  is a weak equivalence by assumption,  $p'_n$  is a weak equivalence by 2-out-of-3.

By the application (c)  $\Longrightarrow$  (a) for fibrant Y, the map  $q': X'^{F[1]} \to X' \times_{Y'} Y'^{F[1]}$ is a trivial fibration. Hence q, being the restriction of q' to  $X \times_Y Y^{F[1]}$ , is a trivial fibration as well.

**2.1.4. Remarks.** (a) By Lemma 2.1.3 (c), a morphism  $f : X \to Y$  is a left fibration if and only if it satisfies the RLP with respect to cofibrations

$$\begin{split} (F[n] \times \Lambda^i[m]) \sqcup_{(\partial F[n] \times \Lambda^i[m])} (\partial F[n] \times \Delta[m]) &\hookrightarrow \Box[n,m] \\ (F[n] \times \partial \Delta[m]) \sqcup_{(F[0] \times \partial \Delta[m])} (F[0] \times \Delta[m]) &\hookrightarrow \Box[n,m]. \end{split}$$

In particular, a morphism  $f : X \to Y$  is a left fibration if and only if for every morphism  $\tau : \Box[n,m] \to Y$ , the pullback  $\tau^*(f) : \tau^*(X) \to \Box[n,m]$  is a left fibration.

(b) It also can be deduced from Lemma 2.1.3 (c) that if  $f : X \to Y$  is a left fibration and Y is a (complete) Segal space, then X is a (complete) Segal space as well. We will not use this fact.

**Lemma 2.1.5.** A morphism  $f: X \to Y$  of left fibrations over Z is a weak equivalence if and only if the map of fibers  $f_z: (X_z)_0 \to (Y_z)_0$  is a weak equivalence for each  $z \in Z$ .

*Proof.* Notice that  $f_0 : X_0 \to Y_0$  is a morphism between fibrations over  $f_0$ . Thus  $f_0$  is a weak equivalence if and only if the induced map  $f_z : (X_z)_0 \to (Y_z)_0$  between fibers is a weak equivalence for all  $z \in Z_0$  (by Lemma 1.2.9 (c)). Thus it remains

to show that if  $f_0$  is a weak equivalence, then  $f_n$  is a weak equivalence for each n. We have a commutative diagram

$$\begin{array}{cccc} X_n & \stackrel{f_n}{\longrightarrow} & Y_n \\ & & \downarrow \\ X_0 \times_{Z_0} Z_n & \stackrel{\widetilde{f}_0}{\longrightarrow} & Y_0 \times_{Z_0} Z_n, \end{array}$$

whose vertical maps are trivial fibrations by Lemma 2.1.3 (c). Since  $f_0$  is a weak equivalence, while  $X_0 \to Z_0$  and  $Y_0 \to Z_0$  are fibrations, the map  $\tilde{f}_0$  is a weak equivalence by Corollary 3.1.5. Hence  $f_n$  is a weak equivalence by 2-out-of-3.  $\Box$ 

**2.1.6.** Undercategory. (a) For  $X \in sSp$  and  $x \in X$ , we set  $x \setminus X := \{x\} \times_X X^{F[1]} \to X$ , where  $X \to X^{F[1]}$  is induced by  $s^0 : F[1] \to F[0]$ , and put  $id_x := s_0(x) \in \{x\} \times_{X_0} X_1 = (x \setminus X)_0$ .

(b) We claim that the projection  $\operatorname{pr}_2 : \operatorname{id}_x \setminus (x \setminus X) \to x \setminus X$  has a section r such that  $r(\operatorname{id}_x) = \operatorname{id}_{\operatorname{id}_x}$ . Indeed, set  $A := (F[1] \times \{0\}) \cup (\{0\} \times F[1]) \subset F[1] \times F[1]$ . Then  $\operatorname{id}_x \setminus (x \setminus X) \subset X^{F[1] \times F[1]}$  can be written as  $\{x\} \times_{X^A} X^{F[1] \times F[1]}$ . Therefore the map  $m : [1] \times [1] \to [1]$  defined by m(i, j) := ij induces a map  $m : F[1] \times F[1] \to F[1]$  such that m(A) = 0. Hence m induces a map  $r : x \setminus X \to \operatorname{id}_x \setminus (x \setminus X)$ , which satisfies  $r(\operatorname{id}_x) = \operatorname{id}_{\operatorname{id}_x}$  and  $\operatorname{pr}_2 \circ r = \operatorname{Id}$ .

The following result is one of the main steps in the proof of the Yoneda lemma.

**Proposition 2.1.7.** For every left fibration  $\pi : E \to X$  and  $x \in X$ , the evaluation map  $\operatorname{ev}_{\operatorname{id}_x} : \operatorname{Map}_X(x \setminus X, E) \to \operatorname{Map}_X(\{\operatorname{id}_x\}, E) = (E_x)_0$ , induced by the inclusion  $\{\operatorname{id}_x\} \hookrightarrow x \setminus X$ , is a trivial fibration.

*Proof.* Since  $\{\mathrm{id}_x\} \hookrightarrow x \setminus X$  is a cofibration, while  $E \to X$  is a fibration, the map  $\mathrm{ev}_{\mathrm{id}_x}$  is a fibration (see 1.3.7 (c)). Therefore it remains to show that for each  $\alpha \in E_x$ , the Kan complex  $\mathrm{Map}_X(x \setminus X, E)_\alpha := \mathrm{ev}_{\mathrm{id}_x}^{-1}(\alpha)$  is contractible (by Lemma 1.2.9 (b)). Using remark 1.3.12, it suffices to show that the identity map of  $\mathrm{Map}_X(x \setminus X, E)_\alpha$  factors through a contractible Kan complex.

Since  $E \to X$  is a left fibration, the projection  $E^{F[1]} \to E \times_X X^{F[1]}$  is a trivial fibration. Thus  $\alpha \setminus E \to x \setminus X$ , being its fiber over  $\alpha \in E$ , is a trivial fibration. Therefore the evaluation map  $\operatorname{ev}'_{\operatorname{id}_x} : \operatorname{Map}_X(x \setminus X, \alpha \setminus E) \to ((\alpha \setminus E)_{\operatorname{id}_x})_0$  is a fibration between contractible Kan complexes. Hence  $\operatorname{ev}'_{\operatorname{id}_x}$  is a weak equivalence, thus a trivial fibration. Therefore its fiber  $\operatorname{ev}'_{\operatorname{id}_x}(\operatorname{id}_\alpha) = \operatorname{Map}_{x \setminus X}(x \setminus X, \alpha \setminus E)_{\operatorname{id}_\alpha}$  is a contractible Kan complex.

Note that the projection  $pr_2 : \alpha \setminus E \to E$  induces a projection

 $\rho: \operatorname{Map}_{x \setminus X}(x \setminus X, \alpha \setminus E)_{\operatorname{id}_{\alpha}} \to \operatorname{Map}_X(x \setminus X, E)_{\alpha}.$ 

Thus it remains to show that  $\rho$  has a section.

The natural morphism  $\operatorname{Map}(x \setminus X, E) \to \operatorname{Map}((x \setminus X)^{F[1]}, E^{F[1]})$  induces a morphism  $s' : \operatorname{Map}_X(x \setminus X, E)_{\alpha} \to \operatorname{Map}_{x \setminus X}(\operatorname{id}_x \setminus (x \setminus X), \alpha \setminus E)$ . By 2.1.6, the projection  $\operatorname{pr}_2 : \operatorname{id}_x \setminus (x \setminus X) \to x \setminus X$  has a section r such that  $r(\operatorname{id}_x) = \operatorname{id}_{\operatorname{id}_x}$ . Then

$$r^* \circ s' : \operatorname{Map}_X(x \setminus X, E)_{\alpha} \to \operatorname{Map}_{x \setminus X}(\operatorname{id}_x \setminus (x \setminus X), \alpha \setminus E) \to \operatorname{Map}_{x \setminus X}(x \setminus X, \alpha \setminus E)$$

has an image in  $\operatorname{Map}_{x \setminus X}(x \setminus X, \alpha \setminus E)_{\operatorname{id}_{\alpha}}$  and is a section of  $\rho$ .

**Corollary 2.1.8.** Let  $\pi : E \to X$  be a left fibration,  $x \in X$ , and let f and g be maps  $x \setminus X \to E$  over X such that  $f(\operatorname{id}_x) \sim g(\operatorname{id}_x) \in E_x$ . Then  $f \sim_X g$ . In particular, f is a weak equivalence if and only if g is a weak equivalence.

*Proof.* Since  $ev_{id_x}$  is a trivial fibration (by Proposition 2.1.7), the induced map  $\pi_0(ev_{id_x})$  is a bijection by 1.2.8 (b).

**2.1.9. Remarks.** Let X be a Segal space. (a) Then  $(\delta_{01}, \delta_{12}) : X_2 \to X_1 \times_{X_0} X_1$ and its pullback  $\delta_{12} : X_0 \times_{s_0, X_1, \delta_{01}} X_2 \to X_1$  are trivial fibrations.

(b) The map  $\delta_{02} : X_0 \times_{s_0, X_1, \delta_{01}} X_2 \to X_1$  is a fibration. Indeed,  $\delta_{12}$  is a pullback of the map  $(\delta_{01}, \delta_{02}) : X_2 \to X_1 \times_{X_0} X_1$ , induced by the inclusion  $\delta^{01}F[1] \cup \delta^{02}F[1] \hookrightarrow F[2]$ . Thus it is a fibration, because X is fibrant.

(c) The map  $\delta_{02}$  from (b) is a weak equivalence. Indeed, the map  $r = (\delta_0, \delta_{001})$ :  $X_1 \to X_0 \times_{s_0, X_1, \delta_{01}} X_2$  satisfy  $\delta_{12} \circ r = \delta_{02} \circ r = \text{Id.}$  Since  $\delta_{12}$  is a weak equivalence (by (a)), we deduce that r and  $\delta_{02}$  are weak equivalences by 2-out-of-3.

**Lemma 2.1.10.** Let X be a Segal space and  $x \in X$ . Then  $x \setminus X \to X$  is a left fibration.

*Proof.* Since X is fibrant, the projection  $X^{F[1]} \to X^{\partial F[1]} = X \times X$  is fibration, hence its pullback  $x \setminus X \to X$  is a fibration. It remains to show that the map  $(x \setminus X)^{F[1]} \to (x \setminus X) \times_X X^{F[1]}$  is a weak equivalence, or, equivalently, that the map  $((x \setminus X)^{F[1]})_n \to (x \setminus X)_n \times_{X_n} (X^{F[1]})_n$  is a weak equivalence for all n.

 $\begin{array}{l} ((x \setminus X)^{F[1]})_n \to (x \setminus X)_n \times_{X_n} (X^{F[1]})_n \text{ is a weak equivalence for all } n. \\ ((x \setminus X)^{F[1]})_n \to (x \setminus X)_n \times_{X_n} (X^{F[1]})_n = \operatorname{Map}(F[m] \times F[n], X) = (X^{F[n]})_m, \text{ we can} \\ \text{rewrite the last map in the form } (x \setminus X^{F[n]})_1 \to (x \setminus X^{F[n]})_0 \times_{(X^{F[n]})_0} (X^{F[n]})_1. \\ \text{Since } X^{F[n]} \text{ is also a Segal space (see 1.4.6), we can replace } X \text{ by } X^{F[n]}. \text{ It remains} \\ \text{to show that the map } (x \setminus X)_1 \to (x \setminus X)_0 \times_{X_0} X_1 \text{ is a trivial fibration.} \end{array}$ 

Using decomposition  $F[1] \times F[1] = F[2] \sqcup_{F[1]} F[2]$ , we get a decomposition  $(X^{F[1]})_1 = X_2 \times_{\delta_{02}, X_1, \delta_{02}} X_2$ . Hence we get a decomposition

(2.1) 
$$(x \setminus X)_1 = (\{x\} \times_{X_1, \delta_{01}} X_2) \times_{(x \setminus X)_0} (\{x\} \times_{X_0, \delta_0} X_2),$$

which identifies the map  $(x \setminus X)_1 \to (x \setminus X)_0 \times_{X_0} X_1$  with a composition

$$(x \setminus X)_1 \xrightarrow{J} \{x\} \times_{X_0, \delta_0} X_2 \xrightarrow{g} (x \setminus X)_0 \times_{X_0} X_1.$$

We claim that f and g are trivial fibrations. Since g is a pullback of  $(\delta_{01}, \delta_{12})$ :  $X_2 \to X_1 \times_{X_0} X_1$ , while f is a pullback of  $\delta_{02} : \{x\} \times_{X_1, \delta_{01}} X_2 \to (x \setminus X)_0$ , hence a pullback of  $\delta_{02} : X_0 \times_{s_0, X_1, \delta_{01}} X_2 \to X_1$ , both assertions follow from 2.1.9.  $\Box$ 

### 2.2. The $\infty$ -category of spaces.

**2.2.1.** Overcategories. (a) For each  $K \in sSp$  we denote by [K] the category of "bisimplexes of K". Explicitly, the set objects of [K] is the disjoint union  $\sqcup_{n,m}K_{n,m}$  and for every  $a \in K_{n,m}$  and  $b \in K_{n',m'}$  the set of morphisms  $\operatorname{Mor}_{[K]}(a, b)$  is the set of  $\tau \in \operatorname{Mor}_{\Delta \times \Delta}([n', m'], [n, m])$  such that  $\tau^*(a) = b$ .

(b) Note that we have a natural isomorphism of categories  $sSp/K \to \operatorname{Fun}([K], Set)$ . Namely, each map  $f: X \to K$  defines a functor  $[K] \to Set$ , which sends  $a \in K_{n,m}$  to  $f_{n,m}^{-1}(a) \subset X_{n,m}$ . Conversely, every  $\phi : [K] \to Set$  gives rise to  $X_{\phi} \in sSp/K$ , where  $(X_{\phi})_{n,m} := \sqcup_{a \in K_{n,m}} \phi(a)$  with obvious transition maps.

(c) Every map  $\phi : L \to K$  in sSp induces a functor  $[\phi] : [L] \to [K]$ . Then the bijection of (b) identifies  $\phi^* : sSp/K \to sSp/L$  with the pullback functor  $[\phi]^* : \operatorname{Fun}([K], Set) \to \operatorname{Fun}([L], Set)$ . **2.2.2.** Universes. From now on we fix an infinite set  $\mathcal{U}$ , which we call a *universe*.

(a) Let  $Set_{\mathcal{U}} \subset Set$  be the category of subsets of  $\mathcal{U}$ , and let  $Set_{|\mathcal{U}|}$  the category of sets of cardinality  $\leq |\mathcal{U}|$ . Then category  $Set_{\mathcal{U}}$  is small, and the natural embedding  $Set_{\mathcal{U}} \to Set_{|\mathcal{U}|}$  is an equivalences of categories.

(b) We set  $Sp_{\mathcal{U}} := \operatorname{Fun}(\Delta^{op}, Set_{\mathcal{U}}) \subset Sp$  and  $sSp_{\mathcal{U}} := \operatorname{Fun}(\Delta^{op}, Sp_{\mathcal{U}}) \subset sSp$ .

(c) More generally, for every  $K \in sSp$ , we denote by  $(sSp/K)_{\mathcal{U}} \subset sSp/K$  (resp.  $(sSp/K)_{|\mathcal{U}|} \subset sSp/K$ ) the full subcategory of morphisms  $f: X \to K$  such that fibers of all  $f_{n,m}: X_{n,m} \to K_{n,m}$  belong to  $Set_{\mathcal{U}}$  (resp.  $Set_{|\mathcal{U}|}$ ).

(d) Bijection of 2.2.1 (b) induces a bijection between  $(sSp/K)_{\mathcal{U}}$  (resp.  $(sSp/K)_{|\mathcal{U}|}$ ) and functors  $[K] \to Set_{\mathcal{U}}$  (resp.  $[K] \to Set_{|\mathcal{U}|}$ ). In particular, category  $(sSp/K)_{\mathcal{U}}$ is small, and the inclusion  $(sSp/K)_{\mathcal{U}} \to (sSp/K)_{|\mathcal{U}|}$  is an equivalence of categories.

(e) We denote by  $(LFib/K)_{\mathcal{U}}$  the set of left fibrations  $X \to K$ , belonging to  $(sSp/K)_{\mathcal{U}}$ . By (d), 2.2.1 (c) and Lemma 2.1.2 (a), for every map  $\phi : L \to K$ , the pullback functor  $\phi^* : sSp/K \to sSp/L$  maps  $(LFib/K)_{\mathcal{U}}$  to  $(LFib/L)_{\mathcal{U}}$ .

**2.2.3.** Main construction. (a) Let  $\mathfrak{S}_{\mathcal{U}} \in sSp$  be the simplicial space such that •  $(\mathfrak{S}_{\mathcal{U}})_{n,m}$  is the set of left fibrations  $(LFib/\Box[n,m])_{\mathcal{U}}$ ;

• for every  $a \in (\mathfrak{S}_{\mathcal{U}})_{n,m}$  with the corresponding left fibration  $E_a \to \Box[n,m]$  and every  $\nu : [n',m'] \to [n,m]$ , we have  $E_{\nu^*(a)} = \nu^*(E_a)$  (use remark 2.2.2 (e)).

(b) Consider the "universal left fibration"  $p_{\mathcal{U}} : \mathcal{E}_{\mathcal{U}} \to \mathfrak{S}_{\mathcal{U}}$ , where  $(\mathcal{E}_{\mathcal{U}})_{n,m}$  is defined to be the disjoint union  $\sqcup_{a \in (\mathfrak{S}_{\mathcal{U}})_{n,m}} (E_a)_{n,m}$ , and  $p_{\mathcal{U}}$  is the map, which maps each  $(E_a)_{n,m}$  to  $a \in (\mathfrak{S}_{\mathcal{U}})_{n,m}$ .

**Lemma 2.2.4.** The map  $p_{\mathcal{U}} : \mathcal{E}_{\mathcal{U}} \to \mathfrak{S}_{\mathcal{U}}$  is a left fibration. For each  $K \in sSp$ , the map  $\phi \mapsto \phi^*(p_{\mathcal{U}})$  defines a bijection between  $\operatorname{Hom}_{sSp}(K,\mathfrak{S}_{\mathcal{U}})$  and  $(LFib/K)_{\mathcal{U}}$ .

*Proof.* By construction, for every  $a \in (\mathfrak{S}_{\mathcal{U}})_{n,m} = \text{Hom}(\Box[n,m],\mathfrak{S}_{\mathcal{U}})$ , the pullback  $a^*(p_{\mathcal{U}})$  equals  $E_a \to \Box[n,m]$ . In particular, each  $a^*(p_{\mathcal{U}})$  is a left fibration. Thus p is a left fibration by remark 2.1.4 (a).

Next notice that for every  $\phi: K \to \mathfrak{S}_{\mathcal{U}}$ , the pullback  $\phi^*(p_{\mathcal{U}}): \phi^*(\mathcal{E}_{\mathcal{U}}) \to K$  is a left fibration, satisfying  $a^*(\phi^*(p_{\mathcal{U}})) = (\phi \circ a)^*(p_{\mathcal{U}}) \in (LFib/\Box[n,m])_{\mathcal{U}}$  for each  $a: \Box[n,m] \to K$ . Thus  $\phi^*(p_{\mathcal{U}}) \in (LFib/K)_{\mathcal{U}}$ .

Conversely, every  $E \in (LFib/K)_{\mathcal{U}}$  defines a map  $\phi_E : K \to \mathfrak{S}_{\mathcal{U}}$ , which sends  $a \in K_{n,m} = \operatorname{Hom}(\Box[n,m],K)$  to the left fibration  $a^*(E) \to \Box[n,m]$  in  $(LFib/\Box[n,m])_{\mathcal{U}}$ . Then the map  $E \mapsto \phi_E$  is inverse to  $\phi \mapsto \phi^*(p_{\mathcal{U}})$ .

**2.2.5. Remarks.** (a) The main result of this subsection (Theorem 2.2.11) asserts that  $\mathfrak{S}_{\mathcal{U}}$  is a complete Segal space. It is our model for the  $\infty$ -category of spaces, or, more formally, the  $(\infty, 1)$ -category of  $(\infty, 0)$ -categories.

(b) It can be shown that every inclusion  $i : \mathcal{U} \hookrightarrow \mathcal{V}$  of infinite sets induces a fully faithful map  $i : \mathfrak{S}_{\mathcal{U}} \hookrightarrow \mathfrak{S}_{\mathcal{V}}$  of complete Segal spaces.

(c) One can show (see [KV2]) that  $\mathfrak{S}_{\mathcal{U}}$  is equivalent to the fibrant replacement  $N^f(Sp_{\mathcal{U}}, W)$  of the simplicial space  $N(Sp_{\mathcal{U}}, W)$ , associated by Rezk ([Re1, 3.3]) to the pair  $(Sp_{\mathcal{U}}, W)$ , where W denotes weak equivalences.

(d) One can also consider the "large"  $(\infty, 1)$ -category of  $(\infty, 0)$ -categories  $\widehat{\mathfrak{S}}$  such that  $\widehat{\mathfrak{S}}_{n,m}$  is the class of all left fibrations  $E \to \Box[n,m]$ .

(e) In [KV2] we generalize 2.2.3 and construct the  $(\infty, n+1)$ -category of  $(\infty, n)$ -categories.

**2.2.6.** Notation. (a) For every  $n \geq 1$  denote by  $\mathfrak{S}_{\mathcal{U}}^{(n)} \in sSp$  the simplicial space such that  $(\mathfrak{S}^{(n)})_{m,k}$  is the set of diagrams  $\phi : E^{(0)} \xrightarrow{\phi_1} \dots \xrightarrow{\phi_n} E^{(n)}$  over  $\Box[m,k]$ , where each  $E^{(i)} \to \Box[m,k]$  belongs to  $(LFib/\Box[m,k])_{\mathcal{U}}$ .

(b) To every map  $\mu : [m] \to [n]$  we associate morphism  $\mu^* : \mathfrak{S}_{\mathcal{U}}^{(n)} \to \mathfrak{S}_{\mathcal{U}}^{(m)}$ , which sends diagram  $\phi : E^{(0)} \xrightarrow{\phi_1} \dots \xrightarrow{\phi_n} E^{(n)}$  to a diagram  $\mu^*(\phi) : E^{(\mu(0))} \to \dots \to E^{(\mu(m))}$ , whose morphisms are compositions of the  $\phi_i$ 's.

(c) Let  $\mathfrak{S}_{\mathcal{U}}^{we} \subset \mathfrak{S}_{\mathcal{U}}^{(1)}$  be a simplicial subspace such that  $(\mathfrak{S}^{(we)})_{m,k} \subset (\mathfrak{S}^{(1)})_{m,k}$  consists of diagrams consists of diagrams  $E^{(0)} \xrightarrow{\phi} E^{(1)}$ , where  $\phi$  is a weak equivalence (use Lemma 1.3.11 (b)).

(d) We have a natural projection  $\mathfrak{S}_{\mathcal{U}}^{(n)} \to (\mathfrak{S}_{\mathcal{U}})^{n+1}$ , which maps a diagram  $\phi$  as in (a) to the (n+1)-tuple  $E^{(0)}, \ldots, E^{(n)}$ .

**2.2.7. Remarks.** (a) Note that for every  $X, Y \in sSp/K$ , to give a map  $\phi \in \operatorname{Hom}_{K}(X,Y)$  is the same as to give maps  $\tau^{*}(\phi) \in \operatorname{Hom}_{\Box[n,m]}(\tau^{*}(X),\tau^{*}(Y))$  for all  $\phi : \Box[n,m] \to K$ , compatible with compositions. Using this observation and Lemma 2.2.4, we conclude that for every  $K \in sSp$  we have a natural bijection between  $\operatorname{Hom}_{sSp}(K,\mathfrak{S}_{\mathcal{U}}^{(n)})$  and set of diagrams  $\phi : E^{(0)} \xrightarrow{\phi_{1}} \ldots \xrightarrow{\phi_{n}} E^{(n)}$  of left fibrations from  $(sSp/K)_{\mathcal{U}}$ .

(b) By definition, a map  $\phi \in \operatorname{Hom}_{sSp}(K, \mathfrak{S}_{\mathcal{U}}^{(1)})$  belongs to  $\operatorname{Hom}_{sSp}(K, \mathfrak{S}_{\mathcal{U}}^{(we)})$  if and only if  $\phi(a) \in (\mathfrak{S}_{\mathcal{U}}^{(we)})_{n,m}$  for every  $a \in K_{n,m}$ . Moreover, by Lemma 2.1.5, it happens if and only if  $\phi(a) \in (\mathfrak{S}_{\mathcal{U}}^{(we)})_{0,0}$  for every  $a \in K_{0,0}$ . Using Lemma 2.1.5 again, we see that under the bijection of (a) elements of  $\operatorname{Hom}_{sSp}(K, \mathfrak{S}_{\mathcal{U}}^{(we)}) \subset$  $\operatorname{Hom}_{sSp}(K, \mathfrak{S}_{\mathcal{U}}^{(1)})$  correspond to weak equivalences  $\phi : E^{(0)} \to E^{(1)}$ .

The following two propositions and a corollary will be shown in Section 4.

**Proposition 2.2.8.** (a) The simplicial space  $\mathfrak{S}_{\mathcal{U}} \in sSp$  is Ready fibrant.

- (b) The projections  $\mathfrak{S}_{\mathcal{U}}^{(n)} \to (\mathfrak{S}_{\mathcal{U}})^{n+1}$  and  $\mathfrak{S}_{\mathcal{U}}^{(we)} \to (\mathfrak{S}_{\mathcal{U}})^2$  are fibrations.
- (c) Both compositions  $\mathfrak{S}_{\mathcal{U}}^{(we)} \to (\mathfrak{S}_{\mathcal{U}})^2 \xrightarrow{p_i} \mathfrak{S}_{\mathcal{U}}$  are trivial fibrations.
- (d)  $(\mathfrak{S}_{\mathcal{U}}^{we})_0 \subset (\mathfrak{S}_{\mathcal{U}}^{(1)})_0$  is a union of a connected components.

**Proposition 2.2.9.** (a) There exists a homotopy equivalence  $(\mathfrak{S}_{\mathcal{U}})^{\Delta[1]} \to \mathfrak{S}_{\mathcal{U}}^{(we)}$ over  $(\mathfrak{S}_{\mathcal{U}})^2$ .

(b) For every  $n \in \mathbb{N}$  there exists a "natural" homotopy equivalence  $\psi_{\mathcal{U}}^{(n)} : \mathfrak{S}_{\mathcal{U}}^{(n)} \to (\mathfrak{S}_{\mathcal{U}})^{F[n]}$  over  $(\mathfrak{S}_{\mathcal{U}})^{n+1}$ , defined uniquely up to a homotopy.

(c) Moreover, for every map  $\mu : [m] \to [n]$  the diagram

(2.2) 
$$\begin{array}{c} \mathfrak{S}_{\mathcal{U}}^{(n)} \xrightarrow{\psi^{(n)}} (\mathfrak{S}_{\mathcal{U}})^{F[n]} \\ \mu^* \downarrow \qquad \mu^* \downarrow \\ \mathfrak{S}_{\mathcal{U}}^{(m)} \xrightarrow{\psi^{(m)}} (\mathfrak{S}_{\mathcal{U}})^{F[m]} \end{array}$$

is homotopy commutative, that is,  $\mu^* \circ \psi^{(n)} \sim \psi^{(m)} \circ \mu^*$ .

**Corollary 2.2.10.** Let  $K \in sSp$ , let  $\alpha, \beta \in \text{Hom}(K, \mathfrak{S}_{\mathcal{U}})$ , and let  $E_{\alpha} \to K$  and  $E_{\beta} \to K$  be the corresponding left fibrations. Then  $\alpha \sim \beta$  in  $(\mathfrak{S}_{\mathcal{U}})^K$  if and only if the left fibrations  $E_{\alpha}$  and  $E_{\beta}$  are homotopy equivalent over K.

Now we are ready to prove one of the main results of this work.

### **Theorem 2.2.11.** $\mathfrak{S}_{\mathcal{U}}$ is a complete Segal space.

*Proof.* We denote  $\mathfrak{S}_{\mathcal{U}}$  simply by  $\mathfrak{S}$ . Then  $\mathfrak{S}$  is fibrant by Proposition 2.2.8 (a).

To show that  $\mathfrak{S}$  is Segal, we have to prove that for every  $n \geq 2$  the morphism  $\varphi_n : \mathfrak{S}_n \to \mathfrak{S}_1 \times \mathfrak{S}_0 \dots \times \mathfrak{S}_0 \mathfrak{S}_1$  is a weak equivalence. Applying Proposition 2.2.9 (c) to  $(\delta^{01}, \dots, \delta^{n-1,n}) : [n] \to [1] \times \dots \times [1]$ , we get a homotopy commutative diagram

$$\begin{array}{cccc} \mathfrak{S}^{(n)} & \xrightarrow{\psi^{(n)}} & \mathfrak{S}^{F[n]} \\ & & \downarrow & & \downarrow \\ \mathfrak{S}^{(1)} \times_{\mathfrak{S}} \dots \times_{\mathfrak{S}} \mathfrak{S}^{(1)} & \xrightarrow{\psi^{(1)} \times \dots \times \psi^{(1)}} \mathfrak{S}^{F[1]} \times_{\mathfrak{S}} \dots \times_{\mathfrak{S}} \mathfrak{S}^{F[1]}. \end{array}$$

We want to show that the right vertical arrow is a weak equivalence, which implies the Segal conditions by passing to the zero spaces. The top horizontal arrow is a weak equivalence by Proposition 2.2.9 (b). The bottom horizontal arrow is a equivalence by Proposition 2.2.9 (b) together with the observation that  $\mathfrak{S}^{(1)} \to \mathfrak{S}$ and  $\mathfrak{S}^{F[1]} \to \mathfrak{S}$  are fibrations (use Proposition 2.2.8 and Corollary 3.1.5). Next, since the left vertical arrow is a bijection, while diagram is homotopy commutative, the right vertical arrow is a weak equivalence, by 2-out-of-3.

To show that  $\mathfrak{S}$  is complete, we have to show that  $\delta_0 : \mathfrak{S}_{heq} \to \mathfrak{S}_0$  is a trivial fibration (by Lemma 1.4.5 (b)). Since  $p_0 : \mathfrak{S}^{(we)} \to \mathfrak{S}$  is a trivial fibration by Proposition 2.2.8 (c), it is enough to show that the map  $\psi := (\psi^{(1)})_0 : (\mathfrak{S}^{(1)})_0 \to (\mathfrak{S}^{F[1]})_0 = \mathfrak{S}_1$  from Proposition 2.2.9 (b) induces an equivalence  $(\mathfrak{S}^{(we)})_0 \to \mathfrak{S}_{heq}$ .

Both  $(\mathfrak{S}^{(we)})_0 \subset (\mathfrak{S}^{(1)})_0$  and  $\mathfrak{S}_{heq} \subset \mathfrak{S}_1$  are unions of connected components (by Proposition 2.2.8 (d) and 1.4.4 (b)). Since  $\psi$  is a weak equivalence, it remains to show that  $\pi_0(\psi)$  induces a bijection  $\pi_0((\mathfrak{S}^{(we)})_0) \to \pi_0(\mathfrak{S}_{heq})$ .

By Proposition 2.2.9 (c), we have the following homotopy commutative diagram

Recall that in Lemma 1.4.5 (a) we introduced unions of connected components  $\mathfrak{S}'_1 \subset \mathfrak{S}_1$  and  $\mathfrak{S}'_3 \subset \mathfrak{S}_3$  and showed that  $\delta_{12}(\mathfrak{S}'_3) = \mathfrak{S}_{heq}$ .

Similarly, we define  $(\mathfrak{S}^{(1)})'_0 \subset (\mathfrak{S}^{(1)})_0$  to be the union of connected components, intersecting  $s_0(\mathfrak{S}_0)$ , and set  $(\mathfrak{S}^{(3)})'_0 := \delta_{02}^{-1}((\mathfrak{S}^{(1)})'_0) \cap \delta_{13}^{-1}((\mathfrak{S}^{(1)})'_0) \subset (\mathfrak{S}^{(3)})_0$ . We claim that  $\delta_{12}((\mathfrak{S}^{(3)})'_0) = (\mathfrak{S}^{(we)})_0$ . Indeed, since  $s_0(\mathfrak{S}_0) \subset \mathfrak{S}^{(we)}$ , it follows

We claim that  $\delta_{12}((\mathfrak{S}^{(3)})_0') = (\mathfrak{S}^{(we)})_0$ . Indeed, since  $s_0(\mathfrak{S}_0) \subset \mathfrak{S}^{(we)}$ , it follows from Proposition 2.2.8 (d) that for every  $\widetilde{\phi} \in (\mathfrak{S}^{(3)})_0'$  we have  $\delta_{02}(\widetilde{\phi}), \delta_{13}(\widetilde{\phi}) \in \mathfrak{S}^{(we)}$ . In other words, if  $\widetilde{\phi}$  corresponds to a diagram  $E^{(0)} \xrightarrow{\phi_1} E^{(1)} \xrightarrow{\phi_2} E^{(2)} \xrightarrow{\phi_3} E^{(3)}$ , then  $\phi_2 \circ \phi_1$  and  $\phi_3 \circ \phi_2$  are weak equivalences. Therefore  $\phi_2$  have left and right homotopy inverses. Hence  $\phi_2$  is a weak equivalence, thus  $\phi_2 = \delta_{12}(\widetilde{\phi}) \in \mathfrak{S}^{(we)}$ .

Conversely, every  $\phi \in \mathfrak{S}^{(we)}$  corresponds to a homotopy equivalence  $\phi : E^{(0)} \to E^{(1)}$  (by Lemma 1.3.11 (b)), thus there exists a diagram  $\tilde{\phi} : E^{(1)} \xrightarrow{\phi'} E^{(0)} \xrightarrow{\phi} E^{(1)} \xrightarrow{\phi'} E^{(0)}$  such that  $\phi' \circ \phi \sim \operatorname{Id}_{E^{(0)}}$  and  $\phi \circ \phi' \sim \operatorname{Id}_{E^{(1)}}$ . By definition,  $\tilde{\phi}$  corresponds to an element of  $(\mathfrak{S}^{(3)})'_0$  and  $\delta_{12}(\tilde{\phi}) = \phi$ .

Now we are ready to show the assertion. Since  $\psi^{(3)}$  is a weak equivalence, the induced map  $\pi_0(\mathfrak{S}^{(3)})_0) \to \pi_0(\mathfrak{S}_3)$  is a bijection. Next, using definitions of  $(\mathfrak{S}^{(3)})'_0$  and  $\mathfrak{S}'_3$  and the homotopy commutativity of the interior and right inner squares of  $(2.3), \psi^{(3)}$  induces a bijection  $\pi_0((\mathfrak{S}^{(3)})'_0) \to \pi_0(\mathfrak{S}'_3)$ . Finally, since  $\mathfrak{S}_{heq} = \delta_{12}(\mathfrak{S}'_3), (\mathfrak{S}^{(we)})_0 = \delta_{12}((\mathfrak{S}^{(3)})'_0)$ , and the left inner square of (2.3) is homotopy commutative,  $\psi^{(1)}$  induces a bijection  $\pi_0(\mathfrak{S}^{(we)}) \to \pi_0(\mathfrak{S}_{heq})$ .

# 2.3. The Yoneda embedding.

**2.3.1. The opposite simplicial space.** (a) For every map  $\tau : [n] \to [m]$  in  $\Delta$ , we denote by  $\iota(\tau) : [n] \to [m]$  the map  $\iota(\tau)(n-i) := m - \tau(i)$ . Then  $\iota$  defines a functor  $\Delta \to \Delta$ , hence a functor  $\iota^* : sSp = \operatorname{Hom}(\Delta^{op}, Sp) \to \operatorname{Hom}(\Delta^{op}, Sp) = sSp$ .

(b) For every  $X \in sSp$ , we set  $X^{op} := \iota^*(X) \in sSp$ . Explicitly, we have  $(X^{op})_n = X_n$  for all n, and for every  $\tau : [n] \to [m]$  the map  $\tau^* : (X^{op})_m \to (X^{op})_n$  is the map  $\iota(\tau)^* : X_m \to X_n$ .

(c) Note that if X is a (complete) Segal space, then  $X^{op}$  is also such, and we have equality of homotopy categories  $\operatorname{Ho}(X^{op}) = (\operatorname{Ho} X)^{op}$ . Therefore we call  $X^{op}$  the opposite simplicial space.

**2.3.2. The twisted arrow category.** (a) Consider the functor  $\mu : \Delta \to \Delta$  such that  $\mu([n]) = [2n+1]$ , and for every  $\tau : [n] \to [m]$  in  $\Delta$  the map  $\mu(\tau) : [2n+1] \to [2m+1]$  is defined by formulas  $\mu(\tau)(n-i) := m - \tau(i)$  and  $\mu(\tau)(n+1+j) = (m+1+\tau(j))$  for  $i, j = 0, \ldots, n$ .

(b) For every  $X \in sSp$ , we define simplicial space  $\mathcal{M}(X) := \mu^*(X) \in sSp$ . Explicitly, we have  $\mathcal{M}(X)_n = X_{2n+1}$  for all n, and for every  $\tau : [n] \to [m]$  the map  $\tau^* : \mathcal{M}(X)_m \to \mathcal{M}(X)_n$  is the map  $\mu(\tau)^* : X_{2m+1} \to X_{2n+1}$ .

(c) We have natural morphisms  $\iota \to \mu$  and  $\operatorname{Id} \to \mu$  of functors  $\operatorname{Hom}(\Delta, \Delta)$  which correspond to maps  $e^0 : [n] \to [2n+1]$  and  $e^{n+1} : [n] \to [2n+1]$ , respectively. These maps corresponds to a morphism  $\pi_X : \mathcal{M}(X) \to X^{op} \times X$  in sSp.

**2.3.3. Remarks.** (a) Note that  $\Delta$  is equivalent to the category  $\Delta'$  of finite totally ordered sets, and functors  $\iota, \mu : \Delta \to \Delta$  correspond to functors  $\iota, \mu : \Delta' \to \Delta'$  defined by  $\iota(P) = P^{op}$  and  $\mu(P) = P^{op} * P$ , the "join" of  $P^{op}$  and P.

(b) It can be shown (using Lemma 2.3.4 and 2.1.4 (b)) that if X is a (complete) Segal space, then  $\mathcal{M}(X)$  is a (complete) Segal space as well. In this case, the space of objects  $\mathcal{M}(X)_0$  equals the space of morphisms  $X_1$ , and for every  $\alpha : x \to y$  and  $\alpha' : x' \to y'$  in  $\mathcal{M}(X)_0 = X_1$ , the mapping space map<sub> $\mathcal{M}(X)$ </sub> $(\alpha, \alpha')$  can be intuitively thought as the space of triples  $(\beta, \beta', \gamma)$ , where  $\beta : x' \to x$  and  $\beta' : y \to y'$  belong to  $X_1$ , and  $\gamma$  is a path between  $\beta' \circ \alpha \circ \beta$  and  $\alpha'$ .

From now on in this subsection we always assume that X is a Segal space.

**Lemma 2.3.4.** The map  $\pi_X : \mathcal{M}(X) \to X^{op} \times X$  is a left fibration.

*Proof.* To show that  $\pi_X$  is a fibration, we have to check that the induced map

(2.4) 
$$\mathcal{M}(X)_n \to \mathcal{M}(X)_{\partial n} \times_{(X^{op} \times X)_{\partial n}} (X^{op} \times X)_n$$

is a fibration for every  $n \ge 0$ . Recall that  $\mathcal{M}(X)_n = X_{2n+1} = \operatorname{Map}(F[2n+1], X)$ . Since  $\partial F[n] = \bigcup_{i=0}^n d^i F[n-1]$  we get that  $\mathcal{M}(X)_{\partial n} = \operatorname{Map}(F[2n+1]', X)$ , where  $F[2n+1]' \subset F[2n+1]$  is the union  $\bigcup_{i=0}^n d^{i,2n+1-i}(F[2n-1]) \subset F[2n+1]$ . Thus morphism (2.4) can be identified with the morphism

 $\operatorname{Map}(F[2n+1], X) \to \operatorname{Map}(F[2n+1]' \sqcup_{(e^0 \partial F[n] \cup e^{n+1} \partial F[n])} (e^0 F[n] \cup e^{n+1} F[n]), X).$ 

Since  $F[2n+1]' \cap (e^0 F[n] \cup e^{n+1} F[n]) = e^0 \partial F[n] \cup e^{n+1} \partial F[n] \subset F[2n+1]$ , the natural map  $F[2n+1]' \sqcup_{(e^0 \partial F[n] \cup e^{n+1} \partial F[n])} (e^0 F[n] \cup e^{n+1} F[n]) \to F[2n+1]$  is a cofibration. Since X is fibrant, the map (2.4) is a fibration.

It remains to show that the fibration  $\mathcal{M}(X)_n \to \mathcal{M}(X)_0 \times_{(X^{op} \times X)_0} (X^{op} \times X)_n$ or, equivalently,  $X_{2n+1} \to X_n \times_{X_0} X_1 \times_{X_0} X_n$  is a weak equivalence. Since X is a Segal space, thus both the composition

$$X_{2n+1} \to X_n \times_{X_0} X_1 \times_{X_0} X_n \to (X_1 \times_{X_0} \dots \times_{X_0} X_1) \times_{X_0} X_1 \times_{X_0} (X_1 \times_{X_0} \dots \times_{X_0} X_1)$$
  
and the second morphism are trivial fibrations, this follows from 2-out-of-3.  $\Box$ 

**2.3.5. Remark.** It follows from Lemma 2.3.4 and Lemma 2.1.2 (a) that left fibration  $\pi_X$  induces a left fibration  $\{x\} \times_{X^{op}} \mathcal{M}(X) \to X$  for every  $x \in X$ . Notice that  $(\{x\} \times_{X^{op}} \mathcal{M}(X))_0 = \{x\} \times_{X_0} X_1 = (x \setminus X)_0$ .

**Lemma 2.3.6.** Let X be a Segal space and  $x \in X$ . There exists a weak equivalence  $\widetilde{\phi} : x \setminus X \to \{x\} \times_{X^{op}} \mathcal{M}(X)$  of left fibrations over X such that  $\widetilde{\phi}(\mathrm{id}_x) \sim \mathrm{id}_x$ .

*Proof.* It will suffice to construct a homotopy equivalence  $\psi : \{x\} \times_{X^{op}} \mathcal{M}(X) \to x \setminus X$  over X such that  $\psi(\mathrm{id}_x) \sim \mathrm{id}_x$  and to take  $\phi$  to be its homotopy inverse.

To construct  $\psi$ , we will construct a simplicial space  $x \setminus X$  over X and maps  $\psi' : \widetilde{x \setminus X} \to x \setminus X$  and  $\psi'' : \widetilde{x \setminus X} \to \{x\} \times_{X^{op}} \mathcal{M}(X)$  over X such that  $(\widetilde{x \setminus X})_0 = (x \setminus X)_0$ ,  $\psi'_0 = \psi'_0 = \text{Id}$  and  $\psi'$  is a trivial cofibration. Since  $x \setminus X \to X$  is a fibration, the map  $\psi''$  extends to a map  $\psi : \{x\} \times_{X^{op}} \mathcal{M}(X) \to x \setminus X$  over X.

In this case,  $\psi_0 = \text{Id}$  would be a weak equivalence, so  $\psi$  would be a weak equivalence by Lemma 2.1.5, hence a homotopy equivalence by Lemma 1.3.11 (b).

For every map  $\tau : [n] \to [m]$ , we denote by  $\tau'$  the map  $[n+1] \to [m+1]$  defined by  $\tau'(0) = 0$  and  $\tau'(i+1) = \tau(i) + 1$  for all  $i = 0, \ldots, n$ . Consider  $x \setminus X \in sSp$  such that  $(x \setminus X)_n := \{x\} \times_{X_0, \delta_0} X_{n+1}$ , and for every  $\tau$  the map  $\tau^* : (x \setminus X)_m \to (x \setminus X)_n$ is induced by  $\tau'^* : X_{m+1} \to X_{n+1}$ .

Note that projections  $e_1 : X_{n+1} \to X_n$  induce a projection  $x \setminus X \to X$ . Next, map  $[n] \times [1] \to [n+1] : (i,j) \mapsto (i+j)j$  induces maps  $F[n] \times F[1] \to F[n+1]$  and

$$p_n: X_{n+1} = \operatorname{Map}(F[n+1], X) \to \operatorname{Map}(F[n] \times F[1], X) = (X^{F[1]})_n$$

Then  $p_n$ 's give rise to a map  $\psi' : x \setminus X \to x \setminus X$  over X.

Finally, maps  $r: [2n+1] \to [n+1]$ , where r(i) = 0 and r(i+n+1) = i+1 for all  $i = 0, \ldots, n$  induce maps  $X_{n+1} \to X_{2n+1}$  and give rise to a map

 $\psi'': x \setminus X \to \{x\} \times_{X^{op}} \mathcal{M}(X)$  over X, which we claim is a trivial cofibration.

We have to show that  $\psi_n'': \{x\} \times_{X_0} X_{n+1} \to \{x\} \times_{X_n} X_{2n+1}$  is a trivial cofibration for all *n*. Since *X* is Segal, the natural map  $X_{2n+1} \to X_n \times_{X_0} X_{n+1}$  is a trivial fibration, whose pullback  $\pi_n: \{x\} \times_{X_n} X_{2n+1} \to \{x\} \times_{X_0} X_{n+1}$  is a trivial fibration, satisfying  $\pi_n \circ \psi_n'' = \text{Id}$ . Therefore  $\psi_n'': \{x\} \times_{X_0} X_{n+1} \to \{x\} \times_{X_n} X_{2n+1}$  is a trivial cofibration by 2-out-of-3, and the proof is complete.  $\Box$ 

# **2.3.7. The Yoneda embedding.** We fix an infinite set $\mathcal{U}$ , and set $\mathfrak{S} := \mathfrak{S}_{\mathcal{U}}$ .

(a) For every  $X \in sSp_{\mathcal{U}}$ , we set  $\mathfrak{P}(X) := \mathfrak{S}^{X^{op}}$ . Since  $\mathfrak{S}$  is a complete Segal space,  $\mathfrak{P}(X)$  is also a complete Segal space (by 1.4.6), and we call it the  $\infty$ -category of simplicial presheaves on X.

(b) By definition of  $\mathfrak{S}$ , for every Segal space X, the left fibration  $\pi_X : \mathcal{M}(X) \to X^{op} \times X$  corresponds to the morphism  $X^{op} \times X \to \mathfrak{S}$ , hence to the morphism  $j_X : X \to \mathfrak{P}(X)$ . We call  $j_X$  the Yoneda embedding of X.

(c) For every  $x \in X$  and  $\alpha \in \mathfrak{P}(X)$ , we form  $\alpha(x) \in \mathfrak{S}$  and denote by  $E_{\alpha(x)} \to \mathrm{pt}$  the corresponding left fibration.

**2.3.8. Example.** By definition, for every  $x \in X$  element  $j_X(x) \in \mathfrak{P}(X) = \mathfrak{S}^{X^{op}}$  corresponds to the left fibration  $\mathcal{M}(X) \times_X \{x\} = \{x\} \times_X \mathcal{M}(X^{op}) \to X^{op}$ .

**Theorem 2.3.9.** Let X be a Segal space,  $x \in X$  and  $\alpha \in \mathfrak{P}(X)$ .

(a) We have a "natural" weak equivalence  $\operatorname{map}_{\mathfrak{P}(X)}(j_X(x), \alpha) \to (E_{\alpha(x)})_0$ , canonical up to homotopy.

(b) The Yoneda embedding  $j_X : X \to \mathfrak{P}(X)$  is fully faithful (see 1.4.2 (d)).

First we have to introduce certain notation.

**2.3.10.** The universal left fibration over  $X^{op}$ . (a) Let  $\widetilde{E} \to X^{op} \times \mathfrak{P}(X)$  be the left fibration, corresponding to the evaluation map  $\operatorname{ev}_{X^{op}} : X^{op} \times \mathfrak{P}(X) \to \mathfrak{S}$ . Then for every  $\alpha \in \mathfrak{P}(X)$ , the pullback  $\widetilde{E}_{\alpha} := \widetilde{E} \times_{\mathfrak{P}(x)} \{\alpha\}$  is the left fibration over  $X^{op}$ , corresponding to  $\alpha$ . In particular, for every  $x \in X$ , its fiber  $(\widetilde{E}_{\alpha})_x$  is the left fibration  $E_{\alpha(x)} \to \operatorname{pt}$ , corresponding to  $\alpha(x) \in \mathfrak{S}$  (see 2.3.7 (c)).

(b) For every  $x \in X^{op}$ , we set  $\widetilde{E}_x := \{x\} \times_{X^{op}} \widetilde{E}$ . Then  $\widetilde{E}_x \to \mathfrak{P}(X)$  is a left fibration such that  $(\widetilde{E}_x)_{\alpha} = E_{\alpha(x)}$  for every  $\alpha \in \mathfrak{P}(X)$  (by (a)). For every  $y \in X$ we have an equality  $(\widetilde{E}_x)_{j_X(y)} = E_{j_X(y)(x)}$ , thus  $((\widetilde{E}_x)_{j_X(y)})_0 = \max_X(x, y)$ . In particular, we have an element  $\mathrm{id}_x \in \max_X(x, x)$  of  $(\widetilde{E}_x)_{j_X(x)}$ .

(c) By definition, left fibration  $\pi_X : \mathcal{M}(X) \to X^{op} \times X$  corresponds to the composition of  $\mathrm{Id} \times j_X : X^{op} \times X \to X^{op} \times \mathfrak{P}(X)$  and  $\mathrm{ev}_{X^{op}}$ . Therefore we have an equality  $\mathcal{M}(X) = \widetilde{E} \times_{\mathfrak{P}(X)} X$ , hence  $\widetilde{E}_x \times_{\mathfrak{P}(X)} X = \{x\} \times_{X^{op}} \mathcal{M}(X)$ .

**2.3.11. Remarks.** The homotopy equivalence  $\psi^{(1)} : \mathfrak{S}^{F[1]} \to \mathfrak{S}^{(1)}$  over  $\mathfrak{S}^2$  (see Proposition 2.2.9 (b)) induces a homotopy equivalence from  $\mathfrak{P}(X)^{F[1]} = (\mathfrak{S}^{F[1]})^{X^{op}}$ to  $\mathfrak{P}(X)^{(1)} := (\mathfrak{S}^{(1)})^{X^{op}}$  over  $\mathfrak{P}(X)^2$ . Hence for every  $\alpha \in \mathfrak{P}(X)$  it induces a homotopy equivalence  $\psi_{\alpha} : \alpha \setminus \mathfrak{P}(X) \to (\alpha \setminus \mathfrak{P}(X))' := \{\alpha\} \times_{\mathfrak{P}(X)} \mathfrak{P}(X)^{(1)}$  over  $\mathfrak{P}(X)$ . Then, by Proposition 2.2.9 (c), we have  $\psi_{\alpha}(\mathrm{id}_{\alpha}) \sim \mathrm{id}_{\alpha}$ .

**Lemma 2.3.12.** For every  $x \in X$ , there exists a weak equivalence  $\phi : j_X(x) \setminus \mathfrak{P}(X) \to \widetilde{E}_x$  of left fibrations over  $\mathfrak{P}(X)$  such that  $\phi(\operatorname{id}_{j_X(x)}) \sim \operatorname{id}_x \in (\widetilde{E}_x)_{j_X(x)}$ .

*Proof.* We construct  $\phi$  as a composition of weak equivalences over  $\mathfrak{P}(X)$ 

$$j_X(x) \backslash \mathfrak{P}(X) \xrightarrow{\phi'''} (j_X(x) \backslash \mathfrak{P}(X))' \xrightarrow{\phi''} ((x \backslash X^{op}) \backslash \mathfrak{P}(X))' \xrightarrow{\phi'} \widetilde{E}_x$$

By definition, if  $\alpha: K \to \mathfrak{P}(X)$  corresponds to the left fibration  $G \to X^{op} \times K$ , then maps  $K \to ((x \setminus X^{op}) \setminus \mathfrak{P}(X))'$  over  $\alpha$  are in bijection with maps  $\nu: (x \setminus X^{op}) \times K \to G$  over  $X^{op} \times K$ , and maps  $K \to (j_X(x) \setminus \mathfrak{P}(X))'$  over  $\alpha$  are in bijection with maps  $\nu': (\{x\} \times_X \mathcal{M}(X^{op})) \times K \to G$  over  $X^{op} \times K$  (use 2.3.8).

Let  $\phi : x \setminus X^{op} \to \{x\} \times_X \operatorname{Mor}(X^{op})$  be the weak equivalence from Lemma 2.3.6, and we define  $\phi''$  to be the map, which sends  $\nu'$  to  $\nu' \circ \phi$ . Then  $\phi''$  is a homotopy equivalence, because  $\phi$  is such.

Next we observe that maps  $K \to E_x$  over  $\alpha : K \to \mathfrak{P}(X)$  are in bijection with sections s of the left fibration  $G_x := \{x\} \times_{X^{op}} G \to K$ . We define  $\phi'$  to be the map, which sends  $\nu : (x \setminus X^{op}) \times K \to G$  to  $s := \nu|_{\mathrm{id}_x} : K \to G_x$ .

Since  $\phi'$  is a map between left fibrations, to show that  $\phi'$  is a weak equivalence, it remans to show that for every  $\alpha \in \mathfrak{P}(X)$  the induced map  $(\phi'_{\alpha})_0$  is a weak equivalence of simplicial sets (by Lemma 2.1.5). Let  $G \to X^{op}$  be the left fibration corresponding to  $\alpha$ . Then  $(\phi'_{\alpha})_0$  is the trivial fibration  $\operatorname{ev}_{\operatorname{id}_x} : \operatorname{Map}_{X^{op}}(x \setminus X^{op}, G) \to (G_x)_0$  from Proposition 2.1.7.

Finally, we define  $\phi''': j_X(x) \setminus \mathfrak{P}(X) \to (j_X(x) \setminus \mathfrak{P}(X))'$  to be the weak equivalence from 2.3.11, and set  $\phi := \phi' \circ \phi'' \circ \phi'''$ . By construction, we have  $\phi'''(\mathrm{id}_{j_X(x)}) \sim \mathrm{id}_{j_X(x)}, \phi''(\mathrm{id}_{j_X(x)}) = \widetilde{\phi}$ , and  $\phi'(\widetilde{\phi}) = \widetilde{\phi}(\mathrm{id}_x) \sim \mathrm{id}_x$ . Thus  $\phi(\mathrm{id}_{j_X(x)}) \sim \mathrm{id}_x$ .  $\Box$ 

Now we are ready to prove the Yoneda lemma.

Proof of Theorem 2.3.9. (a) By Lemma 2.3.12, there exists a homotopy equivalence  $\phi : j_X(x) \setminus \mathfrak{P}(X) \to \widetilde{E}_x$  of left fibrations over  $\mathfrak{P}(X)$  such that  $\phi(\mathrm{id}_{j_X(x)}) \sim \mathrm{id}_x$ . Since for every  $\alpha \in \mathfrak{P}(X)$  the fiber of  $\widetilde{E}_x$  at  $\alpha$  is  $E_{\alpha(x)}$  (see 2.3.10 (b)),  $\phi$  induces an equivalence  $\phi_\alpha : \mathrm{map}_{\mathfrak{P}(X)}(j_X(x), \alpha) \to (E_{\alpha(x)})_0$ .

(b) The morphism  $j_X : X \to \mathfrak{P}(X)$  induces a morphism  $x \setminus X \to j_X(x) \setminus \mathfrak{P}(X)$ over  $j_X$ . Hence  $j_X$  induces a morphism  $\psi : x \setminus X \to j_X(x) \setminus \mathfrak{P}(X) \times_{\mathfrak{P}(X)} X$  of left fibrations over X. We claim that  $\psi$  is a weak equivalence, hence it induces a weak equivalence of fibers  $\psi_b : \operatorname{map}_X(a, b) \to \operatorname{map}_{\mathfrak{P}(X)}(j_X(a), j_X(b))$  for all  $b \in X$ .

Consider the composition  $\phi \circ \psi : x \setminus X \to \widetilde{E}_x \times_{\mathfrak{P}(X)} X$ , where  $\phi$  is as in the proof of (a). Since  $\widetilde{E}_x \times_{\mathfrak{P}(X)} X = \{x\} \times_{X^{op}} \mathcal{M}(X)$  (see 2.3.10 (c)),  $\phi \circ \psi$  is a map  $x \setminus X \to \{x\} \times_{X^{op}} \mathcal{M}(X)$  over X, which by construction satisfy  $\phi \circ \psi(\mathrm{id}_x) \sim \mathrm{id}_x$ . Therefore  $\phi \circ \psi$  is a weak equivalence by Corollary 2.1.8 and Lemma 2.3.6, hence  $\psi$  is a weak equivalence by 2-out-of-3.

# 3. QUASIFIBRATIONS OF SIMPLICIAL SPACES

### 3.1. Definitions and basic properties.

**Definition 3.1.1.** Let C be a right proper model category. We say that a morphism  $p: X \to B$  in C is a *quasifibration*, if for every weak equivalence  $g: Y \to Z$  over B, its pullback  $p^*(g): p^*(Y) \to p^*(Z)$  (see 1.1.1 (d)) is a weak equivalence over X.

**3.1.2. Remarks.** (a) By definition, any pullback of a quasifibration is a quasifibration, and a composition of quasifibrations is a quasifibration.

(b) When C is a model category of topological spaces, our notion of a quasifibration is stronger than the classical notion. However we will only use this notion for model categories of spaces and simplicial spaces, where no classical notion exists.

(c) After this work was essentially completed, we found that quasifibrations were also studied in a unpublished preprint of Rezk [Re3] under a name of *sharp* morphisms. But we think that our terminology is more suggestive.

#### **Lemma 3.1.3.** (a) Every fibration in C is a quasifibration.

(b) Let  $f: X \to X'$  be a weak equivalence between quasifibrations  $p: X \to B$ and  $p': X' \to B$  over B. Then for every morphism  $\tau: A \to B$  the pullback  $\tau^*(f): X \times_B A \to X' \times_B A$  is a weak equivalence.

(c) Conversely, assume that  $f: X \to X'$  is a weak equivalence over B such that each pullback  $\tau^*(f)$  is a weak equivalence. Then  $p: X \to B$  is a quasifibration if and only if  $p': X' \to B$  is a quasifibration.

*Proof.* (a) follows from the fact that C is right proper.

(b) If  $\tau : A \to B$  is fibration, then  $\tau^*(g)$  is a weak equivalence by (a). If  $\tau$  is a weak equivalence, then  $\tau^*(X) \to X$  and  $\tau^*(X') \to X'$  are weak equivalences,

because  $X \to B$  and  $X' \to B$  are quasifibrations. Therefore  $\tau^*(f) : \tau^*(X) \to \tau^*(X')$  is a weak equivalence by 2-out-of-3. Since every morphism decomposes as a composition of a trivial cofibration and a fibration, the general case follows.

(c) Let  $g: Y \to Z$  be a weak equivalence over B. Then  $X \times_B Y \to X' \times_B Y$ and  $X \times_B Z \to X' \times_B Z$  are weak equivalences by the assumption on f. Hence, by 2-out-of-3,  $p'^*(g): X' \times_B Y \to X' \times_B Z$  is a weak equivalence if and only if  $p^*(g)$  is a weak equivalence. Thus, by definition,  $p: X \to B$  is a quasifibration if and only if  $p': X' \to B$  is a quasifibration.

**Corollary 3.1.4.** Assume that model category C has the property that a pullback of a cofibration is a cofibration, and let  $p' : X' \to B$  be a fibrant replacement of  $p: X \to B$ . Then p is a quasifibration if and only if  $\tau^*(p')$  is a fibrant replacement of  $\tau^*(p)$  for every  $\tau: A \to B$ .

*Proof.* By MC5, p decomposes as  $X \xrightarrow{i} X' \xrightarrow{p'} B$ , where i is a trivial cofibration, and p is a fibration. Then for every map  $\tau : A \to B$ ,  $\tau^*(p')$  is a fibration, while  $\tau^*(i)$  is a cofibration. Thus we have to show that  $X \to B$  is a quasifibration if and only if each  $\tau^*(i)$  is a weak equivalence. Since  $p' : X' \to B$  is a quasifibration by Lemma 3.1.3 (a), the assertion follows from Lemma 3.1.3 (b) and (c).

Corollary 3.1.5. Suppose we are given a commutative diagram

where g and g' are quasifibrations and all vertical morphisms are weak equivalences. Then the induced map  $X' \times_{Z'} Y' \to X \times_Y Z$  is a weak equivalence.

*Proof.* Weak equivalence  $Y' \to Y$  decomposes as composition  $Y' \to Z' \times_Z Y \to Y$ , the second on which is a weak equivalence, because it is a pullback of a weak equivalence  $Z' \to Z$  along a quasifibration  $Y \to Z$ . Therefore by 2-out-of-3  $Y' \to Z' \times_Z Y$  is a weak equivalence between quasifibrations over Z'.

Now map  $X' \times_{Z'} Y' \to X \times_Y Z$  decomposes as composition

$$X' \times_{Z'} Y' \to X' \times_{Z'} (Z' \times_Z Y) = X' \times_Z Y \to X \times_Z Y,$$

the first of which is a weak equivalence, being a pullback of a weak equivalence between quasifibrations (use Lemma 3.1.3 (b)), while the second one is a weak equivalence, since  $Y \to Z$  is a quasifibration, and  $X' \to X$  is a weak equivalence.  $\Box$ 

From now on we assume that C is ether category Sp with Kan model structure (Theorem 1.2.6) or category sSp with Reedy model structure (Theorem 1.3.6).

**Lemma 3.1.6.** Let  $X_B \to B$  be a quasifibration in sSp,  $A \to B$  a cofibration, and  $i: X_A := X_B \times_B A \to Y_A$  a weak equivalence of quasifibrations over A. Then the pushout  $Y_B := X_B \sqcup_{X_A} Y_A$  is a quasifibration over  $B \sqcup_A A = B$ , and the natural map  $j: X_B \to Y_B$  is a weak equivalence.

*Proof.* By Lemma 3.1.3 (c), it is enough to show that for every map  $\tau : B' \to B$ , the pullback  $\tau^*(j)$  is a weak equivalence.

Note that  $\tau^*(j)$  is a pushout of  $\tau^*(i) : \tau^*(X_A) \to \tau^*(Y_A)$  along  $\tau^*(X_A) \to \tau^*(X_B)$ . Since  $A \to B$  a cofibration, the induced maps  $X_A \to X_B$  and  $\tau^*(X_A) \to \tau^*(X_B)$ .

 $\tau^*(X_B)$  are cofibrations. Since *i* is a weak equivalence between quasifibrations,  $\tau^*(i)$  is a weak equivalence by Lemma 3.1.3 (b). Hence the pushout  $\tau^*(j)$  is a weak equivalence, because sSp is left proper.

**Lemma 3.1.7.** A morphism  $p: X \to B$  is a quasifibration in sSp if and only if  $p_n: X_n \to B_n$  is a quasifibration in Sp for every n.

*Proof.* Note that if  $p_n$  is a quasifibration for all n, then for every weak equivalence  $g: Y \to Z$  over B, the corresponding maps  $g_n: Y_n \to Z_n$  are weak equivalences over  $B_n$  for all n. Therefore each  $p^*(g)_n = p_n^*(g_n)$  is a weak equivalence, since  $p_n$  is a quasifibration. Hence  $p^*(g)$  is a weak equivalence, thus p is a quasifibration.

Conversely, assume that p is a quasifibration. Every weak equivalence  $\tilde{g}: \tilde{Y} \to \tilde{Z}$ over  $B_n$  defines a weak equivalence  $g := \tilde{g} \times \mathrm{Id}_{F[n]} : \tilde{Y} \times F[n] \to \tilde{Z} \times F[n]$  over  $B \times F[n]$  such that  $\tilde{g}$  is the restriction of  $g_n$  to  $\mathrm{Id}_{F[n]} \in F[n]_n$ . Since p is a quasifibration, the pullback  $p^*(g)$  is a weak equivalence over F[n]. Thus  $p_n^*(\tilde{g}) = (p^*(g)_n)_{\mathrm{Id}_{F[n]}}$  is a weak equivalence, implying that  $p_n$  is a quasifibration.  $\Box$ 

**Definition 3.1.8.** A map  $f : X \to B$  in sSp is called a *left quasifibration*, if it is quasifibration and the morphism  $X_n \to X_0 \times_{B_0} B_n$ , induced by the inclusion  $\delta^0 : [0] \hookrightarrow [n]$ , is a weak equivalence for all n.

**3.1.9. Remark.** It follows from Lemma 2.1.3 (c) that a fibration  $f : X \to B$  in sSp is a left fibration if and only if it is a left quasifibration.

Lemma 3.1.10. Suppose we have a commutative diagram

$$\begin{array}{cccc} X' & \xrightarrow{g} & X \\ f' \downarrow & & f \downarrow \\ B' & \xrightarrow{h} & B \end{array}$$

in sSp, where g and h are weak equivalences, while f and f' are quasifibrations. Then f is a left if and only if f' is left.

*Proof.* Consider commutative diagram induced by (3.1)

$$\begin{array}{cccc} X'_n & \xrightarrow{g_n} & X_n \\ p'_n \downarrow & & p_n \downarrow \\ X'_0 \times_{B'_0} B'_n & \xrightarrow{g_0 \times_{h_0} h_n} & X_0 \times_{B_0} B_n \end{array}$$

We have to show that  $p_n$  is a weak equivalence if and only if  $p'_n$  is. Since g is a weak equivalence, by 2-out-of-3, it suffices to show that  $g_0 \times_{h_0} h_n$  is a weak equivalence.

By Lemma 3.1.7, maps  $X_0 \to B_0$  and  $X'_0 \to B'_0$  are quasifibrations. Therefore  $g_0 \times_{h_0} h_n$  is a weak equivalence by assumption and Corollary 3.1.5.

Corollary 3.1.11. A fibrant replacement of a left quasifibration is a left fibration.

*Proof.* Apply Lemma 3.1.10 and remark 3.1.9 to the case when B' = B and f is a fibrant replacement of f'.

#### 3.2. Extension of fibrations, and fibrant replacements.

**Lemma 3.2.1.** Suppose we are given a commutative diagram



in sSp such that vertical arrows are fibrations, horizontal arrows are cofibration, and the induced map  $Y_A \to X_A := X_B \times_B A$  is a trivial cofibration.

Then there exists the largest simplicial subspace  $Y_B \subset X_B$  such that  $Y_B \times_B A = Y_A$ . Moreover,  $Y_B \subset X_B$  is a strong deformation retract over B. In particular,  $Y_B \to B$  is a fibration, and the inclusion  $i: Y_B \hookrightarrow X_B$  is a trivial cofibration.

Proof. Consider simplicial subspace  $Y_B \subset X_B$  such that  $(Y_B)_{n,m}$  is the set of all  $\tau \in (X_B)_{n,m} = \operatorname{Hom}(\Box[n,m], X_B)$  such that  $\tau(\tau^{-1}(X_A)) \subset Y_A$ . Then  $Y_B \subset X_B$  is the largest subspace such that  $Y_B \times_B A = Y_A$ . By construction, for every morphism  $C \to B$ , the set  $\operatorname{Hom}_B(C, Y_B)$  can be identified with the set of maps  $f \in \operatorname{Hom}_B(C, X_B)$  such that  $f(C \times_B A) \subset Y_A$ .

Since  $Y_A \subset X_A$  is a trivial cofibration between fibrations over A, it is a strong deformation retract (see 1.3.10 (c)). Thus there exists a map  $g: X_A \times \Delta[1] \to X_A$  over A such that (i)  $g|_{X_A \times \{0\}} = \mathrm{Id}_{X_A}$ ; (ii)  $g(X_A \times \{1\}) \subset Y_A$  and (iii)  $g|_{Y_A \times \Delta[1]} = \mathrm{pr}_1: Y_A \times \Delta[1] \to Y_A \subset X_A$ .

Since  $Y_B \cap X_A = Y_A$ , property (iii) of g implies that g extends to a map  $g' : (X_A \sqcup_{Y_A} Y_B) \times \Delta[1] \to X_B$  over B such that  $g'|_{Y_B \times \Delta[1]} = \text{pr}_1$ . Next property (i) of g implies that g' extends to a morphism

$$g'': ((X_A \sqcup_{Y_A} Y_B) \times \Delta[1]) \sqcup_{(X_A \sqcup_{Y_A} Y_B) \times \{0\}} (X_B \times \{0\}) \to X_B$$

over B such that  $g''|_{X_B \times \{0\}} = \mathrm{Id}_{X_B}$ .

Since  $\{0\} \hookrightarrow \Delta[1]$  is a trivial cofibration, while model category sSp is Cartesian, we conclude that

$$((X_A \sqcup_{Y_A} Y_B) \times \Delta[1]) \sqcup_{(X_A \sqcup_{Y_A} Y_B) \times \{0\}} (X_B \times \{0\}) \hookrightarrow X_B \times \Delta[1]$$

is a trivial cofibration, thus g'' extends to a map  $h: X_B \times \Delta[1] \to X_B$  over B.

Then h satisfies  $h|_{X_B \times \{0\}} = g''|_{X_B \times \{0\}} = \operatorname{Id}_{X_B}, h|_{Y_B \times \Delta[1]} = g'|_{Y_B \times \Delta[1]} = \operatorname{pr}_1$ and  $h(X_A \times \{1\}) = g(X_A \times \{1\}) \subset Y_A$ . Thus, by the construction of  $Y_B$ , we get that  $h(X_B \times \{1\}) \subset Y_B$ . In other words, h realizes  $Y_B$  as a strong deformation retract of  $X_B$  over B. The last assertion follows from 1.3.10 (b).

**Corollary 3.2.2.** Let  $X \to B$  be an quasifibration, and let  $A \hookrightarrow B$  be a cofibration such that  $X \times_B A \to A$  is a fibration. Then there exists a fibrant replacement  $Y \to B$  of  $X \to B$  such that  $X \times_B A = Y \times_B A$ .

*Proof.* Let  $X \xrightarrow{i} Y' \xrightarrow{p} B$  be any decomposition of  $X \to B$ , where *i* is a trivial cofibration, and *p* is a fibration. Since  $X \to B$  is quasifibration, *i* induces a trivial cofibration  $X \times_B A \hookrightarrow Y' \times_B A$  over *A* (by Lemma 3.1.3 (b)).

Let  $Y \subset Y'$  be the largest simplicial subspace such that  $Y \times_B A = X \times_B A$ . Then  $Y \to B$  is a fibration, and  $Y \hookrightarrow Y'$  is a trivial cofibration (see Lemma 3.2.1). Since trivial cofibration *i* factors as a composition  $X \hookrightarrow Y \hookrightarrow Y'$ , the map  $X \hookrightarrow Y$ is a trivial cofibration by 2-out-of-3. **3.2.3.** Proof of Lemma 1.3.11 (d). Composition  $Y_A \to A \to B$  decomposes as a composition of a trivial cofibration  $Y_A \to X_B$  and a fibration  $X_B \to B$ . Now let  $Y_B \subset X_B$  be the largest simplicial subspace such that  $Y_B \times_B A = Y_A$ . Then  $Y_B \to B$  is a fibration by Lemma 3.2.1.

**3.2.4. Notation.** (a) For  $X \in sSp$  and  $n \geq 0$ , we define the *n*-skeleton  $sk_nX$  to be the smallest simplicial subspace  $Y \subset X$  such that  $Y_{m,k} = X_{m,k}$  for all  $m + k \leq n$ . Then  $0 = sk_{-1}X \subset sk_0X \subset \ldots \subset sk_nX \subset \ldots \subset X$  and  $X = \operatorname{colim}_n sk_nX$ .

Then  $0 = sk_{-1}X \subset sk_0X \subset \ldots \subset sk_nX \subset \ldots \subset X$  and  $X = \operatorname{colim}_n sk_nX$ . (b) For every  $m, k \geq 0$ , we denote by  $X_{m,k}^{nd} := X_{m,k} \smallsetminus (sk_{m+k-1}X)_{m,k}$  the set of "non-degenerate bisimplices". Then  $X_{0,0}^{nd} = X_{0,0}$ , and for each n > 0the *n*-th skeleton  $sk_nX$  is naturally isomorphic to the pushout of  $sk_{n-1}X$  and  $\sqcup_{m+k=n,a\in X_{m,k}^{nd}} \Box[m,k]$  over  $\sqcup_{m+k=n,a\in X_{m,k}^{nd}} \partial \Box[m,k]$ .

**Lemma 3.2.5.** Let  $f : X \to K$  be a morphism in  $(sSp/K)_{\mathcal{U}}$  such that either (a)  $K \in sSp_{|\mathcal{U}|}$  or (b) f is a quasifibration. Then f has a fibrant replacement  $f' : X' \to K$  in  $(sSp/K)_{\mathcal{U}}$ .

*Proof.* (a) Since  $(sSp/K)_{\mathcal{U}} \to (sSp/K)_{|\mathcal{U}|}$  is an equivalence of categories, it is enough to show the existence of a fibrant replacement f' in  $(sSp/K)_{|\mathcal{U}|}$ . Since  $K \in sSp_{|\mathcal{U}|}$  and  $|\mathcal{U}| \times |\mathcal{U}| = |\mathcal{U}|$ , we conclude that  $X \in sSp_{|\mathcal{U}|}$ . Moreover, using  $|\mathcal{U}| \times |\mathcal{U}| = |\mathcal{U}|$  again, we get that fibrant replacement  $f' : X' \to K$ , constructed in the proof of Theorem 1.3.6, satisfies  $X' \in sSp_{|\mathcal{U}|}$ , thus  $f' \in (sSp/K)_{|\mathcal{U}|}$ .

(b) By induction on i, we are going to construct a fibrant replacement  $f'[i] : X'[i] \to sk_iK$  of  $f|_{sk_iK} : X|_{sk_iK} \to sk_iK$  such that f'[i] belongs to  $(sSp/sk_iK)_{\mathcal{U}}$  and  $f'[i+1]|_{sk_iK} = f'[i]$ .

Assuming this is done, we set  $f' := \operatorname{colim}_i f'[i] : X' \to K$ . Then f' satisfies  $f'|_{sk_iK} = f'[i]$ , thus f' is a fibration and  $f' \in (sSp/K)_{\mathcal{U}}$ . Moreover, since  $X|_{sk_iK} \to X'[i]$  is a weak equivalence for all i, we get that  $X \to X'$  is a weak equivalence as well, thus f' is a fibrant replacement of f.

Assume that f'[i] was already constructed. Set  $X[i] := X \sqcup_{X|_{sk_iK}} X'[i]$ . Then it follows from Lemma 3.1.6 that  $f[i] : X[i] \to K$  is a quasifibration and that  $X \to X[i]$  is a weak equivalence. By construction,  $f[i]|_{sk_iK} : X[i]|_{sk_iK} \to sk_iK$  is a fibration f'[i], and we want to show that  $f[i]|_{sk_{i+1}K}$  has a fibrant replacement f'[i+1], all of whose fibers are in  $Set_{\mathcal{U}}$ , such that  $f'[i+1]|_{sk_iK} = f[i]|_{sk_iK}$ .

Recall that  $sk_{i+1}K = sk_iK \sqcup_{\sqcup \partial \Box[n,m]} \sqcup \Box[n,m]$ . Thus it remains to show that for every quasifibration  $f: X \to \Box[n,m]$  from  $(sSp/\Box[n,m])_{\mathcal{U}}$ , whose restriction to  $\partial \Box[n,m]$  is a fibration, there exists a fibrant replacement  $f': X' \to \Box[n,m]$  from  $(sSp/\Box[n,m])_{\mathcal{U}}$  such that  $f'|_{\partial \Box[n,m]} = f|_{\partial \Box[n,m]}$ .

Since  $\Box[n,m] \in sSp_{|\mathcal{U}|}$ , it follows from (a) that there exists a fibrant replacement  $f'': X'' \to \Box[n,m]$  of f in  $(sSp/\Box[n,m])_{\mathcal{U}}$ . Moreover, since f is a quasifibration, there exists  $X' \subset X''$  such that  $X'|_{\partial \Box[n,m]} = X|_{\partial \Box[n,m]}$ , and  $f' := f''|_{X'}$  is a fibrant replacement of f (see the proof of Corollary 3.2.2). Then  $f' \in (sSp/\Box[n,m])_{\mathcal{U}}$  and  $f'|_{\partial \Box[n,m]} = f|_{\partial \Box[n,m]}$ .

**Corollary 3.2.6.** In the situation Corollary 3.2.2, assume that the quasifibration  $X \to B$  is in  $(sSp/B)_{\mathcal{U}}$ . Then  $Y \to B$  can also chosen to be in  $(sSp/B)_{\mathcal{U}}$ .

*Proof.* If  $X \to B$  is in  $(sSp/B)_{\mathcal{U}}$ , then  $Y' \to B$  from the proof of Corollary 3.2.2 can be chosen to be in  $(sSp/B)_{\mathcal{U}}$  by Lemma 3.2.5, thus  $Y \to B$  is also in  $(sSp/B)_{\mathcal{U}}$ .  $\Box$ 

#### 4. Complements

#### 4.1. Discrete iterated cylinder.

**4.1.1. Observations.** (a) Every map  $p: X \to B \times F[m]$  in sSp induces a map  $p_n: X_n \to B_n \times F[m]_n$  for all n. Since  $F[m]_n$  decomposes as  $\coprod_{\tau:[n]\to[m]} pt$ , each  $X_n$  decomposes as a disjoint union  $X_n = \coprod_{\tau:[n]\to[m]} X_{\tau}$ , and  $p_n$  decomposes as a disjoint union of  $p_{\tau}: X_{\tau} \to B_n$ .

(b) By definition, all fibers of p belong to  $Set_{\mathcal{U}}$  if and only of all fibers of each  $p_{\tau}$  belong to  $Set_{\mathcal{U}}$ .

(c) Notice that  $p_n$  is a quasifibration if and only if each  $p_{\tau} : X_{\tau} \to B_n$  is a quasifibration. Using Lemma 3.1.7 we conclude that p is a quasifibration if and only if each  $p_{\tau}$  is a quasifibration.

(d) Recall that a quasifibration  $p: X \to B \times F[m]$  is left if and only if each morphism  $g_n: X_n \to X_0 \times_{(B \times F[m])_0} (B \times F[m])_n$  is a weak equivalence. Note that  $g_n$  decomposes as a disjoint union of morphisms  $g_\tau: X_\tau \to X_{\tau|_0} \times_{B_0} B_n$ . Thus f is left if and only if each  $g_\tau$  is a weak equivalence.

(e) A morphism  $f: X \to Y$  between left quasifibrations over  $B \times F[m]$  is a weak equivalence if and only if  $f|_i: X|_i \to Y|_i$  is a weak equivalence over B for each  $i = 1 \dots, m$ .

*Proof.* By definition, f is a weak equivalence if and only if  $f_n : X_n \to Y_n$  is a weak equivalence for all n, which by (a) is equivalent to the assertion that  $f_\tau : X_\tau \to Y_\tau$  is a weak equivalence for all  $\tau : [n] \to [m]$ . Similarly,  $f|_i$  is a weak equivalence if and only if  $f_\tau : X_\tau \to Y_\tau$  is a weak equivalence for all  $\tau : [n] \to \{i\} \subset [m]$ . This implies "the only if" assertion.

To show the converse, notice that by (d) and 2-out-of-3,  $f_{\tau}$  is a weak equivalence if and only if  $f_{\tau|_0}: X_{\tau|_0} \times_{B_0} B_n \to Y_{\tau|_0} \times_{B_0} B_n$  is a weak equivalence. Since by (b) the maps  $X_{\tau|_0} \to B_0$  and  $Y_{\tau|_0} \to B_0$  are quasifibrations, if follows from Corollary 3.1.5 that f is a weak equivalence if each  $f_{\tau|_0}: X_{\tau|_0} \to Y_{\tau|_0}$  is a weak equivalence or, equivalently, if  $f_{\tau}: X_{\tau} \to Y_{\tau}$  is a weak equivalence for all  $\tau: [0] \to [m]$ .  $\Box$ 

**4.1.2.** Discrete iterated cylinder. For  $B \in sSp$  and a sequence of morphisms  $f: K^{(0)} \xrightarrow{f_1} \ldots \xrightarrow{f_m} K^{(m)}$  in sSp/B, we define recursively a discrete iterated cylinder  $Cyl^{disc}(f) \to B \times F[m]$  and morphisms  $\iota_j: K^{(j)} \times e^j F[m-j] \to Cyl^{disc}(f)$  over  $B \times F[m]$  for all  $j = 0, \ldots, m$  as follows.

If m = 0, we set  $Cyl^{disc}(f) := K^{(0)}$ , and put  $\iota_0 = \text{Id}$ . If  $m \ge 1$ , we denote by f(1) the sequence  $f(1) : K^{(1)} \xrightarrow{f_2} \dots \xrightarrow{f_m} K^{(m)}$ , and assume by induction that we have defined an iterated cone  $Cyl^{disc}(f(1)) \to B \times F[m-1]$  and a morphism  $\iota_j : K^{(j)} \times e^j F[m-j] \to e^1 Cyl^{disc}(f(1))$  for all  $j = 1, \dots, m$ .

We define  $Cyl^{disc}(f) \to B \times F[m]$  to be the pushout

(4.1) 
$$Cyl^{disc}(f) := (K^{(0)} \times F[m]) \sqcup_{(K^{(0)} \times e^1 F[m-1])} e^1 Cyl^{disc}(f(1)).$$

where the map  $K^{(0)} \times e^1 F[m-1] \to e^1 Cyl^{disc}(f(1))$  is defined to be the composition

$$K^{(0)} \times e^1 F[m-1] \xrightarrow{f_1} K^{(1)} \times e^1 F[m-1] \xrightarrow{\iota_1} e^1 Cyl^{disc}(f(1))$$

Finally, we define  $\iota_0: K^{(0)} \times F[m] \hookrightarrow Cyl^{disc}(f)$  be the natural embedding.

**Lemma 4.1.3.** If each  $K^{(i)} \to B$  is a (left) quasifibration in  $(sSp/B)_{\mathcal{U}}$ , then  $Cyl^{disc}(f) \to B \times F[m]$  is a (left) quasifibration in  $(sSp/B \times F[m])_{\mathcal{U}}$ .

*Proof.* We are going to apply 4.1.1 to the projection  $p: Cyl^{disc}(f) \to B \times F[m]$ .

We claim that for each  $\tau : [n] \to [m]$ , we have  $Cyl^{disc}(f)_{\tau} = K_n^{(\tau(0))}$ . The proof goes by induction. The assertion is obvious, if m = 0. Next, if  $m \ge 1$ , then (4.1) implies that  $Cyl^{disc}(f)_{\tau}$  equals  $K_n^{(0)}$ , if  $\tau(0) = 0$ , and equals  $Cyl^{disc}(f(1))_{\tau'}$ , where  $\tau' : [n] \to [m-1]$  is given by  $\tau'(i) = \tau(i) - 1$ , if  $\tau(0) \ge 1$ . By induction hypothesis, in the second case  $Cyl^{disc}(f)_{\tau}$  equals  $K_n^{(\tau'(1))} = K_n^{(\tau(0))}$ .

By the proven above and 4.1.1 (a), each fiber of p is a fiber of some  $K_n^{(\tau(0))} \to B_n$ . Thus it belongs to  $Set_{\mathcal{U}}$ , because  $K^{(\tau(0))} \to B$  is in  $(sSp/B)_{\mathcal{U}}$ .

Moreover, since each  $K^{(\tau(0))} \to B$  is a quasifibration, each  $K_n^{(\tau(0))} \to B_n$  is a quasifibration (by Lemma 3.1.7). Therefore p is a quasifibration by 4.1.1 (c). Similarly, each projection  $Cyl^{disc}(f)_{\tau} \to Cyl^{disc}(f)_{\tau|_0} \times_{B_0} B_n$  is simply  $K_n^{(\tau(0))} \to K_0^{(\tau(0))} \times_{B_0} B_n$ . Therefore it is a weak equivalence, because  $K^{(\tau(0))} \to B$  is left. The assertion now follows from 4.1.1 (d).

**Lemma 4.1.4.** Let  $f: X \to Y$  be a map, and  $E \to Y \times F[n]$  be a left fibration.

(a) The map  $p: \mathcal{M}ap_{Y \times F[n]}(X \times F[n], E) \to \mathcal{M}ap_{Y \times F[n]}(X, E)$ , induced by the inclusion  $e^0: F[0] \hookrightarrow F[n]$ , is a trivial fibration.

(b) The map

 $q: \mathcal{M}ap_{Y \times F[n]}(X \times F[n], E) \to \mathcal{M}ap_{Y \times F[n]}(X \times (e^0 F[1] \sqcup_{e^1 F[0]} e^1 F[n-1]), E),$ 

induced by the inclusion  $e^0 F[1] \sqcup_{e^1 F[0]} e^1 F[n-1] \hookrightarrow F[n]$ , is a trivial fibration.

*Proof.* (a) Since  $E \to Y \times F[n]$  is a left fibration, the map  $E^X \to (Y \times F[n])^X$  is a left fibration by Lemma 2.1.2 (b). Then by Lemma 2.1.3 (b), the map

 $E^{X \times F[n]} \to E^X \times_{(Y \times F[n])^X} (Y \times F[n])^{X \times F[n]}$ 

is a trivial fibration. Taking fiber over  $f \times \mathrm{Id}_{F[n]} \in (Y \times F[n])^{X \times F[n]}$ , we get the assertion.

(b) Since  $e^0 F[1] \sqcup_{e^1 F[0]} e^1 F[n-1] \hookrightarrow F[n]$  is a cofibration, the map q is a fibration. Thus it remains to show that q is a weak equivalence. Consider map

$$r: \mathcal{M}ap_{Y \times F[n]}(X \times (e^0 F[1] \sqcup_{e^1 F[0]} e^1 F[n-1]), E) \to \mathcal{M}ap_{Y \times F[n]}(X, E),$$

induced by the inclusion  $F[0] \xrightarrow{e^0} e^0 F[1] \hookrightarrow e^0 F[1] \sqcup_{e^1 F[0]} e^1 F[n-1]$ . Since  $r \circ q = p$ , it is a weak equivalence by (a). Thus it remains to show that r is a weak equivalence. But r can be written as a composition of

$$\mathcal{M}ap_{Y\times F[n]}(X\times (e^0F[1]\sqcup_{e^1F[0]}e^1F[n-1]), E) \to \mathcal{M}ap_{Y\times F[n]}(X\times e^0F[1], E)$$

and  $(e^0)^* : \mathcal{M}ap_{Y \times F[n]}(X \times e^0F[1], E) \to \mathcal{M}ap_{Y \times F[n]}(X, E)$ , so it remains to show that both maps are trivial fibrations. Since the first map is the pullback of the morphism  $\mathcal{M}ap_{Y \times F[n]}(X \times e^1F[n-1], E) \to \mathcal{M}ap_{Y \times F[n]}(X \times e^1F[0], E)$ , both maps are trivial fibrations by (a).

4.2. **Proofs.** In this subsection we prove Propositions 2.2.8, 2.2.9 and Corollary 2.2.10. The most difficult part is Proposition 2.2.9 (b), whose proof is carried out in 4.2.7-4.2.9 and 4.2.11-4.2.13. We denote  $\mathfrak{S}_{\mathcal{U}}$  simply by  $\mathfrak{S}$ .

**4.2.1. Proof of Proposition 2.2.8 (a).** By 1.3.7 (a), we have to show that for every trivial cofibration  $i : A \to B$  in sSp with  $B = \Box[n,m]$  the morphism

 $i^*$ : Hom $(B, \mathfrak{S}) \to$  Hom $(A, \mathfrak{S})$  is surjective. By Lemma 2.2.4 this means that every left fibration  $Y_A \to A$  in  $(sSp/A)_{\mathcal{U}}$  extends to a left fibration  $Y_B \to B$  in  $(sSp/B)_{\mathcal{U}}$ .

Note that composition  $j: Y_A \to A \to B$  belongs to  $(sSp/B)_{\mathcal{U}}$ , and  $B \in sSp_{|\mathcal{U}|}$ . Then j decomposes as a composition of a trivial cofibration  $Y_A \to X_B$  and a fibration  $X_B \to B$  in  $(sSp/B)_{\mathcal{U}}$  (by Lemma 3.2.5 (a)). Now let  $Y_B \subset X_B$  be the largest simplicial subspace such that  $Y_B \times_B A = Y_A$  (see Lemma 3.2.1). Then  $Y_B \to B$  belongs to  $(sSp/B)_{\mathcal{U}}$ , and we claim that  $Y_B \to B$  is a left fibration.

Let  $X_A := X_B \times_A B$ . Since  $A \to B$  is a weak equivalence and the Reedy model structure is proper, the inclusion  $X_A \to X_B$  is a weak equivalence. Since  $Y_A \to X_A \to X_B$  is also a weak equivalence, we conclude that  $Y_A \to X_A$  is a weak equivalence. Then by Lemma 3.2.1, the projection  $Y_B \to B$  is a fibration, while  $Y_A \hookrightarrow Y_B$  is a weak equivalence. Since  $Y_A \to A$  is a left fibration,  $Y_B \to B$  is a left fibration by Lemma 3.1.10.

**4.2.2.** Proof of Proposition 2.2.8 (b). To show that  $\mathfrak{S}^{(n)} \to \mathfrak{S}^{n+1}$  is a fibration we have to show that for every trivial cofibration  $A \hookrightarrow B$ , the map  $(\mathfrak{S}^{(n)})^B \to (\mathfrak{S}^{n+1})^B \times_{(\mathfrak{S}^{n+1})^A} (\mathfrak{S}^{(n)})^A$  is surjective. Using the observation of 2.2.7, we have to show that for every (n + 1)-tuple of left fibrations  $E^{(0)}, \ldots, E^{(n)}$  of over B, every diagram  $E^{(0)}|_A \to \ldots \to E^{(n)}|_A$  over A extends to a diagram  $E^{(0)}|_B \to \ldots \to E^{(n)}|_B$  over B. For this enough to show that each restriction map

 $\operatorname{Hom}_B(E^{(i)}, E^{(i+1)}) \to \operatorname{Hom}_B(E^{(i)}|_A, E^{(i+1)}) = \operatorname{Hom}_A(E^{(i)}|_A, E^{(i+1)}|_A)$ 

is surjective. Since  $E^{(i)} \to B$  is a fibration,  $A \to B$  is a trivial cofibration, and the Reedy model structure is proper, we get that  $E^{(i)}|_A \to E^{(i)}$  is a trivial cofibration. Thus the assertion follows from the fact  $E^{(i+1)} \to B$  is a fibration.

To show that  $\mathfrak{S}^{(we)} \to \mathfrak{S}^2$  is a fibration, we argue as above word-by-word, and note that since  $E^{(i)}|_A \to E^{(i)}$  are trivial cofibrations, it follows from 2-out-of-3 that the morphism  $E^{(0)} \to E^{(1)}$  is a weak equivalence if and only if its restriction  $E^{(0)}|_A \to E^{(1)}|_A$  is a weak equivalence.

**4.2.3.** Proof of Proposition 2.2.8 (c). We have to show that for every cofibration  $A \hookrightarrow B$ , the map  $(\mathfrak{S}^{(we)})^B \to \mathfrak{S}^B \times_{\mathfrak{S}^A} (\mathfrak{S}^{(we)})^A$  is surjective. Let  $E^{(0)} \to B$ belong to  $(LFib/B)_{\mathcal{U}}, E_A^{(1)} \to A$  belong to  $(LFib/A)_{\mathcal{U}}$ , and  $\phi' : E^{(0)}|_A \to E_A^{(1)}$  be a weak equivalence over A. We have to show that  $\phi'$  extends to a weak equivalence  $\phi : E^{(0)} \to E^{(1)}$  over B such that  $E^{(1)} \to B$  belongs to  $(LFib/B)_{\mathcal{U}}$ .

By Lemma 3.1.6, the pushout  $E^{(0)} \sqcup_{E^{(0)}|_A} E^{(1)}_A$  is a quasifibration over B, whose restriction to A is a fibration  $E^{(1)}_A \to A$ . Therefore, by Corollary 3.2.6, there exists a fibrant replacement  $E^{(1)} \to B$  in  $(sSp/B)_{\mathcal{U}}$  such that  $E^{(1)}|_A = E^{(1)}_A$ .

By construction,  $\phi'$  extends to a morphism  $\phi: E^{(0)} \xrightarrow{\phi_1} E^{(0)} \sqcup_{E^{(0)}|_A} E^{(1)}_A \xrightarrow{\phi_2} E^{(1)}$ of fibrations over B. Moreover, since  $\phi'$  is a weak equivalence, its pushout  $\phi_1$  is a weak equivalence, hence  $\phi$  is a weak equivalence as well. Since  $E^{(0)} \to B$  is a left fibration,  $E^{(1)} \to B$  is a left fibration by Lemma 3.1.10.

**4.2.4.** Proof of Proposition 2.2.8 (d). Using observations of 2.2.7 (b), it remains to show that for every  $a \in \mathfrak{S}^{(we)}$ ,  $b \in \mathfrak{S}^{(1)}$  such that  $a \sim b$  in  $\mathfrak{S}^{(1)}$ , we have  $b \in \mathfrak{S}^{(we)}$ . Since  $\mathfrak{S}^{(we)} \to \mathfrak{S} \times \mathfrak{S}$  is a fibration, while  $\mathfrak{S}^{(1)}$  is fibrant (by Proposition 2.2.8 (a),(b)), we may assume that  $a \sim b$  in some fiber of  $\mathfrak{S}^{(1)} \to \mathfrak{S} \times \mathfrak{S}$ .

Then a corresponds to a weak equivalence  $\phi_a : E^{(0)} \to E^{(1)}$ , b corresponds to a morphism  $\phi_b : E^{(0)} \to E^{(1)}$ , assumption  $a \sim b$  means that the maps  $\phi_a$  and  $\phi_b$  are homotopic. Then b is a weak equivalence, hence  $b \in \mathfrak{S}^{(we)}$ .

**4.2.5.** Proof of Proposition 2.2.9 (a). By Lemma 2.2.4, for every  $K \in sSp$  the set  $\operatorname{Hom}(K, \mathfrak{S}^{\Delta[1]}) = \operatorname{Hom}(K \times \Delta[1], \mathfrak{S})$  is in bijection with the set of left fibrations  $E \to K \times \Delta[1]$  in  $(sSp/K \times \Delta[1])_{\mathcal{U}}$ , while the set  $\operatorname{Hom}(K, \mathfrak{S}^2)$  is in bijection with the set of pairs of left fibrations  $E^{(0)} \to K, E^{(1)} \to K$  in  $(sSp/K)_{\mathcal{U}}$ . Moreover, the projection  $\mathfrak{S}^{(we)} \to \mathfrak{S}^2$  sends  $E \to K \times \Delta[1]$  to a pair  $E|_0 := E|_{K \times \{0\}}$  and  $E|_1 := E|_{K \times \{1\}}$ , and the set  $\operatorname{Hom}(K, \mathfrak{S}^{(we)})$  is in bijection with the set of weak equivalences  $\phi: E^{(0)} \to E^{(1)}$  of left fibrations over K (by 2.2.7).

By Lemma 1.3.11 (c) for every left fibration  $E \to K \times \Delta[1]$  there exists a weak equivalence  $\phi : E|_0 \to E|_1$  of left fibrations over K. We take  $K := \mathfrak{S}^{\Delta[1]}$  and E be the left fibration, corresponding to  $\mathrm{Id}_K$ , then  $\phi$  gives rise to the morphism  $\psi : \mathfrak{S}^{\Delta[1]} \to \mathfrak{S}^{(we)}$  over  $\mathfrak{S}^2$ .

Moreover,  $p_0 : \mathfrak{S}^{(we)} \to \mathfrak{S}$  is a trivial fibration by Proposition 2.2.8 (c), while  $\delta_0 : \mathfrak{S}^{\Delta[1]} \to \mathfrak{S}$  is a trivial fibration, because  $\mathfrak{S}$  is fibrant (see Proposition 2.2.8 (a)), and  $\Delta[0] \to \Delta[1]$  is a trivial cofibration. Then  $\psi$  is a weak equivalence by 2-out-of-3, hence a homotopy equivalence by Lemma 1.3.11 (b).

**4.2.6. Proof of Corollary 2.2.10.** The homotopy equivalence  $\mathfrak{S}^{(we)} \to \mathfrak{S}^{\Delta[1]}$  over  $\mathfrak{S}^2$  from Proposition 2.2.9 (a) induces a homotopy equivalence

$$\operatorname{Map}(K, \mathfrak{S}^{(we)}) \to \operatorname{Map}(K, \mathfrak{S}^{\Delta[1]}) = \operatorname{Map}(\Delta[1], \mathfrak{S}^K)$$

over Map $(K, \mathfrak{S}^2)$ , hence a homotopy equivalence between fibers over  $(\alpha, \beta) \in$ Map $(K, \mathfrak{S}^2)$ . Since fiber Map $(K, \mathfrak{S}^{(we)})_{\alpha,\beta} \neq \emptyset$  means that  $E_{\alpha}$  and  $E_{\beta}$  are homotopy equivalent over K (by 2.2.7(b)), while Map $(K, \mathfrak{S}^{\Delta[1]})_{\alpha,\beta} \neq \emptyset$  means that  $\alpha \sim \beta$  in  $\mathfrak{S}^K$ , we get the assertion.

**4.2.7.** Construction of  $\psi^{(n)}$ . By 2.2.7, the identity map  $\mathrm{Id}_{\mathfrak{S}^{(n)}}$  corresponds to a diagram  $\phi : E^{(0)} \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_n} E^{(n)}$  of left fibrations over  $\mathfrak{S}^{(n)}$  in  $(sSp/\mathfrak{S}^{(n)})_{\mathcal{U}}$ . To define a morphism  $\psi^{(n)} : \mathfrak{S}^{(n)} \to \mathfrak{S}^{F[n]}$  over  $\mathfrak{S}^{n+1}$ , we have to construct a left fibration  $E \to \mathfrak{S}^{(n)} \times F[n]$  in  $(sSp/\mathfrak{S}^{(n)} \times F[n])_{\mathcal{U}}$  such that  $E|_i := E|_{\mathfrak{S}^{(n)} \times \{i\}} = E^{(i)}$ .

Consider iterated discrete cylinder  $p: Cyl^d(\phi) \to \mathfrak{S}^{(n)} \times F[n]$  (see 4.1.2). Then p is a left quasifibration in  $(sSp/\mathfrak{S}^{(n)} \times F[n])_{\mathcal{U}}$  (see Lemma 4.1.3) such that the restriction  $Cyl^d(\phi)|_i = E^{(i)}$  is a fibration over  $\mathfrak{S}^{(n)}$ . Thus, by Corollary 3.2.6, there exists a fibrant replacement  $p': E \to \mathfrak{S}^{(n)} \times F[n]$  of p in  $(sSp/\mathfrak{S}^{(n)} \times F[n])_{\mathcal{U}}$  such that  $E|_i = Cyl^d(\phi)|_i$  for all i. Then p' is a left fibration by Corollary 3.1.11.

**4.2.8.** Uniqueness of  $\psi^{(n)}$ . Notice that if E' is another fibrant replacement of  $Cyl^d(\phi) \to \mathfrak{S}^{(n)} \times F[n]$  such that  $E'|_i = Cyl^d(\phi)|_i$  for all i, then there exists a weak equivalence  $E \to E'$  over  $\mathfrak{S}^{(n)} \times F[n]$ , which is identity over each  $\{i\} \in F[n]$ . Now it follows from Corollary 2.2.10 that the morphism  $\psi'^{(n)} : \mathfrak{S}^{(n)} \to \mathfrak{S}^{F[n]}$ , corresponding to E', is homotopic to  $\psi^{(n)}$  over  $\mathfrak{S}^{F[n]}$ .

**4.2.9.** Modular interpretation of  $\psi^{(n)}$ . Note that a morphism  $\varphi : K \to \mathfrak{S}^{(n)}$  corresponds to the diagram  $\varphi^*(\phi)$  of left fibrations over K, while the composition  $\psi^{(n)} \circ \varphi : K \to \mathfrak{S}^{F[n]}$  corresponds to the left fibration  $\varphi^*(E)$ . Recall that  $Cyl^d(\phi) \to K \times F[n]$  is a quasifibration and that E is a fibrant replacement of  $Cyl^d(\phi)$  such that  $E|_i = E^{(i)}$  for all i. Hence it follows from Corollary 3.1.4 that  $\varphi^*(E)$  is a

fibrant replacement of  $\varphi^*(Cyl^d(\phi)) = Cyl^d(\varphi^*(\phi))$  such that  $\varphi^*(E)|_i = \varphi^*(E^{(i)})$  for all *i*.

**4.2.10.** Proof of Proposition 2.2.9 (c). By 2.2.7, the composition  $\mu^* \circ \psi^{(n)}$ :  $\mathfrak{S}^{(n)} \to \mathfrak{S}^{F[m]}$  corresponds to the left fibration  $\mu^*(E) \to \mathfrak{S}^{(n)} \times F[m]$ , which is as in 4.2.9 is a fibrant replacement of  $\mu^*(Cyl^d(\phi)) = Cyl^d(\mu^*(\phi))$ . Similarly,  $\psi^{(m)} \circ \mu^* : \mathfrak{S}^{(n)} \to \mathfrak{S}^{F[m]}$  also corresponds to a fibrant replacement of  $Cyl^d(\mu^*(\phi))$ . Since all fibrant replacement are weakly equivalent, two compositions are homotopic by Corollary 2.2.10.

It remains to show that  $\psi = \psi^{(n)}$  is a homotopy equivalence over  $\mathfrak{S}^{n+1}$ .

**4.2.11. Reduction.** By Lemma 1.3.11 (a), we have to show that for every map  $\eta$ :  $M \to \mathfrak{S}^{n+1}$ , the map  $\pi_0(\psi/\eta) : \pi_0(\operatorname{Map}_{\mathfrak{S}^{n+1}}(M,\mathfrak{S}^{(n)})) \to \pi_0(\operatorname{Map}_{\mathfrak{S}^{n+1}}(M,\mathfrak{S}^{F[n]}))$ , induced by  $\psi$ , is a bijection.

Let  $\eta$  corresponds to an (n + 1)-tuple  $H^{(0)}, \ldots, H^{(n)}$  of left fibrations over M. Then  $\varphi \in \operatorname{Hom}_{\mathfrak{S}^{n+1}}(M, \mathfrak{S}^{(n)})$  corresponds to diagrams  $\varphi : H^{(0)} \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_n} H^{(n)}$  over M, and  $\tau \in \operatorname{Hom}_{\mathfrak{S}^{n+1}}(M, \mathfrak{S}^{F[n]})$  corresponds to left fibrations  $H \to M \times F[n]$  such that  $H|_i = H^{(i)}$  for all i.

Using Corollary 2.2.10 (a) and 4.2.9, we see that  $\psi \circ \varphi \sim \tau$  in  $\operatorname{Map}_{\mathfrak{S}^{n+1}}(M, \mathfrak{S}^{F[n]})$ if and only if there exists a weak equivalence  $\nu : Cyl^d(\varphi) \to H$  over  $M \times F[n]$  such that  $\nu|_i : H^{(i)} \to H|_i = H^{(i)}$  is the identity. Moreover, by 4.1.1 (e), this happens if and only if there exists a morphism  $\nu : Cyl^d(\varphi) \to H$  over  $M \times F[n]$  such that  $\nu|_i : H^{(i)} \to H|_i = H^{(i)}$  is  $\operatorname{Id}_{H^{(n)}}$  for all i.

**4.2.12.** Proof of surjectivity of  $\pi_0(\psi/\eta)$ . We have to show that for every left fibration  $H \to M \times F[n]$  such that  $H|_i = H^{(i)}$  for all *i* there exists a diagram  $\varphi$  and a morphism  $\nu : Cyl^d(\varphi) \to H$  over  $M \times F[n]$  such that each  $\nu|_i : H^{(i)} \to H|_i = H^{(i)}$  is the identity. We construct  $\varphi$  and  $\nu$  by induction on *n*. If n = 0, then  $\varphi$  is empty,  $Cyl^d(\varphi) = H^{(0)} = H$ , so  $\nu = \operatorname{Id}_H$  does the job.

Assume that n > 0. By induction hypothesis, there exists a diagram  $\varphi^{(1)} : H^{(1)} \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} H^{(n)}$  over M and a morphism  $e^1 Cyl^d(\varphi(1)) \to H|_{e^1 F[n-1]} \subset H$  over  $M \times F[n]$  such that  $\nu|_i : H^{(i)} \to H|_i = H^{(i)}$  is the identity for all i > 0. In particular, we have a morphism  $\nu[1] : H^{(1)} \times e^1 F[n-1] \to e^1 Cyl^d(\varphi(1)) \to H$  over  $M \times F[n]$  such that  $\nu[1]|_1 = \mathrm{Id}_{H^{(1)}}$ .

Since  $Cyl^d(\varphi) = (H^{(0)} \times F[n]) \sqcup_{(H^{(0)} \times e^1 F[n-1])} e^1 Cyl^d(\varphi(1))$ , it remains to construct a morphism  $\varphi_1 : H^{(0)} \to H^{(1)}$  over M and a morphism  $\nu[0] : H^{(0)} \times F[n] \to H$  over  $M \times F[n]$  such that  $\nu[0]|_0 = \mathrm{Id}_{H^{(0)}}$ , and restriction  $\nu[0]|_{e^1 F[n-1]}$  decomposes as a composition  $H^{(0)} \times e^1 F[n-1] \xrightarrow{\varphi_1} H^{(1)} \times e^1 F[n-1] \xrightarrow{\nu[1]} H$ .

Since  $H \to M \times F[n]$  is a left fibration, the inclusion  $H^{(0)} = H|_0 \hookrightarrow H$  extends to a morphism  $\nu'[0] : H^{(0)} \times e^0 F[1] \to H|_{e^0 F[1]} \subset H$  over  $M \times F[n]$  (see Lemma 4.1.4 (a)). Denote  $\nu'[0]|_1 : H^{(0)} \to H^{(1)}$  by  $\varphi_1$ , and define  $\nu''[0] : H^{(0)} \times e^1 F[n-1] \to H|_{e^1 F[n-1]} \subset H$  to be the composition  $\nu[1] \circ \varphi_1$ . Then  $\nu'[0]$  and  $\nu''[0]$  define a morphism  $H^{(0)} \times (e^0 F[1] \sqcup_{e^1 F[0]} e^1 F[n-1]) \to H$  over  $M \times F[n]$ , which by Lemma 4.1.4 (b) can be extended to all of  $H^{(0)} \times F[n]$ .

**4.2.13.** Proof of injectivity of  $\pi_0(\psi/\eta)$ . Fix a left fibration  $H \to M \times F[n]$ , and consider all diagrams  $\varphi : H^{(0)} \xrightarrow{\varphi_1} \dots \xrightarrow{\varphi_n} H^{(n)}$  over M for which there exists a morphism  $\nu : Cyl^d(\varphi) \to H$  over  $M \times F[n]$  such that each  $\nu|_j$  is the identity. We have to show that each  $\pi_0(\varphi_i) \in \pi_0(\operatorname{Map}_M(H^{(j)}, H^{(j+1)}))$  only depends on H.

Consider canonical embedding  $\iota_j : H^{(j)} \times e^j F[n-j] \to Cyl^d(\varphi)$  (see 4.1.2). Then the composition  $\nu \circ \iota_j : H^{(j)} \times e^j F[n-j] \to H$  is such that  $(\nu \circ \iota)|_j : H^{(j)} \to H^{(j)}$  is  $\mathrm{Id}_{H^{(j)}}$ , while  $(\nu \circ \iota)|_{j+1} : H^{(j)} \to H^{(j+1)}$  is  $\varphi_j$ . Thus it remains to show that each  $\pi_0(\nu \circ \iota_j) \in \pi_0(\mathrm{Map}_{M \times F[n]}(H^{(j)} \times e^j F[n-j], H))$  only depends on H. Since the restriction map  $\mathrm{Map}_{M \times F[n]}(H^{(j)} \times e^j F[n-j], H) \to \mathrm{Map}_{M \times F[n]}(H^{(j)} \times \{j\}, H)$ is a trivial fibration (by Lemma 4.1.4 (b)), while  $(\nu \circ \iota_j)|_j = \mathrm{Id}_{H^{(j)}}$ , the assertion follows from 1.2.8 (b).

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