LSZ reduction formula in many-dimensional theory with space-space noncommutativity

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An analogue of the Lehmann—Symanzik—Zimmermann reduction formula is obtained for the case of noncommutative space-space theory. Some consequences of the reduction formula and Haag's theorem are discussed.

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1 Introduction

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It is well known that in conventional quantum field theory the LSZ reduction formula allows effective calculation of the scattering amplitudes from the Green functions [\[1\]](#page-8-0). In the present paper we analyze the applicability of this formula in the framework of noncommutative quantum field theory (NC-QFT). We will consider the case of a neutral scalar field in many-dimensional theory with space-space noncommutativity, so that the temporal variable commutes with the spatial ones.

Let us consider the general case of $SO(1, d)$ -invariant theory with $d+1$ commutative coordinates (including time) and an arbitrary even number l of noncommutative ones. The commutation relations between l noncommutative coordinates have the form

$$
\left[\hat{x}^i, \hat{x}^j\right] = i\theta^{ij}, \quad i, j = 1, \dots, l,\tag{1}
$$

where θ^{ij} — real antisymmetric $l \times l$ matrix. As we said, the rest $(d+1)$ variables commute with each other and all \hat{x}^j from [\(1\)](#page-1-0).

In order to formulate the theory in commutative space-time, we use the Weyl ordered symbol [\[2,](#page-9-0) [3\]](#page-9-1) $\varphi(x)$ of the noncommutative field operator $\Phi(\hat{x})$:

$$
\varphi(x) = \frac{1}{(2\pi)^l} \int d^l k \int \text{Tr} \, e^{ik(x-\hat{x})} \Phi(\hat{x}), \tag{2}
$$

and the corresponding multiplication law $\varphi_1\star\varphi_2$ between the two symbols in the Weyl–Moyal–Groenewold form:

$$
(\varphi_1 \star \varphi_2)(x) = \left[e^{\frac{i}{2}\theta^{\mu\nu}\partial'_{\mu}\partial''_{\nu}} \varphi_1(x') \varphi_2(x'') \right]_{x'=x''=x}.
$$
 (3)

Relation [\(3\)](#page-1-1) admits further generalization: for the symbols (fields) taken at different points one can define twisted tensor product [\[3,](#page-9-1) [4\]](#page-9-2)

$$
\varphi(x_1) \star \ldots \star \varphi(x_n) = \prod_{a < b} \exp\left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x_a^{\mu}} \frac{\partial}{\partial x_b^{\nu}}\right) \varphi(x_1) \ldots \varphi(x_n),
$$

\n
$$
a, b = 1, 2, \ldots n.
$$
\n(4)

Thus the algebra of field operators is deformed, and it is not clear whether one can apply the standard LSZ formula for the noncommutative fields or not.

2 Commutation Relations for Creation and Annihilation Operators in NCQFT

As in conventional field theory, a free real scalar field in NCQFT admits a normal mode expansion:

$$
\varphi(x) = \varphi^+(x) + \varphi^-(x),
$$

$$
\varphi^{\pm}(x) = \frac{1}{(2\pi)^{(d+l)/2}} \int \frac{d\vec{k}}{\sqrt{2\omega(\vec{k})}} e^{\pm ikx} a^{\pm}(\vec{k}) \Big|_{k^0 = \omega(\vec{k})},
$$
(5)

where $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$, $\vec{k}^2 = \vec{k_c}^2 + \vec{k}_{nc}^2$, $\vec{k_c}$ commutative part of the $(d+l)$ -dimensional vector \vec{k}, \vec{k}_{nc} noncommutative part of the same vector.

Let us obtain commutation relations for the creation and annihilation operators a^{\pm} directly from the assumption that the canonical quantization of a real scalar field in NCQFT is defined by the relations

$$
[\varphi(x) \star \partial_0 \varphi(y)]|_{x^0 = y^0} = i\delta(\vec{x} - \vec{y}),
$$

$$
[\varphi(x) \star \varphi(y)]|_{x^0 = y^0} = 0, \quad [\partial_0 \varphi(x) \star \partial_0 \varphi(y)]|_{x^0 = y^0} = 0.
$$
 (6)

Performing an inverse Fourier transform one can get the expression for a^{\pm} from (5) :

$$
a^{\pm}(\vec{k}) = \frac{1}{(2\pi)^{(d+l)/2}} \int d\vec{x} e^{\mp ikx} \left[\sqrt{\frac{k_0}{2}} \varphi(x) \mp \frac{i}{\sqrt{2k_0}} \partial_0 \varphi(x) \right] \Big|_{k_0 = \omega(\vec{k})} . \tag{7}
$$

Let us take the operator product $a^-(\vec{k})a^+(\vec{q})$ and multiply it by $e^{\frac{i}{2}\theta^{\mu\nu}k_{\mu}q_{\nu}}$ $e^{\frac{i}{2}\theta^{\mu\nu}ik_{\mu}(-i)q_{\nu}}$. Expanding the phase factor in a series, we get

$$
e^{\frac{i}{2}\theta^{\mu\nu}ik_{\mu}(-i)q_{\nu}} a^{-}(\vec{k})a^{+}(\vec{q}) = \sum_{n=0}^{\infty} \frac{(i/2)^{n}}{n!} (\theta^{\mu\nu}ik_{\mu}(-i)q_{\nu})^{n} a^{-}(\vec{k})a^{+}(\vec{q}) =
$$

$$
= \frac{1}{(2\pi)^{(d+l)}} \int \int d\vec{x} d\vec{y} \sum_{n=0}^{\infty} \frac{(i/2)^{n}}{n!} (\theta^{\mu\nu}ik_{\mu}(-i)q_{\nu})^{n} e^{ikx} e^{-iqy} \times \qquad (8)
$$

$$
\times \left(\sqrt{\frac{k_{0}}{2}}\varphi(x) + \frac{i}{\sqrt{2k_{0}}} \partial_{0}\varphi(x)\right) \left(\sqrt{\frac{q_{0}}{2}}\varphi(y) - \frac{i}{\sqrt{2q_{0}}} \partial_{0}\varphi(y)\right).
$$

Next, let us replace momenta k and q with the derivatives, using the relation

$$
(ik_{\mu}(-i)q_{\nu})^n e^{ikx} e^{-iqy} = (\partial_{\mu}\partial_{\nu})^n e^{ikx} e^{-iqy}, \qquad (9)
$$

.

and perform integration by parts, so that the derivatives act on the field φ in each term of the series. Thus we obtain \star -product of the field operators:

$$
e^{\frac{i}{2}\theta^{\mu\nu}k_{\mu}q_{\nu}}a^{-}(\vec{k})a^{+}(\vec{q}) = \frac{1}{(2\pi)^{(d+l)}}\int\int d\vec{x}\,d\vec{y}\,e^{ikx}\,e^{-iqy}\times
$$

$$
\times\left(\sqrt{\frac{k_{0}}{2}}\varphi(x) + \frac{i}{\sqrt{2k_{0}}}\partial_{0}\varphi(x)\right)\star\left(\sqrt{\frac{q_{0}}{2}}\varphi(y) - \frac{i}{\sqrt{2q_{0}}}\partial_{0}\varphi(y)\right)\Big|_{k_{0}=\omega(\vec{k}),\,q_{0}=\omega(\vec{q})}
$$
(10)

Similarly,

$$
e^{\frac{i}{2}\theta^{\mu\nu}q_{\mu}k_{\nu}}a^{+}(\vec{q})a^{-}(\vec{k}) = \frac{1}{(2\pi)^{(d+l)}}\int\int d\vec{x}\,d\vec{y}\,e^{ikx}\,e^{-iqy}\times
$$

$$
\times\left(\sqrt{\frac{q_{0}}{2}}\varphi(y) - \frac{i}{\sqrt{2q_{0}}}\partial_{0}\varphi(y)\right)\times\left(\sqrt{\frac{k_{0}}{2}}\varphi(x) + \frac{i}{\sqrt{2k_{0}}}\partial_{0}\varphi(x)\right)\Big|_{k_{0}=\omega(\vec{k}),\,q_{0}=\omega(\vec{q})}.
$$
\n(11)

Now, taking the fields $\varphi(x)$ and $\varphi(y)$ at equal moments of time $x^0 = y^0$ and subtracting (11) from (10) , with the use of (6) we obtain:

$$
e^{\frac{i}{2}\theta^{\mu\nu}k_{\mu}q_{\nu}}a^{-}(\vec{k})a^{+}(\vec{q}) - e^{\frac{i}{2}\theta^{\mu\nu}q_{\mu}k_{\nu}}a^{+}(\vec{q})a^{-}(\vec{k}) = \delta(\vec{k} - \vec{q}), \qquad (12)
$$

or, in a more convenient form:

$$
a^{-}(\vec{k})a^{+}(\vec{q}) = e^{-i\theta^{\mu\nu}k_{\mu}q_{\nu}}a^{+}(\vec{q})a^{-}(\vec{k}) + e^{-\frac{i}{2}\theta^{\mu\nu}k_{\mu}q_{\nu}}\delta(\vec{k}-\vec{q}).
$$
 (13)

In the same way we get

$$
a^{\pm}(\vec{k})a^{\pm}(\vec{q}) = e^{i\theta^{\mu\nu}k_{\mu}q_{\nu}}a^{\pm}(\vec{q})a^{\pm}(\vec{k}).
$$
\n(14)

Commutation relations [\(13\)](#page-3-2) and [\(14\)](#page-3-3) are equivalent to the ones obtained in [\[5\]](#page-9-3) from general group-theoretical considerations involving the twisted Poincaré symmetry.

3 Analogue of the LSZ reduction formula for spacespace NCQFT

In [\[4\]](#page-9-2) it was proposed that the expression for the noncommutative Wightman functions has the following form:

$$
W_{\star}(x_1,\ldots,x_n)=\langle 0|\varphi(x_1)\star\ldots\star\varphi(x_n)|0\rangle, \qquad (15)
$$

where \star -product of fields taken at independent points is given by [\(4\)](#page-1-2).

In accordance with [\(15\)](#page-4-0) we suppose that the noncommutative Green functions are

$$
G_{\star}(x_1,\ldots,x_n)=\langle 0|T(\varphi(x_1)\star\ldots\star\varphi(x_n))|0\rangle, \qquad (16)
$$

where we defined time-ordered \star -product of fields as straightforward generalization of the usual T-product:

$$
T(\varphi_1(x_1) \star \ldots \star \varphi_n(x_n)) = \varphi_{\sigma_1}(x_{\sigma_1}) \star \ldots \star \varphi_{\sigma_n}(x_{\sigma_n}),
$$

\n
$$
x_{\sigma_1}^0 > x_{\sigma_2}^0 > \ldots > x_{\sigma_n}^0.
$$
\n(17)

Below we extend the classical proof of the LSZ formula [\[1,](#page-8-0) [6,](#page-9-4) [7\]](#page-9-5) to the case of space-space NCQFT.

Let us single out the variable x_1 and consider the expression

$$
\lim_{p_1^0 \to \omega(\vec{p}_1)} (p_1^2 - m^2) \int dx_1^0 d\vec{x}_1 e^{-ip_1 x_1} \langle 0 | T(\varphi(x_1) \star \ldots \star \varphi(x_n)) | 0 \rangle. \tag{18}
$$

Dividing integration over dx_1^0 into three parts

$$
\int (\ldots) dx_1^0 = \int_{-\infty}^{-\tau} (\ldots) dx_1^0 + \int_{-\tau}^{\tau} (\ldots) dx_1^0 + \int_{\tau}^{+\infty} (\ldots) dx_1^0, \qquad (19)
$$

we denote the summands as $I_1(\tau)$, $I_2(\tau)$, and $I_3(\tau)$ respectively.

Using expression $(p_1^2 - m^2)e^{-ip_1x_1} = (\Box_1 - m^2)e^{-ip_1x_1}$ and performing integration by parts, we get:

$$
I_1(\tau) = \int d\vec{x}_1 e^{i\vec{p}_1 \vec{x}_1 + i\omega(\vec{p}_1)\tau} \langle 0|T(\varphi(x_2) \star \ldots \star \varphi(x_n)) \star
$$

$$
\star (i\omega(\vec{p}_1) - \frac{\partial}{\partial \tau})\varphi(-\tau, \vec{x}_1)|0\rangle - \int_{-\infty}^{-\tau} dx_1^0 \int d\vec{x}_1 e^{i\vec{p}_1 \vec{x}_1 - i\omega(\vec{p}_1)x_1^0} \times (20)
$$

$$
\times \langle 0|T(\varphi(x_2) \star \ldots \star \varphi(x_n)) \star (\square_1 - m^2)\varphi(x_1)|0\rangle.
$$

Here $\Box_1 \equiv \frac{\partial^2}{\partial (x_1^1)}$ $\frac{\partial^2}{\partial (x_1^1)^2} + \frac{\partial^2}{\partial (x_1^2)}$ $\frac{\partial^2}{\partial (x_1^2)^2} + \frac{\partial^2}{\partial (x_1^3)}$ $\frac{\partial^2}{\partial (x_1^3)^2} - \frac{\partial^2}{\partial (x_1^0)}$ $\frac{\partial^2}{\partial (x_1^0)^2}$, and τ is taken sufficiently large so that the permutation of $\varphi(x_1)$ to the last position on the right is possible.

Next, we use the Fourier-expression for $\varphi(-\tau, \vec{x}_1)$:

$$
\varphi(-\tau, \vec{x}_1) = \frac{1}{(2\pi)^{(d+l)/2}} \int dk_0 d\vec{k} \, e^{-ik_0\tau} \, e^{-i\vec{k}\vec{x}_1} \tilde{\varphi}(k). \tag{21}
$$

All derivatives in the ^{*}-product will act on the factor $e^{-i\vec{k}\vec{x}_1}$ in the Fourier expansion of $\varphi(x_1)$. Therefore, additional factor $N(k_{nc})$ will appear. Note that $N(k_{nc})$ depends only on the noncommutative part of \vec{k} .

Let us also take into account the asymptotic representation for the field φ :

$$
\lim_{t \to -\infty} \int dk_0 e^{it(k_0 - \omega(\vec{k}))} \tilde{\varphi}(k) = \frac{1}{\sqrt{2\omega(\vec{k})}} a_{in}^+(\vec{k}). \tag{22}
$$

Taking the limit $\tau \to \infty$, we obtain:

$$
I_{1} = \lim_{\tau \to \infty} I_{1}(\tau) = i(2\pi)^{(d+l)/2} \int dk_{0} (k_{0} + \omega(\vec{p}_{1})) \delta(k_{0} - \omega(\vec{p}_{1})) \times
$$

$$
\times \langle 0|T(\varphi(x_{2}) \star \ldots \star \varphi(x_{n})) N(p_{1,nc}) \frac{a_{in}^{+}(\vec{p}_{1})}{\sqrt{2\omega(\vec{p}_{1})}}|0\rangle =
$$

\n= $i(2\pi)^{(d+l)/2} \sqrt{2\omega(\vec{p}_{1})} N(p_{1,nc}) \langle 0|T(\varphi(x_{2}) \star \ldots \star \varphi(x_{n})) a_{in}^{+}(\vec{p}_{1})|0\rangle.$ (23)

In this limit the second term of the expression (20) is equal to null.

Similar calculations for $I_3(\tau)$ will give:

$$
I_3 = i(2\pi)^{(d+l)/2} \sqrt{2\omega(\vec{p}_1)} N(p_{1,nc}) \langle 0 | a_{out}^+(\vec{p}_1) T(\varphi(x_2) \star \ldots \star \varphi(x_n)) | 0 \rangle = 0.
$$
\n(24)

As to the second summand in [\(19\)](#page-4-2), $I_2(\tau)$ can be presented as

$$
\int_{-\infty}^{\infty} dx_1^0 e^{-ip_1^0 x_1^0} \chi(x_1^0, \tau) F(x_1),
$$

$$
\chi(x_1^0, \tau) = \begin{cases} 1, & |x_1^0| \leq \tau; \\ 0, & |x_1^0| > \tau. \end{cases}
$$
 (25)

The integrand contains a generalized function with compact support, so its Fourier-transform is a smooth function and doesn't have a pole. For this reason

$$
\lim_{p_1^0 \to \omega(\vec{p}_1)} (p_1^2 - m^2) \int_{-\tau}^{\tau} dx_1^0 \int d\vec{x}_1 e^{-ip_1 x_1} \langle 0 | T(\varphi(x_1) \star \ldots \star \varphi(x_n)) | 0 \rangle = 0.
$$
\n(26)

Following similar limiting procedure over x_1 and x_2 consecutively, we obtain

$$
\lim_{p_2^0 \to \omega(\vec{p}_2) p_1^0 \to \omega(\vec{p}_1)} (p_2^2 - m^2)(p_1^2 - m^2) \int dx_1 \int dx_2 e^{-ip_2 x_2 - ip_1 x_1} \times
$$

$$
\times \langle 0|T(\varphi(x_1) \star \ldots \star \varphi(x_n))|0\rangle = \left[i(2\pi)^{(d+l)/2}\right]^2 \sqrt{2\omega(\vec{p}_1)} \sqrt{2\omega(\vec{p}_2)} \times (27)
$$

$$
\times N(p_{2,nc})N(p_{1,nc}) \langle 0|T(\varphi(x_3) \star \ldots \star \varphi(x_n))a_{in}^+(\vec{p}_2)a_{in}^+(\vec{p}_1)|0\rangle.
$$

Now let us replace the secong procedure (over x_2) with the one corresponding to transition to the bottom sheet of the mass hyperboloid, that is $\lim_{p_2^0 \to -\omega(\vec{p}_2)}$ $(p_2^2 - m^2)$. Making use of the asymptotic representation

$$
\lim_{t \to \pm \infty} \int dk_0 e^{it(k_0 + \omega(\vec{k}))} \tilde{\varphi}(k) = \frac{1}{\sqrt{2\omega(\vec{k})}} a_{in(out)}^-(-\vec{k}), \tag{28}
$$

we obtain:

$$
\lim_{p_2^0 \to -\omega(\vec{p}_2)} \lim_{p_1^0 \to \omega(\vec{p}_1)} (p_2^2 - m^2)(p_1^2 - m^2) \int dx_1 \int dx_2 e^{-ip_2x_2 - ip_1x_1} \times
$$

\n
$$
\times \langle 0|T(\varphi(x_1) \star \ldots \star \varphi(x_n))|0\rangle = (i(2\pi)^{(d+l)/2})^2 \sqrt{2\omega(\vec{p}_1)} \sqrt{2\omega(\vec{p}_2)} \times
$$

\n
$$
\times [N(p_{2,nc})N(p_{1,nc}) \langle 0|a_{out}(-\vec{p}_2)T(\varphi(x_3) \star \ldots \star \varphi(x_n))a_{in}^+(\vec{p}_1)|0\rangle -
$$

\n
$$
-\tilde{N}(p_{2,nc})\tilde{N}(p_{1,nc}) \langle 0|T(\varphi(x_3) \star \ldots \star \varphi(x_n))a_{in}^-(-\vec{p}_2)a_{in}^+(\vec{p}_1)|0\rangle].
$$
\n(29)

The first term here is the contribution of $I_3(\tau)$, which is not equal to zero in this limit. Let us make the substitution $\vec{p}_2 \rightarrow -\vec{p}_2$. We consider the scattering processes in which \vec{p}_1 — incoming momentum, \vec{p}_2 — outcoming momentum, and $\vec{p}_1 \neq \vec{p}_2$. In accordance with [\(13\)](#page-3-2) we can commute $a_{in}^-(\vec{p}_2)$ and $a_{in}^{+}(\vec{p}_1)$ in the second term of [\(29\)](#page-6-0) so that $a_{in}^{-}(\vec{p}_2)$ can act on the vacuum state and give null.

We can repeat the above-mentioned procedure n times — until nothing is left under the time-ordered \star -product. Now the additional factor $N(p_{1,nc}) \times$

 $\ldots \times N(p_{n,nc})$ can be expressed in explicit form: each derivative ∂_{μ} in [\(4\)](#page-1-2) should be replaced with ip_{μ} , and we have:

$$
N(p_{1,nc}) \times \ldots \times N(p_{n,nc}) = \exp\left[-\frac{i}{2} \theta_{\mu\nu} \sum_{a\n
$$
a, b = 1, \ldots, n,
$$
\n(30)
$$

where $\left|\rule{0pt}{13pt}\right._{-p_{out}}$ means that outcoming momenta should be taken with the minus sign (as the result of the substitution $\vec{p}\rightarrow -\vec{p}$ we made earlier).

The final expression for the scattering amplitude:

$$
\langle 0|a_{out}^{-}(\vec{p}_1) \dots a_{out}^{-}(\vec{p}_k) a_{in}^{+}(\vec{p}_{k+1}) \dots a_{in}^{+}(\vec{p}_n)|0\rangle = \left[\frac{1}{i(2\pi)^{(d+l)/2}}\right]^n \times
$$

$$
\times \exp\left[\frac{i}{2} \theta_{\mu\nu} \sum_{a\n(31)
$$

where $G_{\star}(p_1, \ldots, p_n)$ — Fourier transform of the noncommutative Green function:

$$
G_{\star}(p_1, \ldots, p_n) = \int dx_1 \ldots dx_n \exp\left[-i \sum_{j=1}^n p_j x_j\right] G_{\star}(x_1, \ldots, x_n). \quad (32)
$$

Relation [\(31\)](#page-7-0) is a noncommutative analogue of the LSZ reduction formula. This result corresponds to the one obtained in [\[8\]](#page-9-6), authors of which didn't use the \star -product between the fields taken at different points and considered the Green functions with the usual time-ordered product of noncommutative fields. The difference between the two results is the additional phase-factor [\(30\)](#page-7-1) due to the chosen form of the Green function [\(16\)](#page-4-3).

4 Consequences

Now we can extend to NCQFT the considerations that were originally proposed in [\[9\]](#page-9-7) for the case of commutative theory.

Suppose that we have two noncommutative $SO(1, d)$ -invariant theories on Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 respectively, related by a unitary transformation. Let φ_1 and φ_2 be two irreducible sets of field operators defined in \mathcal{H}_1 and \mathcal{H}_2 . Let $\langle p'_1, \ldots, p'_n | p_1, \ldots, p_m \rangle_i$, $i = 1, 2$ be inelastic scattering amplitudes of the process $m \to n$ for the fields φ_1 and φ_2 respectively. In accordance with the reduction formula [\(31\)](#page-7-0)

$$
\langle p'_1, \dots, p'_n | p_1, \dots, p_m \rangle_i \sim
$$

$$
\sim \int dx_1 \dots dx_{n+m} \exp\{i(-p_1 x_1 - \dots - p_m x_m + p'_1 x_{m+1} + \dots + p'_n x_{n+m})\} \times
$$

$$
\times \prod_{j=1}^{n+m} (\square_j - m^2) \langle 0 | T(\varphi_i(x_1) \star \dots \star \varphi_i(x_{n+m})) | 0 \rangle,
$$

$$
i = 1, 2.
$$
 (33)

Let us also take into account the results obtained for the generalized Haag's theorem in the context of noncommutative theory [\[10,](#page-9-8) [11\]](#page-9-9). Namely, it was shown that in two $SO(1, d)$ -invariant theories, related by a unitary transformation, the two-, three, \dots , $d+1$ -point Wightman functions coincide:

$$
\langle 0|\varphi_1(x_1)\star\ldots\star\varphi_1(x_s)|0\rangle = \langle 0|\varphi_2(x_1)\star\ldots\star\varphi_2(x_s)|0\rangle,
$$

2\leq s\leq d+1. (34)

From [\(33\)](#page-8-1) and [\(34\)](#page-8-2) it follows that the amplitudes $\langle p'_1, \ldots, p'_n | p_1, \ldots, p_m \rangle_1$ and $\langle p'_1, \ldots, p'_n | p_1, \ldots, p_m \rangle_2$ coincide in the two theories if

$$
m + n \leq d + 1. \tag{35}
$$

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