On Stability of Hyperbolic Thermoelastic Reissner-Mindlin-Timoshenko Plates

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Abstract

In the present article, we consider a thermoelastic plate of Reissner-Mindlin-Timoshenko type with the hyperbolic heat conduction arising from Cattaneo's law. In the absense of any additional mechanical dissipations, the system is often not even strongly stable unless restricted to the rotationally symmetric case, etc. We present a well-posedness result for the linear problem under general mixed boundary conditions for the elastic and thermal parts. For the case of a clamped, thermally isolated plate, we show an exponential energy decay rate under a full damping for all elastic variables. Restricting the problem to the rotationally symmetric case, we further prove that a single frictional damping merely for the bending compoment is sufficient for exponential stability. To this end, we construct a Lyapunov functional incorporating the Bogovskiĭ operator for irrotational vector fields which we discuss in the appendix.

MOS subject classification: 35L55; 35Q74; 74D05; 93D15; 93D20

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a boundary $\Gamma := \partial\Omega$. We consider a thermoelastic Reissner-Mindlin-Timoshenko plate of a uniform thickness $h > 0$ such that its midplane occupies the domain Ω when being in a reference state free of any elastic or thermal stresses. The heat propagation is modeled by means of the Cattaneo's law (viz. [5]). With w denoting the vertical displacement of the midplane, ψ, φ the after-bending-angles of vertical filaments being perpendicular to the midplane in the reference state, θ the thermal moment and q the moment of the heat flux, respectively, the symmetrized form of Reissner-Mindlin-Timoshenko equations reads as

$$\rho_1 w_{tt} - K(w_{x_1} + \psi)_{x_1} - K(w_{x_2} + \varphi)_{x_2} = 0 \text{ in } (0, \infty) \times \Omega, \tag{1.1}$$

$$\rho_2 \psi_{tt} - D(\psi_{x_1x_1} + \tfrac{1-\mu}{2}\psi_{x_2x_2} + \tfrac{1+\mu}{2}\varphi_{x_1x_2}) + K(\psi + w_{x_1}) + \gamma\theta_{x_1} = 0 \text{ in } (0, \infty) \times \Omega, \tag{1.2}$$

$$\rho_2 \varphi_{tt} - D(\varphi_{x_2x_2} + \tfrac{1-\mu}{2}\varphi_{x_1x_1} + \tfrac{1+\mu}{2}\psi_{x_1x_2}) + K(\varphi + w_{x_2}) + \gamma\theta_{x_2} = 0 \text{ in } (0, \infty) \times \Omega, \tag{1.3}$$

$$\rho_3 \theta_t + \kappa \operatorname{div} q + \beta\theta + \gamma(\psi_{tx_1} + \varphi_{tx_2}) = 0 \text{ in } (0, \infty) \times \Omega, \tag{1.4}$$

$$\tau_0 q_t + \delta q + \kappa \nabla\theta = 0 \text{ in } (0, \infty) \times \Omega. \tag{1.5}$$

A physical deduction of the model can be found in [26, Kapitel 1]. See also [17, Chapter 1] for the case of purely elastic Reissner-Mindlin-Timoshenko plates or thermoelastic Kirchhoff-Love plates with

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parabolic heat conduction. Note that, in contrast to the heat equation in uniformly thick bodies, $\beta\theta$-term naturally arises in the model since θ is the thermal moment and not the temperature.

Structurally viewed, Reissner-Mindlin-Timoshenko Equations (1.1)–(1.5) can be interpreted as a 2D Lamé system (1.2)–(1.3) for the filament angles $(\psi, \varphi)'$ coupled to the wave equation (1.1) for the bending component w and the Cattaneo system (1.4)–(1.5) for the thermal moment θ and the moment of the heat flux q. Since neither mechanical, no thermal dissipation due to the lack of a direct coupling to the Cattaneo system is present in Equation (1.1), one expects the decay properties of (1.1)–(1.5) to be not better than those of classical or hyperbolic 2D thermoelasticity. The latter have been investigated by numerous authors. Whereas the thermal dissipation arising from the parabolic heat equation leads (with "few" exceptions) to the strong stability when coupled with a Lamé system in a bounded domain of $\mathbb{R}^n$ – as shown by Dafermos in [6], no uniform decay can usually be expected (cp. [20]). Reducing the problem to the case of rotationally symmetric solutions, Jiang and Racke [14, Theorem 4.2] showed an exponential decay of the second-order energy, also in the nonlinear situation (cf. [14, Theorem 7.3]). A similar result was latter obtained by Racke in [28] for the linear 2D and 3D hyperbolic thermoelasticity.

As a matter of fact, Reissner-Mindlin-Timoshenko plates and Timoshenko beams have a certain degree of similarity with Kirchhoff-Love plates and Euler-Bernoulli beams. The latter also describe the bending of an elastic plate or a beam under the assumption that the linear filaments remain perpendicular to mid-plane even after the plate's deformation. This model can be shown to be a limit (in a certain sense) of the Timoshenko model as the shear correction factor $K \to \infty$ (cf. [16]). Numerous mathematical results on Kirchhoff plates are known in the literature. In his monograph [16], Lagnese studied various boundary feedback stabilizers furnishing uniform or strong stability for the Kirchhoff-Love plate coupled with a parabolic heat equation in a bounded domain with or without assumptions on the geometry. Avalos and Lasiecka exploited further in [1] a multiplier technique, interpolation tools and regularity results to obtain exponential stability of a thermoelastic Kirchhoff-Love plate without any additional boundary dissipation in the presence or absense of rotational inertia. Another important development in this field was made by Lasiecka and Triggiani (see, e.g., [19]) who showed the analyticity of underlying semigroup for all combinations of natural boundary conditions. Implying the maximal L^2-regularity property, this became an important tool for studying nonlinear plates, e.g., the von Kármán model, which was done by Avalos et al. in [2]. It should though be pointed out that this approach is not directly applicable to the case of coupling with the hyperbolic Cattaneo's heat conduction system which destroys the analyticity of the semigroup. Nonetheless, in an analogous situation of a partly hyperbolic systems such as the full von Kármán one, Lasiecka [18] obtained the existence of weak and regular solutions and showed their uniform stability in the presense of a mechanical damping only for the solenoidal part for the in-plane displacements. A similar study has then later been carried out in [3] by Benabdallah and Lasiecka for the full von Kármán model incorporating rotational inertia.

Turning back to Reissner-Mindlin-Timoshenko plates, we once again refer to the monograph [16] of Lagnese in which he addressed the question of uniform (in particular, exponential) and strong stabilization of purely elastic plates by the means of boundary feedbacks. For the following choice of stabilizing feedbacks on a portion $\Gamma_1 \neq \emptyset$ of the boundary

$$\begin{aligned}
w = \psi = \varphi = 0 \quad &\text{in } (0, \infty) \times \Gamma_0, \\
K\left(\tfrac{\partial w}{\partial \nu} + \nu_1 \psi + \nu_2 \varphi\right) = m_1 \ &\text{in } (0, \infty) \times \Gamma_1, \\
D\left(\nu_1 \psi_{x_1} + \mu \nu_1 \varphi_{x_2} + \tfrac{1-\mu}{2}(\psi_{x_2} + \varphi_{x_1})\nu_2\right) = m_2 \ &\text{in } (0, \infty) \times \Gamma_1, \\
D\left(\nu_2 \varphi_{x_2} + \mu \nu_2 \psi_{x_1} + \tfrac{1-\mu}{2}(\psi_{x_2} + \varphi_{x_1})\nu_1\right) = m_3 \ &\text{in } (0, \infty) \times \Gamma_1,
\end{aligned}$$

the purely elastic Reissner-Mindlin-Timoshenko plate

$$\begin{aligned}
\rho_1 w_{tt} - K(w_{x_1} + \psi)_{x_1} - K(w_{x_2} + \varphi)_{x_2} &= 0 \text{ in } (0,\infty) \times \Omega, \\
\rho_2 \psi_{tt} - D(\psi_{x_1x_1} + \tfrac{1-\mu}{2}\psi_{x_2x_2} + \tfrac{1+\mu}{2}\varphi_{x_1x_2}) + K(\psi + w_{x_1}) + \gamma\theta_{x_1} + d_1\psi_t &= 0 \text{ in } (0,\infty) \times \Omega, \\
\rho_2 \varphi_{tt} - D(\varphi_{x_2x_2} + \tfrac{1-\mu}{2}\varphi_{x_1x_1} + \tfrac{1+\mu}{2}\psi_{x_1x_2}) + K(\varphi + w_{x_2}) + \gamma\theta_{x_2} + d_2\varphi_t &= 0 \text{ in } (0,\infty) \times \Omega
\end{aligned}$$

was proved to be strongly stable (i.e., the energy was shown to vanish as $t \to \infty$) if $\Gamma_0 \neq \varnothing$ und $(m_1, m_2, m_3)' = -F(w_t, \psi_t, \varphi_t)'$ where $F \in L^\infty(\Gamma_1, \mathbb{R}^{3\times 3})$ is a symmetric positive semidefinite matrix function which is additionally positive definite on a connected nontrivial portion of Γ_1, etc. Under the geometric condition stating that $(\Omega, \Gamma_0, \Gamma_1)$ is "star complemented—star shaped" and some additional assumptions on F, even uniform stability has been shown.

Similar results were also obtained by Muñoz Rivera and Portillo Oquendo in [23] auch for the boundary conditions of memory-type

$$\begin{aligned}
w = \psi = \varphi &= 0 \text{ in } (0,\infty) \times \Gamma_0, \\
w + \int_0^t g_1(t-s) K(\tfrac{\partial w}{\partial \nu} + \nu_1\psi + \nu_2\varphi)(s)\mathrm{d}s &= 0 \text{ in } (0,\infty) \times \Gamma_1, \\
\psi + \int_0^t g_1(t-s) D(\nu_1\psi_{x_1} + \mu\nu_1\varphi_{x_2} + \tfrac{1-\mu}{2}(\psi_{x_2} + \varphi_{x_1})(s)\mathrm{d}s &= 0 \text{ in } (0,\infty) \times \Gamma_1, \\
\varphi + \int_0^t g_1(t-s) D(\nu_2\varphi_{x_2} + \mu\nu_2\psi_{x_1} + \tfrac{1-\mu}{2}(\psi_{x_2} + \varphi_{x_1})\nu_1)(s)\mathrm{d}s &= 0 \text{ in } (0,\infty) \times \Gamma_1
\end{aligned}$$

with exponential kernels g_1, g_2, g_3.

In [7], Fernández Sare studied a linear Reissner-Mindlin-Timoshenko plate with a damping for both angle components

$$\rho_1 w_{tt} - K(w_{x_1} + \psi)_{x_1} - K(w_{x_2} + \varphi)_{x_2} = 0 \text{ in } (0,\infty) \times \Omega, \tag{1.6}$$

$$\rho_2 \psi_{tt} - D(\psi_{x_1x_1} + \tfrac{1-\mu}{2}\psi_{x_2x_2} + \tfrac{1+\mu}{2}\varphi_{x_1x_2}) + K(\psi + w_{x_1}) + \gamma\theta_{x_1} + d_1\psi_t = 0 \text{ in } (0,\infty) \times \Omega, \tag{1.7}$$

$$\rho_2 \varphi_{tt} - D(\varphi_{x_2x_2} + \tfrac{1-\mu}{2}\varphi_{x_1x_1} + \tfrac{1+\mu}{2}\psi_{x_1x_2}) + K(\varphi + w_{x_2}) + \gamma\theta_{x_2} + d_2\varphi_t = 0 \text{ in } (0,\infty) \times \Omega. \tag{1.8}$$

He proved that the system is polynomially stable under Dirichlet boundary conditions on all three variables. For a particular choice of boundary conditions in a rectangular configuration $\Omega = (0, L_1) \times (0, L_2)$, a resolvent criterion was exploited to show that the system is not exponentially stable.

Muñoz Rivera und Racke considered in [24] an nonlinear Timoshenko-beam coupled to a parabolic heat equation

$$\begin{aligned}
\rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x &= 0 \text{ in } (0,\infty) \times (0, L), \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x &= 0 \text{ in } (0,\infty) \times (0, L), \\
\rho_3 \theta_t - \kappa\theta_{xx} + \gamma\psi_{tx} &= 0 \text{ in } (0,\infty) \times (0, L)
\end{aligned}$$

subject to mixed boundary conditions $\varphi = \psi = \theta_x = 0$ or $\varphi = \psi_x = \theta = 0$. Both in the linear case, i.e., $\sigma(r, s) = kr + s$, and the nonlinear case, i.e., for a smooth stress function σ satisfying $\nabla\sigma = (k, k)'$, $\nabla^2\sigma = 0$, but in the latter case only for sufficiently small initial data, the energy was shown to decay exponentially if the condition $\frac{\rho_1}{k} = \frac{\rho_2}{b}$ holds true. For the linear situation, this condition was even shown to be necessary for the exponential stability. It should though be pointed out that the latter proportionality condition, being mathematically fully sound, is physically not possible.

Surprisingly, this result could not be carried over to the case of Cattaneo heat conduction. Namely, Fernández Sare and Racke showed in [8] that the purely hyperbolic system

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \text{ in } (0,\infty)\times(0,L), \tag{1.9}$$
$$\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0 \text{ in } (0,\infty)\times(0,L), \tag{1.10}$$
$$\rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0 \text{ in } (0,\infty)\times(0,L), \tag{1.11}$$
$$\tau_0 q_t + \delta q + \kappa\theta_x = 0 \text{ in } (0,\infty)\times(0,L) \tag{1.12}$$

is not exponentially stable even under the assumption $\frac{\rho_1}{k} = \frac{\rho_2}{b}$. This motivated Messaoudi et al. to introduce a frictional damping for the bending component. In [22], they replaced Equation (1.9) with the damped equation

$$\rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x + \mu\varphi_t = 0 \text{ in } (0,\infty)\times(0,L)$$

for some $\mu > 0$. Under this additional mechanical dissipation, they proved that both linear and nonlinear systems are stable under the boundary conditions $\varphi = \psi = q = 0$ und $\varphi_x = \psi = q = 0$ independent of whether the relation $\frac{\rho_1}{k} = \frac{\rho_2}{b}$ holds or not.

The impact of thermal coupling on the strong stability of a Reissner-Mindlin-Timoshenko plate has also been studied by Grobbelaar in her papers [10], [11] and [12]. In [10], the author considered a stuctural 3D acoustic model with a 2D plate interface and proved a strong asymptotic stability for the radially symmetric case. A similar result was later obtained in [11] for a rotationally symmetric Reissner-Mindlin-Timoshenko plate with hyperbolic heat conduction due to Cattaneo. To this end, both articles employed Benchimol's spectral criterion. The arguments can be directly carried over to the case of classical Fourier heat conduction being a formal limit Cattaneo's system as the relaxation parameter $\tau \to 0$. In her recent article [12], Grobbelaar proved a polynimal decay rate of $t^{-1/4}$ in the rotationally symmetric case for the Reissner-Mindlin-Timoshenko system coupled to the classical Fourier heat conduction under Dirichlet boundary conditions on w and θ as well as free boundary conditions on ψ and φ.

In the present article, we consider the linear Reissner-Mindlin-Timosheko plate equations (1.1)–(1.5) in a bounded domain. The paper is structured as follows. In the first section, we exploit the semigroup theory to show that the initial-boundary value problem (1.1)–(1.5) subject to corresponding initial conditions as well as homogeneous Dirichlet and Neumann boundary conditions on both elastic and thermal variables on different portions of the boundary is well-posed. In the second section, we prove the lack of strong stability for this problem provided Γ is smooth for a particular set of boundary conditions. We further show that a mechanical damping for all three variables w, φ and ψ leads to an exponential decay rate under Dirichlet boundary conditions for the elastic and Neumann boundary conditions for the thermal part of the system. Restricting the domain Ω and the data to the rotationally symmetric case, we prove that a single mechanical damping on w is enough to exponentially stabilize the system. This is a generalization of Messaoudi's et al. stability results from [22] to a multi-dimensional situation. In the appendix, we finally present a brief discussion on Bogovskiĭ operator for irrotational vector fields and show its continuity.

2 Existence and uniqueness of classical solutions

In the following, unless specified otherwise, we assume the boundary Γ to be Lipschitzian and satisfy $\Gamma = \bar\Gamma_1 \cup \bar\Gamma_2 = \bar\Gamma_3 \cup \bar\Gamma_4$ with $\Gamma_1 \neq \varnothing$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_3 \cap \Gamma_4 = \emptyset$ and Γ_k, $k = 1, \dots, 4$, being relatively open. Let the plate be clamped at Γ_1 and hinged at Γ_2. Further, let it be held at the reference temperature on Γ_3 and be thermally insulated on Γ_4. Then, the boundary conditions read as

$$w = \psi = \varphi = 0 \text{ on } (0,\infty)\times\Gamma_1, \tag{2.1}$$
$$K(\tfrac{\partial w}{\partial \nu} + \nu_1\psi + \nu_2\varphi) = 0 \text{ on } (0,\infty)\times\Gamma_2, \tag{2.2}$$
$$D(\nu_1\psi_{x_1} + \mu\nu_1\varphi_{x_2} + \tfrac{1-\mu}{2}(\psi_{x_2}+\varphi_{x_1})\nu_2) - \gamma\theta\nu_1 = 0 \text{ on } (0,\infty)\times\Gamma_2, \tag{2.3}$$
$$D(\nu_2\varphi_{x_2} + \mu\nu_2\psi_{x_1} + \tfrac{1-\mu}{2}(\psi_{x_2}+\varphi_{x_1})\nu_1) - \gamma\theta\nu_2 = 0 \text{ on } (0,\infty)\times\Gamma_2, \tag{2.4}$$
$$\theta = 0 \text{ on } (0,\infty)\times\Gamma_3, \tag{2.5}$$
$$q\cdot\nu = 0 \text{ on } (0,\infty)\times\Gamma_4, \tag{2.6}$$

where $\nu = (\nu_1,\nu_2)'$ denotes the outer unit normal vector to Γ and $(\cdot)'$ stands for the usial matrix transposition.

Using the standard notation from the Theory of elasticity (cf. [14, p. 8]), we introduce the generalized gradient and the corresponding boundary symbol

$$\mathcal{D} := \begin{pmatrix} \partial_1 & 0 \\ 0 & \partial_2 \\ \partial_2 & \partial_1 \end{pmatrix}, \quad \mathcal{N} := \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \\ \nu_2 & \nu_1 \end{pmatrix},$$

respectively. With this notation, we can easily conclude

$$D\begin{pmatrix} \psi_{x_1x_1} + \frac{1-\mu}{2}\psi_{x_2x_2} + \frac{1+\mu}{2}\varphi_{x_1x_2} \\ \varphi_{x_2x_2} + \frac{1-\mu}{2}\varphi_{x_1x_2} + \frac{1+\mu}{2}\psi_{x_1x_2} \end{pmatrix} = \mathcal{D}'S\mathcal{D}v,$$
$$D\begin{pmatrix} \nu_1\psi_{x_1} + \mu\nu_1\varphi_{x_2} + \frac{1-\mu}{2}(\psi_{x_2}+\varphi_{x_1})\nu_2 \\ \nu_2\varphi_{x_2} + \mu\nu_2\psi_{x_1} + \frac{1+\mu}{2}(\psi_{x_2}+\varphi_{x_1})\nu_1 \end{pmatrix} = \mathcal{N}'S\mathcal{D}v,$$

where $v := (\psi,\varphi)'$ and

$$S := D\begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{pmatrix}. \tag{2.7}$$

With μ satisfying $\mu \in (-1,1)$, the symmetric matrix S is positive definite since $\sigma(S) = \{D\frac{1-\mu}{2}, D(1-\mu), D(1+\mu)\}$ due to the fact

$$\det(S-\lambda I) = (D\,\tfrac{1-\mu}{2} - \lambda)((D-\lambda)^2 - \mu^2D^2) = (D\,\tfrac{1-\mu}{2} - \lambda)(D-\lambda-\mu D)(D-\lambda+\mu D).$$

By the virtue of physical condition $\mu \in (0,\frac{1}{2})$, the latter is not an actual restriction. Throughout this section, we assume S to be an arbitrary symmetric, positive definite matrix, i.e., $S \in \mathrm{SPD}(\mathbb{R}^3)$.

With the notations above, Equations (1.1)–(1.5) can be equivalently written as

$$\rho_1 w_{tt} - K\mathrm{div}\,(\nabla w + v) = 0 \text{ in } (0,\infty)\times\Omega, \tag{2.8}$$
$$\rho_2 v_{tt} - \mathcal{D}'S\mathcal{D}v + K(v+\nabla w) + \gamma\nabla\theta = 0 \text{ in } (0,\infty)\times\Omega, \tag{2.9}$$
$$\rho_3\theta_t + \kappa\mathrm{div}\,q + \beta\theta + \gamma\mathrm{div}\,v_t = 0 \text{ in } (0,\infty)\times\Omega, \tag{2.10}$$
$$\tau_0 q_t + \delta q + \kappa\nabla\theta = 0 \text{ in } (0,\infty)\times\Omega \tag{2.11}$$

with the boundary conditions (2.1)–(2.6) transformed to

$$w = |v| = 0 \text{ on } (0,\infty)\times\Gamma_1, \tag{2.12}$$
$$(\nabla w + v)\cdot\nu = 0 \text{ on } (0,\infty)\times\Gamma_2, \tag{2.13}$$
$$\mathcal{N}'S\mathcal{D}v - \gamma\theta\nu = 0 \text{ on } (0,\infty)\times\Gamma_2, \tag{2.14}$$
$$\theta = 0 \text{ on } (0,\infty)\times\Gamma_3, \tag{2.15}$$
$$q\cdot\nu = 0 \text{ on } (0,\infty)\times\Gamma_4 \tag{2.16}$$

and initial conditions

$$w(0,\cdot) = w^0,\ w_t(0,\cdot) = w^1,\ v(0,\cdot) = v^0,\ v_t(0,\cdot) = v^1,\ \theta(0,\cdot) = \theta^0,\ q(0,\cdot) = q^0, \tag{2.17}$$

where $v^0 = (\psi^0, \varphi^0)'$, $v^1 = (\psi^1, \varphi^1)'$.

2.1 Well-Posedness

We further exploit the semigroup theory to obtain the classical well-posedness of Reissner-Mindlin-Timoshenko equations. To this end, we transform Equations (2.8)–(2.17) into the Cauchy problem

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(t) &= \mathcal{A} V(t) \text{ für } t \in (0,\infty), \\ V(0) &= V^0 \end{aligned}$$

on a Hilbert space $\mathcal{H}$. According to [25, Theorem 1.3], the latter is well-posed if and only if $\mathcal{A}$ is an infinitesimal generator of a strongly continuous semigroup on $\mathcal{H}$.

We set $V := (w, v, w_t, v_t, \theta, q)'$ and formally define the differential operator

$$A := \rho^{-1} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ K\triangle & K\mathrm{div} & 0 & 0 & 0 & 0 \\ -K\nabla & \mathcal{D}'S\mathcal{D} - K & 0 & 0 & -\gamma\nabla & 0 \\ 0 & 0 & 0 & -\gamma\mathrm{div} & -\beta & -\kappa\mathrm{div} \\ 0 & 0 & 0 & 0 & -\kappa\nabla & -\delta \end{pmatrix}$$

with $\rho := \mathrm{diag}(1, 1, \rho_1, \rho_2, \rho_3, \tau_0)$. To introduce the functional analytic settings, we consider the Hilbert space

$$\mathcal{H} := (H^1_{\Gamma_1}(\Omega))^3 \times (L^2(\Omega))^3 \times (L^2(\Omega))^3$$

equipped with the scalar product

$$\begin{aligned} \langle V, W\rangle_{\mathcal{H}} := \rho_1 \langle V^3, W^3\rangle_{L^2(\Omega)} + \rho_2 \langle V^4, W^4\rangle_{(L^2(\Omega))^2} + K\langle \nabla V^1 + V^2, \nabla W^1 + W^2\rangle_{(L^2(\Omega))^2} + \\ \langle \mathcal{D}V^2, S\mathcal{D}W^2\rangle_{(L^2(\Omega))^3} + \rho_3\langle V^5, W^5\rangle_{(L^2(\Omega))^2} + \tau_0\langle V^6, W^6\rangle_{L^2(\Omega)}. \end{aligned}$$

Here, we define for a relatively open set $\Gamma_0 \subset \Gamma$

$$H^1_{\Gamma_0}(\Omega) = \mathrm{cl}\left(\{u \in \mathcal{C}^\infty(\Omega) \,|\, \mathrm{supp}(u) \cap \Gamma_0 = \emptyset\}, \|\cdot\|_{H^1(\Omega)}\right).$$

Note that due to the Lipschitz continuity of Γ, there exists a linear, continuous operator $T\colon H^1(\Omega) \to H^{1/2}(\Gamma)$. Thus, the notation $u|_{\Gamma_0} = 0$ is also legitimate.

The following theorem implies that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is equivalent with the standard product topology on $\mathcal{H}$, i.e., $\mathcal{H}$ is complete. The proof is a direct consequence of an analogous result in [17] (cf. also [21] for the case of domains with a strict cone property).

Lemma 1. *There exist constants $C_{\mathcal{K},1}, C_{\mathcal{K},2}, C_{\mathcal{K}} > 0$ such that*

$$C_{\mathcal{K},1}\|v\|_{(H^1(\Omega))^2} \leq \|\sqrt{S}\mathcal{D}v\|_{(L^2(\Omega))^3} \leq C_{\mathcal{K},2}\|v\|_{(H^1(\Omega))^2}$$

and

$$\|\sqrt{S}\mathcal{D}v\|^3_{(L^2(\Omega))^2} + K\|\nabla w + v\|^2_{(L^2(\Omega))^2} \geq C_{\mathcal{K}}\left(\|v\|^2_{(H^1(\Omega))^2} + \|w\|^2_{H^1(\Omega)}\right)$$

holds for any $(w, v) \in (H^1_{\Gamma_1}(\Omega))^3$.

We introduce the operator

$$\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}, \quad V \longmapsto AV,$$

where

$$\begin{aligned} D(\mathcal{A}) &= \{V \in \mathcal{H} \,|\, AV \in \mathcal{H}, V \text{ satisfies the generalized Neumann boundary conditions (2.18)–(2.20)}\} \\ &= \{V \in \mathcal{H} \,|\, V^1, V^3 \in H^1_{\Gamma_1(\Omega)}, V^2, V^4 \in (H^1_{\Gamma_1(\Omega)})^2, \triangle V^1 \in L^2(\Omega), \mathcal{D}^T S \mathcal{D} V^2 \in (L^2(\Omega))^2, \\ &\qquad V^5 \in H^1_{\Gamma_3}(\Omega), \operatorname{div} V^6 \in L^2(\Omega), \\ &\qquad V \text{ satisfies the generalized Neumann boundary conditions (2.18)–(2.20)}\} \end{aligned}$$

with the generalized Neumann boundary conditions given by

$$\langle \triangle V^1 + \operatorname{div} V^2, \phi\rangle_{L^2(\Omega)} + \langle \nabla V^1 + V^2, \nabla\phi\rangle_{(L^2(\Omega))^2} = 0 \text{ for all } \phi \in H^1_{\Gamma_1}(\Omega) \tag{2.18}$$

$$\langle \mathcal{D}^T S \mathcal{D} V^2 - \gamma \nabla V^5, \phi\rangle_{(L^2(\Omega))^2} + \langle S\mathcal{D}V^2, \mathcal{D}\phi\rangle_{(L^2(\Omega))^3} - \gamma\langle V^5, \operatorname{div}\phi\rangle_{L^2(\Omega)} = 0 \text{ for all } \phi \in (H^1_{\Gamma_1}(\Omega))^2 \tag{2.19}$$

$$\langle \operatorname{div} V^6, \phi\rangle_{L^2(\Omega)} + \langle V^6, \nabla\phi\rangle_{(L^2(\Omega))^2} = 0 \text{ for all } \phi \in H^1_{\Gamma_3}(\Omega). \tag{2.20}$$

Obviously, $D(\mathcal{A})$ is a linear subspace of $\mathcal{H}$.

Thus,

$$V_t = \mathcal{A}V, \quad V(0) = V^0 \tag{2.21}$$

is a generalization of (2.8)–(2.17) since any classically differentiable solution to (2.8)–(2.17) solves the abstract Cauchy problem (2.21). Here, $V^0 := (w^0, v^0, w^1, v^1, \theta^0, q^0)'$ is assumed to be an element of $D(\mathcal{A})$.

The following theorem characterizes $\mathcal{A}$ as an infinitesimal generator of a strongly continuous semigroup of bounded linear operators on $\mathcal{H}$.

Theorem 2. *The following statements hold true for* $\mathcal{A}$*.*

1. $D(\mathcal{A})$ *is dense in* $\mathcal{H}$.
2. $\mathcal{A}$ *is a closed operator.*
3. $\operatorname{im}(\lambda - \mathcal{A}) = \mathcal{H}$ *for any* $\lambda > 0$.
4. $\mathcal{A}$ *is dissipative.*

Proof. 1. The fact that $D(\mathcal{A})$ is a dense subspace of $\mathcal{H}$ is a direct consequence of the inclusion

$$(\mathcal{C}^\infty(\Omega))^9 \cap \mathcal{H} \subset D(\mathcal{A}).$$

Note that the generalized Neumann boundary conditions (2.18)–(2.20) are satisfied per defintion.

2. The proof of the closedness of $\mathcal{A}$ is also standard. We select an arbitrary sequence $(V_n)_{n\in\mathbb{N}} \subset D(\mathcal{A})$ such that $V_n \to V \in \mathcal{H}$ and $\mathcal{A}V_n \to F \in \mathcal{H}$ as $n \to \infty$ and show that $V \in D(\mathcal{A})$ and $\mathcal{A}V = F$ (cf. [26] for the case $\Gamma_2 = \Gamma_3 = \emptyset$).

   Taking into account $((L^2(\Omega))^9)' \subset \mathcal{H}'$, the strong convergence in $\mathcal{H}$ implies the weak convergence in $(L^2(\Omega))^9$, i.e.,

   $$\langle \mathcal{A}V_n, \Phi\rangle_{(L^2(\Omega))^9} \to \langle F, \Phi\rangle_{(L^2(\Omega))^9} \text{ as } n \to \infty$$

   for any $\Phi \in (L^2(\Omega))^9$. With a proper selection of Φ, the problem can be projected onto a corresponding component. The proof will be made by means of a proper selection of Φ.

There generally holds for $V \in D(\mathcal{A})$

$$\mathcal{A}V = \rho^{-1}\begin{pmatrix} V^3 \\ V^4 \\ K\triangle V^1 + K\operatorname{div} V^2 \\ -K\nabla V^1 + \mathcal{D}'S\mathcal{D}V^2 - KV^2 - \gamma\nabla V^5 \\ -\gamma\operatorname{div} V^4 - \beta V^5 - \kappa\operatorname{div} V^6 \\ -\kappa\nabla V^5 - \delta V^6 \end{pmatrix}.$$

We consider the following cases:

i) First, we select $\Phi = (\phi, 0, 0, 0, 0, 0)'$, $\phi \in H^1_{\Gamma_1}(\Omega)$ to obtain

$$\langle F^1, \phi\rangle_{L^2(\Omega)} = \langle F, \Phi\rangle_{(L^2(\Omega))^9} \leftarrow \langle \mathcal{A}V_n, \Phi\rangle_{(L^2(\Omega))^9} = \tfrac{1}{\rho_1}\langle V_n^3, \phi\rangle_{L^2(\Omega)} \to \tfrac{1}{\rho_1}\langle V^3, \phi\rangle_{L^2(\Omega)}.$$

Therefore, $\frac{1}{\rho_1}V^3 = F^1$, i.e., $(\mathcal{A}V)^1 = F^1$. Taking into account $F^1 \in H^1_{\Gamma_1}(\Omega)$, we conclude $V^3 \in H^1_{\Gamma_1}(\Omega)$.

ii) Letting $\Phi = (0, \phi, 0, 0, 0, 0)$, $\phi \in (H^1_{\Gamma_1}(\Omega)$, we similarly get $(\mathcal{A}V)^2 = F^2$ und $V^4 \in (H^1_{\Gamma_1}(\Omega))^2$.

iii) Further, we choose $\Phi = (0, 0, \phi, 0, 0, 0)'$, $\phi \in H^1_{\Gamma_1}(\Omega)$. This yields

$$\begin{aligned}\langle F^3, \phi\rangle_{L^2(\Omega)} \leftarrow & \tfrac{1}{\rho_1}\langle K\triangle V_n^1 + K\operatorname{div} V_n^2, \phi\rangle_{L^2(\Omega)} \\ = & -\tfrac{K}{\rho_1}\langle \nabla V_n^1, \nabla\phi\rangle_{(L^2(\Omega))^2} + \langle K\operatorname{div} V_n^2, \phi\rangle_{L^2(\Omega)} \\ \to & \tfrac{K}{\rho_1}\langle \nabla V^1, \nabla\phi\rangle_{(L^2(\Omega))^2} + \langle K\operatorname{div} V^2, \phi\rangle_{L^2(\Omega)}\end{aligned}$$

implying $\triangle V^1 \in L^2(\Omega)$ and $\frac{1}{\rho_1}(K\triangle V^1 + K\operatorname{div} V^2) = F^3$, i.e., $(\mathcal{A}V)^3 = F^3$.

iv) For $\Phi = (0, 0, 0, 0, 0, \phi)'$, $\phi \in (H^1_{\Gamma_3}(\Omega))^2$, we obtain

$$\begin{aligned}\langle F^6, \phi\rangle_{(L^2(\Omega))^2} \leftarrow & \tfrac{1}{\tau_0}\langle -\kappa\nabla V_n^5 - \delta V_n^6, \phi\rangle_{(L^2(\Omega))^2} = \tfrac{\kappa}{\tau_0}\langle V_n^5, \operatorname{div}\phi\rangle_{L^2(\Omega)} - \tfrac{\delta}{\tau_0}\langle V_n^6, \phi\rangle_{(L^2(\Omega))^2} \\ \to & \tfrac{\kappa}{\tau_0}\langle V^5, \operatorname{div}\phi\rangle_{L^2(\Omega)} - \tfrac{\delta}{\tau_0}\langle V^6, \phi\rangle_{(L^2(\Omega))^2}.\end{aligned}$$

Hence, $V^5 \in H^1_{\Gamma_3}(\Omega)$ and $\frac{1}{\tau_0}(-\kappa\nabla V^5 - \delta V^6) = F^6$, i.e., $(\mathcal{A}V)^6 = F^6$.

v) Selecting now $\Phi = (0, 0, 0, \phi, 0, 0)'$, $\phi \in (H^1_{\Gamma_1}(\Omega))^2$, we find

$$\begin{aligned}\langle F^4, \phi\rangle_{(L^2(\Omega))^2} \leftarrow & \tfrac{1}{\rho_2}\langle -K\nabla V_n^1 + \mathcal{D}'S\mathcal{D}V_n^2 - KV_n^2 - \gamma\nabla V_n^5, \phi\rangle_{(L^2(\Omega))^2} \\ = & \tfrac{K}{\rho_2}\langle S\mathcal{D}V_n^2, \mathcal{D}\phi\rangle_{(L^2(\Omega))^3} + \tfrac{1}{\rho_2}\langle -K\nabla V_n^1 - KV_n^2 - \gamma\nabla V_n^5, \phi\rangle_{(L^2(\Omega))^2} \\ \to & \tfrac{K}{\rho_2}\langle S\mathcal{D}V^2, \mathcal{D}\phi\rangle_{(L^2(\Omega))^3} + \tfrac{1}{\rho_2}\langle -K\nabla V^1 - KV^2 - \gamma\nabla V^5, \phi\rangle_{(L^2(\Omega))^2}.\end{aligned}$$

Thus, $\mathcal{D}'S\mathcal{D}V^2 \in (L^2(\Omega))^2$ and $\frac{1}{\rho_2}(-K\nabla V^1 + \mathcal{D}'S\mathcal{D}V^2 - KV^2 - \gamma\nabla V^5) = F^4$, i.e., $(\mathcal{A}V)^4 = F^4$.

vi) Finally, we let $\Phi = (0, 0, 0, 0, \phi, 0)'$ mit $\phi \in H^1_{\Gamma_3}(\Omega)$ and deduce

$$\begin{aligned}\langle F^5, \phi\rangle_{L^2(\Omega)} \leftarrow & \tfrac{1}{\rho_3}\langle -\gamma\operatorname{div} V_n^4 - \beta V_n^5 - \kappa\operatorname{div} V_n^6, \phi\rangle_{L^2(\Omega)} \\ = & \tfrac{\kappa}{\rho_3}\langle V_n^6, \nabla\phi\rangle_{L^2(\Omega)} - \tfrac{1}{\rho_3}\langle \gamma\operatorname{div} V_n^4 + \beta V_n^5, \phi\rangle_{L^2(\Omega)} \\ \to & \tfrac{\kappa}{\rho_3}\langle V_n^6, \nabla\phi\rangle_{L^2(\Omega)} - \tfrac{1}{\rho_3}\langle \gamma\operatorname{div} V^4 + \beta V^5, \phi\rangle_{L^2(\Omega)}\end{aligned}$$

implying that $\operatorname{div} V^6 \in L^2(\Omega)$ and $\frac{1}{\rho_3}(-\gamma\operatorname{div} V^4 - \beta V^5 - \kappa\operatorname{div} V^6) = F^5$ hold true, i.e., $(\mathcal{A}V)^5 = F^5$.

There remains to show that V satisfies the generalized Neumann boundary conditions (2.18)–(2.20). To this end, we proceed as follows.

i) Let $\phi \in H^1_{\Gamma_1}(\Omega)$. Then

$$\begin{aligned}\langle \triangle V^1 + \operatorname{div} V^2, \phi\rangle_{L^2(\Omega)} \leftarrow & \langle \triangle V^1_n + \operatorname{div} V^2_n, \phi\rangle_{L^2(\Omega)}\\ = & -\langle \nabla V^1_n + V^2_n, \nabla\phi\rangle_{(L^2(\Omega))^2} \to \langle \nabla V^1 + V^2, \nabla\phi\rangle_{(L^2(\Omega))^2}.\end{aligned}$$

ii) For $\phi \in (H^1_{\Gamma_1}(\Omega))^2$, we get

$$\begin{aligned}\langle \mathcal{D}'S\mathcal{D}V^2 - \gamma\nabla V^5, \phi\rangle_{(L^2(\Omega))^2} \leftarrow & \langle \mathcal{D}'S\mathcal{D}V^2_n - \gamma\nabla V^5_n, \phi\rangle_{(L^2(\Omega))^2}\\ = & -\langle S\mathcal{D}V^2_n, \mathcal{D}\phi\rangle_{(L^2(\Omega))^2} + \gamma\langle V^5_n, \operatorname{div}\phi\rangle_{L^2(\Omega)}\\ \to & -\langle S\mathcal{D}V^2, \mathcal{D}\phi\rangle_{(L^2(\Omega))^2} + \gamma\langle V^5, \operatorname{div}\phi\rangle_{L^2(\Omega)}.\end{aligned}$$

iii) Choosing an arbitrary $\phi \in H^1_{\Gamma_3}(\Omega)$, we finally obtain

$$\langle \operatorname{div} V^6, \phi\rangle_{L^2(\Omega)} \leftarrow \langle \operatorname{div} V^6_n, \phi\rangle_{(L^2(\Omega))^2} = \langle V^6_n, \nabla\phi\rangle_{L^2(\Omega)} \to -\langle V^6, \nabla\phi\rangle_{(L^2(\Omega))^2}.$$

Alltogether, we have shown that $\mathcal{A}$ is a closed operator.

3. Next, we show $\operatorname{im}(\lambda - \mathcal{A}) = \mathcal{H}$ for all $\lambda > 0$. To this end, we prove that the equation

$$(\lambda - \mathcal{A})V = F \tag{2.22}$$

is solvable for any $F \in \mathcal{H}$. Since $D(\mathcal{A})$ is a dense subset of $\mathcal{H}$ and $\mathcal{A}$ is closed, we can select $F \in D(\mathcal{A})$. Thus, for $F \in D(\mathcal{A})$, we are looking for solutions of

$$\begin{aligned}\lambda V^1 - V^3 &= F^1,\\ \lambda V^2 - V^4 &= F^2,\\ \lambda V^3 - K\triangle V^1 - K\operatorname{div} V^2 &= \rho_1 F^3,\\ \lambda V^4 + K\nabla V^1 - \mathcal{D}'S\mathcal{D}V^2 + KV^2 + \gamma\nabla V^5 &= \rho_2 F^4,\\ \lambda V^5 + \gamma\operatorname{div} V^4 + \beta V^5 + \kappa\operatorname{div} V^6 &= \rho_3 F^5,\\ \lambda V^6 + \kappa\nabla V^5 + \delta V^6 &= \tau_0 F^6.\end{aligned}$$

To eliminate V^3, V^4, we substitute

$$V^3 = \lambda V^1 - F^1, \qquad V^4 = \lambda V^2 - F^2, \qquad V^6 = \tfrac{1}{\lambda+\delta}(-\kappa\nabla V^5 + \tau_0 F^6)$$

and obtain

$$\begin{aligned}\lambda(\lambda + d)V^1 - K\triangle V^1 - K\operatorname{div} V^2 &= G_1,\\ \lambda^2 V^2 + K\nabla V^1 - \mathcal{D}'S\mathcal{D}V^2 + KV^2 + \gamma\nabla V^5 &= G_2,\\ \lambda V^5 + \gamma\lambda\operatorname{div} V^2 + \beta V^5 - \tfrac{\kappa^2}{1+\delta}\triangle V^5 &= G_3\end{aligned} \tag{2.23}$$

with

$$G_1 = \rho_1 F^3 + \lambda F^1, \quad G_2 = \rho_2 F^4 + \lambda F^2, \quad G_3 = \rho_3 F^5 + \gamma\operatorname{div} F^2 + \tfrac{\tau_0\kappa}{\lambda+\delta}\operatorname{div} F^6.$$

To solve the elliptic problem (2.23), we exploit the lemma of Lax & Milgram. We consider the Hilbert space

$$\mathcal{V} := H^1_{\Gamma_1}(\Omega) \times (H^1_{\Gamma_1}(\Omega))^2 \times H^1_{\Gamma_3}(\Omega)$$

equipped with the standard norm and introduce the bilinear form $a\colon \mathcal{V}\times\mathcal{V}\to\mathbb{R}$ via

$$\begin{aligned} a(V,W) :=& \lambda^3\langle V^1, W^1\rangle_{L^2(\Omega)} + \lambda^3\langle V^2, W^2\rangle_{(L^2(\Omega))^2} + (\lambda+\beta)\langle V^5, W^5\rangle_{L^2(\Omega)} \\ & K\lambda\langle \nabla V^1 + V^2, \nabla W^1 + W^2\rangle_{(L^2(\Omega))^2} + \lambda\langle S\mathcal{D}V^2, \mathcal{D}W^2\rangle_{(L^2(\Omega))^3} + \\ & \tfrac{\kappa^2}{\lambda+\delta}\langle \nabla V^5, \nabla W^5\rangle_{(L^2(\Omega))^2} + \gamma\lambda\langle \nabla V^5, W^2\rangle_{(L^2(\Omega))^2} + \gamma\lambda\langle \operatorname{div} V^2, W^5\rangle_{L^2(\Omega)}. \end{aligned} \tag{2.24}$$

After multiplying the equations in (2.23) scalar in $L^2(\Omega)$, $(L^2(\Omega))^2$ and $L^2(\Omega)$ with λV^1, λV^2 and V^3, respectively, summing up the resulting equations and performing a partial integration, we obtain a weak formulation of Equation (2.23) in the form: Determine $V \in \mathcal{V}$ such that

$$a(V,W) = \lambda\langle G^1, W^1\rangle_{L^2(\Omega)} + \lambda\langle G^2, W^2\rangle_{(L^2(\Omega))^2} + \langle G^3, W^5\rangle_{L^2(\Omega)}$$

for any $W \in \mathcal{V}$.

The bilinear form a is continuous and coercive on $\mathcal{V}$ due to the boundary conditions and the Korn's inequality from Theorem 1. The functional

$$\mathcal{V} \ni W \mapsto \lambda\langle G^1, W^1\rangle_{L^2(\Omega)} + \lambda\langle G^2, W^2\rangle_{(L^2(\Omega))^2} + \langle G^3, W^5\rangle_{L^2(\Omega)}$$

is linear and continuous on $\mathcal{V}$. Applying now lemma of Lax & Milgram, we deduce the existence of a weak solution $V \in \mathcal{V}$ to (2.24) which, in its turn, solves (2.23), too.

Letting

$$V^3 = \lambda V^1 - F^1, \qquad V^4 = \lambda V^2 - F^2, \qquad V^6 = \tfrac{1}{\lambda+\delta}(-\kappa\nabla V^5 + \tau_0 F^6),$$

we conclude that $V = (V^1, \dots, V^6)'$ solves Equation (2.22).

Thus, we have shown that $D(\mathcal{A}) \subset \operatorname{im}(\lambda - \mathcal{A})$. Since $D(\mathcal{A})$ is dense in $\mathcal{H}$ and $\operatorname{im}(\mathcal{A})$ is closed in $\mathcal{H}$, we finally obtain $\operatorname{im}(\lambda - \mathcal{A}) = \mathcal{H}$.

□

We can now apply the theorem of Lumer & Phillips to the Cauchy problem (2.21) to obtain the following existence result.

Theorem 3. *Let $V_0 \in D(\mathcal{A})$. There exists then a unique classical solution to Equation (2.21) satisfying*

$$V \in \mathcal{C}^1([0,\infty), \mathcal{H}) \cap \mathcal{C}^0([0,\infty), D(\mathcal{A})).$$

Moreover, if $V_0 \in D(\mathcal{A}^s)$ for a certain $s \in \mathbb{N}$, then we additionally have

$$V \in \bigcap_{k=0}^{s} \mathcal{C}^k([0,\infty), D(\mathcal{A}^{s-k})),$$

where $D(\mathcal{A}^0) := \mathcal{H}$.

3 Exponential stability

In this section, we study the stability properties of Equations (2.8)–(2.11) subject to Dirichlet boundary conditions for the elastic part and Neumann boundary conditions for the thermal part in two situations. First, we look at the case of a frictional damping on all elastic variables. Second, we restrict ourselves to the rotationally symmetric situation but retain only the frictional damping for the bending component w.

For a number $d \geq 0$ and a symmetric, positive semidefinite matrix $D \in \mathbb{R}^{3\times 3}$, we consider thus the problem

$$\rho_1 w_{tt} - K \mathrm{div}\,(v + \nabla w) + d w_t = 0 \text{ in } (0,\infty) \times \Omega, \tag{3.1}$$

$$\rho_2 v_{tt} - \mathcal{D}' S \mathcal{D} v + K(v + \nabla w) + \gamma \nabla \theta + D v_t = 0 \text{ in } (0,\infty) \times \Omega, \tag{3.2}$$

$$\rho_3 \theta_t + \kappa \mathrm{div}\, q + \beta \theta + \gamma \mathrm{div}\, v_t = 0 \text{ in } (0,\infty) \times \Omega, \tag{3.3}$$

$$\tau_0 q_t + \delta q + \kappa \nabla \theta = 0 \text{ in } (0,\infty) \times \Omega \tag{3.4}$$

subject to the boundary conditions

$$w = |v| = 0 \text{ on } (0,\infty) \times \Gamma, \tag{3.5}$$

$$q \cdot \nu = 0 \text{ on } (0,\infty) \times \Gamma \tag{3.6}$$

and the initial conditions

$$w(0,\cdot) = w^0,\ w_t(0,\cdot) = w^1,\ v(0,\cdot) = v^0,\ v_t(0,\cdot) = v^1,\ \theta(0,\cdot) = \theta^0,\ q(0,\cdot) = q^0, \tag{3.7}$$

Despite of the notation abuse, the matrix $D \in \mathrm{SPD}(\mathbb{R}^3)$ should not be confused with constant $D > 0$ consituting the matrix S. The natural first order energy associated with (3.1)–(3.4) reads as

$$\mathcal{E}(t) := \tfrac{\rho_1}{2}\|w_t\|^2_{L^2(\Omega)} + \tfrac{\rho_2}{2}\|v_t\|^2_{(L^2(\Omega))^2} + \tfrac{1}{2}\|\sqrt{S}\mathcal{D}v\|^2_{(L^2(\Omega))^3} + \tfrac{K}{2}\|v + \nabla w\|^2_{(L^2(\Omega))^2} + \tfrac{\rho_3}{2}\|\theta\|^2_{L^2(\Omega)} + \tfrac{\tau_0}{2}\|q\|_{(L^2(\Omega))^2}.$$

3.1 Full mechanical and thermal damping

First, we address the case of a full mechanical and thermal damping, i.e., $d > 0$, $D \in \mathrm{SPD}(\mathbb{R}^2)$, $\beta > 0$. Analogous results for the equations of thermoelasticity with a mechanical damping were proved by Racke in [27] for the case of parabolic heat conduction and by Ritter in [29] for the case of hyperbolic heat conduction due to Cattaneo.

Theorem 4. *Let the parameters satisfy $\rho_1, \rho_2, \rho_3, \tau_0, K, \kappa, \delta, \gamma, d > 0, \beta > 0$, $S \in \mathrm{SPD}(\mathbb{R}^3)$, $D \in \mathrm{SPD}(\mathbb{R}^2)$. There exist then positive constants C and α such that*

$$\mathcal{E}(t) \leq C\mathcal{E}(0)e^{-2\alpha t}$$

holds true for all $t \geq 0$. The latter depend neither on the initial data, nor on t and can be explicitely estimated based on the parameters and the domain Ω.

Proof. To prove the theorem, we want to construct a Lyapunov functional $\mathcal{F}$. Multiplying Equations (3.1) and (3.3) in $L^2(\Omega)$ with w_t and θ, respectively, as well as Equations (3.2) and (3.4) in $(L^2(\Omega))^2$ with v_t and q, respectively, and exploiting the boundary conditions (3.5), (3.6), we find after a partial integration

$$\partial_t \mathcal{E}(t) \leq d \int_\Omega w_t^2 \mathrm{d}x - \lambda \int_\Omega |v_t|^2 \mathrm{d}x - \beta \int_\Omega \theta^2 \mathrm{d}x - \delta \int_\Omega |q|^2 \mathrm{d}x \tag{3.8}$$

with $\lambda := \min \sigma(D) > 0$ denoting the smallest eigenvalue of D. The function $\mathcal{F}$ has thus to be constructed in a way such that $\partial_t \mathcal{F}$ contains a negative multiple of $\mathcal{E}$, in particular, the terms $\int_\Omega |\nabla w|^2 \mathrm{d}x$, $\int_\Omega |\sqrt{S}\mathcal{D}v|^2 \mathrm{d}x$ and $\int_\Omega |\theta|^2 \mathrm{d}x$. We define

$$\mathcal{F}_1(t) := \rho_1 \int_\Omega w_t w \mathrm{d}x, \quad \mathcal{F}_2(t) := \rho_1 \int_\Omega v_t \cdot v \mathrm{d}x$$

with $\cdot$ denoting the standard dot product on $\mathbb{R}^2$ and exploit Equations (3.1), (3.2) und (3.5) to find after a partial integration

$$\begin{aligned}\partial_t\mathcal{F}_1(t) &= \int_\Omega (K\mathrm{div}\ (\nabla w + v) - dw_t)w\mathrm{d}x + \rho_1\int_\Omega w_t^2\mathrm{d}x\\ &= \int_\Omega -K(\nabla w + v)\cdot\nabla w\mathrm{d}x - dw_t w + \rho_1 w_t^2\mathrm{d}x,\\ \partial_t\mathcal{F}_2(t) &= \int_\Omega (\mathcal{D}'S\mathcal{D}v - K(v+\nabla w) - \gamma\nabla\theta_t - Dv_t)\cdot v\mathrm{d}x + \rho_2\int_\Omega |v_t|^2\mathrm{d}x\\ &= \int_\Omega -|\sqrt{S}\mathcal{D}v|^2 - K(v+\nabla w)\cdot v + \gamma\theta_t\mathrm{div}\ v - Dv_t\cdot v + \rho_2|v_t|^2\mathrm{d}x.\end{aligned} \tag{3.9}$$

Using now Young's inequality, the first Poincaré's and well as Korn's inequality, we can estimate for arbitrary $\varepsilon, \varepsilon' > 0$ the functionals in (3.9) as follows:

$$\begin{aligned}\partial_t\mathcal{F}_1(t) &\le \int_\Omega -K|\nabla w|^2 + \tfrac{K}{2}|\nabla w|^2 + \tfrac{K}{2}|v|^2 + \tfrac{d\varepsilon}{2}w^2 + \big(\tfrac{d}{2\varepsilon} + \rho_1\big)w_t^2\mathrm{d}x\\ &\le \int_\Omega -\big(\tfrac{K}{2} - \tfrac{C_{\mathcal{P}}d\varepsilon}{2}\big)|\nabla w|^2 + \tfrac{K}{2}|v|^2 + \big(\tfrac{d}{2\varepsilon} + \rho_1\big)w_t^2\mathrm{d}x,\\ \partial_t\mathcal{F}_2(t) &\le \int_\Omega -|\sqrt{S}\mathcal{D}v|^2 - K|v|^2 + \tfrac{K(1+\varepsilon')}{2}|v|^2 + \tfrac{K}{2(1+\varepsilon')}|\nabla w|^2 + \tfrac{\gamma\varepsilon}{2}|\mathrm{div}\ v|^2\\ &\qquad + \tfrac{\gamma}{2\varepsilon}\theta^2 + \tfrac{\|D\|\varepsilon}{2}|v|^2 + \big(\tfrac{\|D\|}{2\varepsilon} + \rho_2\big)|v_t|^2\mathrm{d}x\\ &\le \int_\Omega -\big(1 - \tfrac{K\varepsilon'}{2C_{\mathcal{K},1}} - \tfrac{(\gamma+\|D\|)\varepsilon}{2C_{\mathcal{K},1}}\big)|\sqrt{S}\mathcal{D}v|^2 - \tfrac{K}{2}|v|^2 + \tfrac{K}{2(1+\varepsilon')}|\nabla w|^2\\ &\qquad + \tfrac{\gamma}{2\varepsilon}\theta^2 + \big(\tfrac{\|D\|}{2\varepsilon} + \rho_2\big)|v_t|^2\mathrm{d}x,\end{aligned} \tag{3.10}$$

where $C_{\mathcal{P}}$ denotes the Poincaré's constant and $C_{\mathcal{K},1}$ stands for the Korn's constant from Lemma 1. We let

$$\mathcal{F}(t) := \mathcal{F}_1(t) + \mathcal{F}_2(t) + N\mathcal{E}(t)$$

and combine Equations (3.8) and (3.10) to obtain

$$\begin{aligned}\partial_t\mathcal{F}(t) \le C_{w_t}\int_\Omega w_t\mathrm{d}x + C_{v_t}\int_\Omega |v_t|^2\mathrm{d}x + C_\vartheta\int_\Omega \vartheta_t^2\mathrm{d}x + C_q\int_\Omega |q|^2\mathrm{d}x +\\ C_{\nabla w}\int_\Omega |\nabla w|\mathrm{d}x + C_{\sqrt{S}\mathcal{D}v}\int_\Omega |\sqrt{S}\mathcal{D}v|^2\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned} C_{w_t} &= Nd - \tfrac{d}{2\varepsilon} + \rho_1, & C_{v_t} &= N\lambda - \tfrac{\|D\|}{2\varepsilon} + \rho_2 - \tfrac{\gamma}{2\varepsilon},\\ C_\vartheta &= N\beta - \tfrac{\gamma}{2\varepsilon} + \rho_3, & C_q &= N\delta,\\ C_{\nabla w} &= \big[\tfrac{K}{2} - \tfrac{K}{2(1+\varepsilon')}\big] - \tfrac{C_{\mathcal{P}}d\varepsilon}{2}, & C_{\sqrt{S}\mathcal{D}v} &= \big[1 - \tfrac{K\varepsilon'}{2C_{\mathcal{K},1}}\big] - \tfrac{(\gamma+\|D\|)\varepsilon}{2C_{\mathcal{K},1}}.\end{aligned} \tag{3.11}$$

Now, we select $\varepsilon' > 0$ to be sufficiently small such that the terms in the brackets from Equation (3.11) become positive. Further, we fix a small $\varepsilon > 0$ to assure for $C_{\nabla w} > 0$ and $C_{\sqrt{S}\mathcal{D}v} > 0$. Finally, we pick a sufficiently large $N > 0$ such that all constants in (3.11) become positive. Thus,

$$C_{\min} := \min\{C_{w_t}, C_{v_t}, C_\theta, C_q, C_{\nabla w}, C_{\sqrt{S}\mathcal{D}v}\} > 0.$$

Using now the Korn's inequality from Lemma 1, we obtain

$$\partial_t\mathcal{F}(t) \le -2C_{\min}\cdot\frac{\min\{1, C_{\mathcal{K}}\}}{\max\{1,\rho_1,\rho_2,\rho_3,\tau 0\}}\mathcal{E}(t) =: -\tilde{C}\mathcal{E}(t). \tag{3.12}$$

Taking into account

$$|(\mathcal{F}_1 + \mathcal{F}_2)(t)| \leq \frac{\max\left\{1,\rho_1,\rho_2\right\}}{\min\{1,C_{\mathcal{K}}\}}\mathcal{E}(t) =: \hat{C}\mathcal{E}(t)$$

we conclude

$$\beta_1\mathcal{E}(t) \leq \mathcal{L}(t) \leq \beta_2\mathcal{E}(t) \text{ for } t \geq 0$$

with $\beta_1 = N - \hat{C}$, $\beta_2 = N + \hat{C}$. If neccessary, we increase N to make β_1 positive. Gronwall's inequality now yields

$$\mathcal{E}(t) \leq \tfrac{1}{\beta_1}\mathcal{L}(t) \leq \tfrac{1}{\beta_1}\mathcal{E}(0)e^{-\frac{C}{\beta_2}t} =: C\mathcal{E}(0)e^{-2\alpha t} \text{ for all } t \geq 0$$

with $C, \alpha > 0$. This means that E decays exponentially. □

Remark 5. *As a matter of fact, the constant β must be positive in physical settings. Assuming*

$$\int_\Omega \theta_0 \mathrm{d}x = 0$$

and using the functional $\mathcal{F}_4$ from the proof of Theorem 7, our arguments can easily be carried over to the case $\beta = 0$. In contrast to Ritter's approach in [29], no second order energy is required.

3.2 Lack of strong stability in smooth domains

To justify the necessity of a frictional damping for both w and v, we prove next that Equations (3.1)–(3.7) even lack a strong stability for $d > 0$ and $D = 0$ when considered in a bounded domain Ω with a smooth boundary Γ. In this case, the domain Ω contains a ray of geometrical optics perpendicularly reflected from Γ and one could theoretically perform constructions similar to those in [6] or [20] to prove a non-uniform decay rate even for a bigger class of domains. For simplicity, we restrict ourselves to the case of a smooth boundary allowing for the definition of Helmholz projection. We will namely show the imposibility of stabilizing the solenoidal part of v.

To avoid a trivial null space, we impose for simplicity the following boundary conditions:

$$w = |v| = 0 \text{ on } (0,\infty) \times \Gamma, \tag{3.13}$$

$$\theta = 0 \text{ on } (0,\infty) \times \Gamma. \tag{3.14}$$

It should though be pointed out that a similar result would also hold under any natural boundary conditions on w, θ and q provided Dirichlet boundary conditions are imposed on v on the whole of Γ.

Theorem 6. *Let the boundary Γ be of class $\mathcal{C}^2$ and let $D = 0$. Problem (3.1)–(3.4), (3.7), (3.13), (3.14) is not strongly stable, in particular, not uniformly stable.*

Proof. Equations (3.1)–(3.4), (3.7), (3.13), (3.14) can be rewritten in the evolution form. Theorem 3 yields then the existence of unique solution $V = (w, v, w_t, v_t, \theta, q)'$ given as an application of the strongly continuous semigroup of linear bounded operators to the initial data.

Now, we want to select the initial data such that the solution component v remains solenoidal, i.e., $\operatorname{div} v = \operatorname{div} v_t = 0$. Since Γ is smooth, there exists the Helmholtz-projection (cf. [30])

$$P \colon (L^2(\Omega))^2 \to L^2_\sigma(\Omega)$$

into the Hilbert space

$$L^2_\sigma(\Omega) = \{u \in (L^2(\Omega))^2 \,|\, \langle u, \nabla\varphi\rangle_{(L^2(\Omega))^2} = 0 \text{ for all } \varphi \in L^1_{\mathrm{loc}}(\Omega) \text{ such that } \nabla\varphi \in (L^2(\Omega))^2\}.$$

P is an orthogonal operator and $L^2_\sigma(\Omega)$ is closed.

Applying the operator P to Equation (3.2) and exploiting the representation

$$\mathcal{D}'S\mathcal{D}v = D\tfrac{1-\mu}{2}\triangle v + D\tfrac{1+\mu}{2}\nabla \mathrm{div}\, v,$$

we obtain an equation for $u := Pv$

$$\begin{aligned} \rho_2 u_{tt} - D\tfrac{1-\mu}{2}P\triangle u + Ku = 0 \quad &\text{in } (0,\infty)\times\Omega, \\ u = 0 \quad &\text{in } (0,\infty)\times\Gamma, \\ u(t,\cdot) = u_0 := Pv_0,\ u_t(t,\cdot) = u_1 := Pv_1 \ &\text{in } \Omega. \end{aligned} \tag{3.15}$$

Equation (3.15) has a strong resemblance to the Klein-Gordon-Equation with an unbounded selfadjoint Dirichlet-Stokes-Operator $DP\triangle$. We define the operator

$$\mathcal{A}\colon D(\mathcal{A}) \subset L^2_\sigma(\Omega) \to L^2_\sigma(\Omega), \quad u \mapsto D\tfrac{1-\mu}{2}P\triangle u + Ku,$$

where

$$D(\mathcal{A}) = (H^2(\Omega)\cap H^1_0(\Omega))^2 \cap L^2_\sigma(\Omega).$$

It is known (see, e.g., [15]) that the spectrum $\sigma(D\frac{1+\mu}{2}P\triangle)$ of $DP\triangle$ purely consists of a discrete point spectrum

$$\sigma(D\tfrac{1+\mu}{2}P\triangle) = \sigma_p(D\tfrac{1+\mu}{2}P\triangle) = \{\lambda_k \,|\, k\in\mathbb{N}\}$$

with the eigenvalues λ_k, $k\in\mathbb{N}$, of finite multiplicity satisfying $0<\lambda_1\le\lambda_k\le\lambda_{k+1}\to\infty$ as $k\to\infty$. Hence, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\mu_k \,|\, k\in\mathbb{N}\}$ with $\mu_k = \lambda_k + K$ for $k\in\mathbb{N}$.

Let $\nu^*\in\sigma(\mathcal{A})$ and let $u^*\in D(\mathcal{A})$ be the eigenfunction corresponding to ν^* with $\|u^*\|_{(L^2(\Omega))^2} = 1$. We set $u_0 := u^*$, $u_1 := 0$ and find that

$$u(t) := \cos\big(\sqrt{\tfrac{\nu^*}{\rho_2}}t\big)v^*, \quad t\in\mathbb{R},$$

is a solution of (3.15). The energy associated with u reads as

$$\begin{aligned} \mathcal{E}_1(t) &= \rho_2\|v_t\|_{(L^2(\Omega))^2} + D\|\nabla v\|_{(L^2(\Omega))^2} = \nu^*\cos^2\big(\sqrt{\tfrac{\nu^*}{\rho_2}}t\big) + D\nu^*\cos^2\big(\sqrt{\tfrac{\nu^*}{\rho_2}}t\big) \\ &= \nu^*(1+D)\cos^2\big(\sqrt{\tfrac{\nu^*}{\rho_2}}t\big) \nrightarrow 0 \text{ for } t\to\infty. \end{aligned}$$

Thus, $(w,v,\theta,q)' = (0,\cos\big(\sqrt{\frac{\nu^*}{\rho_2}}t\big)v^*,0,0)'$ is a solution of the original problem (3.1)–(3.7), (3.13), (3.14) for the initial conditions

$$w^0 = w^1 = 0, \quad v^0 = u^*, v^1 = 0, \quad \theta^0 = 0, \quad q^0 = 0,$$

which satisfies

$$\begin{aligned} \mathcal{E}(t) &= \rho_2\|v_t\|_{(L^2(\Omega))^2} + D\|\nabla v_1\|^2_{(L^2(\Omega))^2} + D\|\nabla v_2\|^2_{(L^2(\Omega))^2} + D\tfrac{1+\mu}{2}\|\mathrm{div}\, v\|^2_{L^2(\Omega)} \\ &= \rho_2\|v_t\|_{(L^2(\Omega))^2} + D\|\nabla v_1\|^2_{(L^2(\Omega))^2} + D\|\nabla v_2\|^2_{(L^2(\Omega))^2} = \mathcal{E}_1(t) \nrightarrow 0 \text{ for } t\to\infty, \end{aligned}$$

where $v = (v_1, v_2)'$. Hence, $\mathcal{E}$ does not decay. □

3.3 Exponential stability for rotationally symmetric plates

As we have seen before, a single frictional damping for w is not sufficiently strong to stabilize a thermoelastic Reissner-Mindlin-Timoshenko plate (3.1)–(3.4) for general data. Motivated by Racke's result in [28], we reduce the problem to the rotationally symmetric case making thus the vector field v irrotational. Though arguments similar to those of Jiang and Racke in [14, Theorem 4.2] and Racke in [28] made for the system of classical or hyperbolic thermoelasticity could be adopted in our case, we decided to propose our own approach incorporating the Bogovskiĭ operator and, to some extent, being a generalization of the method applied by Messaoudi et al. in [22] to a one-dimensional Timoshenko-beam. In addition to its technical novelty, a direct benefit of our approach lies in the fact that we only need to consider a first and not a second order energy. We would like to mention that the spectral approach of Grobbelaar (cf. [11, 12]) seems also to be applicable to our problem. At the same time, we do not require the assumption of simple connectedness on Ω.

We study Equations (3.1)–(3.4) subject to the boundary conditions

$$w = |v| = 0 \text{ on } (0,\infty)\times\Gamma, \tag{3.16}$$

$$q\cdot\nu = 0 \text{ on } (0,\infty)\times\Gamma \tag{3.17}$$

and the initial conditions

$$w(0,\cdot) = w^0,\ w_t(0,\cdot) = w^1,\ v(0,\cdot) = v^0,\ v_t(0,\cdot) = v^1,\ \theta(0,\cdot) = \theta^0,\ q(0,\cdot) = q^0. \tag{3.18}$$

For the solution given in Theorem 3, we assume the vanishing mean value for θ

$$\int_\Omega \theta \mathrm{d}x = 0 \text{ in } (0,\infty) \tag{3.19}$$

as well as the vanishing rotation for v

$$\mathrm{rot} v = \partial_2 v_1 - \partial_1 v_2 = 0 \text{ in } (0,\infty). \tag{3.20}$$

Theorem 7. *Let $\Omega\subset\mathbb{R}^2$ be a rotationally symmetric bounded domain. Let the parameters satisfy $\rho_1,\rho_2,\rho_3,\tau_0,K,\kappa,\delta,\gamma,d>0$, $\beta\geq 0$, $D=0$ and the matrix S come from Equation (2.7). Further, let the data $w^0,w^1,v^0,v^1,\theta^0,q^0$ be radially symmetric in the sense of [14, Definition 4.4] and satisfy*

$$\int_\Omega \theta_0 \mathrm{d}x = 0.$$

There exist then positive constants C and α such that for the energy E

$$\mathcal{E}(t) \leq C\mathcal{E}(0)e^{-2\alpha t}$$

holds true for all $t\geq 0$. The latter depend neither on the initial data, nor on t and can be explicitely estimated based on the parameters and the domain Ω.

Proof. Theorem 3 applied for the case $\Gamma_2=\Gamma_3=\emptyset$ yields the existence of a unique classical solution. After a straighforward modification, [14, Lemma 4.6] implies that the solution remains rotationally symmetric for all times $t\geq 0$.

Without loss of generality, we may assume $\beta=0$. Indeed, denoting with $\mathcal{E}_\beta$ the natural energy associated with the system subject to some fixed initial conditions for a given $\beta\geq 0$ and assuming the existence of constants C and α independent of the initial data such that

$$\mathcal{E}_0(t) \leq C\mathcal{E}_0(0)e^{-2\alpha t} \text{ for } t\geq 0,$$

we take into account

$$\partial_t \mathcal{E}_\beta(t) \le \partial_t \mathcal{E}_0(t) - \beta \int_\Omega \theta^2 \mathrm{d}x \le \partial_t \mathcal{E}_0(t) \text{ for } t \ge 0$$

as well as $\mathcal{E}_\beta(0) = \mathcal{E}_0(0)$ to conclude

$$\mathcal{E}_\beta(t) \le \mathcal{E}_0(t) \le C\mathcal{E}_0(0)e^{-2\alpha t} \text{ for } t \ge 0.$$

Thus, we let $\beta = 0$. As already mentioned, some of the following steps are motivated by the one-dimensional proof of Messaoudi et al. from [22].

Multiplying (3.1) and (3.3) in $L^2(\Omega)$ with w_t and θ, respectively, as well as (3.2) and (3.4) in $(L^2(\Omega))^2$ with v_t and q, respectively, and employing integration by parts, we find

$$\partial_t \mathcal{E}(t) = -d \int_\Omega w_t^2 \mathrm{d}x - \delta \int_\Omega |q|^2 \mathrm{d}x.$$

With the solution $u \in H_0^1(\Omega)$ to the Poisson equation

$$\begin{aligned} -\triangle u &= \operatorname{div} v \text{ in } \Omega, \\ u &= 0 \text{ auf } \Gamma, \end{aligned}$$

we obtain

$$\int_\Omega |\nabla u|^2 \mathrm{d}x = -\int_\Omega v \cdot \nabla u \mathrm{d}x.$$

Young's inequality further yields

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \le \tfrac{1}{2} \int_\Omega |v|^2 \mathrm{d}x + \tfrac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x$$

and, therefore,

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \le \int_\Omega |v|^2 \mathrm{d}x. \tag{3.21}$$

Similarly,

$$\int_\Omega |\nabla u_t|^2 \mathrm{d}x \le \int_\Omega |v_t|^2 \mathrm{d}x. \tag{3.22}$$

We define the functional

$$\mathcal{F}_1(t) := \int_\Omega \left(\rho_1 w_t u + \rho_2 v_t v - \tfrac{\gamma \tau_0}{\kappa} v q \right) \mathrm{d}x.$$

Taking into account Equation (3.1), we use partial integration to obtain

$$\begin{aligned} \partial_t \int_\Omega \rho_1 w_t u \mathrm{d}x &= \rho_1 \int_\Omega (w_{tt} u + w_t u_t) \mathrm{d}x \\ &= \ K \int_\Omega \operatorname{div}\, (\nabla w + v) \cdot u \mathrm{d}x - d \int_\Omega w_t u \mathrm{d}x + \rho_1 \int_\Omega w_t u_t \mathrm{d}x \\ &= -K \int_\Omega (\nabla w + v) \cdot \nabla u \mathrm{d}x - d \int_\Omega w_t u \mathrm{d}x + \rho_1 \int_\Omega w_t u_t \mathrm{d}x \\ &= \ K \int_\Omega w \triangle u \mathrm{d}x - K \int_\Omega v \cdot \nabla u \mathrm{d}x - d \int_\Omega w_t u \mathrm{d}x + \rho_1 \int_\Omega w_t u_t \mathrm{d}x \\ &= -K \int_\Omega w \operatorname{div}\, v \mathrm{d}x + K \int_\Omega |\nabla u|^2 \mathrm{d}x - d \int_\Omega w_t u \mathrm{d}x + \rho_1 \int_\Omega w_t u_t \mathrm{d}x. \end{aligned}$$

By the virtue of Equation (3.2), we similarly get

$$\begin{aligned}
\partial_t \int_\Omega \rho_2 v_t \cdot v \mathrm{d}x &= \rho_2 \int_\Omega v_{tt} \cdot v \mathrm{d}x + \rho_2 \int_\Omega |v_t|^2 \mathrm{d}x \\
&= \int_\Omega \mathcal{D}' S \mathcal{D} v \cdot v \mathrm{d}x - K \int_\Omega (v + \nabla w) \cdot v \mathrm{d}x - \gamma \int_\Omega \nabla\theta \cdot v \mathrm{d}x + \rho_2 \int_\Omega |v_t|^2 \mathrm{d}x \\
&= - \int_\Omega |\sqrt{S}\mathcal{D} v|^2 \mathrm{d}x - K \int_\Omega |v|^2 \mathrm{d}x + K \int_\Omega w \mathrm{div}\; v \mathrm{d}x - \gamma \int_\Omega \nabla\theta \cdot v \mathrm{d}x + \rho_2 \int_\Omega |v_t|^2 \mathrm{d}x
\end{aligned}$$

as well as

$$\partial_t \int_\Omega -\tfrac{\gamma\tau_0}{\kappa} v \cdot q \mathrm{d}x = -\tfrac{\gamma\tau_0}{\kappa} \int_\Omega v_t \cdot q \mathrm{d}x + \tfrac{\gamma\delta}{\kappa} \int_\Omega v \cdot q \mathrm{d}x + \gamma \int_\Omega v \cdot \nabla\theta \mathrm{d}x.$$

Finally, we conclude

$$\begin{aligned}
\partial_t \mathcal{F}_1(t) =& K \int_\Omega |\nabla u|^2 \mathrm{d}x - K \int_\Omega |v|^2 \mathrm{d}x - d \int_\Omega w_t u \mathrm{d}x + \rho_1 \int_\Omega w_t u_t \mathrm{d}x + \\
& \rho_2 \int_\Omega |v_t|^2 \mathrm{d}x - \int_\Omega |\sqrt{S}\mathcal{D} v|^2 \mathrm{d}x - \tfrac{\gamma\tau_0}{\kappa} \int_\Omega v_t \cdot q \mathrm{d}x + \tfrac{\gamma\delta}{\kappa} \int_\Omega v \cdot q \mathrm{d}x.
\end{aligned}$$

Using now the first Poincaré's inequality, Young's inequality and Korn's inequality from Lemma 1 as well as the estimates from Equations (3.21) and (3.22), we obtain

$$\begin{aligned}
\partial_t \mathcal{F}_1(t) \leq\; & \tfrac{d}{2} \int_\Omega \left(\varepsilon_1 u^2 + \tfrac{1}{\varepsilon_1} w_t^2\right) \mathrm{d}x + \tfrac{\rho_1}{2} \int_\Omega \left(\varepsilon_1 u_t^2 + \tfrac{1}{\varepsilon_1} w_t^2\right) \mathrm{d}x + \rho_2 \int_\Omega |v_t|^2 \mathrm{d}x \\
& - \int_\Omega |\sqrt{S}\mathcal{D} v|^2 \mathrm{d}x + \tfrac{\gamma\tau_0}{2\kappa} \int_\Omega \left(\varepsilon_1 |v_t|^2 + \tfrac{1}{\varepsilon_1} |q|^2\right) \mathrm{d}x + \tfrac{\gamma\delta}{2\kappa} \int_\Omega \left(\varepsilon_1 |v|^2 + \tfrac{1}{\varepsilon_1} |q|^2\right) \mathrm{d}x \\
\leq\; & \tfrac{d}{2} \int_\Omega \left(\varepsilon_1 \tfrac{C_\mathcal{P}}{C_{\mathcal{K},1}} |\sqrt{S}\mathcal{D} v|^2 + \tfrac{1}{\varepsilon_1} w_t^2\right) \mathrm{d}x + \tfrac{\rho_1}{2} \int_\Omega \left(C_\mathcal{P} \varepsilon_1 |v_t|^2 + \tfrac{1}{\varepsilon} w_t^2\right) \mathrm{d}x + \rho_2 \int_\Omega |v_t|^2 \mathrm{d}x \\
& - \int_\Omega |\sqrt{S}\mathcal{D} v|^2 \mathrm{d}x + \tfrac{\gamma\tau_0}{2\kappa} \int_\Omega \left(\varepsilon_1 |v_t|^2 + \tfrac{1}{\varepsilon_1} |q|^2\right) \mathrm{d}x + \tfrac{\gamma\delta}{2\kappa} \int_\Omega \left(\tfrac{\varepsilon_1}{C_{\mathcal{K},1}} |\sqrt{S}\mathcal{D} v|^2 + \tfrac{1}{\varepsilon_1} |q|^2\right) \mathrm{d}x \\
\leq\; & \tfrac{\rho_1 + d}{2\varepsilon_1} \int_\Omega w_t^2 \mathrm{d}x + \left[\rho_2 - \tfrac{\varepsilon_1}{2}\left(\rho_1 C_\mathcal{P} + \tfrac{\gamma\tau_0}{\kappa}\right)\right] \int_\Omega |v_t|^2 \mathrm{d}x \\
& - \left[1 - \tfrac{\varepsilon_1}{2C_{\mathcal{K},1}} \left(d C_\mathcal{P} + \tfrac{\gamma\delta}{\kappa}\right)\right] \int_\Omega |\sqrt{S}\mathcal{D} v|^2 \mathrm{d}x + \tfrac{\gamma(\tau_0 + \delta)}{2\kappa\varepsilon_1} \int_\Omega |q|^2 \mathrm{d}x
\end{aligned} \tag{3.23}$$

with the Poincaré's constant $C_\mathcal{P} = C_\mathcal{P}(\Omega) > 0$ and an arbitrary small number $\varepsilon_1 > 0$ to be fixed later. Here, we estimated

$$\int_\Omega |u|^2 \mathrm{d}x \leq C_\mathcal{P} \int_\Omega |\nabla u|^2 \mathrm{d}x \leq C_\mathcal{P} \int_\Omega |v|^2 \mathrm{d}x \leq \tfrac{C_\mathcal{P}}{C_{\mathcal{K},1}} \int_\Omega |\sqrt{S}\mathcal{D} v|^2 \mathrm{d}x.$$

Next, we consider the functional

$$\mathcal{F}_2(t) := \rho_1 \int_\Omega w_t w \mathrm{d}x$$

and use Equation (3.1) to find

$$\begin{aligned}
\partial_t \mathcal{F}_2(t) &= \rho_1 \int_\Omega w_t^2 \mathrm{d}x + K \int_\Omega \mathrm{div}\; (v + \nabla w) \cdot w \mathrm{d}x - d \int_\Omega w_t w \mathrm{d}x \\
&= \rho_1 \int_\Omega w_t^2 \mathrm{d}x - K \int_\Omega |\nabla w|^2 \mathrm{d}x - K \int_\Omega v \cdot \nabla w \mathrm{d}x - d \int_\Omega w_t w \mathrm{d}x.
\end{aligned}$$

The latter can be estimated as

$$\partial_t \mathcal{F}_2(t) \le -K \int_\Omega |\nabla w|^2 \mathrm{d}x + \tfrac{K}{2} \int_\Omega \left(\varepsilon_2 |\nabla w|^2 + \tfrac{1}{\varepsilon_2}|v|^2\right) \mathrm{d}x +$$
$$\tfrac{d}{2} \int_\Omega \left(\varepsilon_2 w^2 + \tfrac{1}{\varepsilon_2} w_t^2\right) \mathrm{d}x + \rho_1 \int_\Omega w_t^2 \mathrm{d}x \tag{3.24}$$
$$\le -\left(K - \tfrac{\varepsilon_2 C_{\mathcal{P}}}{2}(K+d)\right) \int_\Omega |\nabla w|^2 \mathrm{d}x + \tfrac{K}{\varepsilon_2 \mathcal{C}_{\mathcal{K},1}} \int_\Omega |\sqrt{S}\mathcal{D}v|^2 \mathrm{d}x + \left(\tfrac{d}{2\varepsilon_2} + \rho_1\right) \int_\Omega w_t^2 \mathrm{d}x.$$

Exploiting the fact

$$0 = \rho_3 \int_\Omega \theta_t \mathrm{d}x + \kappa \int_\Omega \operatorname{div}\, q \mathrm{d}x + \gamma \int_\Omega \operatorname{div}\, v_t \mathrm{d}x$$
$$= \rho_3 \partial_t \int_\Omega \theta \mathrm{d}x + \kappa \int_\Gamma q \cdot \nu \mathrm{d}x + \gamma \int_\Gamma v_t \cdot \nu \mathrm{d}x \;=\; \rho_3 \partial_t \int_\Omega \theta \mathrm{d}x,$$

we easily see

$$\int_\Omega \theta(t,x) \mathrm{d}x \equiv \int_\Omega \theta(0,x)\mathrm{d}x = \int_\Omega \theta_0(x) \mathrm{d}x = 0.$$

This enables us to apply the second Poincaré's inequality to θ. Using now the definition of Bogowskiĭ operator $\mathcal{B}_{\mathrm{rot}}$ from Theorem 9, we introduce the following functional

$$\mathcal{F}_3(t) := \rho_2 \rho_3 \int_\Omega \mathcal{B}_{\mathrm{rot}} \theta \cdot v_t \mathrm{d}x.$$

Exploiting Equations (3.2) and (3.3), we obtain

$$\partial_t \mathcal{F}_3(t) = \rho_2\rho_3 \int_\Omega \mathcal{B}_{\mathrm{rot}}\theta_t \cdot v_t \mathrm{d}x + \rho_2\rho_3 \int_\Omega \mathcal{B}_{\mathrm{rot}}\theta \cdot v_{tt}\mathrm{d}x$$
$$= \rho_2 \int_\Omega \mathcal{B}_{\mathrm{rot}}(-\kappa \operatorname{div}\, q - \gamma \operatorname{div}\, v_t) \cdot v_t \mathrm{d}x + \rho_3 \int_\Omega \mathcal{B}_{\mathrm{rot}}\theta \cdot (\mathcal{D}' S \mathcal{D} v - K(v + \nabla w) - \gamma \nabla \theta)\mathrm{d}x$$
$$= -\rho_2 \kappa \int_\Omega (\mathcal{B}_{\mathrm{rot}} \operatorname{div}\, q)\cdot v_t \mathrm{d}x - \rho_2\gamma \int_\Omega |v_t|^2 \mathrm{d}x - \rho_3 \int_\Omega (\mathcal{D}\mathcal{B}_{\mathrm{rot}}\theta)\cdot (S\mathcal{D}v)\mathrm{d}x$$
$$- \rho_3 K \int_\Omega \mathcal{B}_{\mathrm{rot}}\theta \cdot (v + \nabla w)\mathrm{d}x + \rho_3 \gamma \int_\Omega \operatorname{div}\, \mathcal{B}_{\mathrm{rot}}\theta \cdot \theta \mathrm{d}x$$
$$= -\rho_2 \kappa \int_\Omega (\mathcal{B}_{\mathrm{rot}} \operatorname{div}\, q)\cdot v_t \mathrm{d}x - \rho_2\gamma \int_\Omega |v_t|^2 \mathrm{d}x - \rho_3 \int_\Omega (\mathcal{D}\mathcal{B}_{\mathrm{rot}}\theta)\cdot (S\mathcal{D}v)\mathrm{d}x$$
$$- \rho_3 K \int_\Omega \mathcal{B}_{\mathrm{rot}}\theta \cdot (v + \nabla w)\mathrm{d}x - \rho_3 \gamma \int_\Omega \theta^2 \mathrm{d}x.$$

We would like to stress that the injectivity of Bogowskiĭ operator was essential here for us to be able to reconstruct v_t from $\mathcal{B}_{\mathrm{rot}}\operatorname{div}\, v_t$. In general, this is not possible unless the vector field is irrotational and vanishes on Γ. Using the Young's inequality and exploiting the continuity of Bogowskiĭ operator, we can estimate

$$\partial_t \mathcal{F}_3(t) \le -\rho_2\gamma \int_\Omega |v_t|^2 \mathrm{d}x + \tfrac{\rho_2\kappa}{2} \int_\Omega \left(\varepsilon_3 |v_t|^2 + \tfrac{C'_{\mathcal{B}_{\mathrm{rot}}}}{\varepsilon_3}|q|^2\right) \mathrm{d}x$$
$$+ \tfrac{\rho_3 \|S\|}{2} \int_\Omega \left(\varepsilon_3' |S\mathcal{D}v|^2 + \tfrac{C_{\mathcal{B}}}{\varepsilon_3'}\theta^2\right) \mathrm{d}x + \tfrac{\rho_3 K}{2} \int_\Omega \left(\varepsilon_3'|v|^2 + \tfrac{C_{\mathcal{K},2}C_{\mathcal{B}}}{\varepsilon_3'}\theta^2\right)\mathrm{d}x$$
$$+ \tfrac{\rho_3 K}{2} \int_\Omega \left(\varepsilon_3' |\nabla w|^2 + \tfrac{c_{\mathcal{B}}}{\varepsilon_3'}\theta^2\right)\mathrm{d}x + \rho_3\gamma \int_\Omega \theta^2 \mathrm{d}x$$
$$\le -\left(\rho_2\gamma - \tfrac{\rho_2\kappa\varepsilon_3}{2}\right) \int_\Omega |v_t|^2 \mathrm{d}x + \left(\tfrac{\rho_3 2\|S\|\varepsilon_3'}{2} + \tfrac{\rho_3 K C_P \varepsilon_3'}{2C_{\mathcal{K},1}}\right) \int_\Omega |\sqrt{S}\mathcal{D}v|^2 \mathrm{d}x + \tfrac{\rho_3 K \varepsilon_3'}{2} \int_\Omega |\nabla w|^2 \mathrm{d}x$$

$$\left(\rho_3\gamma - \tfrac{\rho_3(\|S\|+K)C_{\mathcal{B}}}{2\varepsilon_3'} - \tfrac{\rho_3 K C_{\mathcal{B}}}{2\varepsilon_3'}\theta^2\right)\int_\Omega \theta^2 \mathrm{d}x + \tfrac{\rho_2\kappa C'_{\mathcal{B}_{\mathrm{rot}}}}{2\varepsilon_3}\int_\Omega |q|^2\mathrm{d}x \tag{3.25}$$

for arbitrary positive ε_3 and ε_3'. The constants $C_{\mathcal{B}_{\mathrm{rot}}}$ and $C'_{\mathcal{B}_{\mathrm{rot}}}$ occuring above come from Theorem 9 and Theorem 11.

Finally, we define

$$\mathcal{F}_4(t) := -\tau_0\rho_3\int_\Omega q\cdot\mathcal{B}_{\mathrm{rot}}\theta\mathrm{d}x$$

and obtain

$$\begin{aligned}\partial_t\mathcal{F}_4(t) &= -\rho_3\int_\Omega(-\delta q-\kappa\nabla\theta)\cdot\mathcal{B}_{\mathrm{rot}}\theta\mathrm{d}x - \tau_0\int_\Omega q\cdot\mathcal{B}_{\mathrm{rot}}(-\kappa\mathrm{div}\ q-\gamma\mathrm{div}\ v_t)\mathrm{d}x\\ &= \rho_3\delta\int_\Omega q\cdot\mathcal{B}\theta\mathrm{d}x - \rho_3\kappa\int_\Omega\theta^2\mathrm{d}x + \tau_0\kappa\int_\Omega q\cdot\mathcal{B}\mathrm{div}\ q\mathrm{d}x + \tau_0\gamma\int_\Omega q\cdot v_t\mathrm{d}x\end{aligned}$$

since

$$\int_\Omega\nabla\theta\cdot\mathcal{B}_{\mathrm{rot}}\theta\mathrm{d}x = -\int_\Omega\theta\mathrm{div}\ \mathcal{B}_{\mathrm{rot}}\theta\mathrm{d}x = -\int_\Omega\theta^2\mathrm{d}x.$$

This yields the estimate

$$\begin{aligned}\partial_t\mathcal{F}_4 &\le -\rho_3\kappa\int_\Omega\theta^2\mathrm{d}x + \tfrac{\rho_3\delta}{2}\int_\Omega\left(\varepsilon_4 C_{\mathcal{B}_{\mathrm{rot}}}\theta^2 + \tfrac{1}{\varepsilon_4}|q|^2\right)\mathrm{d}x\\ &\quad + \tau_0\kappa(1+C'_{\mathcal{B}_{\mathrm{rot}}})\int_\Omega|q|^2\mathrm{d}x + \tfrac{\tau_0\gamma}{2}\int_\Omega\left(\varepsilon_4'|v_t|^2+\tfrac{1}{\varepsilon_4'}|q|^2\right)\mathrm{d}x\\ &= \left(-\rho_3\kappa + \tfrac{\varepsilon_4\rho_3\delta C_{\mathcal{B}_{\mathrm{rot}}}}{2}\right)\int_\Omega\theta^2\mathrm{d}x + \tfrac{\varepsilon_4'\tau_0\gamma}{2}\int_\Omega|v_t|^2\mathrm{d}x\\ &\quad + \left((1+C'_{\mathcal{B}_{\mathrm{rot}}})\tau_0\kappa + \tfrac{\rho_3\delta}{2\varepsilon_4} + \tfrac{\tau_0\gamma}{2\varepsilon_4'}\right)\int_\Omega|q|^2\mathrm{d}x.\end{aligned} \tag{3.26}$$

For positive N, N_4, we define the auxiliary functional $\mathcal{F}$ by the means of

$$\mathcal{F}(t) := N\mathcal{E}(t) + \mathcal{F}_1(t) + \mathcal{F}_2(t) + \mathcal{F}_3(t) + N_4\mathcal{F}_4(t).$$

Using now the estimates for $\partial_t\mathcal{F}_1$, $\partial_t\mathcal{F}_2$, $\partial_t\mathcal{F}_3$ and $\partial_t\mathcal{F}_4$ from Equations (3.23)–(3.26), we obtain

$$\begin{aligned}\partial_t\mathcal{L}(t) \le &- C_{w_t}\int_\Omega w_t^2\mathrm{d}x - C_{\nabla w}\int_\Omega|\nabla w|^2\mathrm{d}x - C_{v_t}\int_\Omega|v_t|^2\mathrm{d}x - C_{\sqrt{S}\mathcal{D}v}\int_\Omega|\sqrt{S}\mathcal{D}'v|^2\mathrm{d}x\\ &- C_\theta\int_\Omega\theta^2\mathrm{d}x - C_q\int_\Omega|q|^2\mathrm{d}x\end{aligned}$$

with the constants

$$\begin{aligned}C_{w_t} &= dN - \tfrac{\rho_1+d}{2\varepsilon_1} - \left(\tfrac{d}{2\varepsilon_2}+\rho_1\right),\\ C_{\nabla w} &= \left[K - \tfrac{\varepsilon_2 C_{\mathcal{P}}}{2}(K+d)\right] - \tfrac{\rho_3 K\varepsilon_3'}{2},\\ C_{v_t} &= \left[\rho_2 - \tfrac{\varepsilon_1}{2}\left(\rho_1 C_{\mathcal{P}} + \tfrac{\gamma\tau_0}{\kappa}\right)\right] + \left[\rho_2\gamma - \tfrac{\rho_2\kappa\varepsilon_3}{2}\right] - N_4\tfrac{\varepsilon_4'\tau_0\gamma}{2},\\ C_{\sqrt{S}\mathcal{D}v} &= \left[1 - \tfrac{\varepsilon_1}{2C_{\mathcal{K},1}}\left(dC_{\mathcal{P}} + \tfrac{\gamma\delta}{\kappa}\right)\right] - \left(\tfrac{\rho_3 2\|S\|\varepsilon_3'}{2} + \tfrac{\rho_3 K C_P\varepsilon_3'}{2C_{\mathcal{K},1}}\right),\\ C_\theta &= \left(\rho_3\gamma - \tfrac{\rho_3(\|S\|+K)C_{\mathcal{B}}}{2\varepsilon_3'} - \tfrac{\rho_3 K C_{\mathcal{B}}}{2\varepsilon_3'}\theta^2\right) + N_4\left[\rho_3\kappa - \tfrac{\varepsilon_4\rho_3\delta C_{\mathcal{B}_{\mathrm{rot}}}}{2}\right],\\ C_q &= \tau_0 N - \tfrac{\gamma(\tau_0+\delta)}{2\kappa\varepsilon_1} - \tfrac{\rho_2\kappa C'_{\mathcal{B}_{\mathrm{rot}}}}{2\varepsilon_3} - N_4\left((1+C'_{\mathcal{B}_{\mathrm{rot}}})\tau_0\kappa + \tfrac{\rho_3\delta}{2\varepsilon_4} + \tfrac{\tau_0\gamma}{2\varepsilon_4'}\right).\end{aligned}$$

Now, we select $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 > 0$ sufficiently small for all bracket terms in $C_{\sqrt{S}\mathcal{D}v}$, $C_{\nabla w}$, C_{v_t} and C_θ to be positive. Next, we choose $\varepsilon_3' > 0$ so small that $C_{\nabla w}$ and $C_{\sqrt{S}\mathcal{D}v}$ become positive. Then we fix a sufficiently large $N_4 > 0$ to assure for $C_\theta > 0$. We further pick a small $\varepsilon_4' > 0$ to make C_{v_t} positive. Finally, we choose $N > 0$ to be sufficiently large to guarantee the positivity of C_{w_t} and C_q. Therefore, we get

$$C_{\min} := \min\{C_{w_t}, C_{\nabla w}, C_{v_t}, C_{\sqrt{S}\mathcal{D}v}, C_\theta, C_q\} > 0.$$

Taking into account Young's inequality

$$|\nabla w + v|^2 \leq \tfrac{1}{2}(|\nabla w|^2 + |\nabla v|^2),$$

Korn's inequality immediately yields the estimate

$$\begin{aligned}|\nabla w| + |\sqrt{S}\mathcal{D}v|^2 &\geq |\nabla w| + \tfrac{1}{2}C_{\mathcal{K},1}|v|^2 + \tfrac{1}{2}|\sqrt{S}\mathcal{D}v|^2 \geq \min\{2, C_{\mathcal{K},1}\}|\nabla w + v|^2 + \tfrac{1}{2}|\sqrt{S}\mathcal{D}v|^2 \\ &\geq \min\{\tfrac{1}{2}, C_{\mathcal{K},1}\}(|\nabla w + v|^2 + |\sqrt{S}\mathcal{D}v|^2).\end{aligned}$$

Hence, we get

$$\partial_t \mathcal{F}(t) \leq -2\frac{\min\left\{1, \min\left\{\frac{1}{2}, C_{\mathcal{K},1}\right\}^{-1}\right\}}{\max\{1, \rho_1, \rho_2, \rho_3, \tau_0, K\}}\mathcal{E}(t) =: C\mathcal{E}(t).$$

On the other hand, we can estimate

$$\begin{aligned}|\mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 + N_4\mathcal{F}_4|(t) \leq \tfrac{1}{2}\int_\Omega \Big(&\rho_1(w_t^2 + |u|^2) + \rho_2(|v_t|^2 + |v|^2) + \tfrac{\gamma\tau_0}{\kappa}(|v|^2 + |q|^2) + \\ &\rho_1(w_t^2 + w^2) + \rho_2\rho_3(|\mathcal{B}_{\text{rot}}\theta|^2 + |v_t|^2) + \tau_0\rho_3(|q|^2 + |\mathcal{B}_{\text{rot}}\theta|^2)\Big)\mathrm{d}x \\ \leq \tfrac{1}{2}\Big(&2\rho_1\|w_t\|^2_{L^2(\Omega)} + \rho_1\|w\|^2_{H^1(\Omega)} + \rho_2\|v_t\|^2_{(L^2(\Omega)^2)} + (\rho_2 + \tfrac{\gamma\tau_0}{\kappa})\|v\|_{(H^1(\Omega))^2} \\ &+ C_{\mathcal{B}_{\text{rot}}}(\rho_2\rho_3 + \tau_0\rho_3)\|\theta\|^2_{L^2(\Omega)} + (\tfrac{\gamma\tau_0}{\kappa} + \tau_0\rho_3)\|q\|^2_{(L^2(\Omega))^2}\Big) \\ \leq \tfrac{1}{2}\Big(&2\rho_1\|w_t\|^2_{L^2(\Omega)} + \rho_2\|v_t\|^2_{(L^2(\Omega)^2)} + \\ &\tfrac{\max\{\rho_1, (\rho_2 + \frac{\gamma\tau_0}{\kappa})\}}{C_{\mathcal{K}}}(K\|\nabla w + v\|^2_{(L^2(\Omega))^2} + \|\sqrt{S}\mathcal{D}v\|^2_{(L^2(\Omega))^2}) \\ &+ C_{\mathcal{B}_{\text{rot}}}(\rho_2\rho_3 + \tau_0\rho_3)\|\theta\|^2_{L^2(\Omega)} + (\tfrac{\gamma\tau_0}{\kappa} + \tau_0\rho_3)\|q\|^2_{(L^2(\Omega))^2}\Big) \leq \hat{C}\mathcal{E}(t).\end{aligned}$$

Letting now $\alpha_1 := N - \frac{\max\{\rho_1, \rho_2, C_{\mathcal{K}}^{-1}\}}{\min\{\rho_1, \rho_2, \rho_3\}}$ and $\alpha_2 := N + \frac{\max\{\rho_1, \rho_2, C_{\mathcal{K}}^{-1}\}}{\min\{\rho_1, \rho_2, \rho_3\}}$, we obtain the following equivalence between $\mathcal{E}$ and $\mathcal{F}$

$$\alpha_1\mathcal{E}(t) \leq \mathcal{F}(t) \leq \alpha_2\mathcal{E}(t) \text{ for } t \geq 0.$$

If necessary, we increase the constant N to assure for the positivity of α_1. Thus, both C, α_1 and α_2 are positive. Exploiting Gronwall's inequality, we obtain the following estimate for $\mathcal{E}$

$$\mathcal{E}(t) \leq \tfrac{1}{\alpha_1}\mathcal{F}(t) \leq \tfrac{1}{\alpha_1}\mathcal{E}(0)e^{-\frac{C}{\alpha_2}t} =: C\mathcal{E}(0)e^{-2\alpha t} \text{ for } t \geq 0$$

meaning an exponential decay of $\mathcal{E}$. □

Appendices

A The divergence problem and the Bogowskiĭ operator

In various applications of partial differential equations, e.g., when studying Navier-Stokes equations, there arises a so-called "divergence problem": For a given function f, determine a vector field u such that its divergence coincides with f. We refer to [9] for a rather general solution of this problem in bounded domains. It has namely been shown that the solution map $\mathcal{B}\colon f \mapsto u$, called the Bogowskiĭ-operator, is a bounded linear operator between $W_0^{s,p}(\Omega)$ and $W_0^{s+1,p}(\Omega)$ for $p \in (0,\infty)$, $s \in \big(-2+\frac{1}{p},\infty\big)$.

For our application, we want to additionally guarantee that the solution u is irrotational. To this end, we exploit the following result from [13].

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ be a domain with a smooth boundary and let $\nu\colon \Omega \to \mathbb{R}^n$ denote the outer unit normal vector on $\partial\Omega$. There exists then a function $u \in H^1(\Omega,\mathbb{R}^n)$ satisfying $\nu \otimes u = u \otimes \nu$ on $\partial\Omega$ and*

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\operatorname{div} u\|^2_{L^2(\Omega)} + \tfrac{1}{2}\|\nabla u - (\nabla u)'\|^2_{L^2(\Omega)} + (n-1)\int_{\partial\Omega} |u|^2 H_n \mathrm{d}S, \tag{A.1}$$

where $H_n\colon \partial\Omega \to \mathbb{R}$, $x \mapsto H_n(x)$ denotes the mean curvature of $\partial\Omega$ with respect to the outer normal vector. In $n = 2,3$, Equation (A.1) reduces to

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\operatorname{div} u\|^2_{L^2(\Omega)} + \|\mathrm{rot} u\|^2_{L^2(\Omega)} + (n-1)\int_{\partial\Omega} |u|^2 H_n \mathrm{d}S, \tag{A.2}$$

where

$$\mathrm{rot} u = \begin{pmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix} \textit{ for } n = 3 \textit{ and } \mathrm{rot} u = \partial_{x_1} u_2 - \partial_{x_2} u_1 \textit{ for } n = 2.$$

For $u \in H_0^1(\Omega,\mathbb{R}^n)$, the second term in (A.1) and (A.2) vanishes and no assumptions on $\partial\Omega$ are required:

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\operatorname{div} u\|^2_{L^2(\Omega)} + \|\nabla u - (\nabla u)'\|^2_{L^2(\Omega)}. \tag{A.3}$$

In the following, we assume $n = 2$. We define the space

$$H^1_{0,\mathrm{rot}}(\Omega) = \big\{u \in (H^1_0(\Omega))^2 \,|\, \nabla u = (\nabla u)'\big\} = \big\{u \in (H^1_0(\Omega))^2 \,|\, \mathrm{rot} u = 0\big\}$$

equipped with the standard inner product of $(H^1_0(\Omega))^2$. Since $H^1_{0,\mathrm{rot}}(\Omega)$ is a closed subspace of $(H^1_0(\Omega))^2$, $H^1_{0,\mathrm{rot}}(\Omega)$ is a Hilbert space. We prove the following theorem.

Theorem 9. *The mapping*

$$\operatorname{div} : H^1_{0,\mathrm{rot}}(\Omega) \to L^2(\Omega)/\{1\}$$

is an isomorphism with an inverse $\operatorname{div}^{-1} = \mathcal{B}_{\mathrm{rot}}$

$$\mathcal{B}_{\mathrm{rot}}\colon L^2(\Omega)/\{1\} \to H^1_{0,\mathrm{rot}}(\Omega)$$

in the sense

$$\operatorname{div} \mathcal{B}_{\mathrm{rot}} = \mathrm{id}_{L^2(\Omega)/\{1\}} \textit{ and } \mathcal{B}_{\mathrm{rot}} \operatorname{div} = \mathrm{id}_{H^1_{0,\mathrm{rot}}(\Omega)}.$$

Furthermore, the exists $C_{\mathcal{B}} > 0$ such that

$$\|\mathcal{B}_{\mathrm{rot}} f\|_{(H^1(\Omega))^2} \le C_{\mathcal{B}_{\mathrm{rot}}} \|f\|_{L^2(\Omega)}$$

holds true for all $f \in L^2_*(\Omega)$.

Proof. The linearity of div is obvious For each $u \in H^1_{0,\mathrm{rot}}(\Omega)$, we have $\operatorname{div} u \in L^2(\Omega)$ and thus

$$\int_\Omega \operatorname{div} u \mathrm{d}x = \int_\Gamma u \cdot \nu \mathrm{d}\Gamma = 0,$$

meaning $\operatorname{div} u \in L^2(\Omega)/\{1\}$. The continuity is also trivial since

$$\|\operatorname{div} u\|_{L^2(\Omega)} \leq \sqrt{2}\|\nabla u\|_{L^2(\Omega)} \leq \sqrt{2}\|u\|_{H^1(\Omega)}.$$

The operator div is injective. Indeed, let $u_1, u_2 \in H^1_{0,\mathrm{rot}}(\Omega)$. Let $\operatorname{div} u_1 = \operatorname{div} u_2$. Then, using Poincaré inequality,

$$0 = \|\operatorname{div} u_1 - \operatorname{div} u_2\|_{L^2(\Omega)} \geq \|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)} \geq \tfrac{1}{C_{\mathcal{P}}}\|u_1 - u_2\|_{L^2(\Omega)},$$

i.e., $u_1 = u_2$.

To explicitely construct the operator $\mathcal{B}_{\mathrm{rot}}$, we follow the variational approach. For $f, g \in L^2(\Omega)/\{1\}$, we consider a boundary value problem for $\varphi, \psi \in H^1(\Omega)/\{1\}$:

$$\begin{aligned} -\operatorname{div}(\nabla\varphi + \mathrm{rot}'\psi) &= f \text{ in } \Omega, \\ -\mathrm{rot}(\nabla\varphi + \mathrm{rot}'\psi) &= g \text{ in } \Omega, \\ \nu \cdot (\nabla\varphi + \mathrm{rot}'\psi) &= 0 \text{ on } \Gamma, \\ \nu^{\perp} \cdot (\nabla\varphi + \mathrm{rot}'\psi) &= 0 \text{ on } \Gamma, \end{aligned} \tag{A.4}$$

where $\nu^{\perp} := (\nu_2, -\nu_1)'$, $\mathrm{rot}' := (\partial_{x_2}, -\partial_{x_1})'$. We multiply the equations with $\tilde{\varphi}, \tilde{\psi} \in H^1(\Omega)/\{1\}$, sum up the resulting identities, take into account the boundary conditions and apply a partial integration to find

$$-\int_\Omega \operatorname{div}(\nabla\varphi + \mathrm{rot}'\psi)\tilde{\varphi}\mathrm{d}x - \int_\Omega \mathrm{rot}(\nabla\varphi + \mathrm{rot}'\psi)\tilde{\psi}\mathrm{d}x = \int_\Omega (\nabla\varphi + \mathrm{rot}'\psi)\cdot(\nabla\tilde{\varphi} + \mathrm{rot}\tilde{\psi})\mathrm{d}x$$

This lead to the following operator equation

$$\mathcal{A}(\varphi, \psi)' = (f, g)', \tag{A.5}$$

where

$$\mathcal{A}\colon D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}, \quad (\varphi, \psi)' \mapsto \begin{pmatrix} -\operatorname{div}(\nabla\varphi + \mathrm{rot}'\psi) \\ -\mathrm{rot}(\nabla\varphi + \mathrm{rot}'\psi) \end{pmatrix}$$

and

$$D(\mathcal{A}) = \Big\{(\varphi, \psi)' \in \mathcal{V} \,\Big|\, \exists (f_1, f_2)' \in \mathcal{H}\ \forall (\tilde{\varphi}, \tilde{\psi})' \in \mathcal{V} : B(\varphi, \psi; \tilde{\varphi}, \tilde{\psi}) = \int_\Omega f_1\tilde{\varphi} + f_2\tilde{\psi}\mathrm{d}x\Big\}$$

with the bilinear form

$$B\colon \mathcal{V} \times \mathcal{V} \to \mathbb{R}, \quad (\phi, \psi, \tilde{\phi}, \tilde{\psi})' \mapsto \int_\Omega (\nabla\varphi + \mathrm{rot}'\psi)\cdot(\nabla\tilde{\varphi} + \mathrm{rot}\tilde{\psi})\mathrm{d}x.$$

Here, we introduced the Hilbert spaces

$$\mathcal{H} := (L^2(\Omega)/\{1\}) \times (L^2(\Omega)/\{1\}), \quad \mathcal{V} := (H^1(\Omega)/\{1\}) \times (H^1(\Omega)/\{1\})$$

equipped with the standard inner products of $L^2(\Omega) \times L^2(\Omega)$ and $H^1(\Omega) \times H^1(\Omega)$, respectively. Since $\mathcal{A}$ has a nontrivial kernel, we consider the operator given as its restriction onto the closed subspace

$$\tilde{\mathcal{V}} = \{(\varphi, \psi)' \in \mathcal{V} \,|\, \forall (\tilde{\varphi}, \tilde{\psi})' \in \mathcal{V} : \int_\Omega \nabla\varphi \cdot \mathrm{rot}'\tilde{\psi}\mathrm{d}x = \int_\Omega \nabla\tilde{\varphi}\cdot\mathrm{rot}'\psi\mathrm{d}x = 0\}$$

of $\mathcal{V}$ and denote it as

$$\tilde{\mathcal{A}}\colon D(\tilde{\mathcal{A}}) := D(\mathcal{A}) \cap \tilde{\mathcal{V}} \subset \mathcal{H} \to \mathcal{H}.$$

Equation (A.5) reduces then to

$$\tilde{\mathcal{A}}(\varphi, \psi)' = (f, g)'. \tag{A.6}$$

We multiply Equation (A.6) scalar in $\mathcal{H}$ with $(\tilde{\varphi}, \tilde{\psi})' \in \tilde{\mathcal{V}}$ to find after a partial integration the weak formulation of (A.6): Determine an element $(\varphi, \psi)' \in \tilde{\mathcal{V}}$ such that

$$B(\varphi, \psi; \hat{\varphi}, \hat{\psi}) = F(\hat{\varphi}, \hat{\psi}) \text{ for all } (\hat{\varphi}, \hat{\psi})' \in \tilde{\mathcal{V}}, \tag{A.7}$$

where

$$B\colon \tilde{\mathcal{V}} \times \tilde{\mathcal{V}} \to \mathbb{R}, \quad (\phi, \psi, \hat{\phi}, \hat{\psi})' \mapsto \int_\Omega (\nabla\varphi + \mathrm{rot}'\psi) \cdot (\nabla\hat{\varphi} + \mathrm{rot}\hat{\psi}) \mathrm{d}x,$$
$$F\colon \tilde{\mathcal{V}} \to \mathbb{R}, \quad (\hat{\phi}, \hat{\psi})' \mapsto \int_\Omega \hat{\varphi} f \mathrm{d}x + \int_\Omega \hat{\psi} g \mathrm{d}x.$$

The bilinear form B and the linear functional F are continuous on $\tilde{\mathcal{V}} \times \tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}$, respectively. The bilinear form B is symmetrical. By the virtue of second POINCARÉ's inequality, we obtain

$$\begin{aligned} B(\varphi, \psi) &= \|\nabla\varphi\|^2_{L^2(\Omega)} + 2\langle \nabla\varphi, \mathrm{rot}'\psi\rangle + \|\mathrm{rot}'\psi\|^2_{L^2(\Omega)} \\ &= \|\nabla\varphi\|^2_{L^2(\Omega)} + \|\mathrm{rot}'\psi\|^2_{L^2(\Omega)} = \|\nabla\varphi\|^2_{L^2(\Omega)} + \|\nabla\psi\|^2_{L^2(\Omega)} \\ &\geq \tfrac{1}{2}(1 + \tfrac{1}{C_{\mathcal{P}}})(\|\varphi\|^2_{H^1(\Omega)} + \|\psi\|^2_{H^1(\Omega)}) = \tfrac{1}{2}(1 + \tfrac{1}{C_{\mathcal{P}}})\|(\varphi, \psi)'\|^2_{\tilde{\mathcal{V}}} =: b\|(\varphi, \psi)'\|^2_{\tilde{\mathcal{V}}}, \end{aligned}$$

i.e., B is coercive. The lemma of Lax & Milgram yields the existence of a unique solution $(\varphi, \psi)' \in \tilde{\mathcal{V}}$ to Equation (A.7). There further holds

$$b\|(\varphi, \psi)'\|^2_{\tilde{\mathcal{V}}} \leq B(\varphi, \psi) \leq \tfrac{b}{2}\|(\varphi, \psi)'\|^2_{\mathcal{H}} + \tfrac{1}{2b}\|(f, g)'\|^2_{\mathcal{H}} \leq \tfrac{b}{2}\|(\varphi, \psi)'\|^2_{\tilde{\mathcal{V}}} + \tfrac{1}{2b}\|(f, g)'\|^2_{\mathcal{H}},$$

i.e.,

$$\|(\varphi, \psi)'\|^2_{\tilde{\mathcal{V}}} \leq \tfrac{1}{b}\|(f, g)'\|^2_{\mathcal{H}}.$$

Exploiting the trivial identities

$$\mathrm{div}\ \mathrm{rot}'\varphi = 0, \quad \mathrm{rot}\, \nabla\varphi = 0, \text{ etc., in } (\mathcal{C}_0^\infty(\Omega))'$$

and the definition of $\mathcal{V}$, we find

$$\int_\Gamma \nu\varphi \cdot \hat{\mathrm{rot}}'\psi \mathrm{d}x = \int_\Omega \nabla\varphi \cdot \mathrm{rot}'\hat{\psi} \mathrm{d}x = 0, \quad \int_\Gamma \nu^\perp\psi \cdot \nabla\hat{\varphi} \mathrm{d}\Gamma = \int_\Gamma \mathrm{rot}'\psi \cdot \nabla\hat{\varphi} \mathrm{d}x = 0, \text{ etc.}$$

for all $(\varphi, \psi)' \in \tilde{\mathcal{V}}$ and $(\hat{\varphi}, \hat{\psi})' \in \mathcal{V}$. Hence,

$$\begin{aligned} &-\int_\Omega \mathrm{div}\,(\nabla\varphi + \mathrm{rot}'\psi)\hat{\varphi} + \mathrm{rot}(\nabla\varphi + \mathrm{rot}'\psi)\hat{\psi} \mathrm{d}x \\ &\qquad = B(\varphi, \psi; \hat{\varphi}, \hat{\psi}) - \int_\Gamma \nu \cdot (\nabla\varphi + \mathrm{rot}'\psi)\hat{\varphi} + \nu^\perp \cdot (\nabla\varphi + \mathrm{rot}'\psi)\hat{\varphi} \mathrm{d}\Gamma \end{aligned}$$

holds true for all $(\hat{\varphi}, \hat{\psi})' \in \mathcal{V}$ and, in particular, the solution $(\varphi, \psi)' \in \tilde{\mathcal{V}}$ of (A.7). Therefore, $(\varphi, \psi)' \in D(\tilde{\mathcal{A}})$. Thus, we have shown that $\tilde{\mathcal{A}}$ is invertible and its inverse $\tilde{\mathcal{A}}^{-1}\colon \mathcal{H} \to D(\tilde{\mathcal{A}})$ is continuous:

$$\|\tilde{\mathcal{A}}^{-1}(f, g)'\|_{\mathcal{V}} \leq \tfrac{1}{b}\|(f, g)'\|^2_{\mathcal{H}}$$

Let $f \in L^2(\Omega)/\{1\}$. We define $(\phi, \psi) := \tilde{\mathcal{A}}^{-1}(f, 0)'$, $u := \nabla\varphi + \mathrm{rot}'\psi$ and obtain by construction

$$\begin{aligned} \mathrm{div}\, u &= \triangle\varphi = f \text{ in } \Omega, \\ \mathrm{rot} u &= \mathrm{rot} 0 = 0 \text{ in } \Omega, \\ u &= \nabla\varphi + \mathrm{rot}\psi = 0 \text{ on } \Gamma, \end{aligned} \tag{A.8}$$

i.e., $u \in H^1_{\mathrm{rot}}(\Omega)$ with $\operatorname{div} u = f$. Thus, there exists a continuous inverse

$$\mathcal{B}_{\mathrm{rot}} \colon L^2(\Omega)/\{1\} \to H^1_{0,\mathrm{rot}}(\Omega), \quad f \mapsto u$$

of div such that

$$\begin{aligned}
\|\mathcal{B}_{\mathrm{rot}} f\|_{(H^1(\Omega))^2} &= \|\mathcal{B}_{\mathrm{rot}} f\|^2_{(L^2(\Omega))^2} + \|\nabla \mathcal{B}_{\mathrm{rot}} f\|^2_{(L^2(\Omega))^{2\times 2}} \\
&= \|\nabla\varphi + \mathrm{rot}'\psi\|^2_{(L^2(\Omega))^2} + \|\operatorname{div} \mathcal{B}_{\mathrm{rot}} f\|^2_{L^2(\Omega)} \\
&\le 2\|\nabla\varphi\|^2_{(L^2(\Omega))^2} + 2\|\mathrm{rot}'\psi\|^2_{(L^2(\Omega))^2} + \|f\|^2_{L^2(\Omega)} \\
&\le (\tfrac{2}{b} + 1)\|f\|^2_{L^2(\Omega)} =: C_{\mathcal{B}_{\mathrm{rot}}}\|f\|_{L^2(\Omega)}.
\end{aligned}$$

This finishes the proof. ◻

Corollary 10. *The operator $\mathcal{B}_{\mathrm{rot}}$ can be extended to a linear continuous operator*

$$\mathcal{B}_{\mathrm{rot}} \colon (H^1(\Omega))' \to (L^2(\Omega))^2.$$

(Cp. also [4, 9] for the rotational case.)

Proof. Due to the coercivity of the bilinear form B, the operator $\tilde{\mathcal{A}}$ defined in the proof of Theorem 9 strictly positive. According to [31, Section 3.4], it is possible to define square roots

$$\tilde{\mathcal{A}}^{-1/2} \in L(\mathcal{H},\mathcal{H}) \text{ and } \tilde{\mathcal{A}}^{1/2} \colon D(\tilde{\mathcal{A}}^{1/2}) := \operatorname{im} \tilde{\mathcal{A}}^{-1/2} \to \mathcal{H}$$

of $\tilde{\mathcal{A}}^{-1}$ and $\tilde{\mathcal{A}}$, respectively. Further, there exists a continuous continuation of $\tilde{\mathcal{A}}^{-1}$

$$\tilde{\mathcal{A}}^{-1} \in L(D(\tilde{\mathcal{A}}^{-1/2}), D(\tilde{\mathcal{A}}^{1/2})),$$

where $D(\tilde{\mathcal{A}}^{-1/2}) = D(\tilde{\mathcal{A}}^{1/2})'$. Hence,

$$\tilde{\mathcal{B}}_{\mathrm{rot}} \colon D(\tilde{\mathcal{A}}^{-1/2}) \to (L^2(\Omega))^2, \quad f \mapsto \nabla\varphi + \mathrm{rot}'\psi \text{ with } (\varphi,\psi)' := \tilde{\mathcal{A}}^{-1}(f,0)' \in \tilde{\mathcal{V}}$$

represents a continuous continuation of $\mathcal{B}_{\mathrm{rot}}$ onto $D(\tilde{\mathcal{A}}^{-1/2})$. Since $(H^1(\Omega))' \subset D(\tilde{\mathcal{A}}^{-1/2})$ and the norms of $(H^1(\Omega))'$ und $D(\tilde{\mathcal{A}}^{-1/2})$ are equivalent, the claim follows. ◻

Let us now consider a vector field $u \in (H^1(\Omega))^2$ with $u \cdot \nu = 0$ on Γ. Unfortunately, the identity

$$\mathcal{B}_{\mathrm{rot}} \operatorname{div} u = u$$

does not hold in general since u is not necessarily an element of $H^1_{0,\mathrm{rot}}(\Omega)$. Nevertheless, the following estimate holds true.

Theorem 11. *Let $u \in H^1(\Omega)$ satisfy $u \cdot \nu = 0$ on Γ. There exists then a constant $C'_{\mathcal{B}_{\mathrm{rot}}} > 0$ such that*

$$\|\mathcal{B}_{\mathrm{rot}} \operatorname{div} u\|_{L^2(\Omega)} \le C'_{\mathcal{B}_{\mathrm{rot}}}\|u\|_{(L^2(\Omega))^2}$$

for any $u \in (H^1(\Omega))^2$.

Proof. We can estimate

$$\|\mathcal{B}_{\mathrm{rot}} \operatorname{div} u\|_{(L^2(\Omega))^2} \le C_{\mathcal{B}_{\mathrm{rot}}}\|\operatorname{div} u\|_{H^{-1}(\Omega)}.$$

Further, we find

$$\int_\Omega \operatorname{div} u f \,\mathrm{d}x = -\int_\Omega u\nabla f \,\mathrm{d}x + \int_{\partial\Omega} u\cdot\nu f \,\mathrm{d}\Gamma = -\int_\Omega u\nabla f\,\mathrm{d}x \tag{A.9}$$

for all $f \in H^1(\Omega)$ and therefore

$$\begin{aligned}\|\operatorname{div} u\|_{H^{-1}(\Omega)} &= \sup_{\|f\|_{H^1(\Omega)}=1} \left| \int_\Omega \operatorname{div} u f \, \mathrm{d}x \right| = \sup_{\|f\|_{H^1(\Omega)}=1} \left| \int_\Omega u \nabla f \, \mathrm{d}x \right| \\ &\leq \sup_{\|f\|_{H^1(\Omega)}=1} \|u\|_{(L^2(\Omega))^2} \|f\|_{H^1(\Omega)} = \|u\|_{(L^2(\Omega))^2}.\end{aligned}$$

This yields

$$\|\mathcal{B}_{\mathrm{rot}} \operatorname{div} u\|_{L^p(\Omega)} \leq C'_{\mathcal{B}_{\mathrm{rot}}} \|u\|_{L^p(\Omega)} \text{ for all } u \in H^1(\Omega) \tag{A.10}$$

with $C'_{\mathcal{B}_{\mathrm{rot}}} = C_{\mathcal{B}_{\mathrm{rot}}}$. □

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