KAM-TORI NEAR AN ANALYTIC ELLIPTIC FIXED POINT

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Dedicated to our friend Alain Chenciner

ABSTRACT. We study the accumulation of an elliptic fixed point of a real analytic Hamiltonian by quasi-periodic invariant tori.

We show that a fixed point with Diophantine frequency vector ω_0 is always accumulated by invariant complex analytic KAM-tori. Indeed, the following alternative holds: If the Birkhoff normal form of the Hamiltonian at the invariant point satisfies a Rüssmann transversality condition, the fixed point is accumulated by real analytic KAM-tori which cover positive Lebesgue measure in the phase space (in this part it suffices to assume that ω_0 has rationally independent coordinates). If the Birkhoff normal form is degenerate, there exists an analytic subvariety of complex dimension at least $d + 1$ passing through 0 that is foliated by complex analytic KAM-tori with frequency ω_0.

This is an extension of previous results obtained in [EFK] to the case of an elliptic fixed point.

CONTENTS

1. Introduction

Let $\omega_0 \in \mathbb{R}^d$ and let

$$(*) \quad \begin{cases} H(x, y) = \langle \omega_0, r \rangle + \mathcal{O}^3(x, y) \\ r = (r_1, \ldots, r_d), \quad r_j = \frac{1}{2}(x_j^2 + y_j^2) \end{cases}$$

be a real analytic function defined in a neighborhood of $(0, 0)$. The Hamiltonian system associated to H is given by the vector field $X_H = (\partial_y H, -\partial_x H)$, namely

$$\begin{cases} \dot{x} = \partial_y H(x, y) \\ \dot{y} = -\partial_x H(x, y). \end{cases}$$

The flow of X_H has a fixed point $\mathcal{P}_0 = \{(0, 0)\}$. We are interested in the study of whether this Hamiltonian system admits, besides $\mathcal{P}_0$, other invariant sets. More precisely, we shall try to find real analytic *KAM-tori for X_H* in a neighborhood of $\mathcal{P}_0$, that is, real analytic Lagrangian tori invariant under X_H on each one of which the flow of X_H is conjugated to a translation flow $\varphi \mapsto \varphi + t\omega$; it is usually required (and we shall follow this requirement) that $\omega \in \mathbb{T}^d$, the *frequency vector*, is in some Diophantine set $DC(\kappa, \tau)$ $(\kappa, \tau > 0)$ defined by the property

$$(1.1) \qquad |\langle k, \omega \rangle| \geq \frac{\kappa}{|k|^\tau} \quad \forall k \in \mathbb{Z}^d \smallsetminus \{0\}.$$

We will say that $\omega_0 \in \mathbb{R}^d$ is irrational when its coordinates are rationally independent.

We call the complexification of a real analytic KAM-torus a *complex analytic KAM-torus for X_H*, that is, a complex analytic Lagrangian toric manifold invariant under (the complexification of) X_H on which the flow is conjugated to a translation flow $\varphi \mapsto \varphi + t\omega$. Note that there are complex analytic KAM-tori that are not the complexification of any real KAM-torus. Invariant complex analytic toric manifolds were studied in different settings related to KAM theory (see for example [Sto]).

Notice that the problem of finding real or complex analytic KAM-tori in a neighborhood of the invariant fixed point $\mathcal{P}_0$ can be given various more or less strong forms. For example, one can ask for finding a set of KAM tori whose Lebesgue density in the phase space tend to one in the neighborhood of $\mathcal{P}_0$. We shall call this KAM stability.

In classical KAM theory, an elliptic fixed point $\mathcal{P}_0$ is shown to be KAM-stable under the hypothesis that ω_0 is irrational (or just sufficiently non resonant) and that H satisfies a Kolmogorov non degeneracy condition of its Hessian matrix at $\mathcal{P}_0$. Further development of

the theory allowed to relax the non degeneracy condition. In this paper we prove KAM stability of $\mathcal{P}_0$ under the Rüssmann transversality condition on the Birkhoff normal form of H at $\mathcal{P}_0$.

We note that for non singular perturbative theory of analytic Hamiltonians it is known that the Rüssmann condition is necessary and sufficient for KAM stability – survival after perturbation of a positive measure set of KAM-tori – of analytic integrable Hamiltonian systems (see [R] and [Sev]). We stress however that the study of the dynamics in a neighborhood of an elliptic fixed point, or near a given invariant torus, is a singular perturbation problem and that, therefore, the latter results do not apply *per se*.

The problem is more tricky if no nondegeneracy conditions are imposed on the Hamiltonian. In the analytic setting, no examples are known of an elliptic fixed point $\mathcal{P}_0$ with ω_0 irrational that is not KAM stable. It was conjectured by M. Herman in his ICM98-lecture [H] that for analytic Hamiltonians, KAM stability holds in the neighborhood of a KAM torus $\mathcal{T}_0$ or of an elliptic fixed point $\mathcal{P}_0$ if their frequency is assumed to be Diophantine. The conjecture is known to be true in two degrees of freedom $d = 2$ [R], but remains open in general.

In this paper, we show that a fixed point with Diophantine frequency vector ω_0 of an analytic Hamiltonian is always accumulated by complex analytic KAM-tori. We previously obtained a similar result in the neighborhood of an invariant analytic torus with Diophantine frequency vector [EFK]. In the latter setting, the tori obtained were real analytic, but in the context of elliptic fixed points our method does not necessarily yield real analytic tori.

The current paper follows the same strategy as in [EFK]and provides the necessary modifications required by the absence of nice action-angle coordinates in the neighborhood of the fixed point.

An advantage however of the elliptic fixed point case, compared to that of an invariant torus, is that the Birkhoff normal form can be defined and conjugations up to any order can be performed under the sole condition that ω_0 is irrational. This is why we obtain the KAM stability of any irrational fixed point under the Rüssmann transversality condition, a result that we could not obtain for an invariant torus with irrational frequency, except in 2 degrees of freedom (see [EFK], Sections 2 and 9).

1.1. **Statement of the result.** Our main theorem is the following.

Theorem 1.1. *Let* $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ *be a real analytic function of the form* (∗) *and assume that* ω_0 *is Diophantine. Then, the origin is accumulated by infinitely many complex analytic KAM-tori for* X_H.

Let N_H be the *Birkhoff Normal Form* of H – for the Birkhoff Normal Form at a Diophantine, and more generally an irrational elliptic equilibrium, one can consult for example [SM].

We say that N_H is *j-degenerate* if there exist j orthonormal vectors $\gamma_1, \dots, \gamma_j$ such that for every $r \sim 0 \in \mathbb{R}^d$

$$\langle \partial_r N_H(r), \gamma_i \rangle = 0 \quad \forall\, 1 \leq i \leq j,$$

but no $j + 1$ orthonormal vectors with this property. Since $\omega_0 \neq 0$ clearly $j \leq d-1$. A 0-degenerate N_H is also said to be *non-degenerate*.

Our Main Theorem is the consequence of Theorems 1.2 and 1.4 below.

Theorem 1.2. *Let* $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ *be a real analytic function of the form* (∗) *and assume that* ω_0 *is Diophantine. If* N_H *is* j-*degenerate, then there exists an analytic subvariety containing* 0 *of complex dimension* $d+j$ *foliated by invariant complex analytic KAM-tori for* X_H *with translation vector* ω_0.

A stronger result is known when N_H is $(d - 1)$-degenerate. Indeed Rüssmann [R] (in a different setting) proved

Theorem 1.3. *If* ω_0 *is Diophantine and* N_H *is* $(d-1)$-*degenerate, then a full neighborhood of* $0 \in \mathbb{R}^{2d}$ *is foliated by real analytic KAM-tori for* X_H *with translation vector in* $\mathbb{R}\omega_0$.

Our proof of Theorem 1.2 (see Section 5.3) will also yield Rüssmann's result.

In the nondegenrate case we will prove the following.

Theorem 1.4. *Let* $H : (\mathbb{R}^{2d}, 0) \to \mathbb{R}$ *be a real analytic function of the form* (∗) *and assume that* ω_0 *is irrational. If* N_H *is non-degenerate, then in any neighborhood of* $0 \in \mathbb{R}^{2d}$ *the set of real analytic KAM-tori for* X_H *is of positive Lebesgue measure and density one at* 0.

1.2. **Strategy of the proof.** We adopt a similar strategy to the one of [EFK] that was inspired by previous works of Herman and Moser. The basis is a counter term KAM-theorem in which a Hamiltonian as in (∗) is conjugated, for any action value $c \sim 0 \in \mathbb{R}^d$ and any ω in some fixed Diophantine class, to a Hamiltonian that has an invariant torus at the action equal to c with frequency ω up to a correction term $\Lambda(c, \omega)$. Furthermore, for every value $c \in \mathbb{R}^d$ in the neighborhood of 0

of the action variable there exists a unique frequency $\Omega(c)$ that cancels the counter term : $\Lambda(c, \Omega(c)) = 0$. We call the map $c \mapsto \Omega(c)$ the *frequency map*. If $\Omega(c)$ is Diophantine this yields an invariant KAM-torus with frequency $\Omega(c)$. One can show that the jets of the function $\Omega(c)$ are given by those of the gradient of the Birkhoff normal form when the latter is well defined (which is the case if ω_0 is irrational since we are dealing with fixed points). The following alternative then holds : either the BNF is non degenerate and the function Ω takes Diophantine values on a set of Lebesgue density 1 at $\mathcal{P}_0$, which yields KAM stability; or the BNF is degenerate and we can use the analytic dependance of the counter term on the action variable to show the existence of a direction (after a coordinate change in the action variable) that spans a complex analytic subvariety foliated by complex analytic KAM-tori with translation vector ω_0.

Let us briefly explain why we do not necessarily obtain real analytic tori by our method in this context of elliptic fixed point. In the normal form expression we look for a change of variable Z defined in a neighborhood of the origin containing the torus $\mathcal{T} := \{x_j^2 + y_j^2 = c_j^2, j = 1, \ldots d\}$ such that

$$(1.2) \quad H \circ Z(x, y) = \Gamma + \sum_{i=1}^{d} \frac{\omega_j}{2}(x_j^2 + y_j^2 - c_j^2) + \sum_{j=1}^{d}(x_j^2 + y_j^2 - c_j^2)F_j(x, y)$$

where $F = (F_1, \ldots, F_d)$ is null on $x^2 + y^2 - c^2 = 0$ (Γ is a constant which is unimportant). The torus $\mathcal{T}$ is then invariant by the flow of $H \circ Z$ (hence $Z^{-1}\mathcal{T}$ is invariant by H).

Since H is real analytic it has a holomorphic extension to a polydisk $\mathbb{D}_\rho^{2d} \subset \mathbb{C}^{2d}$. Notice that the extension of H to $\mathbb{D}_\rho^{2d}$ thus satisfies $\overline{H(x, y)} = H(\bar{x}, \bar{y})$. It will be convenient to make the following change of variables: for $x, y \in \mathbb{C}^d$, define $z_j = \frac{1}{2}(x_j + \sqrt{-1}y_j)$, $w_j = \frac{1}{2}(x_j - \sqrt{-1}y_j)$ so that $r_j := \frac{1}{2}(x_j^2 + y_j^2) = z_j w_j$, $j = 1, \ldots, d$. One has $dz \wedge dw = -\sqrt{-1}dx \wedge dy$. Notice that a function $(x, y) \mapsto f(x, y)$ is real analytic (hence satisfies $\overline{f(x, y)} = f(\bar{x}, \bar{y})$) if and only if $\tilde{f}(z, w) := f(x, y)$ satisfies the symmetry $\overline{\tilde{f}(z, w)} = \tilde{f}(\bar{w}, \bar{z})$. We then say that $\tilde{f}$ is σ-*symmetric* (where σ is the involution $\sigma(z, w) = (\bar{w}, \bar{z})$). If f depends real holomorphically on an extra complex parameter c and smoothly on an extra real parameter ω, then $\tilde{f} : \mathbb{D}_\rho^d \times \mathbb{D}_\rho^d \times \mathbb{D}_\delta^d \times B(0, 1) \to \mathbb{C}$ satisfies $\overline{\tilde{f}(z, w, c, \omega)} = \tilde{f}(\bar{w}, \bar{z}, \bar{c}, \omega)$, i.e. it is symmetric with respect to the involution $\sigma(z, w, c) = (\bar{w}, \bar{z}, \bar{c})$. By a slight abuse of notation we shall call also this property σ-*symmetry*. One can define a similar notion of σ-symmetry for diffeomorphisms (see Section 1.3).

Equation (1.2) is then equivalent to finding $\omega \in \mathbb{R}^{2d}$, $c \in \mathbb{R}^d$, an exact symplectic change of coordinates $\tilde{Z}$ for $dz \wedge dw$ and maps $\tilde{F}_j$ null on $zw - c^2 = 0$ such that $\tilde{Z}$ and the $\tilde{F}_j$ are σ-symmetric and

$$\tilde{H} \circ \tilde{Z}(z, w) = \Gamma + \sum_{i=1}^{d} \omega_j(z_j w_j - c_j^2) + \sum_{j=1}^{d} (z_j w_j - c^2) \tilde{F}_j(z, w).$$

The searched for torus $\mathcal{T}$ then corresponds in the (z, w)-coordinates to $\{z_j w_j = c_j^2, j = 1, \ldots, d\} \cap \{(z, w) \in \mathbb{C}^{2d} : \sigma(z, w) = (z, w)\}$.

The strategy of the proof is then to find for some values of $c \in \mathbb{R}^d$ and $\omega \in \mathbb{R}^d$ such a normal form.

However, in the j-degenerate case ($j \neq 0$), it will only be possible to do so for some $c^2 := (c_1^2, \ldots, c_d^2)$ (but not necessarily c itself) in $\mathbb{R}^d$ and consequently it will not be possible to ensure that the searched for tori $\mathcal{T}$ are *real*. We obtain instead complex analytic KAM-tori for X_H.

1.3. **Notations.** We denote by $\mathbb{D}_\delta^d$ the polydisk in $\mathbb{C}^d$ with radius δ. More generally if $d = (d_1, \ldots, d_n)$ and $\delta = (\delta_1, \ldots, \delta_n)$, then

$$\mathbb{D}_\delta^d = \mathbb{D}_{\delta_1}^{d_1} \times \cdots \times \mathbb{D}_{\delta_n}^{d_n}.$$

Let $f : \mathbb{D}_\delta^e \to \mathbb{C}$ be a holomorphic function. We denote by $\partial_{z_i} f$ the partial derivate of f with respect to z_i and we use the usual multi-index notation like $\partial_z^\alpha f$. If $z = (z', z'')$ we say that

$$f \in \mathcal{O}^j(z')$$

if and only if $\partial_{z'}^{\alpha'} f(\varphi, 0, z'') = 0$ for all $|\alpha'| < j$. We shall also use the same notations for $\mathbb{C}^n$-valued functions $f = (f_1, \ldots, f_n)$ with the absolute value replaced by $|f| = \max_i |f_i|$ (or some other norm on $\mathbb{C}^n$).

σ-*symmetry.* Let σ be the involution $(z, w, c) \mapsto (\bar{w}, \bar{z}, \bar{c})$ on $\mathbb{C}^d \times \mathbb{C}^d \times \mathbb{C}^d$. A holomorphic function $f : \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \to \mathbb{C}$ is σ-*symmetric* if, and only if, $f \circ \sigma(z, w, c) = \overline{f(z, w, c)}$. This means precisely that it takes real values on the subspace $\{(z, w, c) = \sigma(z, w, c)\}$. A (local) mapping F preserves this subspace if and only if

$$\sigma^{-1} \circ F \circ \sigma = F$$

– we say then that F is σ-symmetric.

We let $\mathcal{C}^{\omega,\sigma}(\mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d)$ denote the space of σ-symmetric holomorphic functions $f : \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \to \mathbb{C}$ provided with the norm

$$|f|_\delta = \sup_{z \in \mathbb{D}_\delta^e} |f(z)|.$$

Formal power series. Let $z = (z_1, \ldots, z_n)$. An element

$$f \in \mathbb{C}[[z]]$$

is a formal power series

$$f = f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$$

whose coefficients $a_\alpha \in \mathbb{C}$ (possibly vector valued). The notion of σ-symmetry carries over to this more general framework. We denote by

$$[f]_j(z) = \sum_{|\alpha|=j} a_\alpha z^\alpha,$$

the homogenous component of degre j, and

$$[f]^j = \sum_{i \leq j} [f]_i.$$

Parameters. Let B be an open subset of some euclidean space. Define

$$\mathcal{C}^{\omega,\infty}(\mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d, B)$$

(or for short $\mathcal{C}_\delta^{\omega,\infty}$) to be the set of $\mathcal{C}^\infty$ functions (possibly vector valued)

$$f : \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times B \ni (z, \omega) \mapsto f(z, \omega)$$

such that for all $\omega \in B$ [1]

$$f_\omega : \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \ni (\omega) \mapsto f(z, \omega)$$

is a holomorphic function. If in addition, this map is σ-symmetric, we shall write $f \in \mathcal{C}^{\omega,\sigma,\infty}(\mathbb{D}_\delta^d \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d, B)$. We define

$$||f||_{\delta,s} = \sup_{|\alpha| \leq s} |\partial_\omega^\alpha f_\omega|_\delta.$$

(κ, τ)-*flat functions.* A $\mathcal{C}^\infty$ function $f : \mathbb{D}^d \times B \to \mathbb{C}$, $(z, \omega) \mapsto f(z, \omega)$ is (κ, τ)-*flat* if, for any set of indices α, β,

$$\partial_z^\alpha \partial_\omega^\beta f(z, \omega) = 0$$

whenever $\omega \in DC(\kappa, \tau)$.

Tensorial notations. When $(v_1, \ldots, v_{m)} \mapsto B(v_1, \cdots, v_m)$ is a m-multilinear form on a vector space V, we shall often see it as a linear form on the m-th tensorial product $V^{\otimes m}$, and use the corresponding tensorial notations. Also, we denote by $\otimes_{sym}$ the symmetrized tensor product

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[1] we apologize for the double use of ω

$v_1 \otimes \cdots \otimes v_m = \sum_\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}$ where the sum is on all the permutations of $\{1, \ldots, d\}$.

2. Power series expansion

2.1. Expansion with Non Resonant functions. Let

$$f(z, w) = \sum_{\alpha, \beta \in \mathbb{N}} f_{\alpha, \beta} z^\alpha w^\beta$$

be some holomorphic function defined on a polydisk of $(\mathbb{C}^2, 0)$ – or more generally a formal power series. We have

$$f(z, w) = \sum_{n=0}^{\infty} (zw)^n \sum_{\alpha\beta=0} f_{\alpha, \beta} z^\alpha w^\beta$$

and since in the last sum in the previous expression $\alpha = 0$ or $\beta = 0$ we can find analytic g_n and h_n, $n \in \mathbb{Z}$ such that

$$f(z, w) = \sum_{n=0}^{\infty} (zw)^n (g_n(z) + h_n(w)).$$

A similar procedure or a simple induction argument show that if f is now analytic in some polydisk $\mathbb{D}_\rho^{2d}$ of $(\mathbb{C}^{2d}, 0)$ then

$$f(z, w) = \sum_{\underline{n} \in \mathbb{N}^d} (z_1 w_1)^{n_1} \cdots (z_d w_d)^{n_d} \sum_{(\alpha, \beta) \in \mathcal{NR}} f_{\underline{n}, \alpha, \beta} z^\alpha w^\beta$$

where $\mathcal{NR}$ is the set of $(\alpha, \beta) \in (\mathbb{N}^d)^2$ such that for all $i = 1, \ldots, d$, $\alpha_i \beta_i = 0$. A power series of the form $h(z, w) = \sum_{(\alpha, \beta) \in \mathcal{NR}} h_{\alpha, \beta} z^\alpha w^\beta$ will be called *non-resonant* and we denote by $\widehat{\mathcal{NR}}$ the vector space of all non-resonant functions. Notice that we allow for the existence of constant terms in this definition. We can also say that

$$f(z, w) = \sum_{\underline{n} \in \mathbb{N}^d} (z_1 w_1)^{n_1} \cdots (z_d w_d)^{n_d} \sum_{\underline{\varepsilon} \in \{0,1\}^d} f_{\underline{n}, \underline{\varepsilon}} (r_1^{\varepsilon_1}, \ldots, r_d^{\varepsilon_d})$$

where $f_{\underline{n}, \underline{\varepsilon}}$ are holomorphic in $\mathbb{D}_\rho^{2d}$ and where we have used the notation $r_i^\varepsilon = z_i$ if $\varepsilon = 0$ and $r_i^\varepsilon = w_i$ if $\varepsilon = 1$.

The following fact will be useful:

Lemma 2.1. *For any $(\alpha, \beta) \in (\mathbb{N}^d)^2$ there is a unique decomposition of the form $(\alpha, \beta) = (\underline{n}, \underline{n}) + (\alpha', \beta')$ where $\underline{n} \in \mathbb{N}^d$ and $(\alpha', \beta') \in \mathcal{NR}$ (this means that for any $i = 1, \ldots, d$, $\alpha_i = \underline{n}_i + \alpha'_i$ and $\beta_i = \underline{n}_i + \beta'_i$).*

Proof.

To prove the existence of such a decomposition just take $\underline{n}_i = \min(\alpha_i, \beta_i)$, $i = 1, \ldots, d$. To prove uniqueness we observe that if for some i $\underline{n}_i \neq \underline{\tilde{n}}_i$, for example $\underline{n}_i > \underline{\tilde{n}}_i$, then $\alpha'_i > \tilde{\alpha}_i$ and $\beta'_i > \tilde{\beta}_i$ a contradiction with $(\alpha', \beta') \in \mathcal{NR}$.

$\square$

The preceding discussion provides the following decomposition

Lemma 2.2. *If $f(z,w)$ is holomorphic on some polydisk $\mathbb{D}_\delta^d \times \mathbb{D}_\delta^d$ there exists a unique decomposition*

$$(2.3) \qquad f(z,w) = \sum_{\underline{n} \in \mathbb{N}^d} (z_1 w_1)^{n_1} \cdots (z_d w_d)^{n_d} \tilde{f}_{\underline{n}}(z,w)$$

where $\tilde{f}_{\underline{n}} \in \widehat{\mathcal{NR}}$ are holomorphic on $\mathbb{D}_\delta^d \times \mathbb{D}_\delta^d$ – the series converges uniformly on compact sub domains of $\mathbb{D}_\delta^d \times \mathbb{D}_\delta^d$. Furthermore f is σ-symmetric if and only if all the $\tilde{f}_{\underline{n}}$ are.

Proof. To prove uniqueness, one just have to prove that if f is null, the same is true of all the series $f_{\underline{n}}$. This is done by looking at the coefficients of the right hand side of (2.3) and by using Lemma 2.1. The σ-symmetry of the $f_{\underline{n}}$ comes from the uniqueness. $\square$

If now f depends (or not) on a parameter $c = (c_1, \ldots, c_d)$, by writing $z_j w_j = c_j + (z_j w_j - c_j)$ in (2.3) we get an expansion uniformly converging on small compact neighborhoods of 0:

$$(2.4) \qquad f(z,w,c) = \sum_{\underline{n} \in \mathbb{N}^d} (z_1 w_1 - c_1)^{n_1} \cdots (z_d w_d - c_d)^{n_d} f_{\underline{n}}(z,w,c).$$

We again notice that each $f_{\underline{n}}(\cdot, \cdot, c)$ is non-resonant (for any fixed c) and

$$(2.5) \qquad f_{\underline{n}}(z,w,c) = \sum_{\underline{k} \geq \underline{n}} \binom{\underline{k}}{\underline{n}} c^{(\underline{k}-\underline{n})} \tilde{f}_{\underline{k}}(z,w,c).$$

We shall still denote by $\widehat{\mathcal{NR}}$ the set of functions $f(z,w,c)$ which are non-resonant for each fixed c.

Lemma 2.3. *If f is σ-symmetric, there exists a unique decomposition of the form (2.4) where each $f_{\underline{n}}$ is non-resonant and σ-symmetric.*

Proof. We have to prove that if in (2.4) f is equal to 0 then all the $f_{\underline{n}}$ are null. If $f_{\underline{n}}(z,w,c) = \sum_{\alpha',\beta',\gamma'} f_{\underline{n},\alpha',\beta',\gamma'} z^{\alpha'} w^{\beta'} c^{\gamma'}$, the coefficient of

$z^\alpha w^\beta$ in (2.4) is the sum

$$\sum_{\substack{\underline{k},\alpha',\beta',\gamma' \\ (\underline{k},\underline{k})+(\alpha',\beta')=(\alpha,\beta)}} \sum_{\underline{n}\geq\underline{k}} \binom{\underline{n}}{\underline{k}}(-1)^{|\underline{n}-\underline{k}|}c^{\gamma'+(\underline{n}-\underline{k})}f_{\underline{n},\alpha',\beta',\gamma'}$$

Since in the last sum $(\alpha',\beta') \in \mathcal{NR}$, the decomposition $(\alpha,\beta) = (\underline{k},\underline{k})+(\alpha',\beta')$ is unique by Lemma 2.1, and thus the last sum is just (the summation is in $\underline{n}$)

$$\sum_{\substack{\underline{n}\geq\underline{k} \\ \gamma'}} \binom{\underline{n}}{\underline{k}}(-1)^{|\underline{n}-\underline{k}|}c^{\gamma'+(\underline{n}-\underline{k})}f_{\underline{n},\alpha',\beta',\gamma'}.$$

By assumption, for any $\underline{k}$, any $(\alpha',\beta') \in \mathcal{NR}$, this has to be equal to zero for any c in a neighborhood of zero. Multiplying the last sum by $(d+c)^{\underline{k}}$ and making the summation on all $\underline{k} \geq 0$ one gets

$$\sum_{\substack{\underline{n} \\ \gamma'}} c^{\gamma'} d^{\underline{n}} f_{\underline{n},\alpha',\beta',\gamma'} = 0.$$

This being true for all c, d in a neighborhood of 0 one has $f_{\underline{n}} = 0$.

The σ-symmetry of the $f_{\underline{n}}$ comes from the uniqueness.

$\square$

Remark Lemmas 2.2 and 2.3 hold in the case of formal series in $\mathbb{C}[[z,w,c]]$.

If $p \in \mathbb{N}$, we shall denote

$$(2.6) \qquad f_p(z,w,c) = \sum_{|\underline{n}|=p}(zw-c)^{\underline{n}}f_{\underline{n}}(z,w,c)$$

We shall use the following notations. We have seen that $f(z,w,c)$ can be written under the form

$$(2.7) \quad f(z,w,c) = f^{(0)}(z,w,c) + \langle f^{(1)}(z,w,c), (zw-c)\rangle +$$
$$\langle(zw-c), f^{[2]}(z,w,c)(zw-c)\rangle$$

or

$$(2.8) \quad f(z,w,c) = f^{(0)}(z,w,c) + \langle f^{(1)}(z,w,c), (zw-c)\rangle +$$
$$\langle(zw-c), f^{(2)}(z,w,c)(zw-c)\rangle + f^{[3]}(z,w,c)(zw-c)^{\otimes 3}$$

where $f^{(0)}(z,w,c)$ and $f_j^{(1)}(z,w,c)$ $(j = 1,\ldots,d)$, $f_{i,j}^{(2)}(z,w,c)$ $(1 \leq i,j \leq d)$ are in $\widehat{\mathcal{NR}}$ and where the notations $f_{ij}^{(2)}(z,w,c)$ and $f_{ij}^{[2]}(z,w,c)$ $(1 \leq i,j \leq d)$ denotes respectively the sums $\sum_{\underline{n}=\lambda_i+\lambda_j}f_{\underline{n}}(z,w,c)$ and

$\sum_{\underline{n} \geq \lambda_i + \lambda_j} (zw - c)^{\underline{n} - \lambda_i - \lambda_j} f_{\underline{n}}(z, w, c)$ with $\lambda_k \in \mathbb{N}^d$ $(k = i, j)$ denoting the multiindex $\lambda_k(l) = \delta_{kl}$ (Kronecker's symbol); $f^{[(3)]}$ is defined similarly. We shall call the decompositions (2.7) and (2.8) the *canonical decomposition* of f (up to order 2 or 3).

Lemma 2.4. *If*

$$f(z, w, c) = a_0(z, w, c) + \langle a_1(z, w, c), (zw - c) \rangle + \mathcal{O}((zw - c)^2)$$

with a_0 and a_1 in $\widehat{\mathcal{NR}}$ then $a_0 = f^{(0)}$ and $a_1 = f^{(1)}$.

Proof. Let us denote $g(z, w, c)$ the $\mathcal{O}((zw - c)^2)$ of the statement of the lemma. The function g can be written

$$g(z, w, c) = \sum_{\alpha \in \mathbb{N}^d, |\alpha| = 2} h_\alpha(z, w, c)(zw - c)^\alpha$$

and each h_α can be decomposed $h_\alpha(z, w, c) = \sum_{\underline{n} \in \mathbb{N}^d} h_{\alpha, \underline{n}}(z, w, c)(zw - c)^{\underline{n}}$ where all the $h_{\alpha, \underline{n}}$ are in $\widehat{\mathcal{NR}}$ and so $g(z, w, c) = \sum_{\underline{m} \in \mathbb{N}^d, |\underline{m}| \geq 2} (zw - c)^{\underline{m}} g_{\underline{m}}(z, w, c)$ where each $g_{\underline{m}}(z, w, c) := \sum_{\alpha + \underline{n} = m, |\alpha| = 2, \underline{n} \in \mathbb{N}^d} h_{\alpha, n}(z, w, c)$ is non resonant. The uniqueness given by Lemma 2.3 concludes the proof. $\qquad\square$

2.2. **The operators $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{D}^\omega$.** We now define the operator $\mathcal{M}$ by

$$(\mathcal{M}f)(z, w) = \sum_{\alpha = \beta} f_{\alpha, \beta} z^\alpha w^\beta$$

(diagonal terms). If $\mathcal{M}f = f$ we say that f is *diagonal*.

Observe that if $f(z, w, c) = \sum_{\underline{n} \in \mathbb{N}^d} (zw - c)^{\underline{n}} f_{\underline{n}}(z, w, c)$ where all the $f_{\underline{n}}$ are in $\widehat{\mathcal{NR}}$ then

$$\mathcal{M}f = \sum_{\underline{n} \in \mathbb{N}^d} (zw - c)^{\underline{n}} f_{\underline{n}}(0, 0, c).$$

Let us introduce the following differential operators

$$\mathcal{D}_i f = (\partial_{z_i} f) z_i - (\partial_{w_i} f) w_i, \qquad \mathcal{D}f = (\mathcal{D}_1 f, \dots, \mathcal{D}_d f),$$

and if $\omega \in \mathbb{R}^d$

$$\mathcal{D}^\omega = \langle \omega, \mathcal{D} \rangle = \omega_1 \mathcal{D}_1 + \cdots + \omega_d \mathcal{D}_d.$$

Notice that

$$\mathcal{D}_i(z^\alpha w^\beta) = (\alpha_i - \beta_i)(z^\alpha w^\beta), \qquad \mathcal{D}^\omega(z^\alpha w^\beta) = \langle \omega, \alpha - \beta \rangle (z^\alpha w^\beta).$$

All these definitions extend to the case when $f(z, w, c)$ depends on c; the derivatives are taken w.r.t. (z, w) and c is then seen as a parameter.

Since $\mathcal{D}(zw - c)^{\underline{n}} = 0$, we observe that

(2.9)
$$\mathcal{D}f_p = \sum_{|\underline{n}|=p}(zw-c)^{\underline{n}}\mathcal{D}f_{\underline{n}}(z,w,c), \qquad \mathcal{D}^\omega f_p = \sum_{|\underline{n}|=p}(zw-c)^{\underline{n}}\mathcal{D}^\omega f_{\underline{n}}(z,w,c).$$

Let us mention the following, easy to prove, but important properties:

Lemma 2.5. *Let $f(z,w,c)$ be a formal series expansion.*

(1) *If f is diagonal, so are $z\partial_z f$, $w\partial_w f$, $\mathcal{D}f$ and $\mathcal{D}^\omega f$.*
(2) *If $[\cdot]_j$ denotes the homogeneous polynomial part of total degree j in (z,w,c), the operators $z\partial_z$, $w\partial_w$, $\mathcal{D}$, $\mathcal{D}^\omega$ commute with $[\cdot]_j$.*
(3) *If g is either $z\partial_z f$, $w\partial_w f$, $\mathcal{D}f$ or $\mathcal{D}^\omega f$ then $[g]_0 = 0$.*
(4) *If f is σ-symmetric then $\mathcal{M}f$, $\sqrt{-1}\mathcal{D}f$ and $\sqrt{-1}\mathcal{D}^\omega f$ are σ-symmetric.*
(5) *Assume that $\omega \in \mathbb{R}^d$ is irrationaland let $g \in \mathbb{C}[[z,w,c]]$ be a σ-symmetric power series. Then the equation*
$$\mathcal{D}^\omega f = \sqrt{-1}g$$
has a σ-symmetric solution $f \in \mathbb{C}[[z,w,c]]$ if and only if $\mathcal{M}g = 0$, and the solution f is unique modulo the addition of any diagonal series expansion in $\mathbb{C}[[z,w,c]]$.

3. Formal Normal Forms

3.1. **Exact symplectic mappings and generating functions.** Let $Z : (z,w) \mapsto (z',w')$ be a holomorphic mapping of $(\mathbb{C}^{2d}, 0)$ endowed with the canonical symplectic form $dz \wedge dw$. Since $d(zdw) = dz \wedge dw$, Z is *symplectic* if and only if the one-form $Z^*(zdw) - zdw$ is closed. By definition, Z is said to be *exact*, or *exact symplectic* if and only if the one-form $Z^*(zdw) - zdw$ is exact. (Since we are on a simply connected domain symplectic implies exact symplectic.) Under general conditions – Z is $\mathcal{C}^1$-close to the identity mapping for example – there exists a holomorphic function $f : (\mathbb{C}^{2d}, 0) \to \mathbb{C}$ such that

(3.10) $(z',w') = Z(z,w) \iff \begin{cases} z' = z + \partial_{w'}f(z,w') \\ w = w' + \partial_z f(z,w') \end{cases}$

The construction of f is the following: since Z is exact, there exists a holomorphic function $g : (\mathbb{C}^{2d}, 0) \to \mathbb{C}$ such that $Z^*(zdw) - zdw = dg$ and we define f by

(3.11) $f(z,w') = g(z,w) - z(w' - w), \quad w' = \phi(z,w'),$

where $w' = \phi(z, w')$ is determined (by the implicit function theorem) from $(z', w') = Z(z, w)$.

A function f like in (3.10) is called a *generating function* for Z – it is unique up to an additive constant. Conversely, any holomorphic function $f : (\mathbb{C}^{2d}, 0) \to \mathbb{C}$ is under general conditions – f is $\mathcal{C}^2$-close to zero for example – the generating function for a unique exact holomorphic mapping. (This is a straight forward verification which can be found in most books on symplectic dynamics/geometry, for example in [SM].)

If Z depends holomorphically on some parameters c, then its generating function depends holomorphically on c, and conversely. This correspondence also preserves reality – Z is real holomorphic if, and only if, it's generating function is real holomorphic – but σ-symmetry is not preserved.

These properties carry over to the setting of (formal) mappings of the form

$$(*) \quad Z(z, w, c^2) - (z, w) \in \mathbb{C}[[z, w, c^2]] \cap \mathcal{O}^2(z, w, c).$$

Lemma 3.1.

(a) *The set of mappings of the form $(*)$ is a group under composition, and the set of (formally) exact mappings of the form $(*)$ is a subgroup.*

(b) *A (formally) exact mapping of the form $(*)$ has a unique (formal) generating function of the form*

$$f(z, w', c^2) \in \mathbb{C}[[z, w', c^2]] \cap \mathcal{O}^3(z, w', c).$$

(c) *Any function*

$$f(z, w', c^2) \in \mathbb{C}[[z, w', c^2]] \cap \mathcal{O}^3(z, w', c)$$

is the generating function of a unique (formally) exact mapping of the form $(*)$.

Proof.   (a) is a direct computation on formal power series. (b) follows since a closed (formal) one-form is exact, which gives us a formal g: notice that "low order" terms (in z, w, c) of g do not depend on "high order" terms of Z. By truncating g at some order N we can apply the formula (3.11) which gives an f_N: notice that "low order" terms of f_N do not depend on "high order" terms of g. Therefore this defines a formal generating function by letting $N \to \infty$. (c) follows by truncating f (in z, w, c) at some order N and and then define Z_N by the formula (3.10): notice that "low order" terms of Z_N do not depend on "high order" terms of f. Therefore this defines a (formally) exact mapping Z by letting $N \to \infty$. $\square$

3.2. **Formal Normal Forms.** Assume that $H(z, w)$ is a formal Hamiltonian in $\mathbb{C}[[z, w]]$ of the form

$$H(z, w) = \langle \omega_0, zw \rangle + \mathcal{O}^3(z, w)$$

with a vector $\omega_0 \in \mathbb{R}^d$ which is rationally independent. It is a classical result that there exist a unique $N \in \mathbb{R}[[r]]$, the Birkhoff Normal Form of H, and a (formally) exact mapping of the form

$$(*) \quad Z(z, w) - (z, w) \in \mathbb{C}[[z, w]] \cap \mathcal{O}^2(z, w)$$

such that

$$H \circ Z(z, w) = N(zw) = \langle \omega_0, zw \rangle + \mathcal{O}^3(zw).$$

If $c \in \mathbb{C}^d$ is an extra formal parameter one can write

$$H \circ Z(z, w) = N(zw) = N(c^2) + \langle \nabla N(c^2), zw - c^2 \rangle + \mathcal{O}^2(zw - c^2).$$

The aim of the following proposition is to prove that such a representation is unique.

Proposition 3.2. *If there exist a formal series*

$$f(z, w', c^2) \in \mathbb{C}[[z, w', c^2]] \cap \mathcal{O}^3(z, w', c^2)$$

and formal series $\Gamma(c^2), \Omega(c^2) \in \mathbb{C}[[c]]$,

$$\Gamma(c^2) = \langle \omega_0, c^2 \rangle + \mathcal{O}^2(c^2), \quad \Omega(c^2) = \omega_0 + \mathcal{O}(c^2)$$

such that

$$H(z, w) = \Gamma(c^2) + \langle \Omega(c^2), z'w' - c^2 \rangle +$$
$$\langle z'w' - c^2 \rangle, F(z', w', c^2)(z'w' - c^2) \rangle,$$

where

$$z' = z + \partial_{w'} f(z, w', c^2), \quad w = w' + \partial_z f(z, w', c^2),$$

then $\Gamma(c^2) = N(c^2)$ *and* $\Omega(c^2) = \nabla N(c^2)$, *i.e. the series* Γ *and* Ω *are unique.*

Proposition 3.2 will be the consequence of the following two Lemmata.

Lemma 3.3. *Let* $H(z, w, c^2)$ *be a formal Hamiltonian depending on* c^2 *of the form*

$$H(z, w, c^2) = \Gamma(c^2) + \langle \Omega(c^2), zw - c^2 \rangle + \mathcal{O}^2(zw - c^2)$$

where $\Omega(0) = \omega_0$ *is rationally independent. Then there exists a formal series*

$$f(z, w', c^2) \in \mathbb{C}[[z, w', c^2]] \cap \mathcal{O}^2(zw' - c^2)$$

and formal series $G_k(c^2) \in \mathbb{C}[[c]]$, $k \geq 2$, such that

$$H(z, w, c^2) = \Gamma(c^2) + \langle \Omega(c^2), (z'w' - c^2) \rangle +$$

$$\sum_{n=2}^{\infty} G_k(c^2) \cdot (z'w' - c^2)^{\otimes n},$$

where

$$z' = z + \partial_{w'} f(z, w', c^2), \quad w = w' + \partial_z f(z, w', c^2).$$

Proof.

Let us write

$$H(z, w, c^2) = \Gamma(c^2) + \langle \Omega(c^2), zw - c^2 \rangle + F(z, w, c^2) \cdot (zw - c^2)^{\otimes 2}$$

and denote $\Delta = zw - c^2$, $\tilde{\Delta} = zw' - c^2$ and $\Delta' = z'w' - c^2$. We have

$$\Delta = zw' + z\partial_z f(z, w', c^2) - c^2$$

$$= \tilde{\Delta} + z\partial_z f(z, w', c^2)$$

and

$$\Delta' = zw' + w'\partial_{w'} f(z, w', c^2) - c^2$$

$$= \tilde{\Delta} + w'\partial_{w'} f(z, w', c^2)$$

We thus have to construct $f = \mathcal{O}^3(z, w', c^2)$ and the G_k such that

$$(3.12) \quad \langle \Omega(c^2), \mathcal{D}f(z, w', c^2)) \rangle =$$

$$- F(z, w' + \partial_z f(z, w', c^2), c^2) \cdot (\tilde{\Delta} + z\partial_z f(z, w', c^2))^{\otimes 2} +$$

$$\sum_{n=2}^{\infty} G_n(c^2) \cdot (\tilde{\Delta} + w'\partial_{w'} f(z, w', c^2))^{\otimes n}.$$

To do this we proceed by induction. If L is a formal function of the variables (z, w', c^2) denote by $[L]_j$ its homogeneous part of total degree j in (z, w', c). Taking the $[\cdot]_j$ part of equation (3.12) we get

$$(3.13) \quad \sum_{k+l=j} \langle [\Omega(c^2)]_k, [\mathcal{D}f(z, w', c^2))]_l \rangle =$$

$$- [F(z, w' + \partial_z f(z, w', c^2), c^2) \cdot (\tilde{\Delta} + z\partial_z f(z, w', c^2))^{\otimes 2}]_j +$$

$$\sum_{n=2}^{\infty} [G_n(c^2) \cdot (\tilde{\Delta} + w'\partial_{w'} f(z, w', c^2))^{\otimes n}]_j.$$

Since $[f]_0 = [f]_1 = [f]_2 = 0$, (3.13) can be written

$$(3.14) \quad \langle \Omega(0), [\mathcal{D}f(z, w', c^2))]_j \rangle = T_{1,j} + T_{2,j} + T_{3,j} +$$
$$\sum_{n \geq s2} [G_n(c^2)]_{j-2n} \cdot (\tilde{\Delta})^{\otimes n}$$

where

$$(3.15) \qquad T_{1,j} = - \sum_{\substack{k+l=j \\ 2 \leq l < j}} \langle [\Omega(c^2)]_k, [\mathcal{D}f(z, w', c^2))]_l \rangle$$

$$(3.16) \quad T_{2,j} = -[F(z, w' + \partial_z f(z, w', c^2), c^2) \cdot (\tilde{\Delta} + z\partial_z f(z, w', c^2))^{\otimes 2}]_j$$

(3.17)
$$T_{3,j} = \sum_{n=2}^{\infty} \sum_{k+l=j} [G_n(c^2)]_k \cdot [(\tilde{\Delta} + w'\partial_{w'} f(z, w', c^2))^{\otimes n}]_l - \sum_{n=2}^{\infty} [G_n(c^2)]_{j-2n} \cdot (\tilde{\Delta})^{\otimes n}$$

Notice that the term $T_{2,j} := [F(z, w' + \partial_z f(z, w', c^2), c^2) \cdot (\tilde{\Delta} + z\partial_z f(z, w', c^2))^{\otimes 2}]_j$ is a linear combination of terms of the form $[F(z, w' + \partial_z f(z, w', c^2), c^2)]_k \cdot \tilde{\Delta}^{\otimes m_1} \otimes_{sym} [z\partial_z f(z, w', c^2)]_{l_2}^{\otimes m_2}$ with $k + 2m_1 + m_2 l_2 = j$ and $m_1 + m_2 = 2$; hence $l_2 \leq j - 1$. Also, since $[z\partial_z f]_0 = 0$ one has $l_2 \geq 1$ and thus $k \leq j - 2$. The term $[F(z, w' + \partial_z f(z, w', c^2), c^2)]_k$ depends on f only through its coefficients of total degree $\leq k + 1$ and thus less or equal to $j - 1$. In conclusion the term $T_{2,j}$ depends on f only through its coefficients of total degree $k \leq j - 1$.

A similar analysis shows that the same is true for the term $T_{3,j}$ and of course, for the term $T_{1,j}$. In conclusion, all the terms $T_{1,j}, T_{2,j}, T_{3,j}$ in equality (3.14), except $\langle \Omega(0), [\mathcal{D}f(z, w', c^2))]_j \rangle$ depend on f only through its coefficients of total degree less or equal to $j - 1$.

Moreover, by assumption on f, the derivatives $z\partial_z f(z, w', c^2)$ and $w'\partial_{w'} f(z, w', c^2) \in \mathcal{O}^1(\tilde{\Delta})$ and, hence, $T_{2,j}$ and $T_{3,j} \in \mathcal{O}^2(\tilde{\Delta})$. Since $\mathcal{D}f(z, w', c^2)) \in \mathcal{O}^2(\tilde{\Delta})$, also $T_{1,j} \in \mathcal{O}^2(\tilde{\Delta})$.

Finally, since $[w'\partial_{w'} f]_j = 0$ for $j = 0, 1, 2$ we see that the term $T_{3,j}$ depends on the $[G_n]_l$ only for $l < j - 2n$.

We can now construct by induction $[f]_j$ and the $[G_n(c^2)]_{j-2n}$ for all the n such that $2n \leq j$. For $j = 3$ it is enough to choose $[f]_3 = 0$. Then assuming we have constructed $[f]_k$ and $[G_n(c^2)]_k$ for all $k \leq j - 2n$, $3 \leq k \leq j - 1$, we can find $[f]_j$ and $[G_n]_{j-2n}$ such that (3.14) holds: indeed, we define

$$\sum_{n \geq 2} [G_n(c^2)]_{j-2n} \cdot (\tilde{\Delta})^{\otimes n} = -\mathcal{M}(T_{1,j} + T_{2,j} + T_{3,j})$$

and we apply item 5 of Lemma 2.5 with $\omega = \Omega(0)$. $\qquad \square$

The second statement is about uniqueness.

Lemma 3.4. *Assume that there exist a formal series*

$$f(z, w', c^2) \in \mathbb{C}[[z, w', c^2]] \cap \mathcal{O}^2(z, w', c)$$

and formal series $\Gamma(c^2), \Omega(c^2), F_k(c^2) \in \mathbb{C}[[c]]$ *such that*

$$N(zw) = \Gamma(c^2) + \langle \Omega(c^2), z'w' - c^2 \rangle + \sum_{n=2}^{\infty} F_n(c^2) \cdot (z'w' - c^2)^{\otimes n},$$

where

$$z' = z + \partial_{w'} f(z, w', c^2), \quad w = w' + \partial_z f(z, w', c^2),$$

then $\Gamma(c^2) = N(c^2)$ *and* $\Omega(c^2) = \nabla N(c^2)$ *(thus they are unique).*

Proof.   Let us denote $\tilde{\Delta} = zw' - c^2$ and $\Delta' = z'w' - c^2$. By assumption

$$N(zw' + z\partial_z f(z, w', c^2)) = \Gamma(c^2) + \langle \Omega(c^2), \Delta' \rangle + \sum_{n=2}^{\infty} F_n(c^2) \cdot (\Delta')^{\otimes n}$$

and using the fact that

$$zw' + z\partial_z f(z, w', c^2) = zw' + w'\partial f(z, w', c^2) + \mathcal{D}f(z, w', c^2)$$
$$= c^2 + \tilde{\Delta} + \mathcal{D}f(z, w', c^2) + w'\partial_{w'} f(z, w', c^2)$$

and

(3.18) $$\Delta' = \tilde{\Delta} + w'\partial_{w'} f(z, w', c^2)$$

we have

$$N(c^2 + \tilde{\Delta} + \mathcal{D}f(z, w', c^2) + w'\partial_{w'} f(z, w', c^2)) =$$

$$\Gamma(c^2) + \langle \Omega(c^2), \tilde{\Delta} + w'\partial_{w'} f(z, w', c^2) \rangle + \sum_{n=2}^{\infty} F_n(c^2) \cdot (\tilde{\Delta} + w'\partial_{w'} f(z, w', c^2))^{\otimes n}$$

Using Taylor formula

$$N(c^2) + \nabla N(c^2) \cdot (\tilde{\Delta} + \mathcal{D}f(z, w', c^2) + w'\partial_{w'} f(z, w', c^2)) +$$
$$\sum_{|m| \geq 2} \partial^m N(c^2) \cdot (\tilde{\Delta} + \mathcal{D}f(z, w', c^2) + w'\partial_{w'} f(z, w', c^2))^{\otimes m} =$$

$$\Gamma(c^2) + \langle \Omega(c^2), \tilde{\Delta} + w'\partial_{w'} f(z, w', c^2) \rangle + \sum_{n=2}^{\infty} F_n(c^2) \cdot (\tilde{\Delta} + w'\partial_{w'} f(z, w', c^2))^{\otimes n}$$

so

(3.19)
$$\nabla N(c^2)\cdot\mathcal{D}f(z,w',c^2)+\sum_{|m|\geq 2}\partial^m N(c^2)\cdot(\tilde{\Delta}+\mathcal{D}f(z,w',c^2)+w'\partial_{w'}f(z,w',c^2))^{\otimes m}=$$

$$(\Gamma(c^2)-N(c^2))+\langle\Omega(c^2)-\nabla N(c^2),\tilde{\Delta}+w'\partial_{w'}f(z,w',c^2)\rangle+$$

$$\sum_{n=2}^{\infty}F_n(c^2)\cdot(\tilde{\Delta}+w'\partial_{w'}f(z,w',c^2)))^{\otimes n}.$$

We denote by $[\cdot]_j$ the homogeneous polynomial part of total degree j in the (z,w',c) variables. Using that $[f]_0=[f]_1=0$, it follows readily from equation (3.19) that

$$\Gamma(c^2)=\langle\omega,c^2\rangle+\mathcal{O}^2(c^2),\quad \Omega(c^2)=\omega+\mathcal{O}(c^2).$$

We now prove that $\mathcal{D}f(z,w',c^2)=0$. We shall prove by induction on j that for any $j\geq 0$, $\mathcal{D}[f]_j=0$. By assumption this is true for $j=0,1$. Let us assume this is true for all $1\leq k\leq j-1$. By taking the $[\cdot]_j$ in equation (3.19) and using items 2 and 3 of Lemma 2.5 ($[\mathcal{D}f]_k=\mathcal{D}[f]_k$, $[w'\partial_{w'}f]_k=w'\partial_{w'}[f]_k$), the fact that $[f]_l=0$ for $0\leq l\leq 1$ and $[\Omega(c^2)-\nabla N(c^2)]_0=0$ we get

(3.20)
$$\nabla N(0)\cdot\mathcal{D}[f]_j(z,w',c^2)=[(\Gamma(c^2)-N(c^2))]_j+\langle[\Omega(c^2)-\nabla N(c^2)]_{j-2},\tilde{\Delta}\rangle+$$
$$S_{1,j}+S_{2,j}+S_{3,j}+S_{4,j}$$

where

(3.21)
$$S_{1,j}=-\sum_{\substack{k+l=j\\l\geq 2,k\geq 1}}[\nabla N(c^2)]_k\cdot\mathcal{D}[f]_l(z,w',c^2)$$

(3.22) $S_{2,j}=-[\sum_{|m|\geq 2}\partial^m N(c^2)\cdot(\tilde{\Delta}+\mathcal{D}f(z,w',c^2)+w'\partial_{w'}f(z,w',c^2))^m]_j$

(3.23)
$$S_{3,j}=\sum_{\substack{k+l=j\\l\geq 2,k\geq 1}}\langle[\Omega(c^2)-\nabla N(c^2)]_k,w'\partial_{w'}[f]_l(z,w',c^2)\rangle$$

(3.24)
$$S_{4,j}=\sum_{n=2}^{\infty}[F_n(c^2)\cdot(\tilde{\Delta}+w'\partial_{w'}f(z,w',c^2)))^{\otimes n}]_j.$$

We also observe that the sum $S_{2,j}:=[\sum_{|m|\geq 2}\partial^m N(c^2)\cdot(\tilde{\Delta}+\mathcal{D}f(z,w',c^2)+w'\partial_{w'}f(z,w',c^2))^m]_j$ is a linear combination of multilinear terms of the

form $[\partial^m N(c^2)]_{l_1} \cdot \tilde{\Delta}^{\otimes_{sym} m_1} \otimes_{sym} [\mathcal{D}f(z, w', c^2)]_{l_2}^{\otimes m_2} \otimes_{sym} [w'\partial_{w'}f(z, w', c^2)]_{l_3}^{\otimes m_3}$
with $l_1 + 2m_1 + m_2 l_2 + m_3 l_3 = j$ and $m_1 + m_2 + m_3 = m \geq 2$; hence
$\max(m_2, m_3) \leq j - 1$. But the induction assumption implies that
$[f]_m$ is diagonal for $m \leq j - 1$, hence the same is true for $\mathcal{D}[f]_m$ and
$w'\partial_{w'}[f]_m$. In conclusion, the sum $S_{2,j}$ is diagonal. A similar argument
shows that the sum $S_{4,j} := \sum_{n=2}^{\infty}[F_n(c^2) \cdot (\tilde{\Delta} + w'\partial_{w'}f(z, w', c^2)))^{\otimes n}]_j$
is diagonal, as well as all the other terms of the equation (3.20) with
the possible exception of the term $\langle \nabla N(0), \mathcal{D}[f]_j \rangle$. It then follows that
$\langle \nabla N(0), \mathcal{D}[f]_j \rangle$ is diagonal. By Lemma 2.5 this forces $\mathcal{D}[f]_j = 0$. This
completes the induction and proves that $\mathcal{D}f = 0$. Now equation (3.19)
reads

$$(3.25) \quad (\Gamma(c^2) - N(c^2)) + \langle \Omega(c^2) - \nabla N(c^2), \Delta' \rangle =$$

$$= \sum_{|m| \geq 2} \partial^m N(c^2) \cdot (\Delta')^{\otimes m} - \sum_{n=2}^{\infty} F_n(c^2) \cdot (\Delta')^{\otimes n}.$$

hence

$$(3.26) \qquad (\Gamma(c^2) - N(c^2)) + \langle \Omega(c^2) - \nabla N(c^2), \Delta' \rangle = \mathcal{O}^2(\Delta')$$

and Lemma 2.4 concludes the proof. $\qquad\square$

Proof. We can now prove Proposition 3.2. Using Lemma 3.1 we
can assume that H is under Birkhoff Normal Form and we then apply
consecutively Lemmas 3.3 and 3.4.

Indeed, using Lemma 3.1(c) there is, by assumption, a (formally)
exact mapping Z_1 in $(*)$ such that

$$H \circ Z_1(z, w, c^2) = \Gamma(c^2) + \langle \Omega(c^2), zw - c^2 \rangle +$$
$$+ \langle zw - c^2), F(z, w, c^2)(zw - c^2) \rangle,$$

Using Lemmas 3.3 and 3.1(c) there is a (formally) exact mapping Z_2
in $(*)$ such that

$$H \circ Z_1 \circ Z_2(z, w, c^2) = \Gamma(c^2) + \langle \Omega(c^2), (z'w' - c^2) \rangle +$$
$$+ \sum_{n=2}^{\infty} G_k(c^2) \cdot (z'w' - c^2)^{\otimes n}.$$

By the Birkhoff normal form there is a (formally) exact mapping Z_3 in
$(*)$ such that

$$H \circ Z_3(z, w) = N(zw) = \langle \omega_0, zw \rangle + \mathcal{O}^3(zw).$$

Hence $W = Z_3^{-1} \circ Z_1 \circ Z_2$ is (formally) exact in $(*)$, by Lemma 3.1(a), and

$$N \circ W(z, w, c^2) = \Gamma(c^2) + \langle \Omega(c^2), (z'w' - c^2) \rangle +$$
$$+ \sum_{n=2}^{\infty} G_k(c^2) \cdot (z'w' - c^2)^{\otimes n}.$$

By Lemma 3.1(b), W has a generating function and now the proposition follows from Lemma 3.4. $\square$

4. A KAM COUNTER TERM THEOREM AND THE FREQUENCY MAP

The proof of Theorem 1.1 relies on the following fundamental result:

Proposition 4.1. *Given $0 < \kappa < 1$ and $\tau > d - 1$. Then, for all $s \in \mathbb{N}$, there exist non-negative constants (only depending on s and τ)*

$$\alpha(s) \geq (s - t) + \alpha(t), \quad s \geq t \geq 0,$$

such that if

$$H(z, w) = N^q(zw) + \mathcal{O}^{2q+1}(z, w) \in \mathcal{C}^\omega(\mathbb{C}^{2d}, 0), \quad q \geq \alpha(1) + 1,$$

is σ-symmetric with

$$N^q(r) = \langle \omega_0, r \rangle + \mathcal{O}^2(r),$$

then there exist $\delta > 0$ and for any $\eta < \delta$ a σ-symmetric function

$$\Lambda = \Lambda(c, \omega) \in \mathcal{C}^{\omega,\infty}(\mathbb{D}_\eta^d \times B)$$

and a symplectic and σ-symmetric diffeomorphism

$$(Z_{c,\omega} - \mathrm{id})(z, w) \in \mathcal{C}^{\omega,\infty}(\mathbb{D}_\eta^{2d} \times \mathbb{D}_\eta^d \times B) \cap \mathcal{O}^2(z, w, c)$$

such that

$$(4.27) \quad (H + \langle \omega + \Lambda(c, \omega), \cdot \rangle) \circ Z_{c,\omega}(z, w)$$
$$= \langle \omega, zw - c \rangle + \mathcal{O}^2(zw - c) + g(z, w, c, \omega)$$

(modulo an additive constant that depends on c, ω) with g (κ, τ)-flat and $g \in \mathcal{O}^2(z, w, c)$.

    Moreover,

     (i) *for any $s \in \mathbb{N}$ there exists a constant C_s, only depending on s, H, τ such that*

$$\|\Lambda + \partial_r N^q\|_{\eta,s} + \|Z - \mathrm{id}\|_{\eta,s} \leq C_s \eta^q (\frac{1}{\kappa\eta})^{\alpha(s)}$$

(ii) *there exists a constant C, only depending on H, τ, such that*

$$\delta \geq \frac{1}{C} \kappa^{\frac{\alpha(1)}{q - \alpha(1)}}$$

(iii) *if*

$$\omega_0 \in DC(2\kappa, \tau)$$

then the mapping

$$\mathbb{D}_{\delta'}^{d+1} \ni (c, \lambda) \mapsto \Lambda(c, (1 + \lambda)\omega_0) \in \mathbb{C}^d$$

is holomorphic and σ-symmetric for some $0 < \delta' < \delta$

(iv)

$$(Z_{c^2, \omega} - \mathrm{id})(z, w) \in \mathcal{O}^{2q}(z, w, c).$$

Remark. *Notice that this proposition (except part (iii)) does not require that ω_0 is Diophantine. Notice also that, a priori, Λ, Z and g depend on κ and on η.*

Remark. *It is also the case that $g(z, w, c^2, \omega)$ and $\Lambda(c^2, \omega) + \nabla N^q(c^2)$ are in $\mathcal{O}^{2q}(z, w, c)$, but we shall not use this fact.*

This proposition follows from the local Normal Form Theorem 6.4 applied to the Hamiltonian $\tilde{H}(z, w, c) = H(z, w) - N^q(c) - \langle \nabla N^q(c), zw - c \rangle$ in a similar way as Proposition 4.2 of [EFK]. Let us discuss this a bit, but for full details we refer to [EFK].

If we write $F(z, w) = H(z, w) - N^q(zw) = \mathcal{O}^{2q+1}(z, w)$, then (with the notation of Section 2)

$$H(z, w) = N^q(c^2) + \langle \nabla N^q(c^2), zw - c^2 \rangle + \mathcal{O}^2(zw - c^2) +$$

$$F^{(0)}(z, w, c^2) + \langle F^{(1)}(z, w, c^2), zw - c^2 \rangle + F^{[2]}(z, w, c^2) \cdot (zw - c^2)^{\otimes 2}$$

where $F^{(0)}(z, w, c^2) = \mathcal{O}^{2q+1}(z, w, c)$ and $F^{(1)}(z, w, c^2) = \mathcal{O}^{2q-1}(z, w, c)$. Hence

$$(4.28) \quad \tilde{H}(z, w, c) = F^{(0)}(z, w, c) +$$

$$+ \langle F^{(1)}(z, w, c), zw - c^2 \rangle + \mathcal{O}^2(zw - c).$$

On domains where $\max(|z|, |w|, |c|) < \eta$, $\tilde{H}^{(0)}(z, w, c, \omega)$ is of order η^{q+1} and $\tilde{H}^{(1)}(z, w, c, \omega)$ is of order η^q. Using Lemma 6.1 we obtain that $[\tilde{H}]_{\eta, 0}$ is of order η^{q-b}, where b is a constant only depending on τ and d. If we take h equal $\frac{\eta}{4}$, say, then the smallness assumption (6.37) is fulfilled for any

$$\eta \leq \frac{1}{C} \kappa^{\frac{a}{q - a - b}}$$

– this gives the estimate of δ in (ii).

If we call $\tilde{\Lambda}(c, \omega)$ the counter term Λ obtained by applying Theorem 6.4 to $\tilde{H}(z, w, c)$, we then get the conjugacy equation (4.27) with $\Lambda(c, \omega) := \tilde{\Lambda}(c, \omega) - \nabla N^q(c)$. Since $\tilde{\Lambda}$ is small (as quantified (6.38) in Theorem 6.4) we get also the first half of the inequality given in item(i). The second half of the inequality in(i) also follows from (6.38).

Item(iii) follows from the last part of Theorem 6.4.

Since $\tilde{H}(z, w, c) = \mathcal{O}^{2q+1}(z, w, c)$ mod $\mathcal{O}^2(zw - c)$ we have(iv) by Theorem 6.4.

Corollary 4.2. *Given $0 < \kappa < 1$ and $\tau > d - 1$ and non-negative constants $\alpha(s)$ as in Proposition 4.1, if*

$$H(z, w) = N^q(zw) + \mathcal{O}^{2q+1}(z, w) \in \mathcal{C}^\omega(\{0\}), \quad q \geq \alpha(1) + 1,$$

with

$$N^q(r) = \langle \omega_0, r \rangle + \mathcal{O}^2(r),$$

then, for any

$$\eta < \eta_0 = \frac{1}{C'} \kappa^{\frac{\alpha(1)}{q - \alpha(1)}},$$

there exists a unique $\mathcal{C}^\infty$ function $\Omega : \{c \in \mathbb{R}^d : |c| < \frac{\eta}{2}\} \to \mathbb{R}^d$ such that

$$\Omega(c) + \Lambda(c, \Omega(c)) = 0, \quad \forall c$$

Moreover,

(i) *for any $s \in \mathbb{N}$ there exists a constant C'_s, only depending on s, H, τ such that*

$$\|\Omega - \partial_r N^q\|_{\mathcal{C}^s(|c| < \frac{\eta}{2})} \leq C'_s \eta^q (\frac{1}{\kappa\eta})^{\alpha(s)}$$

(ii) *If $\omega_0 \in \mathrm{DC}(\tau, \kappa)$, the Taylor series of Ω at $c = 0$ is given by $\nabla N_H(c)$.*

The constants C'_s only depend on H, τ.

We call Ω the *frequency map*. The proof of the corollary is almost identical to the one of Corollary 4.3 of [EFK]. Let us therefore only discuss shortly the proof.

The existence of Ω and the estimate (i) follow from (i) of Proposition 4.1 and the implicit function theorem applied to the function Λ.

Point (ii) is a consequence of the following facts: if $(z, w) \mapsto Z_c(z, w) := Z_{c, \Omega(c)}(z, w)$ is the change of variable given in Proposition 4.1, then

$$(4.29) \quad H \circ Z_{c^2}(z, w) = \langle \Omega(c^2), zw - c^2 \rangle + \mathcal{O}^2(zw - c^2) + g(z, w, c^2, \Omega(c^2)).$$

The condition $\omega_0 \in \mathrm{DC}(\tau, \kappa)$ and the fact that g is (κ, τ)-flat show that $g(z, w, c^2, \Omega(c^2)) = \mathcal{O}^\infty(z, w, c^2)$ and hence, one has in $\mathbb{C}[[z, w, c]]$

$$(4.30) \quad H \circ Z_{c^2}(z, w) = \langle \Omega(c^2), zw - c^2 \rangle + \mathcal{O}^2(zw - c^2).$$

Since $(Z_{c^2} - id)(z, w) = \mathcal{O}^2(z, w, c^2)$, Z_{c^2} has a formal generating function $f(z, w, c^2) \in \mathcal{O}^3(z, w, c^2)$. Proposition 3.2 then shows that in $\mathbb{C}[[z, w, c]]$ one has the identity $\Omega(c^2) = \nabla N(c^2)$.

We shall use the preceding results to prove Theorems 1.2–1.4. In the case the BNF is non degenerate, q is chosen according to the non-degeneracy condition, and it then will follow from (i) that the function $\Omega(c)$ – which depends on κ – takes values in $\mathrm{DC}(\tau, \kappa)$ on a set of positive measure which insures KAM stability. This will be proven in Sections 5.1 and 5.2. In the case of a degenerate BNF, (ii) of Proposition 4.1 as well as the analyticity of $\Lambda(c, \omega)$ in the variable c allows to conclude the proof of Theorem 1.2 and 1.3. This will be carried out in Section 5.3

5. Proof of the main results

This section is devoted to the derivation of Theorems 1.2–1.4, and thus of Theorem 1.1, from Proposition 4.1 and Corollary 4.2.

Consider a real analytic Hamiltonian H of the form $(*)$. By a real symplectic conjugation we can assume, since ω_0 is rationally independent, that H is on Birkhoff normal form up to order $2q + 1$ for any q:

$$H(x, y) = N^q(\frac{1}{2}(x^2 + y^2)) + \mathcal{O}^{2q+1}(x, y).$$

Performing the linear change of variable in Section 1.2 we obtain a σ-symmetric holomorphic Hamiltonian

$$\tilde{H}(z, w) = N^q(zw) + \mathcal{O}^{2q+1}(z, w)$$

of the form treated in Proposition 4.1 and Corollary 4.2.

The linear change of variable is not symplectic and it will change the canonical symplectic structure $dx \wedge dy$ into $\sqrt{-1}dz \wedge dw$. However, any transformation symplectic with respect to $dz \wedge dw$ will also be symplectic with respect to $\sqrt{-1}dz \wedge dw$, so we may just as well study $\tilde{H}$ under a transformation symplectic with respect to $dz \wedge dw$. Then the Birkhoff normal forms N_H and $N_{\tilde{H}}$ are the same and coincide with N^q up to order q.

5.1. **Transversality.** Let us state two lemmas the proof of which can be found in Section 5 of [EFK].

Lemma 5.1. *If $N_H(r)$ is non-degenerate, then there exist $p, \sigma > 0$ such that for any $k \in \mathbb{Z}^d \setminus \{0\}$ there exists a unit vector $u_k \in (\mathbb{R}_+)^d$ such that the series*

$$f_k(r) = \langle \frac{k}{|k|}, \partial_r N_H(r) \rangle$$

is (p, σ)-transverse in direction u_k, i.e.

$$\max_{0 \leq j \leq p} |\partial_t^j f_k(tu_k)_{|t=0}| \geq \sigma.$$

Consider now these p, σ. Let $\Omega \in \mathcal{C}^p(\{|c| < \eta\})$ and assume

$$\|\Omega - [\partial_r N_H]^p\|_{\mathcal{C}^p(\{|c| < \eta\})} \leq \frac{\sigma}{2}.$$

Lemma 5.2. *If N_H is (p, σ)-transverse (in some direction), then*

$$\mathrm{Leb}\{|c| < \eta : |\langle \frac{k}{|k|}, \Omega(c) \rangle| < \varepsilon\} \leq C_p(\frac{\varepsilon}{\sigma})^{\frac{1}{p}} \eta^{d-1}$$

for any η, k, ϵ.

5.2. **Proof of Theorem 1.4.** By Lemma 5.1 we are given p and σ that correspond to the transversality of the Birkhoff normal form $N_H = N_{\tilde{H}}$. We can assume without restriction that $\sigma \leq 1$, and we fix $q = (1 + 2p)\alpha(p) + 1$.

We shall apply (i) of Proposition 4.1 and Corollary 4.2 to $\tilde{H}$ with this q and with

$$\tau = dp + 1 \quad \text{and} \quad 0 < \kappa \leq \sigma^q \leq 1.$$

Now let

(5.31)
$$\eta := \frac{1}{C''}(\frac{\kappa}{\sigma})^{\frac{1}{2p}}.$$

Since $q \geq (1 + 2p)\alpha(1) + 1$ we have $\eta \leq \eta_0$ for all $C'' \geq C'$, where η_0 and C' are defined in Corollary 4.2. Then $\Omega = \Omega_\kappa$ [2] is defined in $\{|c| < \frac{\eta}{2}\}$ and

(5.32)
$$\|\Omega - [\partial_r N_H]^p\|_{\mathcal{C}^p(\{|c| < \frac{\eta}{2}\})} \leq C_p' \eta^q (\frac{1}{\kappa\eta})^{\alpha(p)} + \|[\partial_r N_H]^p - \partial_r N_H^q\|_{\mathcal{C}^p(\{|c| < \frac{\eta}{2}\})}$$

which is

$$\leq \tilde{C}\eta$$

since $q \geq (1 + 2p)\alpha(p) + 1$ – notice that $\tilde{C}$ is independent of $C'' \geq C'$. Finally if C'' is sufficiently large (depending on p, τ, H, thus on q) we have that $\tilde{C}\eta \leq \sigma/2$.

---

[2] not to forget that Ω depends on κ

By Lemma 5.2

$$\text{Leb}\{|c| < \frac{\eta}{2} : |\langle \frac{k}{|k|}, \Omega(c)\rangle| < \varepsilon\} \lesssim (\frac{\varepsilon}{\sigma})^{\frac{1}{p}}\eta^{d-1},$$

hence

$$\text{Leb}\{|c| < \frac{\eta}{2} : \Omega(c) \notin DC(\kappa, \tau)\} \lesssim (\frac{\kappa}{\sigma})^{\frac{1}{p}}\eta^{d-1}$$

$$\lesssim \eta\text{Leb}\{|c| < \frac{\eta}{2}\}$$

(provided κ is sufficiently small). Hence, the set

$$\Sigma_\kappa = \{|c| < \frac{\eta}{2} : \Omega(c) \in DC(\kappa, \tau)\} \cap \mathbb{R}^d_+$$

is of positive measure when κ is sufficiently small and has density 1 at 0 when $\kappa \to 0$. For each $c \in \Sigma_\kappa$

$$\tilde{Z}_{c,\Omega(c),\kappa}(\{zw = c\})$$

is an invariant set for the Hamiltonian system defined by $\tilde{H}$ with respect to the the canonical symplectic structure $dz \wedge dw$, hence also with respect to the symplectic structure $\sqrt{-1}dz \wedge dw$.

Returning to the variables x, y, using the linear transformation defined in Section 1.2, we get for any $c \in \Sigma_\kappa$ a symplectic transformation $Z_{c,\kappa} : (\mathbb{R}^{2d}, 0) \to (\mathbb{R}^{2d}, 0)$ [3] such that

$$Z_{c,\kappa}(\{x^2 + y^2 = c\})$$

is a KAM-torus for the Hamiltonian system defined by H. By (iv) of Proposition 4.1 $Z_{c,\kappa}$ has the form

$$Z_{c,\kappa}(x, y) = (x, y) + \mathcal{O}^q(x, y, c).$$

Let now $W_\kappa : (\mathbb{R}^d_+, 0) \times \mathbb{T}^d \to (\mathbb{R}^d \times \mathbb{R}^d, 0)$ be the mapping

$$(c, \theta) \mapsto Z_{c,\kappa}(c\cos 2\pi\theta, c\sin 2\pi\theta) = (c\cos 2\pi\theta, c\sin 2\pi\theta) + \mathcal{O}^q(c).$$

Then

$$W_\kappa(\{c : c \in \Sigma_\kappa\}, \mathbb{T}^d)$$

is foliated into KAM-tori. By (5.31) and the estimate (i) of Proposition 4.1 we have that the $\mathcal{O}^q(c)$ term in W_κ satisfies the condition (7.51) of Lemma 7.1 of the appendix A, which hence yields that $W_\kappa(\{c : c \in \Sigma_\kappa\}, \mathbb{T}^d)$ has positive measure when κ is sufficiently small and that the union over all $\kappa > 0$ has density 1 at $0 \in \mathbb{R}^d \times \mathbb{R}^d$.

---

[3] $Z_{c,\kappa}$ is real because $\tilde{Z}_{c,\Omega(c),\kappa}(z, w)$ is σ-symmetric

5.3. **Proof of Theorems 1.2 and 1.3.** We shall apply Proposition
4.1 and (ii) of Corollary 4.2 with $q = \alpha(1) + 1$ and

$$q = \alpha(1) + 1, \quad \tau = \tau_0 \quad \text{and} \quad \kappa = \frac{\kappa_0}{2}.$$

Then

$$\Omega(c) + \Lambda(c, \Omega(c)) = 0$$

and

$$\Omega(c) = \partial_r N_H(c) + \mathcal{O}^\infty(c).$$

Since N_H is j-degenerate we have

$$\partial_v^n N_H(0) = 0 \qquad \forall n \geq 0$$

for any $v \in \text{Lin}(\gamma = (\gamma_1, \ldots, \gamma_j))$, where ∂_v is the directional derivative
in direction v. From this we derive that

$$\partial_v^n (\omega_0 + \Lambda(\cdot, \omega_0))_{|c=0} = 0 \qquad \forall n \geq 0.$$

By (iii) of Proposition 4.1 $s \mapsto \Lambda(\langle s, \gamma \rangle, \omega_0)$ is an analytic function in
$s \in \mathbb{R}^j$, $s \sim 0$, it must be identically 0, hence $\Omega(\langle s, \gamma \rangle)$ is identically
ω_0, i.e.

$$\Omega(\langle s, \gamma \rangle) = \omega_0 \in DC(\kappa_0, \tau) \subset DC(\kappa, \tau)$$

for all $|s| \leq s_+$.   Thus we have

$$(5.33) \quad H \circ \tilde{Z}_{c, \omega_0}(z, w) = \langle \omega_0, zw - c \rangle + \mathcal{O}^2(zw - c) + g(z, w, c, \omega_0)$$

for any

$$c \in \Delta = \{c = \langle s, \gamma \rangle : |s| \leq s_+\}.$$

Since everything is analytic in s, (5.33) extends to complex s in some
neighborhood of 0.

Hence, for any $c \in \Delta$ the set

$$\tilde{Z}_{c, \omega_0}(\{zw = c\})$$

is an invariant Lagrangian submanifold for the Hamiltonian system
defined by by $\tilde{H}$ with respect to the the canonical symplectic structure
$dz \wedge dw$, hence also with respect to the symplectic structure $\sqrt{-1} dz \wedge dw$.

The set

$$\bigcup_{c \in \Delta} \{zw = c\} \times \{c\} \subset \mathbb{C}^{2d} \times \mathbb{C}^d$$

is an analytic submanifold of (complex) dimension $d + j$, singular at
the origin. It's image M under the holomorphic diffeomorphism

$$(z, w, c) \mapsto (\tilde{Z}_{c, \omega_0}(z, w), c)$$

is therefore an analytic submanifold of (complex) dimension $d + j$,
singular at the origin. The image of M under the projection on $\{z, w\}$
is a subanalytic set.

Using (iv) of Theorem 4.1, it is easy to find points on (any component of) $M \ni 0$ where this projection, restricted to M, is onto. The image of M under the projection is therefore an analytic subvariety of (complex) dimension $d + j$. This completes the proof of Theorem 1.2.

When N_H is $(d-1)$-degenerate, then

$$\partial_r N_H(c) = \mu(\langle c, \omega_0 \rangle)\omega_0$$

where $\mu(t) = 1 + \mathcal{O}(t)$ is a formal power series in one variable.

Since

$$\mu(\langle c, \omega_0 \rangle)\omega_0 + \Lambda(c, \mu(\langle c, \omega_0 \rangle)\omega_0) = \mathcal{O}^\infty(c),$$

taking $c = t\omega_0$, we have (assuming ω_0 is a unit vector)

(5.34) $$\mu(t)\omega_0 + \Lambda(t\omega_0, \mu(t)\omega_0) = 0$$

modulo a term in $\mathcal{O}^\infty(t)$. Since, by Proposition 4.1 (iii), the lefthand side is analytic in $t\omega_0$ and μ we obtain from (5.34) that $\mu(t)$ is a convergent power series. Then

$$t \mapsto \mu(t)\omega_0 + \Lambda(t\omega_0, \mu(t)\omega_0)$$

is analytic for $t \sim 0$, hence identically zero. We derive from this that

$$\Omega(c) = \mu(\langle c, \omega_0 \rangle)\omega_0,$$

i.e.

$$\Omega(c) \in DC(\kappa, \tau)$$

for all sufficiently small c. Rüssmann's theorem now follows from an argument similar to that of the end of the proof of of Theorem 1.2.

6. The (local) Normal Form Theorem

6.1. **Functional spaces and the operators $\mathcal{P}$ and $\mathcal{L}$.** We come back to the setting and notations of Section 2, but we now consider the general case of functions $f(z, w, c, \omega)$ depending analytically on z, w, c and smoothly on ω. Let $\delta > 0$, and denote by $C_\delta^{\omega, \infty}$ the set of functions $f \in C^{\omega, \infty}(\mathbb{D}_\delta^{2d} \times \mathbb{D}_\delta^d \times B)$ such that $f(z, w, c, \omega) \in \mathcal{O}^2(z, w, c)$.

Let κ, τ be positive numbers and $l : \mathbb{R} \to \mathbb{R}$ a fixed even, non-negative C^∞ function such that $|l| \leq 1$, and $l(x) = 0$ if $|x| \geq 1/2$ and $l(x) = 1$ if $|x| \leq 1/4$. We introduce the cut-off operator $\mathcal{P}$: if $f \in C^{\omega, \infty}(\mathbb{D}_\delta^{2d} \times \mathbb{D}_\delta^d \times B)$, $f(z, w, c, \omega) = \sum_{\alpha, \beta \in \mathbb{N}^d} f_{\alpha, \beta}(c, \omega) z^\alpha w^\beta$ then

$$\mathcal{P}(f)(z, w, c, \omega) = \sum_{\alpha, \beta \in \mathbb{N}^d} f_{\alpha, \beta}(c, \omega) z^\alpha w^\beta l(\langle \alpha - \beta, \omega \rangle \frac{(|\alpha| + |\beta|)^\tau}{\kappa}).$$

A function f such that $\mathcal{P}f = f$ satisfies $\partial_z^\alpha \partial_w^\beta \partial_c^\gamma \partial_\omega^\delta f(z, w, c, \omega) = 0$ for any set of indices $\alpha, \beta, \gamma, \delta$ when $\omega \in DC(\kappa, \tau)$. In particular such a function is (κ, τ)-*flat*.

Notice that $\mathcal{P}$ and $\mathcal{M}$ commute and that $\mathcal{P}$ preserves the space, that we still denote $\widehat{\mathcal{NR}}$, of maps $f(z, w, c, \omega)$ which for each fixed value of ω are in $\widehat{\mathcal{NR}}$.

We now define the linear operator $\mathcal{L} : C^{\omega,\infty}(\mathbb{D}_\delta^{2d} \times \mathbb{D}_\delta^d \times B) \to C^{\omega,\infty}(\mathbb{D}_{\delta'}^{2d} \times \mathbb{D}_\delta^d \times B)$ $(\delta' < \delta)$ by: $\mathcal{L}(f) = u$ if and only if

(6.35)
$$\begin{cases} \mathcal{D}^\omega u(z, w, c, \omega) = f(z, w, c, \omega) - \mathcal{P}(f)(z, w, c, \omega) - \mathcal{M}(f)(z, w, c, \omega) \\ \mathcal{M}(u) = \mathcal{P}(u) = 0. \end{cases}$$

Here is the analogue of Lemma 8.1 of [EFK] the proof being the same (the only modification is to replace d by $2d$).

Lemma 6.1. *One has*

$$\max(\|\mathcal{P}(f)\|_{\delta',s}, \|\mathcal{L}(f)\|_{\delta',s}) \leq C_s (\frac{1}{\kappa})^{s+1} (\frac{1}{\delta - \delta'})^{(\tau+1)s+\tau+2d} \|f\|_{\delta,s}$$

for any $\delta' < \delta$. The constant C_s only depends, besides s, on τ and l.

Since σ-symmetry is an important issue we mention the following obvious lemma (see items 4 and 5 of Lemma 2.5).

Lemma 6.2. *If $f \in C_{\delta,s}^{\omega,\sigma,\infty}$ then $\mathcal{P}f$ and $\sqrt{-1}\mathcal{L}f$ are σ-symmetric.*

Let us also mention the following fact:

Lemma 6.3. *If $f \in C^{\omega,\infty}(\mathbb{D}_\delta^{2d} \times \mathbb{D}_\delta^d \times B)$ then for $j = 0, 1, 2, 3$, $\delta' < \delta$,*

$$\max(\|f^{(j)}\|_{\delta',s}, \|f^{[j]}\|_{\delta',s}) \leq C \frac{1}{(\delta - \delta')^{3d}} \|f\|_{\delta,s}.$$

(The notations $f^{(j)}$, $f^{[j]}$ are defined in (2.7), (2.8)).

Proof. From (2.5) and the fact that $(\delta' < \delta)$

$$\|\tilde{f}_{\underline{k}}\|_{\delta',s} \leq C e^{-2\pi\rho|\underline{k}|} \frac{1}{(\delta - \delta')^{2d}} \|f\|_{\delta,s}$$

we get

$$\|f_{\underline{n}}\|_{\delta',s} \leq e^{-4\pi|\underline{n}|\delta} \left(\sum_{k \geq 0} \frac{k^n}{n!} e^{-2\pi(\delta-\delta')k} \right)^d \frac{1}{(\delta - \delta')^{2d}} \|f\|_{\delta,s}$$

$\square$

We define

$$[f]_{\delta,s} = \max(\|f^{(0)}\|_{\delta,s}, \|f^{(1)}\|_{\delta,s}, \|\mathcal{DL}f^{(0)}\|_{\delta,s}, \|\mathcal{DL}f^{(1)}\|_{\delta,s}).$$

and

$$M_f = \mathcal{M}(f^{(1)} - f^{(2)}\mathcal{DL}f^{(0)}).$$

Remark. Since we shall need it later, we notice that while $\mathcal{L}$ does not preserve σ-symmetry (if f is σ-symmetric, $\sqrt{-1}\mathcal{L}f$ is σ-symmetric) as well as $\mathcal{D}$ (if f is σ-symmetric, $\sqrt{-1}\mathcal{D}f$ is σ-symmetric) the composition $\mathcal{DL}$ preserves σ-symmetry.

Also, we set

$$\{f\}_{\delta,s} = \max(\max_{0 \le j \le 2} \|f^{(j)}\|_{\delta,s}, \max_{j=2,3} \|f^{[j]}\|_{\delta,s}, \|f\|_{\delta,s})$$

Notice that from Lemma 6.3,

$$(6.36) \qquad\qquad \{f\}_{\delta-h,s} \le Ch^{-3d}\|f\|_{\delta,s}.$$

We denote by $\mathcal{E}_\delta^{\omega,\infty}$ the set of (exact) symplectic (with respect to $dz \wedge dw$) σ-symmetric diffeomorphisms defined on $\mathbb{D}_\delta^{2d} \times \mathbb{D}_\delta^d \times B$

$$Z_{c,\omega}(z,w) = \left(\begin{array}{c} z + R(z,w,c,\omega) \\ w + T(z,w,c,\omega) \end{array} \right)$$

with $R,T \in \mathcal{C}^{\omega,\infty}(\mathbb{D}_\delta \times \mathbb{D}_\delta^d \times \mathbb{D}_\delta^d, B)$ and $R,T = \mathcal{O}^2(z,w,c)$. If Z' is another mapping in $\mathcal{E}_\delta^{\omega,\infty}$ then we define

$$\{Z - Z'\}_{\delta,s} = \max(\{R' - R\}_{\delta,s}, \{T' - T\}_{\delta,s}).$$

and

$$(Z \circ Z')_{c,\omega}(z,w) = Z_{c,\omega}(Z'_{c,\omega}(z,w)).$$

We denote by $\mathcal{E}_\delta^{\omega,\sigma,\infty}$ the set of elements of $\mathcal{E}_\delta^{\omega,\infty}$ that are σ-symmetric.

6.2. **Notations.** If h is a positive number we denote by $\mathcal{C}_s(h)$ an expression of the form $C_s \times (h)^{-\alpha(s)}$ where C_s is a constant and $\alpha(\cdot)$ is an increasing real-valued function defined on $\mathbb{N}$. Also, if $(\varepsilon_s)_s$, $(\zeta_s)_s$ are sequences of positive real numbers indexed by $s \in \mathbb{N}$ we use the short hand notation $\langle \varepsilon, \zeta \rangle_s$ (resp. $\langle \varepsilon, \varepsilon, \zeta \rangle_s$) to denote the sum of all possible products $\varepsilon_i \zeta_j$ (resp. $\varepsilon_i \varepsilon_j \zeta_k$) where i, j (resp. i, j, k) take value in $\{0, s\}$ and the value s is taken at most once.

6.3. **The (local) Normal Form Theorem.** Using the preceding notations and the change of coordinates $z = x + \sqrt{-1}y$, $w = x - \sqrt{-1}y$ we are reduced to prove the following result:

Theorem 6.4. *Let $\kappa < 1, \delta < 1$. There exist constants $C, a > 0$ (depending only on τ and d) such that if $H(z, w, c)$ is in $\mathcal{C}_\delta^{\omega,\sigma,\infty}$ (σ-symmetric) and independent of ω and if one assumes that for some $0 < h < \delta/2$*

$$(6.37) \qquad [H]_{\delta,0} \leq C\left((1 + \{H\}_{\delta,0})^{-1}\kappa h\right)^a$$

then, there exist an exact symplectic σ-symmetric change of coordinates $Z(c, \omega, z, w)$ in $\mathcal{E}_{\delta-h}^{\omega,\sigma,\infty}$ and functions $g(c, \omega, z, w), \Gamma(c, \omega), \Lambda(c, \omega), H'$ in $\mathcal{C}_{\delta-h}^{\omega,\sigma,\infty}$, such that

$$(H + \langle \omega + \Lambda(c, \omega), \cdot \rangle) \circ Z = \Gamma(c, \omega) + \langle \omega, zw - c \rangle + H'(z, w, c, \omega) +$$
$$g(z, w, c, \omega)$$

where g is (κ, τ)-flat and $[H']_{\delta-h,0} = 0$. Furthermore, $Z, H', g, \Gamma, \Lambda$ satisfy

$$(6.38) \quad \max\left(\|\Lambda\|_{\delta-h,s}, \{Z - id\}_{\delta-h,s}, \|g\|_{\delta-h,s}, \{H' - H\}_{\delta-h,s}\right)$$
$$\leq C_s((1 + \{H\}_{\delta,0})(\kappa h)^{-1})^{\alpha(s)}[H]_{\delta,0}$$

where C_s and $\alpha(s)$ are constants depending only on s, τ and d.

Furthermore, if $(H - H^{[2]})(z, w, c^2) = \mathcal{O}^{2q+1}(z, w, c)$, then $(Z - id)(z, w, c^2, \omega) = \mathcal{O}^{2q}(z, w, c), g(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$ and $\Lambda(c^2, \omega) = \mathcal{O}^{2q+1}(c)$.

Also, if

$$\omega_0 \in DC(2\kappa, \tau)$$

then $\mathbb{D}_{\delta'}^{2d} \times \mathbb{D}_{\delta'}^d \times \mathbb{D}_{\delta'}^1 \ni (z, w, c, \lambda) \mapsto \Lambda(c, (1 + \lambda)\omega_0), Z(z, w, c, (1 + \lambda)\omega_0)$ and $H'(z, w, c, (1 + \lambda)\omega_0)$ are analytic for some $0 < \delta' \leq \delta$ and $g(z, w, c, (1 + \lambda)\omega_0) = 0$ on $\mathbb{D}_{\delta'}^{2d} \times \mathbb{D}_{\delta'}^d \times \mathbb{D}_{\delta'}^1$.

6.4. **Proof of the local Normal Form Theorem.** The proof of Theorem 6.4 is based on the inductive step described in Lemma 6.7. This lemma is proved in two steps. In a first time we treat the case where $M_H := \mathcal{M}(H^{(1)} - H^{(2)}\mathcal{D}\mathcal{L}H^{(0)})$ is equal to zero and in a second step we show how to reduce to this case by adding a counter term $\langle \Lambda(c, \omega), \cdot \rangle$.

6.4.1. *The case when $M_H = 0$.* In the next Lemma we will prove that if $M_H = 0$, then one can apply a conjugacy to H to reduce its affine part to a quadratically small one.

Lemma 6.5. *Let $H(z,w,c,\omega) \in \mathcal{C}_\delta^{\omega,\sigma,\infty}$ (hence σ-symmetric) and denote*

(6.39)
$$\varepsilon_{\delta,s} = [H]_{\delta,s}, \quad \zeta_{\delta,s} = \{H\}_{\delta,s} + 1.$$

If $M_H = 0$ and if

(6.40)
$$\varepsilon_{\delta,1} \leq \mathcal{C}_1(\kappa h)\zeta_{\delta,1}^{-1}$$

then there exist $Z \in \mathcal{E}_{\delta-h}^{\omega,\sigma,\infty}$ (σ-symmetric), $\tilde{\Gamma}, \tilde{H} \in \mathcal{C}_{\delta-h}^{\omega,\sigma,\infty}$ and a (κ,τ)-flat function, σ-symmetric $\tilde{g}$ such that

$$(H + \langle \omega, \cdot \rangle) \circ Z_{c,\omega}(z,w) = \tilde{\Gamma}(c,\omega) + \langle \omega, zw - c \rangle + \tilde{H}(z,w,c,\omega) + \tilde{g}(z,w,c,\omega),$$

and

$$[\tilde{H}]_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle \varepsilon, \varepsilon, \zeta \rangle_{\delta,s}$$

$$\max\left(\|\tilde{g}\|_{\delta-h,s}, \{\tilde{H}^{[2]} - H^{[2]}\}_{\delta-h,s}, \{Z - id\}_{\delta-h,s} \right) \leq \mathcal{C}_s(\kappa h))\zeta_{\delta,0}\langle \varepsilon, \zeta \rangle_{\delta,s}$$

Furthermore, if $(H - H^{[2]})(z,w,c^2,\omega) = \mathcal{O}^{2q+1}(z,w,c)$, then $(\tilde{H} - \tilde{H}^{[2]})(z,w,c^2,\omega) = O^{2q+1}(z,w,c)$, $(Z - id)(z,w,c^2,\omega) = \mathcal{O}^{2q}(z,w,c)$ and $\tilde{g}(z,w,c^2,\omega) = \mathcal{O}^{2q+1}(z,w,c)$.

Proof. Assumme $Z_k : (z,w) \mapsto (z',w')$ is an exact symplectic change of variable with generating function $k(z,w',c,\omega)$ depending analytically on z,w,c and smoothly on ω: $z' = z + \partial_{w'}k$, $w = w' + \partial_z k$ and denote by H' the hamiltonian defined by $H'(z',w') = H(z,w)$.

With the notations of Subsection 2.2

$$H(z,w,c,\omega) = H^{(0)}(z,w,c,\omega) + \langle zw - c, H^{(1)}(z,w,c,\omega) \rangle +$$

$$\langle zw - c, H^{[2]}(z,w,c,\omega)(zw - c) \rangle$$

$$H'(z,w,c,\omega) = H'^{(0)}(z,w,c,\omega) + \langle zw - c, H'^{(1)}(z,w,c,\omega) \rangle +$$

$$\langle zw - c, H'^{[2]}(z,w,c,\omega)(zw - c) \rangle$$

with $H^{(0)}, H^{(1)}, H'^{(0)}, H'^{(1)}$ in $\widehat{\mathcal{NR}}$. The equality $H'(z',w') = H(z,w)$ is equivalent to the fact that

$$(I) := \langle \omega, (z + \partial_{w'}k)w' \rangle + H^{(0)}(z + \partial_{w'}k, w', c, \omega) +$$

$$\langle (z + \partial_{w'}k)w' - c, H^{(1)}(z + \partial_{w'}k, w', c, \omega) \rangle +$$

$$\langle (z + \partial_{w'}k)w' - c, H^{[2]}(z + \partial_{w'}k, w', c, \omega)((z + \partial_{w'}k)w' - c) \rangle$$

is equal to

$$(II) := \langle \omega, z(w' + \partial_z k) - c \rangle + \Gamma'(c, \omega) + H'^{(0)}(z, w' + \partial_z k, c, \omega) +$$
$$\langle z(w' + \partial_z k) - c, H'^{(1)}(z, w' + \partial_z k, c, \omega) \rangle +$$
$$\langle z(w' + \partial_z k) - c, H'^{[2]}(z, w' + \partial_z k, c, \omega)(z(w' + \partial_z k) - c) \rangle + g'(z, w' + \partial_z k, c, \omega)$$

Since $(z + \partial_{w'} k)w' = z(w' + \partial_z k) - \mathcal{D}k$ we can write

$$0 = (I) - (II) = -\mathcal{D}^\omega k + \langle \omega, c \rangle - \Gamma'(c, \omega) + H^{(0)}(z + \partial_{w'} k, w', c, \omega) +$$
$$\langle (z + \partial_{w'} k)w' - c, H^{(1)}(z + \partial_{w'} k, w', c, \omega) \rangle +$$
$$\langle z(w' + \partial_z k) - c - \mathcal{D}k, H^{[2]}(z + \partial_{w'} k, w', c, \omega)(z(w' + \partial_z k) - c - \mathcal{D}k) \rangle$$
$$- H'^{(0)}(z, w' + \partial_z k, c, \omega) -$$
$$\langle z(w' + \partial_z k) - c, H'^{(1)}(z, w' + \partial_z k, c, \omega) \rangle -$$
$$\langle z(w' + \partial_z k) - c, H'^{[2]}(z, w' + \partial_z k, c, \omega)(z(w' + \partial_z k) - c) \rangle - g'(z, w' + \partial_z k, c, \omega)$$

and then,

$$0 = (I) - (II) = -\mathcal{D}^\omega k + \langle \omega, c \rangle - \Gamma'(c, \omega) + H^{(0)}(z, w', c, \omega) +$$
$$\langle zw' - c, H^{(1)}(z, w', c, \omega) \rangle - 2\langle \mathcal{D}k, H^{[2]}(z, w', c, \omega)(zw' - c) \rangle$$
$$- H'^{(0)}(z, w' + \partial_z k, c, \omega) - \langle z(w' + \partial_z k) - c, H'^{(1)}(z, w' + \partial_z k, c, \omega) \rangle -$$
$$\langle z(w' + \partial_z k) - c, (H'^{[2]}(z, w' + \partial_z k, c, \omega) - H^{[2]}(z + \partial_{w'} k, w', c, \omega))(z(w' + \partial_z k) - c) \rangle$$
$$- g'(z, w', c, \omega) + \mathcal{Q}$$

where $\mathcal{Q}$ is quadratic expression in $(H^{(0)}, H^{(1)}, g', k)$ and their first derivatives and depending on $H^{[2]}$; more precisely

$$\|\mathcal{Q}\|_{\delta-h,s} \le \frac{C_s}{h^{3d}} \bigg((\|H^{(0)}\|_{\delta,s} + \|H^{(1)}\|_{\delta,s} + \|H^{[2]}\|_{\delta,s} \|k\|_{\delta,0}) \|k\|_{\delta,0} +$$
$$(\|H^{(0)}\|_{\delta,0} + \|H^{(1)}\|_{\delta,0} + \|H^{[2]}\|_{\delta,0} \|k\|_{\delta,0}) \|k\|_{\delta,s} \bigg)$$

Finally,

$$0 = (I) - (II) = -\mathcal{D}^\omega k + \langle \omega, c \rangle - \Gamma'(c, \omega) + H^{(0)}(z, w', c, \omega) +$$

$$\langle zw' - c, H^{(1)}(z, w', c, \omega) \rangle - 2\langle \mathcal{D}k, H^{(2)}(z, w', c, \omega)(zw' - c) \rangle$$

$$- H'^{(0)}(z, w' + \partial_z k, c, \omega) - \langle z(w' + \partial_z k) - c, H'^{(1)}(z, w' + \partial_z k, c, \omega) \rangle -$$

$$\langle z(w' + \partial_z k) - c, (H'^{[2]}(z, w' + \partial_z k, c, \omega) - H^{[2]}(z + \partial_{w'} k, w', c, \omega))(z(w' + \partial_z k) - c) \rangle$$

$$+ \quad 2\langle \mathcal{D}k, H^{[3]}(z, w', c, \omega)(zw' - c)^{\otimes 2} \rangle - g'(z, w', c, \omega) + \mathcal{Q}$$

Let us now *define* $\Gamma'(c, \omega)$, $k(z, w', c, \omega) = k^{(0)}(z, w', c, \omega) + \langle (zw' - c), k^{(1)}(z, w', c, \omega) \rangle$ and $g'(z, w', c, \omega) = g'^{(0)}(z, w', c, \omega) + \langle zw' - c, g'^{(1)}(z, w', c, \omega) \rangle$ according to

$$(6.41) \qquad \begin{cases} \Gamma'(c, \omega) & = \langle c, \omega \rangle + \mathcal{M}H^{(0)} \\ g'^{(0)} & = \mathcal{P}(H^{(0)}) \\ g'^{(1)} & = \mathcal{P}(H^{(1)} - \langle \mathcal{D}k^{(0)}, H^{(2)} \rangle) \end{cases}$$

$$(6.42) \qquad \begin{cases} k^{(0)} & = \mathcal{L}(H^{(0)}) \\ k^{(1)} & = \mathcal{L}(H^{(1)} - \langle \mathcal{D}k^{(0)}, H^{(2)} \rangle) \end{cases}$$

and Z, H' by $H'(z', w') = H(z, w)$ and $Z(z, w, c, \omega) = (z', w')$ if and only if $z' = z + \partial_{w'} k(z, w', c, \omega)$ and $w = w' + \partial_z k(z, w', c, \omega)$: observe that

$$(6.43) \qquad \max(\|k\|_{\delta-h,s}, \|g'\|_{\delta-h,s}) \leq \mathcal{C}_s(\kappa h) \langle \varepsilon, \zeta \rangle_{\delta,s}$$

and hence, if the latter quantity is small enough (see the comment preceding equation (6.45), by the Inverse Function Theorem (see Proposition 10.3 of [EFK]), the change of variables $(z, w) \mapsto (z', w')$ and its inverse are well defined. Since by assumption $M_H := \mathcal{M}(H^{(1)} - \langle \mathcal{D}k^{(0)}, H^{(2)} \rangle) = 0$ we have

$$\begin{cases} \mathcal{D}^\omega k^{(0)}(c, \omega, z, w') & = H^{(0)}(z, w'c, \omega) - (\langle \omega, c^2 \rangle - \Gamma'(c, \omega)) \\ & \quad - g'^{(0)}(z, w', c, \omega) \\ \mathcal{D}^\omega k^{(1)}(c, \omega, z, w') & = H^{(1)}(z, w', c, \omega) - \langle \mathcal{D}k^{(0)}, H^{(2)}(z, w', c, \omega) \rangle \\ & \quad - g'^{(1)}(z, w', c, \omega) \end{cases}$$

We then have

$$H'^{(0)}(z, w' + \partial_z k, c, \omega) + \langle z(w' + \partial_z k) - c, H'^{(1)}(z, w' + \partial_z k, c, \omega)\rangle +$$

$$\langle z(w'+\partial_z k)-c, (H'^{[2]}(z, w'+\partial_z k, c, \omega)-H^{[2]}(z+\partial_{w'} k, w', c, \omega))(z(w'+\partial_z k)-c)\rangle$$

$$- 2\langle \mathcal{D}k, H^{[3]}(z, w', c, \omega)(zw' - c)^{\otimes 2}\rangle$$

$$+ 2\langle \mathcal{D}k^{(1)}(z, w', c, \omega)(zw' - c), H^{(2)}(z, w', c, \omega)(zw' - c)\rangle$$

$$= \mathcal{Q}$$

If we define

$$\tilde{\mathcal{Q}} := H'^{(0)}(z, w'+\partial_z k, c, \omega)+\langle z(w'+\partial_z k)-c, H'^{(1)}(z, w'+\partial_z k, c, \omega)\rangle +$$

$$\langle z(w'+\partial_z k)-c, (H'^{[2]}(z, w'+\partial_z k, c, \omega)-H^{[2]}(z+\partial_{w'} k, w', c, \omega))(z(w'+\partial_z k)-c)\rangle$$

$$- 2\langle \mathcal{D}k(z, w' + \partial_z k, c, \omega), H^{[3]}(z, w' + \partial_z k, c, \omega)(z(w' + \partial_z k) - c)^{\otimes 2}\rangle$$

$$-2\langle \mathcal{D}k^{(1)}(z, w'+\partial_z k, c, \omega)(z(w'+\partial_z k)-c), H^{(2)}(z, w'+\partial_z k, c, \omega)(z(w'+\partial_z k)-c)\rangle$$

we see that $\tilde{\mathcal{Q}}$ is still quadratic in $H^{(0)}$, $H^{(1)}$, g', k and their first derivatives:

$$(6.44) \quad \|\tilde{\mathcal{Q}}\|_{\delta-h,s} \leq \frac{C_s}{h^{3d}}\left(\left(\|H^{(0)}\|_{\delta,s} + \|H^{(1)}\|_{\delta,s} + \|H^{[2]}\|_{\delta,s}\|k\|_{\delta,0}\right.\right.$$

$$\left.+ \|H^{[3]}\|_{\delta,s}\|k\|_{\delta,0}\right)\|k\|_{\delta,0}+$$

$$\left(\|H^{(0)}\|_{\delta,0} + \|H^{(1)}\|_{\delta,0} + \|H^{[2]}\|_{\delta,0}\|k\|_{\delta,0} + \|H^{[3]}\|_{\delta,0}\|k\|_{\delta,0}\right)\|k\|_{\delta,s}\right)$$

and thus

$$\|\tilde{\mathcal{Q}}\|_{\delta-h,s} \leq \mathcal{C}_s(\kappa h))\zeta_{\delta,0}\langle\varepsilon, \varepsilon, \zeta\rangle_{\delta,s}.$$

Coming back to the variables (z, w) and setting $\mathcal{Q}'(z, w) = \tilde{\mathcal{Q}}(z, w')$ we see that

$$H'^{(0)}(z, w, c, \omega) + \langle zw - c, H'^{(1)}(z, w, c, \omega)\rangle +$$

$$\langle zw - c, (H'^{[2]}(z, w, c, \omega) - H^{[2]}(z + \partial_{w'} k, w', c, \omega))(zw - c)\rangle$$

$$- 2\langle \mathcal{D}k(z, w, c, \omega), H^{[3]}(z, w, c, \omega)(zw - c)^{\otimes 2}\rangle$$

$$- \langle \mathcal{D}k^{(1)}(z, w, c, \omega)(zw - c), H^{(2)}(z, w, c, \omega)(zw - c)\rangle = \mathcal{Q}'$$

where $\mathcal{Q}'$ is still quadratic in the following sense: from Proposition 10.3 of [EFK] (estimates on composition with the inverse map of the change

of variables), (6.44), (6.42) and Lemmas 6.1 and 6.3 we get, provided $\mathcal{C}_1(\kappa h)\langle\varepsilon,\zeta\rangle_{\delta,1} \leq 1$ (which is the case if $\varepsilon_{\delta,1} \leq \mathcal{C}_1(\kappa h)\zeta_{\delta,1}^{-1}$)

$$(6.45) \qquad \|\mathcal{Q}'\|_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle\varepsilon,\varepsilon,\zeta\rangle_{\delta,s}$$

By Lemma 2.4, $H'^{(0)}$ and $H'^{(1)}$ are uniquely determined by $\mathcal{Q}'$ since they are in $\widehat{\mathcal{NR}}$ and hence are quadratically small: by Lemma 6.3

$$(6.46) \qquad [H']_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle\varepsilon,\varepsilon,\zeta\rangle_{\delta,s}$$

Then, $H'^{[2]} - H^{[2]}$ is of the order of (the derivative of) k:

$$(6.47) \qquad \{H'^{[2]} - H^{[2]}\}_{\delta-h,s} \leq \mathcal{C}_s(\kappa h))\zeta_{\delta,0}\langle\varepsilon,\zeta\rangle_{\delta,s}$$

Finally, in the case $(H - H^{[2]})(z,w,c^2,\omega) = \mathcal{O}^{2q+1}(z,w,c)$, formulas (6.41) and (6.42) show that $g'(z,w,c^2,\omega)$ and $k(z,w,c^2)$ are $\mathcal{O}^{2q+1}(z,w,c)$. Hence also $(Z - id)(z,w,c^2,\omega) = \mathcal{O}^{2q}(z,w,c)$.

We have so far proved that with the choices (6.41) and (6.42)

$$(H+\langle\omega,\cdot\rangle)\circ Z_k(z,w) = \Gamma'(c,\omega)+\langle\omega, zw-c\rangle+H'(z,w,c,\omega)+g'(z,w,c,\omega),$$

where H' satisfies the estimates (6.46) and (6.47).

We are not completely finished with the proof of our Lemma since nothing insures us that the change of variables Z_k we have performed is σ-symmetric. Let us introduce $\tilde{Z}_k$ the time 1-map of the hamiltonian vector field $\sqrt{-1}J\nabla k$. The equations (6.42) show that $\sqrt{-1}k$ is σ-symmetric (see the remark following Lemma 6.3) and thus $\tilde{Z}_k$ is σ-symmetric (see Lemma 7.1). The assumption (6.40) allows to apply Proposition 7.2: we have $\{Z_k^{-1} \circ \tilde{Z}_k - I\}_{\delta-h,s} = \mathcal{C}_s(h)\langle\|k\|, \|k\|\rangle_{\delta,s}$ and thus we can write

(6.48)
$$(H + \langle\omega,\cdot\rangle) \circ \tilde{Z}_k = \tilde{\Gamma}(c,\omega) + \langle\omega, zw - c\rangle + \tilde{H}(z,w,c,\omega) + g'(z,w,c,\omega).$$

where the estimates on composition of Proposition 10.2 of [EFK] and estimates (6.43) show that

$$(6.49) \qquad [\tilde{H}]_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle\varepsilon,\varepsilon,\zeta\rangle_{\delta,s}$$

$$(6.50) \qquad \{\tilde{H}^{[2]} - H^{[2]}\}_{\delta-h,s} \leq \mathcal{C}_s(\kappa h))\zeta_{\delta,0}\langle\varepsilon,\zeta\rangle_{\delta,s}.$$

Equations (6.41), show that g' is σ-symmetric.

$\square$

6.4.2. *Elimination of the mean value M_H.* Here is a lemma similar to Lemma 8.4 of [EFK], that allows to eliminate M_H by adding a term $\langle\Lambda,\cdot\rangle$.

Lemma 6.6. *Let $W \in \mathcal{E}_\delta^{\omega,\sigma,\infty}$ and denote*

$$\eta_s = \{W - \mathrm{id}\}_{\delta,s}.$$

There exists a constant $\mathcal{C}_0$ such that if $\eta_{\delta,0} \leq \mathcal{C}_0 \zeta_{\delta,0}^{-1}$ then there exists $\Lambda \in \mathcal{C}_\delta^{\omega,\infty}$, $\Lambda = \Lambda(c,\omega)$ such that

$$\tilde{H}_\Lambda = H + \langle \Lambda, \cdot \rangle \circ W$$

verifies $M_{\tilde{H}} = 0$ and such that for all $s \in \mathbb{N}$, $0 < h < \delta$

$$\|\Lambda\|_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle \varepsilon, \zeta, \eta + \zeta + \varepsilon\rangle_{\delta,s}$$

and

$$[\tilde{H}_\Lambda - H_\Lambda]_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle \varepsilon, \zeta, \eta + \zeta + \varepsilon\rangle_{\delta,s}.$$

Furthermore, if $(H - H^{[2]})(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$ then $\Lambda(c^2, \omega) = \mathcal{O}^{2q+1}(c)$ and $(\tilde{H}_\Lambda - \tilde{H}_\Lambda^{[2]})(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$.

Proof. Let us denote by $W_{c,\omega} : (z, w) \mapsto (z', w') = (z + R(z, w, c, \omega), w + T(z, w, c, \omega))$. Using $\tilde{H}_\Lambda = H + \langle \Lambda(c,\omega), (z + R)(w + T) - c\rangle$, we now compute the canonical decomposition of $\tilde{H}_\Lambda$ in terms of the canonical decomposition of H and Λ. We have

$$\langle \Lambda, (z + R)(w + T) - c\rangle = \langle \Lambda, zw - c\rangle + \sum_{j=0}^{2} \langle \Lambda, U^{(j)}(zw - c)^{\otimes j}\rangle$$

$$+ \langle \Lambda, U^{[3]}(zw - c)^{\otimes 3}\rangle$$

where $\sum_{j=0}^{2} U^{(j)}(zw - c)^{\otimes j} + U^{[3]}(zw - c)^{\otimes 3}$ is the canonical decomposition of $zT + wR + RT$. Then, for $j = 0, 2$, $\tilde{H}_\Lambda^{(j)} = H^{(j)} + \langle \Lambda, U^{(j)}\cdot\rangle$ and $\tilde{H}_\Lambda^{(1)} = H^{(1)} + \langle \Lambda, (I + U^{(1)})\cdot\rangle$. From this it follows that $\Lambda \mapsto M_{\tilde{H}_\Lambda} = \mathcal{M}(\tilde{H}_\Lambda^{(1)} - \tilde{H}_\Lambda^{(2)}\mathcal{DL}\tilde{H}_\Lambda^{(0)})$ is a map of the form $M_H + a_1 \cdot \Lambda + a_2 \cdot (\Lambda \otimes \Lambda)$ with

$$\|M_H\|_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle \varepsilon, \zeta\rangle_{\delta,s}.$$

$$\max(\|a_1 - I\|_{\delta-h,s}, \|a_2\|_{\delta-h,s}) \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle \eta + \zeta + \epsilon, \eta\rangle_{\delta,s}$$

Now, the first part of the lemma follows from (6.36), the Inverse Function Theorem and the estimates of Section 10 of [EFK].

Furthermore, since $M_H(c^2, \omega) = \mathcal{O}^{2q+1}(c)$, we see that $\Lambda(c^2, \omega) = \mathcal{O}^{2q+1}(c)$ and the last statement of the lemma is proven $\qquad\square$

6.4.3. *The inductive step.* Putting together Lemmas 6.5 and 6.6 we get similarly to Proposition 8.2 of [EFK] the following KAM induction step.

Lemma 6.7. *Let H, g, Γ be in $\mathcal{C}_{\rho,\delta}^{\omega,\sigma,\infty}$ where g is (κ, τ)-flat and W be in $\mathcal{E}_{\delta}^{\omega,\sigma,\infty}$. There are positive constants C and a such that if $\eta_0 \leq C\zeta_0^{-1}$ and*

$$\varepsilon_1 \leq \left(\kappa h \zeta_{\delta,1}^{-1} (1 + \eta_{\delta,1})^{-1} \right)^a \text{ then there exist } Z' \in \mathcal{E}_{\delta-h}^{\omega,\sigma,\infty}, \ \Gamma', \Lambda', g', H' \in$$

$\mathcal{C}_{\delta-h}^{\omega,\sigma,\infty}$ where g' is (κ, τ)-flat such that

$$(H + \Gamma(c, \omega) + g + \langle \omega, \cdot \rangle + (\langle \Lambda'(c, \omega), \cdot \rangle) \circ W) \circ Z'_{c,\omega}(z, w) =$$
$$\Gamma'(c, \omega) + \langle \omega, zw - c \rangle + H'(z, w, c, \omega) + g'(z, w, c, \omega),$$

and for any $s \in \mathbb{N}$, $0 < h < \delta/2$,

$$[H']_{\delta-h,s} \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}\langle \varepsilon, \varepsilon, \zeta \rangle_{\delta,s}$$

$$\max \left(\|\Lambda\|_{0,\delta-h,s}, \|g' - g\|_{\delta-h,s}, \{H'^{[2]} - H^{[2]}\}_{\delta-h,s}, \right.$$

$$\left. \{Z' - id\}_{\delta-h,s}, \{W \circ Z' - W\}_{\delta-h,s} \right) \leq \mathcal{C}_s(\kappa h)\zeta_{\delta,0}^a \langle \varepsilon, \zeta + \eta \rangle_{\delta,s}$$

Furthermore, if $(H - H^{[2]})(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$ and $g(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$, then $(H' - H'^{[2]})(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$ and $(Z' - id)(z, w, c^2, \omega) = \mathcal{O}^{2q}(z, w, c)$, $\Lambda'(c^2, \omega) = \mathcal{O}^{2q+1}(c)$, $g'(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$.

6.4.4. *Convergence of the KAM scheme.* As in Section 8.5 of [EFK] the preceding Lemma 6.7 applied inductively is enough to prove Theorem 6.4. We refer the reader to Sections 8.4 and 8.5 of [EFK] for a proof of this fact.

6.4.5. *End of the proof.* To this point we have proven a theorem, let us call it (T'), which is Theorem 6.4 except the statement on the analyticity with respect to λ when ω is replaced by $(1+\lambda)\omega_0$. This theorem (T') applied to the analytic function $(z, w, (c, \lambda)) \mapsto H(z, w, c, (1 + \lambda)\omega_0)$, $(z, w, c, \lambda) \in \mathbb{D}_{\delta}^{2d} \times \mathbb{D}_{\delta}^{d+1}$, with $s = 0$, completes the proof of Theorem 6.4.

7. APPENDIX

7.1. **Appendix A.** For $\kappa > 0$ we assume given for some $\eta_\kappa > 0$, with $\lim_{\kappa \to 0} \eta_\kappa = 0$, and a family of maps $W_\kappa : \mathbb{R}^d \times \mathbb{T}^d \to \mathbb{R}^d \times \mathbb{R}^d$ that are

of the form

$$W_\kappa(c, \theta) = W(c, \theta) + \epsilon_\kappa$$
$$W(c, \theta) = (c_1 \sin(2\pi\theta_1), c_1 \cos(2\pi\theta_1), \ldots, c_d \sin(2\pi\theta_d), c_d \cos(2\pi\theta_d))$$

where $\epsilon_\kappa(0, \cdot) \equiv 0$ we have for every $\xi \le \eta_\kappa$

(7.51)
$$\max_{x \in \mathcal{C}_d(\xi) \times \mathbb{T}^d} (|\epsilon_\kappa(x)| + |\nabla\epsilon_\kappa(x)|) < \xi^{2d+1}$$

where $\mathcal{C}_d(\xi) := \{c \in \mathbb{R}^d : |c_i| < \xi, \forall i\}$, and assume that $\Sigma_\kappa \subset \mathcal{C}_d(\eta_\kappa)$ are a family of measurable sets such that

$$\lim_{\kappa \to 0} \frac{\mathrm{mes}(\Sigma_\kappa)}{\mathrm{mes}(\mathcal{C}_d(\eta_\kappa))} = 1.$$

Denote $B_{2d}(0, \xi)$ the product $\{x_1^2 + y_1^2 \le \xi\} \times \ldots \times \{x_d^2 + y_d^2 \le \xi\}$. Then we have the following

Lemma 7.1. *Denote by* $\widetilde{\Sigma}_\kappa = W_\kappa(\Sigma_\kappa \times \mathbb{T}^d)$. *Then, for any* $\nu > 0$, *if* κ *is sufficiently small we have*

(7.52)
$$\mathrm{mes}(\widetilde{\Sigma}_\kappa \cap B_{2d}(0, \eta_\kappa))/\mathrm{mes}(B_{2d}(0, \eta_\kappa)) > 1 - \nu$$

Proof.   For $\epsilon > 0$, define $\mathcal{C}_d(\eta_\kappa, \epsilon) := \mathcal{C}_d(\eta_\kappa) \cap \{|c_i| > \epsilon|c_j|, \forall i, j\}$. We also define $\Sigma_{\kappa,\epsilon} = \Sigma_\kappa \cap \mathcal{C}_d(\eta_\kappa, \epsilon)$.

We have that $W(\mathcal{C}_d(\eta_\kappa) \times \mathbb{T}^d) = B_{2d}(0, \eta_\kappa)$. Also, it is not hard to see that if ϵ and then κ are sufficiently small then

(7.53)
$$\mathrm{mes}(W(\Sigma_{\kappa,\epsilon} \times \mathbb{T}^d))/\mathrm{mes}(B_{2d}(0, \eta_\kappa)) > 1 - \nu^2$$

and from (7.51)

$$|\mathrm{Jac}W_\kappa - \mathrm{Jac}W| < \nu^2|\mathrm{Jac}W|$$

on $\mathcal{C}_d(\eta_\kappa, \epsilon) \times \mathbb{T}^d$, which gives

(7.54)
$$\mathrm{mes}(W_\kappa(\Sigma_{\kappa,\epsilon} \times \mathbb{T}^d)) > (1 - \nu^2)\mathrm{mes}(W(\Sigma_{\kappa,\epsilon} \times \mathbb{T}^d))$$

Wa also have that $W_\kappa(\mathcal{C}_d(\eta_\kappa) \times \mathbb{T}^d) \subset B_{2d}(0, \eta_\kappa + o(\eta_\kappa))$. (7.52) hence follows from (7.53) and (7.54) if $\nu \ll 1$.

$\square$

7.2. Appendix B: Generating functions and time-1 map of Hamiltonian flows.

There are two classical methods to construct symplectic diffeomorphisms. The first one, which we have been using throughout the paper, is the generating function method: given $f : (\mathbb{C}^{2d}, 0) \to \mathbb{C}$ we define the symplectomorphism $Z : (\mathbb{C}^{2d}, 0) \to (\mathbb{C}^{2d}, 0)$ *implicitly* by the equations

(7.55)
$$\left\{ Z_f(z, w) = (z', w') \quad \Longleftrightarrow \quad \begin{cases} z' = z + \partial_{w'} f(z, w') \\ w = w' + \partial_z f(z, w'). \end{cases} \right.$$

A second classical method is to use the so-called "Lie method". Given $f : (\mathbb{C}^{2d}, 0) \to \mathbb{C}$, we introduce the hamiltonian flow (ϕ_f^t) defined by the hamiltonian (with respect to the symplectic form $dz \wedge dw$) vector field $J\nabla f$ where $J = \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$ and we let $\tilde{Z}_f$ be the time-1-symplectomorphism ϕ_f^1.

The first method is well adapted in the formal setting but has the drawback of not preserving σ-symmetry. On the other hand, the Lie method preserves σ-symmetry.

Lemma 7.1. *If $\sqrt{-1}f$ is σ-symmetric, then $\tilde{Z}_f$ is σ-symmetric.*

Proof. Let us use the change of variables $(z, w) \mapsto (x, y)$, $z = \frac{1}{2}(x + \sqrt{-1}y)$, $w = \frac{1}{2}(x - \sqrt{-1}y)$. It transforms the symplectic form $dz \wedge dw$ to $-\sqrt{-1}dx \wedge dy$ and the hamiltonian flow $J\nabla f(z, w)$ is transported to $\sqrt{-1}J\nabla \tilde{f}(x, y)$ where $\tilde{f}(x, y) = f(z, w)$. But $\sqrt{-1}\tilde{f}(x, y)$ takes real values when x and y are real (since $\sqrt{-1}f$ is σ-symmetric). Hence its time-1-map has the same property. Coming back to the variables (z, w) shows that $\tilde{Z}_f$ is σ-symmetric. $\square$

Nevertheless, we notice that, Z_f and $\tilde{Z}_f$ differ by a quantity which is quadratic in f (and its derivatives).

Proposition 7.2. *There exists $\xi > 0$ such that if $f \in \mathcal{C}_\delta^{\omega,\infty}$ satisfies for $0 < h < \delta$*

$$\|f\|_{\delta,1} \leq \xi h^2 \tag{7.56}$$

then [4],

$$\{\tilde{Z}_f^{-1} \circ Z_f - Id\}_{\delta-h,s} = \mathcal{C}_s(h)\langle \|f\|, \|f\| \rangle_{\delta,s} \tag{7.57}$$

Furthermore, if $f(z, w, c^2, \omega) = \mathcal{O}^{2q+1}(z, w, c)$ then $\tilde{Z}_f^{-1} \circ Z_f - id = \mathcal{O}^{2q}(z, w, c)$.

Proof. Let us denote by W the local diffeomorphism $(\mathbb{C}^{2d}, 0) \to \mathbb{C}^{2d}$, $W(z, w) = (z + \partial_w f(z, w, c, \omega), w - \partial_z f(z, w, c, \omega))$. Proposition 7.2 will follow from the following two lemmas 7.3 and 7.4

Lemma 7.3. *There exists a constant $\xi > 0$ such that if $0 < h < \delta$ and f satisfy*

$$\|f\|_{\delta,1} \leq \xi h^2 \tag{7.58}$$

then

$$\{W - Z_f\}_{\delta-h,s} = \mathcal{C}_s(h)\langle \|f\|, \|f\| \rangle_{\delta,s} \tag{7.59}$$

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[4]The notations are those of Section 6.2.

Proof. If $Z_f : (z,w) \mapsto (z',w')$ we can write

(7.60)
$$\begin{cases} z' = z + \partial_w f(z,w,c,\omega) + (\partial_{w'} f(z,w',c,\omega) - \partial_w f(z,w,c,\omega)) \\ w' = w - \partial_z f(z,w,c,\omega) + (\partial_z f(z,w,c,\omega) - \partial_z f(z,w',c,\omega)). \end{cases}$$

and using Proposition (10.3) of [EFK] we notice that $(z,w) \mapsto w'(z,w,c,\omega) - w$ has a $\| \cdot \|_{\delta-h,s}$ norm less or equal than $\mathcal{C}_s(h)\|f\|_{\delta,s}$ and by Proposition 10.2 of [EFK] that $(z,w) \mapsto f(z,w',c,\omega) - f(z,w,c,\omega)$ has a $\| \cdot \|_{\delta-h,s}$ norm less or equal than $\mathcal{C}_s(h)\langle\|f\|,\|f\|\rangle_{\delta,s}$. The conclusion then follows. $\square$

Lemma 7.4. *There exists a constant $\xi > 0$ such that if $0 < h < \delta$ and f satisfy*

(7.61)
$$\|f\|_{\delta,0} \leq \xi h^2$$

then one has

(7.62)
$$\{W - \tilde{Z}_f\}_{\delta-h,s} = \mathcal{C}_s(h)\langle\|f\|,\|f\|\rangle_{\delta,s}$$

Proof. If $u_{u_0,c,\omega}(\cdot)$ (we shall denote u for short, $u(t) := (z(t), w(t))$) is a solution of the differential equation $u'(t) = J\nabla f(u(t), c, \omega)$, $u(0) = u_0$, $|u(0)| \leq \delta - h$, $|c| \leq \delta$, $\omega \in B$ one has, as long as the solution $u(\cdot)$ is defined,

(7.63)
$$u(t) = u(0) + \int_0^t J\nabla f(u(s), c, \omega)ds,$$

Let $[0, t_{max})$ a maximal interval of definition of the solution u and, if it exists, $t_* := \inf\{t \in [0, t_{max}) : |u(t)| > \delta - (h/2)\}$. One has, for $0 \leq t < t_*$ and some constant C

(7.64)
$$|u(t)| \leq |u(0)| + Ct\|J\nabla f\|_{\delta,0}$$

(7.65)
$$\leq |u(0)| + Ct(h/2)^{-1}\|f\|_{\delta,0}.$$

Assume that t_* exists and is ≤ 1; then
$$|u(t_*)| \leq \delta - h + Ct(h/2)^{-1}\|f\|_{\delta,0}$$
$$\leq \delta - (3h/4)$$

provided $C(h/2)^{-1}\|f\|_{\delta,0} \leq h/4$, which is the case if the constant ξ in (7.61) is small enough. But, by definition $|u(t_*)| \geq \delta - (h/2)$ which is a contradiction; hence t_* if it exists is > 1.

The theorem on continuous (and differentiable) dependence of the solution of an O.D.E with respect to the initial condition and parameters then shows that $(z,w) \mapsto \phi_f^1(z,w,c,\omega)$ is an analytic diffeomorphism with respect to $(z,w) \in \mathbb{D}_{\delta-h}^{2d}$, analytic with respect to $c \in \mathbb{D}_\delta^{2d}$ and depending smoothly on $\omega \in B$.

Now, the Linearization Theorem for O.D.E. tells us that the derivative $v(\cdot) = \partial_\omega u_{u_0,c,\omega}(\cdot)$ of $u_{u_0,c,\omega}(\cdot)$ with respect to ω satisfies the affine equation

$$(7.66) \quad v'(t) = D_u J \nabla f(u_{u_0,c,\omega}(t), c, \omega) \cdot v(t) + D_\omega J \nabla f(u_{u_0,c,\omega}(t), c, \omega)$$

with initial condition $v(0) = 0$. More generally, $\partial_\omega^\alpha u_{u_0,c,\omega}$ satisfies the differential equation
(7.67)
$$\frac{d}{dt} \partial_\omega^\alpha u_{u_0,c,\omega}(t) = D_u J \nabla f(u_{u_0,c,\omega}(t), c, \omega) \cdot \partial_\omega^\alpha u_{u_0,c,\omega}(t) + G_\alpha(t, u_0, c, \omega)$$

with initial condition $\partial_\omega u_{u_0,c,\omega}(0) = 0$ and where G_α is a finite sum of terms (the number of which depends only on $|\alpha|$ and d) of the form

$$(7.68) \quad D_u^m D_\omega^l J \nabla f(u_{u_0,c,\omega}(t), c, \omega) \cdot (\partial_\omega^{\beta_1} u_{u_0,c,\omega}(t), \dots, \partial_\omega^{\beta_m} u_{u_0,c,\omega}(t))$$

with $|\beta_1| + \cdots + |\beta_m| + l = |\alpha|$, $(m, l) \neq (1, 0)$. Let us now prove by induction on $|\alpha|$ that for any $0 \leq t \leq 1$

$$(7.69) \qquad \|\partial_\omega^\alpha u_\cdot(t)\|_{\delta - h, 0} \leq \mathcal{C}_{|\alpha|}(h(1 + \|f\|_{0,\delta})^{-1})\|f\|_{\delta, |\alpha|}.$$

Assume that there exist a positive valued increasing function $s \mapsto a(s)$ defined for $s \in \mathbb{N}$, $0 \leq s \leq |\alpha| - 1$ and $C > 0$ such that for any $|\beta| < |\alpha|$, any $0 \leq t \leq 1$

$$(7.70) \qquad \|\partial_\omega^\beta u_\cdot(t)\|_{\delta - h, 0} \leq C h^{-a(|\beta|)}(1 + \|f\|_{\delta, 0})^{a(|\beta|)}\|f\|_{\delta, |\beta|}.$$

We get for $|u_0| \leq \delta - h$, $|c| \leq \delta$, $\omega \in B$, $0 \leq t \leq 1$

$$|(7.68)| \leq C_\alpha \delta^{-m-1} \|D_\omega^l f\|_{\delta, 0}(C h^{-a(|\beta_1|)}(1 + \|f\|_{\delta, 0})^{a(|\beta_1|)}\|f\|_{\delta, |\beta_1|}) \cdots$$
$$(C h^{-a(|\beta_m|)}(1 + \|f\|_{\delta, 0}))^{a(|\beta_m|)}\|f\|_{\delta, |\beta_m|}),$$

and using the convexity estimates, see Proposition 10.1 of [EFK]

$$|(7.68)| \leq C_\alpha \delta^{-m-1} \|f\|_{\delta, 0}^{1 - l/|\alpha|} \|f\|_{\delta, |\alpha|}^{l/|\alpha|}$$
$$(C_{\beta_1} h^{-a(|\beta_1|)}(1 + \|f\|_{\delta, 0})^{a(|\beta_1|)}\|f\|_{\delta, 0}^{1 - |\beta_1|/|\alpha|}\|f\|_{\delta, |\alpha|}^{|\beta_1|/|\alpha|}) \cdots$$
$$(C_{\beta_m} h^{-a(|\beta_m|)}(1 + \|f\|_{\delta, 0})^{a(|\beta_m|)}\|f\|_{\delta, 0}^{1 - |\beta_m|/|\alpha|}\|f\|_{\delta, |\alpha|}^{|\beta_m|/|\alpha|})$$

and finally since $|\beta_1| + \cdots + |\beta_m| + l = |\alpha|$

$$(7.71) \qquad |(7.68)| \leq C_\alpha(h^{-1}(1 + \|f\|_{\delta, 0}))^{a(|\alpha|)}\|f\|_{\delta, |\alpha|}$$

provided $a(|\alpha|) \geq m + 1 + a(|\beta_1|) + \cdots + a(|\beta_m|)$. Let us come back to the affine differential equation (7.67) and let $R(t, s)$ be the resolvent of the associated linear differential equation,

$$(7.72) \qquad\qquad v'(t) = D_u J \nabla f(u_{u_0,c,\omega}(t), c, \omega) \cdot v(t).$$

By the variation of constant formula we get

$$\partial_\omega^\alpha u_{u_0,c,\omega}(t) = \int_0^t R(t,s)G_\alpha(t,u_0,c,\omega)ds.$$

For $u_0 \in \mathbb{D}_{\delta-h}^{2d}$, $c \in \mathbb{D}_\delta^{2d}$, $\omega \in B$, $0 \le s \le t \le 1$ we see that $|R(t,s)| \le e^M$ where M is the supremum of the norm of $D_u J\nabla f(u,c,\omega)$ on $\mathbb{D}_{\delta-h/2} \times \mathbb{D}_\delta \times B$. We notice that $M \le \text{const.}\|f\|_{\delta,0}h^{-2}$ and that if the constant ξ in (7.61) is small enough $M \le 1$. Hence, for $0 \le t \le 1$, we get from (7.71)

$$(7.73) \qquad |\partial_\omega^\alpha u_{u_0,c,\omega}(t)| \le C_\alpha e(h^{-1}(1+\|f\|_{\delta,0}))^{a(|\alpha|)}\|f\|_{\delta,|\alpha|}.$$

This complete the proof of (7.69) by induction.

To finish the proof of the Lemma we write

$$(7.74) \quad u_{u_0,c,\omega}(1) - u_0 = \int_0^1 (J\nabla f(u_{u_0,c,\omega}(s),c,\omega) - J\nabla f(u_0,c,\omega))ds$$

and use Proposition 10.2 (i) of [EFK]:

$$\|u.(1) - \cdot\|_{\delta-h,s} \le h^{-1}\int_0^1 \langle \|J\nabla f\|, u.(t)\rangle_{\delta,s}dt$$

which is $\le \mathcal{C}_s(h(1+\|f\|_{\delta,0}^{-1})\langle\|f\|, \|f\|\rangle_{\delta,s}.$ $\qquad\square$

The proof of Proposition 7.2 can now be completed using the estimates of Proposition 10.2 and 10.3 of [EFK] on compositions and inverses of functions. The last statement of the Proposition follows from the validity of a similar statement in Lemma 7.3 and 7.4. $\qquad\square$

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