

ON A CLASS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH JUMPS AND ITS PROPERTIES

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ABSTRACT. We study stochastic differential equations with jumps with no diffusion part. We provide some basic stochastic characterizations of solutions of the corresponding non-local partial differential equations and prove the Harnack inequality for a class of these operators. We also establish key connections between the recurrence properties of these jump processes and the non-local partial differential operator. One of the key results is the regularity of solutions of the Dirichlet problem for a class of operators with weakly Hölder continuous kernels.

CONTENTS

1. Introduction	1
1.1. Notation	3
2. Preliminaries	5
3. Probabilistic representations of solutions of non-local PDE	5
3.1. Three lemmas concerning operators with measurable kernels	6
3.2. A class of operators with weakly Hölder continuous kernels	9
3.3. Some results concerning the fractional Laplacian with drift	11
4. The Harnack property for operators containing a drift term	16
5. Positive recurrence and invariant probability measures	18
5.1. Existence of invariant probability measures	20
5.2. A class of operators with variable order kernels	24
5.3. Mean recurrence times for weakly Hölder continuous kernels	27
6. The Dirichlet problem for weakly Hölder continuous kernels	30
References	39

1. INTRODUCTION

Stochastic differential equations (SDEs) with jumps have received wide attention in stochastic analysis as well as in the theory of differential equations. Unlike continuous diffusion processes, SDEs with jumps have long range interactions and therefore the generators of such processes are non-local in nature. These processes arise in various applications, for instance, in mathematical finance and control [21, 32] and image processing [24]. There have been various studies on such processes from a stochastic analysis viewpoint concentrating on existence, uniqueness, and stability properties of the solution of the stochastic differential equation [1, 8, 19, 20, 27, 29], as well as from a differential equation viewpoint focusing on the existence and regularity of viscosity solutions

Date: November 3, 2021.

2000 Mathematics Subject Classification. Primary 60J45, 60J75, 58F11; Secondary 60H30.

Key words and phrases. Stochastic differential equations with jumps, Harnack inequality, α -stable process, invariant probability measure, invariant density, exit time, Dynkin's formula, ergodicity.

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[5, 6, 15]. One of our objectives in this paper is to establish stochastic representations of solutions of SDEs with jumps via the associated integro-differential operator.

Let us consider a Markov process X in \mathbb{R}^d with generator \mathcal{A} . Let D be a smooth bounded domain in \mathbb{R}^d . We denote the first exit time of the process X from D by $\tau(D) = \inf\{t \geq 0 : X_t \notin D\}$. One can formally say that

$$u(x) := \mathbb{E}_x \left[\int_0^{\tau(D)} f(X_s) ds \right] \quad (1.1)$$

satisfies the following equation

$$\mathcal{A}u = -f \quad \text{in } D, \quad u = 0 \quad \text{in } D^c, \quad (1.2)$$

where \mathbb{E}_x denotes the expectation operator on the canonical space of the process starting at x when $t = 0$. An important question is when can we actually identify the solution of (1.2) as the right hand side of (1.1). When $\mathcal{A} = \Delta + b$, i.e., X is a drifted Brownian motion, one can use the regularity of the solution and Itô's formula to establish (1.1). Clearly then, one standard method to obtain a representation of the mean first exit time from D is to find a classical solution of (1.2) for non-local operators. This is related to the work in [9] where estimates on classical solutions are obtained when $D = \mathbb{R}^d$. The author in [9] also raises questions concerning the existence and regularity of solutions to the Dirichlet problem for non-local operators. We provide a partial answer to these questions in Theorem 6.1.

One of the main results of this paper is the existence of a classical solution of (1.2) for a fairly general class of non-local operators. We focus on operators of the form

$$\mathcal{I}f(x) = b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \pi(x, z) dz, \quad (1.3)$$

where

$$\mathfrak{d}f(x; z) := f(x + z) - f(x) - \mathbf{1}_{\{|z| \leq 1\}} \nabla f(x) \cdot z, \quad (1.4)$$

with $\mathbf{1}_A$ denoting the indicator function of a set A . The kernel π satisfies the usual integrability conditions. When $\pi(x, z) = \frac{k(x, z)}{|z|^{d+\alpha}}$, with $\alpha \in (1, 2)$, and b , k and f are locally Hölder in x with exponent β , and $k(x, \cdot) - k(x, 0)$ satisfies the integrability condition in (3.10), we show in Theorem 3.1 that u defined by (1.1) is the unique solution of (1.2) in $C_{\text{loc}}^{2s+\beta}(D) \cap C(\mathbb{R}^d)$. This result can be extended to include non-zero boundary conditions provided that the boundary data is regular enough. The proof is based on various regularity results concerning the Dirichlet problem, that are of independent interest and can be found in Section 6. For the case $k \equiv 1$, with continuous f and g , we characterize the solution of

$$\mathcal{I}u = -f \quad \text{in } D, \quad u = g \quad \text{in } D^c \quad (1.5)$$

in the viscosity framework. Theorem 3.2, which appears later in Section 3, asserts that

$$u(x) = \mathbb{E}_x \left[\int_0^{\tau(D)} f(X_s) ds + g(X_{\tau(D)}) \right], \quad x \in \mathbb{R}^d,$$

is the unique viscosity solution to (1.5). One of the hurdles in establishing this lies in showing that $\mathbb{E}_x[\tau(\bar{D})] = 0$ whenever $x \in \partial D$. When X is a drifted Brownian motion, this can be easily deduced from the fact that Brownian motion has infinitely many zeros in every finite interval. But similar crossing properties are not known for α -stable processes. We also have to restrict ourselves to the regime $\alpha \in (1, 2)$, so that the jump process ‘dominates’ the drift, and this allows us to establish that $\mathbb{E}_x[\tau(\bar{D})] = 0$ whenever $x \in \partial D$. The proof technique uses an estimate of the first exit time of an α -stable process from a cone [30]. These auxiliary results can be found in Section 3.

Recall that a function h is said to be *harmonic* with respect to X in D if $h(X_{t \wedge \tau(D)})$ is a martingale. One of the important properties of nonnegative harmonic functions for nondegenerate continuous diffusions is the Harnack inequality, which plays a crucial role in various regularity

and stability estimates. The work in [12] proves the Harnack inequality for a class of pure jump processes, and this is further generalized in [10] for non-symmetric kernels that may have variable order. A parabolic Harnack inequality is obtained in [7] for symmetric jump processes associated with the Dirichlet form with a symmetric kernel. In [33] sufficient conditions on Markov processes to satisfy the Harnack inequality are identified. Let us also mention the work in [4, 22, 34] where a Harnack inequality is established for jump processes with a non-degenerate diffusion part. Recently, [25] proves a Harnack type estimate for harmonic functions that are not necessarily nonnegative in all of \mathbb{R}^d .

In this paper we prove a Harnack inequality for harmonic functions relative to the operator \mathcal{I} in (1.3) when k and b are locally bounded and measurable, and either $k(x, z) = k(x, -z)$, or $|\pi(x, z) - \pi(x, 0)|$ is a lower order kernel (Theorem 4.1). The method of the proof is based on verifying the sufficient conditions in [33]. Later we use this Harnack estimate to obtain certain stability results for the process. Let us also mention that the estimates obtained in Section 3 and Section 4 may also be used to establish Hölder continuity for harmonic functions by following a similar method as in [11]. However we don't pursue this here.

In Section 5 we discuss the ergodic properties of the process such as positive recurrence, invariant probability measures, etc. We provide a sufficient condition for positive recurrence and the existence of an invariant probability measure. This is done via imposing a *Lyapunov stability* condition on the generator. Following Has'minskiĭ's method, we establish the existence of a unique invariant probability measure for a fairly large class of processes. We also show that one may obtain a positive recurrent process by using a non-symmetric kernel and no drift (see Theorem 5.3). In this case, the non-symmetric part of the kernel plays the role of the drift. Let us mention here that in [36] the author provides sufficient conditions for stability for a class of jump diffusions and this is accomplished by constructing suitable Lyapunov type functions. However, the class of kernels considered in [36] satisfies a different set of hypotheses than those assumed in this paper, and in a certain way lies in the complement of the class of Lévy kernels that we consider. Stability of 1-dimensional processes is discussed in [35] under the assumption of Lebesgue-irreducibility. Lastly, we want to point out one of the interesting results of this paper, which is the characterization of the mean hitting time of a bounded domain as a viscosity solution of the exterior Dirichlet problem (Theorem 5.4). This is established for the class of operators with weakly Hölder continuous kernels in Definition 3.3.

The organization of the paper is as follows. In Section 1.1 we introduce the notation used in the paper. In Section 2 we introduce the model and assumptions. Section 3 establishes stochastic representations of viscosity solutions. In Section 4 we show the Harnack inequality. Section 5 establishes the connections between the recurrence properties of the process and solutions of the non-local equations. Finally, Section 6 is devoted to the proof of the regularity of solutions to the Dirichlet problem for weakly Hölder continuous kernels. These results are used in Section 5.

1.1. Notation. The standard norm in the d -dimensional Euclidean space \mathbb{R}^d is denoted by $|\cdot|$, and we let $\mathbb{R}_*^d := \mathbb{R}^d \setminus \{0\}$. The set of non-negative real numbers is denoted by \mathbb{R}_+ , \mathbb{N} stands for the set of natural numbers, and $\mathbf{1}_A$ denotes the indicator function of a set A . For vectors $a, b \in \mathbb{R}^d$, we denote the scalar product by $a \cdot b$. We denote the maximum (minimum) of two real numbers a and b by $a \vee b$ ($a \wedge b$). We let $a^+ := a \vee 0$ and $a^- := (-a) \vee 0$. By $[a]$ ($\lceil a \rceil$) we denote the largest (least) integer less than (greater than) or equal to the real number a . For $x \in \mathbb{R}^d$ and $r \geq 0$, we denote by $B_r(x)$ the open ball of radius r around x in \mathbb{R}^d , while B_r without an argument denotes the ball of radius r around the origin. Also in the interest of simplifying the notation we use $B \equiv B_1$, i.e., the unit ball centered at 0.

Given a metric space \mathcal{S} , we denote by $\mathcal{B}(\mathcal{S})$ and $B_b(\mathcal{S})$ the Borel σ -algebra of \mathcal{S} and the set of bounded Borel measurable functions on \mathcal{S} , respectively. The set of Borel probability measures on

\mathcal{S} is denoted by $\mathcal{P}(\mathcal{S})$, $\|\cdot\|_{\text{TV}}$ denotes the total variation norm on $\mathcal{P}(\mathcal{S})$, and δ_x the Dirac mass at x . For any function $f : \mathcal{S} \rightarrow \mathbb{R}^d$ we define $\|f\|_\infty := \sup_{x \in \mathcal{S}} |f(x)|$.

The closure and the boundary of a set $A \subset \mathbb{R}^d$ are denoted by \bar{A} and ∂A , respectively, and $|A|$ denotes the Lebesgue measure of A . We also define

$$\tau(A) := \inf \{s \geq 0 : X_s \notin A\}.$$

Therefore $\tau(A)$ denotes the first exit time of the process X from A . For $R > 0$, we often use the abbreviated notation $\tau_R := \tau(B_R)$.

We introduce the following notation for spaces of real-valued functions on a set $A \subset \mathbb{R}^d$. The space $L^p(A)$, $p \in [1, \infty)$, stands for the Banach space of (equivalence classes) of measurable functions f satisfying $\int_A |f(x)|^p dx < \infty$, and $L^\infty(A)$ is the Banach space of functions that are essentially bounded in A . For an integer $k \geq 0$, the space $C^k(A)$ ($C^\infty(A)$) refers to the class of all functions whose partial derivatives up to order k (of any order) exist and are continuous, $C_c^k(A)$ is the space of functions in $C^k(A)$ with compact support, and $C_b^k(A)$ is the subspace of $C^k(A)$ consisting of those functions whose derivatives up to order k are bounded. Also, the space $C^{k,r}(A)$, $r \in (0, 1]$, is the class of all functions whose partial derivatives up to order k are Hölder continuous of order r . For simplicity we write $C^{0,r}(A) = C^r(A)$. For any $\gamma > 0$, $C^\gamma(A)$ denotes the space $C^{[\gamma], \gamma - [\gamma]}(A)$, under the convention $C^{k,0}(A) = C^k(A)$.

In general if \mathcal{X} is a space of real-valued functions on a domain D , \mathcal{X}_{loc} consists of all functions f such that $f\varphi \in \mathcal{X}$ for every $\varphi \in C_c^\infty(D)$.

For a nonnegative multiindex $\beta = (\beta_1, \dots, \beta_d)$, we let $|\beta| := \beta_1 + \dots + \beta_d$ and $D^\beta := \partial_1^{\beta_1} \dots \partial_d^{\beta_d}$, where $\partial_i := \frac{\partial}{\partial x_i}$.

Let D be a bounded domain with a C^2 boundary. Define $d_x := \text{dist}(x, \partial D)$ and $d_{xy} := \min(d_x, d_y)$. For $u \in C(D)$ and $r \in \mathbb{R}$, we introduce the weighted norm

$$[u]_{0;D}^{(r)} := \sup_{x \in D} d_x^r |u(x)|,$$

and, for $k \in \mathbb{N}$ and $\delta \in (0, 1]$, the seminorms

$$[u]_{k;D}^{(r)} := \sup_{|\beta|=k} \sup_{x \in D} d_x^{k+r} |D^\beta u(x)|$$

$$[u]_{k,\delta;D}^{(r)} := \sup_{|\beta|=k} \sup_{x,y \in D} \left(d_{xy}^{k+\delta+r} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\delta} \right).$$

For $r \in \mathbb{R}$ and $\gamma \geq 0$, with $\gamma + r \geq 0$, we define the space

$$\mathcal{C}_\gamma^{(r)}(D) := \{u \in C^\gamma(D) \cap C(\mathbb{R}^d) : u(x) = 0 \text{ for } x \in D^c, \|u\|_{\gamma;D}^{(r)} < \infty\},$$

where

$$\|u\|_{\gamma;D}^{(r)} := \sum_{k=0}^{[\gamma]-1} [u]_{k;D}^{(r)} + [u]_{[\gamma]-1, \gamma+1-[\gamma];D}^{(r)},$$

under the convention $\|u\|_{0;D}^{(r)} = [u]_{0;D}^{(r)}$. We also use the notation $\|u\|_{k,\delta;D}^{(r)} = \|u\|_{k+\delta;D}^{(r)}$ for $\delta \in (0, 1]$.

It is straightforward to verify that $\|u\|_{\gamma;D}^{(r)}$ is a norm, under which $\mathcal{C}_\gamma^{(r)}(D)$ is a Banach space.

If the distance functions d_x or d_{xy} are not included in the above definitions, we denote the corresponding seminorms by $[\cdot]_{k;D}$ or $[\cdot]_{k,\delta;D}$ and define

$$\|u\|_{C^{k,\delta}(D)} := \sum_{\ell=0}^k [u]_{\ell;D} + [u]_{k,\delta;D}.$$

Thus, $\|u\|_{C^\gamma(D)}$ is well defined for any $\gamma > 0$, by the identification $C^\gamma(D) = C^{[\gamma], \gamma - [\gamma]}(A)$.

We recall the well known interpolation inequalities [23, Lemma 6.32, p. 30]. Let $u \in C^{2,\beta}(D)$. Then for any ε there exists a constant $C = C(\varepsilon, j, k, r)$ such that

$$\begin{aligned} [u]_{j,\gamma;D}^{(0)} &\leq C \|u\|_{0;D}^{(0)} + \varepsilon [u]_{k,\beta;D}^{(0)} \\ [u]_{j,\gamma;D}^{(0)} &\leq C \|u\|_{0;D}^{(0)} + \varepsilon [u]_{k,\beta;D}^{(0)} \end{aligned} \quad j = 0, 1, 2, \quad 0 \leq \beta, \gamma \leq 1, \quad j + \gamma < k + \beta.$$

2. PRELIMINARIES

Let $\alpha \in (1, 2)$. Let $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\pi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be two given measurable functions where π is nonnegative. We define the non-local operator \mathcal{I} as follows:

$$\mathcal{I}f(x) := b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \pi(x, z) dz, \quad (2.1)$$

with $\mathfrak{d}f$ as in (1.4). We always assume that

$$\int_{\mathbb{R}^d} (|z|^2 \wedge 1) \pi(x, z) dz < \infty \quad \forall x \in \mathbb{R}^d.$$

Note that (2.1) is well-defined for any $f \in C_b^2(\mathbb{R}^d)$. Let $\Omega = \mathcal{D}([0, \infty), \mathbb{R}^d)$ denote the space of all right continuous functions mapping $[0, \infty)$ to \mathbb{R}^d , having finite left limits (cádlág). Define $X_t = \omega(t)$ for $\omega \in \Omega$ and let $\{\mathcal{F}_t\}$ be the right-continuous filtration generated by the process $\{X_t\}$. In this paper we always assume that given any initial distribution ν_0 there exists a strong Markov process (X, \mathbb{P}_{ν_0}) that satisfies the martingale problem corresponding to \mathcal{I} , i.e., $\mathbb{P}_{\nu_0}(X_0 \in A) = \nu_0(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and for any $f \in C_b^2(\mathbb{R}^d)$,

$$f(X_t) - f(X_0) - \int_0^t \mathcal{I}f(X_s) ds$$

is a martingale with respect to the filtration $\{\mathcal{F}_t\}$. We denote the law of the process by \mathbb{P}_x when $\nu_0 = \delta_x$. Sufficient conditions on b and π to ensure the existence of such processes are available in the literature. Unfortunately, the available sufficient conditions do not cover a wide class of processes. We refer the reader to [8] for the available results in this direction, as well as to [2, 19, 20, 27, 29]. When $b \equiv 0$, well-posedness of the martingale problem is obtained under some regularity assumptions on π in [1].

Let us mention once more that our goal here is not to study the existence of a solution to the martingale problem. Therefore, we do not assume any regularity conditions on the coefficients, unless otherwise stated. Before we proceed to state our assumptions and results, we recall the Lévy-system formula, the proof of which is a straightforward adaptation of the proof for a purely non-local operator and can be found in [12, Proposition 2.3 and Remark 2.4] (see also [19, 22]).

Proposition 2.1. *If A and B are disjoint Borel sets in $\mathcal{B}(\mathbb{R}^d)$, then for any $x \in \mathbb{R}^d$,*

$$\sum_{s \leq t} \mathbf{1}_{\{X_{s-} \in A, X_s \in B\}} - \int_0^t \int_B \mathbf{1}_{\{X_s \in A\}} \pi(X_s, z - X_s) dz ds$$

is a \mathbb{P}_x -martingale.

3. PROBABILISTIC REPRESENTATIONS OF SOLUTIONS OF NON-LOCAL PDE

The aim in this section is to give a rigorous mathematical justification of the connections between stochastic differential equations with jumps and viscosity solutions to associated non-local differential equations.

Recall the generator in (2.1) where f is in $C_b^2(\mathbb{R}^d)$. We also recall the definition of a viscosity solution [5, 15].

Definition 3.1. Let D be a domain with C^2 boundary. A function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ which is upper (lower) semi-continuous on \bar{D} is said to be a sub-solution (super-solution) to

$$\begin{aligned} \mathcal{I}u &= -f \quad \text{in } D, \\ u &= g \quad \text{in } D^c, \end{aligned}$$

where \mathcal{I} is given by (2.1), if for any $x \in \bar{D}$ and a function $\varphi \in C^2(\mathbb{R}^d)$ such that $\varphi(x) = u(x)$ and $\varphi(z) > u(z)$ ($\varphi(z) < u(z)$) on $\mathbb{R}^d \setminus \{x\}$, it holds that

$$\mathcal{I}\varphi(x) \geq -f(x) \quad (\mathcal{I}\varphi(x) \leq -f(x)), \quad \text{if } x \in D,$$

while, if $x \in \partial D$, then

$$\max(\mathcal{I}\varphi(x) + f(x), g(x) - u(x)) \geq 0 \quad (\min(\mathcal{I}\varphi(x) + f(x), g(x) - u(x)) \leq 0).$$

A function u is said to be a viscosity solution if it is both a sub- and a super-solution.

In Definition 3.1 we may assume that φ is bounded, provided u is bounded. Otherwise, we may modify the function φ by replacing it with u outside a small ball around x . It is evident that every classical solution is also a viscosity solution.

Let f and g be two continuous functions on \mathbb{R}^d , with g bounded. Given a bounded domain D , we let

$$u(x) = \mathbb{E}_x \left[\int_0^{\tau(D)} f(X_s) ds + g(X_{\tau(D)}) \right] \quad \text{for } x \in \mathbb{R}^d, \quad (3.1)$$

where \mathbb{E}_x denotes the expectation operator relative to \mathbb{P}_x . In this section we characterize u as a solution of a non-local differential equation. As usual, we say that b is locally bounded, if for any compact set K , $\sup_{x \in K} |b(x)| < \infty$.

3.1. Three lemmas concerning operators with measurable kernels.

Lemma 3.1. *Let D be a bounded domain. Suppose X is a strong Markov process associated with \mathcal{I} in (2.1), with b locally bounded, and that the integrability conditions*

$$\sup_{x \in K} \int_{\{|z|>1\}} |z| \pi(x, z) dz < \infty, \quad \text{and} \quad \inf_{x \in K} \int_{\mathbb{R}^d} |z|^2 \pi(x, z) dz = \infty \quad (3.2)$$

hold for any compact set K . Then $\sup_{x \in D} \mathbb{E}_x[(\tau(D))^m] < \infty$, for any positive integer m .

Proof. Without loss of generality we assume that $0 \in D$. Otherwise we inflate the domain to include 0. Let $\bar{d} = \text{diam}(D)$ and $M_D = \sup_{x \in D} |b(x)|$. Recall that B_R denotes the ball of radius R around the origin. We choose $R > 1 \vee 2(\bar{d} \vee M_D)$, and large enough so as to satisfy the inequality

$$\inf_{x \in D} \int_{B_R} |z|^2 \pi(x, z) dz > 1 + 2\bar{d}M_D + 2\bar{d} \sup_{x \in D} \int_{\{1 < |z| \leq R\}} |z| \pi(x, z) dz.$$

We let $f \in C_b^2(\mathbb{R}^d)$ be a radially increasing function such that $f(x) = |x|^2$ for $|x| \leq 2R$ and $f(x) = 8R^2$ for $|x| \geq 2R + 1$. Then, for any $x \in D$, we have

$$\begin{aligned} \mathcal{I}f(x) &= b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \pi(x, z) dz \\ &\geq -2\bar{d}M_D + \int_{B_R} (f(x+z) - f(x) - \nabla f(x) \cdot z) \pi(x, z) dz \\ &\quad + \int_{\{1 < |z| \leq R\}} \nabla f(x) \cdot z \pi(x, z) dz + \int_{B_R^c} (f(x+z) - f(x)) \pi(x, z) dz. \end{aligned}$$

Also, for any $|z| \geq R$, it holds that $|x+z| \geq \bar{d} \geq |x|$. Therefore $f(x+z) \geq f(x)$. Hence

$$\mathcal{I}f(x) \geq -2\bar{d}M_D + \int_{\{1 < |z| \leq R\}} \nabla f(x) \cdot z \pi(x, z) dz$$

$$\begin{aligned}
& + \int_{B_R} (f(x+z) - f(x) - \nabla f(x) \cdot z) \pi(x, z) dz \\
& \geq -2\bar{d}M_D - 2\bar{d} \int_{\{1 < |z| \leq R\}} |z| \pi(x, z) dz + \int_{B_R} |z|^2 \pi(x, z) dz \\
& \geq 1.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}_x[f(X_{\tau(D) \wedge t})] - f(x) & = \mathbb{E}_x \left[\int_0^{\tau(D) \wedge t} \mathcal{I}f(X_s) ds \right] \\
& \geq \mathbb{E}_x[\tau(D) \wedge t] \quad \forall x \in D.
\end{aligned}$$

Letting $t \rightarrow \infty$ we obtain $\mathbb{E}_x[\tau(D)] \leq 8R^2$. Since $x \in D$ is arbitrary, this shows that

$$\sup_{x \in D} \mathbb{E}_x[\tau(D)] \leq 8R^2.$$

We continue by using the method of induction. We have proved the result for $m = 1$. Assume that it is true for m , i.e., $M_m := \sup_{x \in D} \mathbb{E}_x[(\tau(D))^m] < \infty$. Let $h(x) = M_m f(x)$ where f is defined above. Then from the calculations above we obtain

$$\mathbb{E}_x[h(X_{\tau(D) \wedge t})] - h(x) \geq \mathbb{E}_x[M_m(\tau(D) \wedge t)] \quad \forall x \in D. \quad (3.3)$$

Denoting $\tau(D)$ by τ we have

$$\begin{aligned}
\mathbb{E}_x[\tau^{m+1}] & = \mathbb{E}_x \left[\int_0^\infty (m+1)(\tau-t)^m \mathbf{1}_{\{t < \tau\}} dt \right] \\
& = \mathbb{E}_x \left[\int_0^\infty (m+1) \mathbb{E}_x[(\tau-t)^m \mathbf{1}_{\{t < \tau\}} \mid \mathcal{F}_{t \wedge \tau}] dt \right] \\
& = \mathbb{E}_x \left[\int_0^\infty (m+1) \mathbf{1}_{\{t \wedge \tau < \tau\}} \mathbb{E}_{X_{t \wedge \tau}}[\tau^m] dt \right] \\
& \leq \sup_{x \in D} \mathbb{E}_x[\tau^m] \mathbb{E}_x \left[\int_0^\infty (m+1) \mathbf{1}_{\{t \wedge \tau < \tau\}} dt \right] \\
& \leq M_m(m+1) \mathbb{E}_x[\tau],
\end{aligned}$$

and in view of (3.3), the proof is complete. \square

Boundedness of solutions to the Dirichlet problem on bounded domains and with zero boundary data is asserted in the following lemma.

Lemma 3.2. *Let b and f be locally bounded functions and D a bounded domain. Suppose π satisfies (3.2). Then there exists a constant C , depending on $\text{diam}(D)$, $\sup_{x \in D} |b(x)|$ and π , such that any viscosity solution u to the equation*

$$\begin{aligned}
\mathcal{I}u & = f \quad \text{in } D, \\
u & = 0 \quad \text{in } D^c,
\end{aligned}$$

satisfies $\|u\|_\infty \leq C \sup_{x \in D} |f(x)|$.

Proof. As shown in the proof of Lemma 3.1, there exists a nonnegative, radially nondecreasing function $\xi \in C_b^2(\mathbb{R}^d)$ satisfying $\mathcal{I}\xi(x) > \sup_{x \in D} |f(x)|$ for all $x \in \bar{D}$. Let $M > 0$ be the smallest number such that $M - \xi$ touches u from above at least at one point. We claim that $M \leq \|\xi\|_\infty$. If not, then $M - \xi(x) > 0$ for all $x \in D^c$. Therefore $M - \xi$ touches u in D from above. Hence by the definition of a viscosity solution we have $\mathcal{I}(M - \xi(x)) \geq f(x)$, or equivalently, $\mathcal{I}\xi(x) \leq -f(x)$,

where $x \in D$ is a point of contact from above. But this contradicts the definition of ξ . Thus $M \leq \|\xi\|_\infty$. Also by the definition of M we have

$$\sup_{x \in D} u(x) \leq \sup_{x \in D} (M - \xi(x)) \leq M \leq \|\xi\|_\infty.$$

The result then follows by applying the same argument to $-u$. \square

Definition 3.2. Let \mathfrak{L}_α denote the class of operators \mathcal{I} of the form

$$\mathcal{I}f(x) := b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz, \quad f \in C_b^2(\mathbb{R}^d), \quad (3.4)$$

with $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow (0, \infty)$ Borel measurable and locally bounded, and $\alpha \in (1, 2)$. We also assume that $x \mapsto \sup_{z \in \mathbb{R}^d} k^{-1}(x, z)$ is locally bounded. The subclass of \mathfrak{L}_α consisting of those \mathcal{I} satisfying $k(x, z) = k(x, -z)$ is denoted by $\mathfrak{L}_\alpha^{\text{sym}}$.

Consider the following *growth condition*: There exists a constant K_0 such that

$$x \cdot b(x) \vee |x| k(x, z) \leq K_0 (1 + |x|^2) \quad \forall x, z \in \mathbb{R}^d. \quad (3.5)$$

It turns out that under (3.5), the Markov process associated with \mathcal{I} does not have finite explosion time, as the following lemma shows.

Lemma 3.3. *Let $\mathcal{I} \in \mathfrak{L}_\alpha$ and suppose that for some constant $K_0 > 0$, the data satisfies the growth condition in (3.5). Let X be a Markov process associated with \mathcal{I} . Then*

$$\mathbb{P}_x \left(\sup_{s \in [0, T]} |X_s| < \infty \right) = 1 \quad \forall T > 0.$$

Proof. Let $\delta \in (0, \alpha - 1)$, and $f \in C^2(\mathbb{R}^d)$ be a non-decreasing, radial function satisfying

$$f(x) = (1 + |x|^\delta) \quad \text{for } |x| \geq 1, \quad \text{and } f(x) \geq 1 \quad \text{for } |x| < 1.$$

We claim that

$$\left| \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| \leq \kappa_0 (1 + |x|^\delta) \quad \forall x \in \mathbb{R}^d, \quad (3.6)$$

for some constant κ_0 . To prove (3.6) first note that since the second partial derivatives of f are bounded over \mathbb{R}^d , it follows that $\left| \int_{|z| \leq 1} \mathfrak{d}f(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right|$ is bounded by some constant. It is easy to verify that, provided $z \neq 0$, then

$$\begin{aligned} \left| |x+z|^\delta - |x|^\delta \right| &\leq 2\delta |z| |x|^{\delta-1}, & \text{if } |x| \geq 2|z|, \\ \left| |x+z|^\delta - |x|^\delta \right| &\leq 8|z|^\delta, & \text{if } |x| < 2|z|, \end{aligned} \quad (3.7)$$

for some constant κ . By the hypothesis in (3.5), for some constant c , we have

$$k(x, z) \leq c(1 + |x|) \quad \forall x \in \mathbb{R}^d. \quad (3.8)$$

Combining (3.7)–(3.8) we obtain, for $|x| > 1$,

$$\begin{aligned} \left| \int_{|z| > 1} \mathfrak{d}f(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| &\leq \int_{1 < |z| \leq \frac{|x|}{2}} 2\delta c (1 + |x|) |x|^{\delta-1} |z| \frac{1}{|z|^{d+\alpha}} dz \\ &\quad + \int_{|z| > \frac{|x|}{2}} 8c (1 + |x|) |z|^\delta \frac{1}{|z|^{d+\alpha}} dz \\ &\leq \kappa(d) \left(\frac{2\delta c}{\alpha - 1} (1 + |x|) |x|^{\delta-1} + \frac{2^{3+\alpha-\delta} c}{\alpha - \delta} (1 + |x|) |x|^{\delta-\alpha} \right) \end{aligned}$$

for some constant $\kappa(d)$, thus establishing (3.6).

By (3.6) and the assumption on the growth of b in (3.5), we obtain

$$|\mathcal{I}f(x)| \leq K_1 f(x) \quad \forall x \in \mathbb{R}^d,$$

for some constant K_1 . Then, by Dynkin's formula, we have,

$$\begin{aligned} \mathbb{E}_x[f(X_{t \wedge \tau_n})] &= f(x) + \mathbb{E}_x \left[\int_0^{t \wedge \tau_n} \mathcal{I}f(X_s) ds \right] \\ &\leq f(x) + K_1 \mathbb{E}_x \left[\int_0^{t \wedge \tau_n} f(X_s) ds \right] \\ &\leq f(x) + K_1 \int_0^t \mathbb{E}_x[f(X_{s \wedge \tau_n})] ds, \end{aligned}$$

where in the second inequality we use the property that f is radial and non-decreasing. Hence, by the Gronwall inequality, we have

$$\mathbb{E}_x[f(X_{t \wedge \tau_n})] \leq f(x) e^{K_1 t} \quad \forall t > 0, \quad \forall n \in \mathbb{N}. \quad (3.9)$$

Since $\mathbb{E}_x[f(X_{t \wedge \tau_n})] \geq f(n) \mathbb{P}_x(\tau_n \leq t)$, we obtain by (3.9) that

$$\begin{aligned} \mathbb{P}_x \left(\sup_{s \in [0, T]} |X_s| \geq n \right) &= \mathbb{P}_x(\tau_n \leq T) \\ &\leq \frac{f(x)}{1 + n^\delta} e^{K_1 T} \quad \forall T > 0, \quad \forall n \in \mathbb{N}, \end{aligned}$$

from which the conclusion of the lemma follows. \square

3.2. A class of operators with weakly Hölder continuous kernels. We introduce a class of kernels whose numerators $k(x, z)$ are locally Hölder continuous in x , and $z \mapsto k(x, z)$ is bounded, locally in x . We call such kernels π weakly Hölder continuous since they have the property that for any f satisfying $\int_{\mathbb{R}^d} \frac{f(z)}{|z|^{d+\alpha}} dz < \infty$ the map $x \mapsto \int_{\mathbb{R}^d} f(z) \frac{k(x, z)}{|z|^{d+\alpha}} dz$ is locally Hölder continuous.

Definition 3.3. Let $\lambda : [0, \infty) \rightarrow (0, \infty)$ be a nondecreasing function that plays the role of a parameter. For a bounded domain D define $\lambda_D := \sup \{\lambda(R) : D \subset B_{R+1}\}$. Let $\mathfrak{I}_\alpha(\beta, \theta, \lambda)$, where $\beta \in (0, 1]$, $\theta \in (0, 1)$, denote the class of operators \mathcal{I} as in (3.4) that satisfy, on each bounded domain D , the following properties:

- (a) $\alpha \in (1, 2)$.
- (b) b is locally Hölder continuous with exponent β , and satisfies

$$|b(x)| \leq \lambda_D, \quad \text{and} \quad |b(x) - b(y)| \leq \lambda_D |x - y|^\beta \quad \forall x, y \in D.$$

- (c) The map $k(x, z)$ is continuous in x and measurable in z and satisfies

$$\begin{aligned} |k(x, z) - k(y, z)| &\leq \lambda_D |x - y|^\beta \quad \forall x, y \in D, \quad \forall z \in \mathbb{R}^d \\ \lambda_D^{-1} &\leq k(x, z) \leq \lambda_D \quad \forall x \in D, \quad \forall z \in \mathbb{R}^d. \end{aligned}$$

- (d) For any $x \in D$, we have

$$\left| \int_{\mathbb{R}^d} (|z|^{\alpha-\theta} \wedge 1) \frac{|k(x, z) - k(x, 0)|}{|z|^{d+\alpha}} dz \right| \leq \lambda_D. \quad (3.10)$$

Remark 3.1. It is evident that if $|k(x, z) - k(x, 0)| \leq \tilde{\lambda}_D |z|^{\theta'}$ for some $\theta' > \theta$, then property (d) of Definition 3.3 is satisfied.

We may view \mathcal{I} as the sum of the translation invariant operator \mathcal{I}_0 defined by

$$\mathcal{I}_0 u(x) := b(x) \cdot \nabla u(x) + \int_{\mathbb{R}^d} \mathfrak{d}u(x; z) \frac{k(x, 0)}{|z|^{d+2s}} dz,$$

which is uniformly elliptic on every bounded domain, and a perturbation that takes the form

$$\tilde{\mathcal{I}} u(x) := \int_{\mathbb{R}^d} \mathfrak{d}u(x; z) \frac{k(x, z) - k(x, 0)}{|z|^{d+2s}} dz.$$

We are not assuming that the numerator k is symmetric, as in the approximation techniques in [13, 16, 28]. Moreover, these operators are not addressed in [17] due to the presence of the drift term.

For operators in the class $\mathfrak{I}_\alpha(\beta, \theta, \lambda)$, we have the following regularity result concerning solutions to the Dirichlet problem.

Theorem 3.1. *Let $\mathcal{I} \in \mathfrak{I}_\alpha(\beta, \theta, \lambda)$, D be a bounded domain with C^2 boundary, and $f \in C^\beta(\bar{D})$. We assume that neither β , nor $2s + \beta$ are integers, and that either $\beta < s$, or that $\beta \geq s$ and*

$$|k(x, z) - k(x, 0)| \leq \tilde{\lambda}_D |z|^\theta \quad \forall x \in D, \forall z \in \mathbb{R}^d,$$

for some positive constant $\tilde{\lambda}_D$. Let \mathbb{E}_x denote the expectation operator corresponding to the Markov process X with generator given by \mathcal{I} . Then $u(x) := \mathbb{E}_x \left[\int_0^{\tau(D)} f(X_s) ds \right]$ is the unique solution in $C^{\alpha+\beta}(D) \cap C(\bar{D})$ to the equation

$$\begin{aligned} \mathcal{I}u &= -f \quad \text{in } D, \\ u &= 0 \quad \text{in } D^c. \end{aligned}$$

Proof. For $\varepsilon > 0$, we denote by D_ε the ε -neighborhood of D , i.e.,

$$D_\varepsilon := \{z \in \mathbb{R}^d : \text{dist}(z, D) < \varepsilon\}. \quad (3.11)$$

Note that for ε small enough, D_ε has a C^2 boundary. Then by Theorem 6.1 there exists $u_\varepsilon \in C^{\alpha+\beta}(D) \cap C(\bar{D})$ satisfying

$$\begin{aligned} \mathcal{I}u_\varepsilon &= -f \quad \text{in } D_\varepsilon, \\ u_\varepsilon &= 0 \quad \text{in } D_\varepsilon^c. \end{aligned}$$

In the preceding equation f stands for the Lipschitz extension of f . We also have the estimate (recall the definition of $\|\cdot\|_{\beta; D}^{(r)}$ in Section 1.1)

$$\|u_\varepsilon\|_{\alpha+\beta; D_\varepsilon}^{(-r)} \leq C_0 \|f\|_{C^\beta(\bar{D}_\varepsilon)},$$

with r some fixed constant in $(0, \frac{\alpha}{2})$. As can be seen from the Lemma 3.2 and the proof of Theorem 6.1, we may select a constant C_0 , that does not depend on ε , for ε small enough. Since $u_\varepsilon = 0$ in D_ε^c , it follows that

$$\|u_\varepsilon\|_{C^r(\mathbb{R}^d)} \leq c_1 \|u_\varepsilon\|_{\alpha+\beta; D_\varepsilon}^{(-r)}$$

for some constant c_1 , independent of ε , for all small enough ε . Hence $u_\varepsilon \rightarrow u$ as $\varepsilon \rightarrow 0$, along some subsequence, and $u \in C^{\alpha+\beta}(D) \cap C(\bar{D})$ by Theorem 6.1. By Itô's formula, we obtain

$$u_\varepsilon(x) = \mathbb{E}_x [u_\varepsilon(X_{\tau(D)})] + \mathbb{E}_x \left[\int_0^{\tau(D)} f(X_s) ds \right].$$

Letting $\varepsilon \searrow 0$, we obtain the result. Uniqueness follows from Theorem 6.1. \square

Theorem 3.1 can be extended to account for non-zero boundary conditions, provided the boundary data is regular enough, say in $C^3(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$.

3.3. Some results concerning the fractional Laplacian with drift. In the rest of this section we consider a smaller class of operators, but the data of the Dirichlet problem is only continuous. We focus on stochastic differential equations driven by a symmetric α -stable process. More precisely, we consider a process X satisfying

$$dX_t = b(X_t) dt + dL_t, \quad (3.12)$$

where L_t is a symmetric α -stable process with generator given by

$$-(-\Delta)^{\alpha/2} f(x) = c(d, \alpha) \int_{\mathbb{R}^d} \mathfrak{D}f(x; z) \frac{1}{|z|^{d+\alpha}} dz, \quad f \in C_b^2(\mathbb{R}^d),$$

with $\alpha \in (1, 2)$, and $c(d, \alpha)$ a normalizing constant. Then the solution of (3.12) is also a solution to the martingale problem for $\bar{\mathcal{I}}$ given by

$$\bar{\mathcal{I}}f(x) := -(-\Delta)^{\alpha/2} f(x) + b(x) \cdot \nabla f(x), \quad \alpha \in (1, 2).$$

The following condition is in effect for the rest of this section unless mentioned otherwise.

Condition 3.1. There exists a positive constant M such that

$$\begin{aligned} |b(x) - b(y)| &\leq M|x - y| \quad \forall x, y \in \mathbb{R}^d, \\ \|b\|_\infty &\leq M. \end{aligned}$$

Under Condition 3.1, equation (3.12) has a unique adapted strong càdlàg solution for any initial condition $X_0 = x \in \mathbb{R}^d$, which is a Feller process [2]. We need the following assertion whose proof is standard, and therefore omitted.

Lemma 3.4. *Assume Condition 3.1 holds and $T > 0$. Let $x_n \rightarrow x$ as $n \rightarrow \infty$ and X^n, X denote the solutions to (3.12) with initial data $X_0^n = x_n, X_0 = x$, respectively. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{s \in [0, T]} |X_s^n - X_s|^2 \right] = 0.$$

The rest of the section is devoted to the proof of the following result.

Theorem 3.2. *Let $D \subset \mathbb{R}^d$ be a bounded domain with C^1 boundary, $f \in C_b(D)$, and $g \in C_b(D^c)$. The function $u(\cdot)$ defined in (3.1) is continuous and bounded, and is the unique viscosity solution to the equation*

$$\begin{aligned} \bar{\mathcal{I}}u &= -f \quad \text{in } D, \\ u &= g \quad \text{in } D^c. \end{aligned} \quad (3.13)$$

The proof of Theorem 3.2 relies on several lemmas which follow. The following lemma is a careful modification of [34, Lemma 2.1].

Lemma 3.5. *Let D be a given bounded domain. There exists a constant $\kappa_1 > 0$ such that for any $x \in D$ and $r \in (0, 1)$*

$$\mathbb{P}_x \left(\sup_{0 \leq s \leq t} |X_s - x| > r \right) \leq \kappa_1 t r^{-\alpha} \quad \forall x \in D,$$

where X satisfies (3.12), and $X_0 = x$.

Proof. Let $f \in C_b^2(\mathbb{R}^d)$ be such that $f(x) = |x|^2$ for $|x| \leq \frac{1}{2}$, and $f(x) = 1$ for $|x| \geq 1$. Let c_1 be a constant such that

$$\begin{aligned} \|\nabla f\|_\infty &\leq c_1, \\ |f(x+z) - f(x) - \nabla f(x) \cdot z| &\leq c_1 |z|^2 \quad \forall x, z \in \mathbb{R}^d. \end{aligned}$$

Define $f_r(y) = f(\frac{y-x}{r})$ where x is a point in D . For $y \in \bar{B}_r(x)$, we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \mathfrak{d}f_r(y; z) \frac{1}{|z|^{d+\alpha}} dz \right| &\leq \left| \int_{|z| \leq r} (f_r(y+z) - f_r(y) - \nabla f_r(y) \cdot z) \frac{1}{|z|^{d+\alpha}} dz \right| \\ &\quad + \left| \int_{|z| > r} (f_r(y+z) - f_r(y)) \frac{1}{|z|^{d+\alpha}} dz \right| \\ &\leq c_1 \frac{1}{r^2} \int_{|z| \leq r} |z|^{2-d-\alpha} dz + 2 \int_{|z| > r} |z|^{-d-\alpha} dz \\ &\leq \frac{c_2}{r^\alpha} \end{aligned}$$

for some constant c_2 . Since $\alpha > 1$, we have

$$|\bar{\mathcal{I}}f_r(y)| \leq \frac{c_3}{r^\alpha} \quad \forall y \in \bar{B}_r(x),$$

where c_3 is a positive constant depending on c_2 and M . Therefore using Itô's formula we obtain

$$\frac{c_3}{r^\alpha} \mathbb{E}_x[\tau(\bar{B}_r(x)) \wedge t] \geq \mathbb{E}_x[f_r(X_{\tau(\bar{B}_r(x)) \wedge t})].$$

Since $f_r = 1$ on $B_r^c(x)$, we have $\mathbb{P}_x(\tau(\bar{B}_r(x)) \leq t) \leq c_3 r^{-\alpha} t$. This completes the proof. \square

Remark 3.2. It is clear from the proof of Lemma 3.5 that the result also holds for operators $\mathcal{I} \in \mathfrak{L}_\alpha$. However, in this case, the constant κ_1 depends also on the local bounds of k and b .

We define the following process

$$Y_t := x + L_t. \tag{3.14}$$

In other words, Y is a symmetric α -stable Lévy process starting at x . It is straightforward to verify using the martingale property that for any measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, we have

$$\mathbb{E}_x[f(Y_t)] = \mathbb{E}_x[f(aY_{a^{-\alpha}t})]. \tag{3.15}$$

We recall the following theorem from [30, Theorem 1].

Theorem 3.3. *Let $\theta \in (0, \pi)$. Let G be a closed cone in \mathbb{R}^d , $d \geq 2$, of angle θ with vertex at 0. For $d = 1$ we let the cone to be the closed half line. Define*

$$\eta(G) = \inf \{t \geq 0 : Y_t \notin G\}.$$

Then there exists a constant $p_\alpha(\theta) > 0$ such that

$$\begin{aligned} \mathbb{E}_x[(\eta(G))^p] &< \infty \quad \text{for } p < p_\alpha(\theta), \\ \mathbb{E}_x[(\eta(G))^p] &= \infty \quad \text{for } p > p_\alpha(\theta), \end{aligned}$$

for all $x \in G \setminus \{0\}$.

The result in [30] is proven for open cones. The statement in Theorem 3.3 follows from the fact that every closed cone is contained in an open cone except for the vertex of the cone and with probability 1 the exit location from an open cone is not the vertex. The following result is also obtained in [19] for $d \geq 2$, using estimates of the transition density. Our proof technique is different, so we present it here.

Lemma 3.6. *Under the process X defined in (3.12), for any bounded domain D , satisfying the exterior cone condition, and $x \in \partial D$, it holds that $\mathbb{P}_x(\tau(\bar{D}) > 0) = 0$.*

Proof. Let $x_0 \in \partial D$ be a fixed point. We consider an open cone G in the complement of \bar{D} at a distance r from the boundary ∂D , such that the distance between the cone and the boundary equals the length of the linear segment connecting x_0 with the vertex of the cone at x_r . In fact, we may choose an angle θ and an axis for the cone that can be kept fixed for all r small enough and the above mentioned property holds. It is quite clear that this can be done for some truncated cone. So first we assume that the full cone G with angle θ and vertex x_r lies in D^c . Let $\eta(G^c)$ denote the first hitting time of G . Since a translation of coordinates does not affect the first hitting time, we may assume that $x_r = 0$. Then from Theorem 3.3, there exists $p \in (0, p_\alpha(\theta))$, satisfying

$$\mathbb{E}_{x_0}[(\eta(G^c))^p] = |x_0|^{\alpha p} \mathbb{E}_{\frac{x_0}{|x_0|}}[(\eta(G^c))^p] < \infty,$$

where we used the property (3.15). Since G is open, by upper semi-continuity we have

$$\sup \{ \mathbb{E}_z[(\eta(G^c))^p] : z \in G^c, |z| = 1 \} < \infty.$$

Therefore we can find a constant $\kappa > 0$ not depending on r (for r small) such that

$$\mathbb{E}_{x_0}[(\eta(G^c))^p] \leq \kappa |r|^{\alpha p}. \quad (3.16)$$

Let $\alpha' \in (1, \alpha)$. Then by (3.16), for any $\varepsilon > 0$, we may choose r small enough so that

$$\mathbb{P}_{x_0}(\eta(G^c) > r^{\alpha'}) \leq \kappa r^{(\alpha - \alpha')p} < \varepsilon. \quad (3.17)$$

Using Condition 3.1 and (3.12), (3.14), we have

$$\sup_{s \in [0, r^{\alpha'}]} |X_s - Y_s| \leq M r^{\alpha'}, \quad (3.18)$$

with probability 1. Hence on $\{\eta(G^c) \leq r^{\alpha'}\}$ we have $|Y_{\eta(G^c)} - X_{\eta(G^c)}| \leq M r^{\alpha'}$ by (3.18). But $Y_{\eta(G^c)} \in G$ and $\text{dist}(x_0, G) = r$. Since $M r^{\alpha'} < r$ for r small enough we have $X_{\eta(G^c)} \in (\bar{D})^c$ on $\{\eta(G^c) \leq r^{\alpha'}\}$. Therefore from (3.17) we obtain

$$\mathbb{P}_{x_0}(\tau(\bar{D}) > r^{\alpha'}) < \varepsilon$$

for all r small enough. This concludes the proof for the case when we can fit a whole cone in D^c near x_0 . For any other scenario we can modify the domain locally around x_0 and deduce that the first exit time from the new domain is 0. We use Lemma 3.5 to assert that with high probability the paths spend r^α amount of time in a ball of radius of order r . Combining these two facts concludes the proof. \square

Remark 3.3. The result of Lemma 3.6 still holds if X satisfies (3.12) in a weak-sense for some locally bounded measurable drift b (see also [20]).

The following corollary follows from Lemma 3.6.

Corollary 3.1. *Under the process X defined in (3.12), for any bounded domain D with C^1 boundary, $\mathbb{P}_x(\tau(D) = \tau(\bar{D})) = 1$ for all $x \in \bar{D}$.*

Lemma 3.7. *Under the process X defined in (3.12), for any bounded domain D with C^1 boundary, and $x \in D$, we have*

$$\mathbb{P}_x(X_{\tau(D)-} \in \partial D, X_{\tau(D)} \in \bar{D}^c) = 0,$$

$$\mathbb{P}_x(X_{s-} \in \partial D, X_s \in D, X_t \in \bar{D} \text{ for all } t \in [0, s]) = 0.$$

Proof. We only prove the first equality, as the proof for the second one follows along the same lines. Condition 3.1 implies that X_t has a density for every $t > 0$ [14]. We let

$$\hat{D}_R := \{z \in D^c : \text{dist}(z, D) \geq R\}.$$

It is enough to prove that $\mathbb{P}_x(X_{\tau(D)-} \in \partial D, X_{\tau(D)} \in \hat{D}_R) = 0$ for every $R > 0$. For any $t > 0$, we obtain by Proposition 2.1 that

$$\begin{aligned} \mathbb{P}_x(X_{t \wedge \tau(D)-} \in \partial D, X_{t \wedge \tau(D)} \in \hat{D}_R) &\leq \mathbb{E}_x \left[\sum_{s \leq t} \mathbf{1}_{\{X_{s-} \in \partial D, X_s \in \hat{D}_R\}} \right] \\ &= c(d, \alpha) \mathbb{E}_x \left[\int_0^t \mathbf{1}_{\{X_s \in \partial D\}} \int_{\hat{D}_R} \frac{1}{|X_s - z|^{d+\alpha}} dz ds \right] \\ &\leq \frac{\kappa}{R^\alpha} \mathbb{E}_x \left[\int_0^t \mathbf{1}_{\{X_s \in \partial D\}} ds \right] \end{aligned}$$

for some constant κ . But the term on the right hand side of the above inequality is 0 by the fact the X_s has density. Hence $\mathbb{P}_x(X_{t \wedge \tau(D)-} \in \partial D, X_{t \wedge \tau(D)} \in \hat{D}_R) = 0$ for any $t > 0$. Letting $t \rightarrow \infty$ completes the proof. \square

Proof of Theorem 3.2. Uniqueness follows by the comparison principle in [18, Corollary 2.9]. Since f and g are bounded, it follows from Lemma 3.1 that u is bounded. Also in view of Lemma 3.6, $u(x) = g(x)$ for $x \in \overline{D^c}$. First we show that u is continuous in D . Let $x_n \rightarrow x$ in D as $n \rightarrow \infty$. To simplify the notation, we let τ^n denote the first exit time from D for the process X^n that starts at x_n . Similarly, τ corresponds to the process X that starts at x . From Lemma 3.4 we obtain

$$\mathbb{E} \left[\sup_{s \in [0, T+1]} |X_s^n - X_s|^2 \right] \xrightarrow{n \rightarrow \infty} 0.$$

Passing to a subsequence we may assume that

$$\sup_{s \in [0, T+1]} |X_s^n - X_s| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \text{ a.s.} \quad (3.19)$$

Recall the definition of D_ε in (3.11). It is evident that, for any $\varepsilon > 0$, (3.19) implies that

$$\liminf_{n \rightarrow \infty} \tau^n \wedge T \leq \tau(D_\varepsilon) \wedge T.$$

Since $\tau(D_\varepsilon) \xrightarrow{\varepsilon \searrow 0} \tau(\bar{D})$ a.s., we obtain

$$\liminf_{n \rightarrow \infty} \tau^n \wedge T \leq \tau(\bar{D}) \wedge T. \quad (3.20)$$

On the other hand, $\tau(\bar{D}) = \tau(D)$ a.s. by Corollary 3.1, and thus we obtain from (3.20) that

$$\liminf_{n \rightarrow \infty} \tau^n \wedge T \leq \tau \wedge T \quad \text{a.s.}$$

The reverse inequality, i.e.,

$$\liminf_{n \rightarrow \infty} \tau^n \wedge T \geq \tau \wedge T \quad \text{a.s.},$$

is evident from (3.19). Hence we have

$$\lim_{n \rightarrow \infty} \tau^n \wedge T = \tau \wedge T, \quad (3.21)$$

with probability 1. It then follows by (3.19) and (3.21) that

$$\mathbb{E} \left[\int_0^{\tau^n \wedge T} f(X_s^n) ds - \int_0^{\tau \wedge T} f(X_s) ds \right] \xrightarrow{n \rightarrow \infty} 0 \quad \forall T > 0.$$

By Lemma 3.1 we can take limits as $T \rightarrow \infty$ to obtain,

$$\mathbb{E} \left[\int_0^{\tau^n} f(X_s^n) ds - \int_0^{\tau} f(X_s) ds \right] \xrightarrow{n \rightarrow \infty} 0. \quad (3.22)$$

Since g is bounded, by Lemma 3.1 we obtain

$$\mathbb{E} \left[\mathbf{1}_{\{\tau \geq T\}} |g(X_{\tau^n}) - g(X_\tau)| \right] \leq 2 \|g\|_\infty \mathbb{P}(\tau \geq T) \xrightarrow{T \rightarrow \infty} 0.$$

From now on we consider a continuous extension of g on \mathbb{R}^d , also denoted by g . We use the triangle inequality

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_\tau)| \right] &\leq \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_{\tau^n})| \right] + \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_\tau)| \right] \\ &\leq \mathbb{E} \left[\mathbf{1}_{\{\tau^n < T+\varepsilon\}} \mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_{\tau^n})| \right] \\ &\quad + 2 \|g\|_\infty \mathbb{P}(\tau^n \geq \tau + \varepsilon) + \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_\tau)| \right]. \end{aligned} \quad (3.23)$$

The second term on the right hand side of (3.23) tends to 0 as $n \rightarrow \infty$, by (3.21). The first term is dominated by

$$\mathbb{E} \left[\left(\sup_{0 \leq t \leq T+\varepsilon} |g(X_t^n) - g(X_t)| \right) \mathbf{1}_{\{\tau^n < T+\varepsilon\}} \mathbf{1}_{\{\tau < T\}} \right],$$

so it also tends to 0 as $n \rightarrow \infty$, by (3.19), and the continuity and boundedness of g . For the third term, we write

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_\tau)| \right] &\leq \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n \wedge T}) - g(X_{\tau \wedge T})| \right] \\ &\quad + \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} |g(X_{\tau^n}) - g(X_{\tau^n \wedge T})| \right]. \end{aligned} \quad (3.24)$$

The first term on the right hand side of (3.24) tends to 0, as $n \rightarrow \infty$, by (3.21), Lemma 3.7, and the continuity and boundedness of g . The second term also tends to 0 as $T \rightarrow \infty$, uniformly in n , by Lemma 3.1. Combining the above, we obtain

$$\mathbb{E} \left[|g(X_{\tau^n}) - g(X_\tau)| \right] \xrightarrow{T \rightarrow \infty} 0. \quad (3.25)$$

By (3.1), (3.22) and (3.25), it follows that $u(x_n) \rightarrow u(x)$, as $n \rightarrow \infty$, which shows that u is continuous.

Next we show that u is a viscosity solution to (3.13). By the strong Markov property of X , for any $t \geq 0$, we have

$$u(x) = \mathbb{E}_x \left[\int_0^{\tau(D) \wedge t} f(X_s) ds + u(X_{\tau(D) \wedge t}) \right]. \quad (3.26)$$

Let $\varphi \in C_b^2(\mathbb{R}^d)$ be such that $\varphi(x) = u(x)$ and $\varphi(z) > u(z)$ for all $z \in \mathbb{R}^d \setminus \{x\}$. Then by (3.26) and Itô's formula we have

$$\begin{aligned} \mathbb{E}_x \left[\int_0^{\tau(D) \wedge t} \mathcal{I}\varphi(X_s) ds \right] &= \mathbb{E}_x [\varphi(X_{\tau(D) \wedge t})] - \varphi(x) \\ &\geq \mathbb{E}_x [u(X_{\tau(D) \wedge t})] - u(x) \\ &= - \mathbb{E}_x \left[\int_0^{\tau(D) \wedge t} f(X_s) ds \right]. \end{aligned}$$

Dividing both sides by t and letting $t \rightarrow 0$ we obtain $\mathcal{I}\varphi(x) \geq -f(x)$ and thus u is a sub-solution. Similarly we can show that u is super-solution and so is a viscosity solution. \square

The following theorem proves the stability of the viscosity solutions over a convergent sequence of domains.

Theorem 3.4. *Let D_n, D be a collection of C^1 domains such that $D_n \rightarrow D$ in the Hausdorff topology, as $n \rightarrow \infty$. Let $f, g \in C_b(\mathbb{R}^d)$. Then $u_n \rightarrow u$, as $n \rightarrow \infty$, where u_n and u are the viscosity solutions of (3.13) in D_n and D , respectively.*

Proof. We only need to establish that for any $T > 0$, $\tau(D_n) \wedge T \rightarrow \tau(D) \wedge T$ with probability 1, and that $X_{\tau(D_n)} \rightarrow X_{\tau(D)}$ on $\{\tau(D) < T\}$, as $n \rightarrow \infty$. This can be shown following the same argument as in the proof of Theorem 3.2. \square

4. THE HARNACK PROPERTY FOR OPERATORS CONTAINING A DRIFT TERM

In this section, we prove a Harnack inequality for harmonic functions. The classes of operators considered are summarized in the following definition.

Definition 4.1. With λ as in Definition 3.3, we let $\mathfrak{L}_\alpha(\lambda)$, denote the class of operators $\mathcal{I} \in \mathfrak{L}_\alpha$ satisfying

$$|b(x)| \leq \lambda_D, \quad \text{and} \quad \lambda_D^{-1} \leq k(x, z) \leq \lambda_D \quad \forall x \in D, z \in \mathbb{R}^d,$$

for a bounded domain D . As in Definition 3.2, the subclass of $\mathfrak{L}_\alpha(\lambda)$ consisting of those \mathcal{I} satisfying $k(x, z) = k(x, -z)$ is denoted by $\mathfrak{L}_\alpha^{\text{sym}}(\lambda)$. Also by $\mathfrak{J}_\alpha(\theta, \lambda)$ we denote the subset of $\mathfrak{L}_\alpha(\lambda)$ satisfying

$$\left| \int_{\mathbb{R}^d} (|z|^{\alpha-\theta} \wedge 1) \frac{|k(x, z) - k(x, 0)|}{|z|^{d+\alpha}} dz \right| \leq \lambda_D \quad \forall x \in D,$$

for any bounded domain D .

A measurable function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be *harmonic* with respect to \mathcal{I} in a domain D if for any bounded subdomain $G \subset D$, it satisfies

$$h(x) = \mathbb{E}_x[h(X_{\tau(G)})] \quad \forall x \in G,$$

where (X, \mathbb{P}_x) is a strong Markov process associated with \mathcal{I} .

Theorem 4.1. *Let D be a bounded domain of \mathbb{R}^d and $K \subset D$ be compact. Then there exists a constant C_H depending on K , D and λ , such that any bounded, nonnegative function which is harmonic in D with respect to an operator $\mathcal{I} \in \mathfrak{L}_\alpha^{\text{sym}}(\lambda) \cup \mathfrak{J}_\alpha(\theta, \lambda)$, $\theta \in (0, 1)$, satisfies*

$$h(x) \leq C_H h(y) \quad \text{for all } x, y \in K.$$

We prove Theorem 4.1 by verifying the conditions in [33] where a Harnack inequality is established for a general class of Markov processes. We accomplish this through Lemmas 4.1–4.3 which follow. Let us also mention that some of the proof techniques are standard but we still add them for clarity. In fact, the Harnack property with non-symmetric kernel is also discussed in [33] under some regularity condition on $k(\cdot, \cdot)$ and under the assumption of the existence of a harmonic measure. Our proof of Lemma 4.1(b) which follows holds under very general conditions, and does not rely on the existence of a harmonic measure. In Lemmas 4.1–4.3, (X, \mathbb{P}_x) is a strong Markov process associated with $\mathcal{I} \in \mathfrak{L}_\alpha^{\text{sym}}(\lambda) \cup \mathfrak{J}_\alpha(\theta, \lambda)$, and D is a bounded domain.

Lemma 4.1. *Let D be a bounded domain. There exist positive constants κ_2 and r_0 such that for any $x \in D$ and $r \in (0, r_0)$,*

- (a) $\inf_{z \in B_{\frac{r}{2}}(x)} \mathbb{E}_z[\tau(B_r(x))] \geq \kappa_2^{-1} r^\alpha$,
- (b) $\sup_{z \in B_r(x)} \mathbb{E}_z[\tau(B_r(x))] \leq \kappa_2 r^\alpha$.

Proof. By Lemma 3.5 and Remark 3.2 there exists a constant κ_1 such that

$$\mathbb{P}_x(\tau(B_r(x)) \leq t) \leq \kappa_1 t r^{-\alpha}, \tag{4.1}$$

for all $t \geq 0$, and all $x \in D_2 := \{y : \text{dist}(y, D) < 2\}$. We choose $t = \frac{r^\alpha}{2\kappa_1}$. Then for $z \in B_{\frac{r}{2}}(x)$, we obtain by (4.1) that

$$\begin{aligned} \mathbb{E}_z[\tau(B_r(x))] &\geq \mathbb{E}_z[\tau(B_{\frac{r}{2}}(z))] \\ &\geq \frac{r^\alpha}{2\kappa_1} \mathbb{P}_z\left(\tau(B_{\frac{r}{2}}(z)) > \frac{r^\alpha}{2\kappa_1}\right) \end{aligned}$$

$$\geq \frac{r^\alpha}{4\kappa_1}.$$

This proves the part (a).

To prove part (b) we consider a radially non-decreasing function $f \in C_b^2(\mathbb{R}^d)$, which is convex in B_4 , and satisfies

$$f(x+z) - f(x) - z \cdot \nabla f(x) \geq c_1 |z|^2 \quad \text{for } |x| \leq 1, |z| \leq 3,$$

for some positive constant c_1 . For an arbitrary point $x_0 \in D$, define $g_r(x) := f(\frac{x-x_0}{r})$. Then for $x \in B_r(x_0)$ and $\mathcal{I} \in \mathfrak{L}_\alpha^{\text{sym}}(\lambda)$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathfrak{d}g_r(x; z) \frac{k(x, z)}{|z|^{\alpha+d}} dz &= \int_{|z| \leq 3r} (g_r(x+z) - g_r(x) - z \cdot \nabla g_r(x)) \frac{k(x, z)}{|z|^{\alpha+d}} dz \\ &\quad + \int_{|z| > 3r} (g_r(x+z) - g_r(x)) \frac{k(x, z)}{|z|^{\alpha+d}} dz \\ &\geq \frac{c_1}{r^2} \lambda_D^{-1} \int_{|z| \leq 3r} |z|^{2-d-\alpha} dz \\ &= c_2 \frac{3^{2-\alpha}}{2-\alpha} \lambda_D^{-1} r^{-\alpha} \end{aligned}$$

for some constant $c_2 > 0$, where in the first equality we use the fact that $k(x, z) = k(x, -z)$, and for the second inequality we use the property that $g(x+z) \geq g(x)$ for $|z| \geq 3r$. It follows that we may choose r_0 small enough such that

$$\mathcal{I}g_r(x) \geq c_3 r^{-\alpha} \quad \text{for all } r \in (0, r_0), x \in B_r(x_0), \text{ and } x_0 \in D,$$

with $c_3 := \frac{c_2}{2} \frac{3^{2-\alpha}}{2-\alpha} \lambda_D^{-1}$.

To obtain a similar estimate for $\mathcal{I} \in \mathfrak{I}_\alpha(\theta, \lambda)$ we fix some $\theta_1 \in (0, \theta \wedge (\alpha - 1))$. Let $\hat{k}(x, z) := k(x, z) - k(x, 0)$. We have

$$\begin{aligned} \int_{\mathbb{R}^d} \mathfrak{d}g_r(x; z) \frac{k(x, z)}{|z|^{\alpha+d}} dz &= \int_{|z| \leq 3r} \mathfrak{d}g_r(x; z) \frac{k(x, z)}{|z|^{\alpha+d}} dz - \int_{3r < |z| < 1} z \cdot \nabla g_r(x) \frac{k(x, z) - k(x, 0)}{|z|^{d+\alpha}} dz \\ &\quad + \int_{|z| > 3r} (g_r(x+z) - g_r(x)) \frac{k(x, z)}{|z|^{\alpha+d}} dz \\ &\geq \frac{c_1}{\lambda_D r^2} \int_{|z| \leq 3r} |z|^{2-d-\alpha} dz - \frac{\|\nabla f\|_\infty}{r} \int_{3r < |z| < 1} |z| \frac{|\hat{k}(x, z)|}{|z|^{d+\alpha}} dz \\ &\geq c_2 \frac{3^{2-\alpha}}{(2-\alpha) \lambda_D r^\alpha} - \frac{\|\nabla f\|_\infty}{r} \int_{3r < |z| < 1} |z|^{\alpha-\theta_1} (3r)^{-\alpha+\theta_1+1} \frac{|\hat{k}(x, z)|}{|z|^{d+\alpha}} dz \\ &\geq c_2 \frac{3^{2-\alpha}}{(2-\alpha) \lambda_D r^\alpha} - \frac{\|\nabla f\|_\infty}{r} \int_{3r < |z| < 1} |z|^{\alpha-\theta} (3r)^{-\alpha+\theta_1+1} \frac{|\hat{k}(x, z)|}{|z|^{d+\alpha}} dz \\ &\geq c_2 \frac{3^{2-\alpha}}{(2-\alpha) \lambda_D r^\alpha} - \kappa(d) 3^{\alpha-\theta_1+1} r^{-\alpha+\theta_1} \lambda_D \|\nabla f\|_\infty \\ &\geq c_4 r^{-\alpha} \quad \forall x \in B_r(x_0), \end{aligned}$$

for some constant $c_4 > 0$ and r small, where in the third inequality we used the fact that $\theta_1 < \alpha - 1$. Thus by Itô's formula we obtain

$$\mathbb{E}_x[\tau(B_r(x_0))] \leq c_4^{-1} r^\alpha \|f\|_\infty \quad \forall x \in B_r(x_0).$$

This completes the proof. \square

Lemma 4.2. *There exists a constant $\kappa_3 > 0$ such that for any $r \in (0, 1)$, $x \in D$ and $A \subset B_r(x)$ we have*

$$\mathbb{P}_z(\tau(A^c) < \tau(B_{3r}(x))) \geq \kappa_3 \frac{|A|}{|B_r(x)|} \quad \forall z \in B_{2r}(x).$$

Proof. Let $\hat{\tau} := \tau(B_{3r}(x))$. Suppose $\mathbb{P}_z(\tau(A^c) < \hat{\tau}) < 1/4$ for some $z \in B_{2r}(x)$. Otherwise there is nothing to prove as $\frac{|A|}{|B_r(x)|} \leq 1$. By Lemma 3.5 and Remark 3.2 there exists $t > 0$ such that $\mathbb{P}_y(\hat{\tau} \leq tr^\alpha) \leq 1/4$ for all $y \in B_{2r}(x)$. Hence using the Lévy-system formula we obtain

$$\begin{aligned} \mathbb{P}_y(\tau(A^c) < \hat{\tau}) &\geq \mathbb{E}_y \left[\sum_{s \leq \tau(A^c) \wedge \hat{\tau} \wedge tr^\alpha} \mathbf{1}_{\{X_{s-} \neq X_s, X_s \in A\}} \right] \\ &= \mathbb{E}_y \left[\int_0^{\tau(A^c) \wedge \hat{\tau} \wedge tr^\alpha} \int_A \frac{k(X_s, z - X_s)}{|z - X_s|^{d+\alpha}} dz ds \right] \\ &\geq \mathbb{E}_y \left[\int_0^{\tau(A^c) \wedge \hat{\tau} \wedge tr^\alpha} \int_A \frac{\lambda_D^{-1}}{(4r)^{d+\alpha}} dz ds \right] \\ &\geq \kappa'_3 r^{-\alpha} \frac{|A|}{|B_r(x)|} \mathbb{E}_y[\tau(A^c) \wedge \hat{\tau} \wedge tr^\alpha] \end{aligned} \quad (4.2)$$

for some constant $\kappa'_3 > 0$, where in the third inequality we use the fact that $|X_s - z| \leq 4r$ for $s < \hat{\tau}$, $z \in A$. On the other hand, we have

$$\begin{aligned} \mathbb{E}_y[\tau(A^c) \wedge \hat{\tau} \wedge tr^\alpha] &\geq tr^\alpha \mathbb{P}_y(\tau(A^c) \geq \hat{\tau} \geq tr^\alpha) \\ &= tr^\alpha [1 - \mathbb{P}_y(\tau(A^c) < \hat{\tau}) - \mathbb{P}_y(\hat{\tau} < tr^\alpha)] \\ &\geq \frac{t}{2} r^\alpha. \end{aligned} \quad (4.3)$$

Therefore combining (4.2)–(4.3), we obtain $\mathbb{P}_z(\tau(A^c) < \hat{\tau}) \geq \frac{t\kappa'_3}{2} \frac{|A|}{|B_r(x)|}$. \square

Lemma 4.3. *There exists positive constants κ_i , $i = 4, 5$, such that if $x \in D$, $r \in (0, 1)$, $z \in B_r(x)$, and H is a bounded nonnegative function with support in $B_{2r}^c(x)$, then*

$$\mathbb{E}_z[H(X_{\tau(B_r(x))})] \leq \kappa_4 \mathbb{E}_z[\tau(B_r(x))] \int_{\mathbb{R}^d} H(y) \frac{k(x, y-x)}{|y-x|^{d+\alpha}} dy,$$

and

$$\mathbb{E}_z[H(X_{\tau(B_r(x))})] \geq \kappa_5 \mathbb{E}_z[\tau(B_r(x))] \int_{\mathbb{R}^d} H(y) \frac{k(x, y-x)}{|y-x|^{d+\alpha}} dy.$$

The proof follows using the same argument as in [33, Lemma 3.5].

Proof of Theorem 4.1. By Lemmas 4.1, 4.2 and 4.3, the hypotheses (A1)–(A3) in [33] are satisfied. Hence the proof follows from [33, Theorem 2.4]. \square

5. POSITIVE RECURRENCE AND INVARIANT PROBABILITY MEASURES

In this section we study the recurrence properties for a Markov process with generator $\mathcal{I} \in \mathfrak{L}_\alpha$ (see Definition 3.2 and 4.1). Many of the results of this section are based on the assumption of the existence of a *Lyapunov function*.

Definition 5.1. We say that the operator \mathcal{I} of the form (3.4) satisfies the Lyapunov stability condition if there exists a $\mathcal{V} \in C^2(\mathbb{R}^d)$ such that $\inf_{x \in \mathbb{R}^d} \mathcal{V}(x) > -\infty$, and for some compact set $\mathcal{K} \subset \mathbb{R}^d$ and $\varepsilon > 0$, we have

$$\mathcal{I}\mathcal{V}(x) \leq -\varepsilon \quad \forall x \in \mathcal{K}^c. \quad (5.1)$$

It is straightforward to verify that if \mathcal{V} satisfies (5.1) for $\mathcal{I} \in \mathfrak{L}_\alpha$, then

$$\int_{|z| \geq 1} |\mathcal{V}(z)| \frac{1}{|z|^{d+\alpha}} dz < \infty. \quad (5.2)$$

Proposition 5.1. *If there exists a constant $\gamma \in (1, \alpha)$ such that*

$$\frac{b(x) \cdot x}{|x|^{2-\gamma} \sup_{z \in \mathbb{R}^d} k(x, z) \vee 1} \xrightarrow{|x| \rightarrow \infty} -\infty,$$

then the operator \mathcal{I} satisfies the Lyapunov stability condition.

Proof. Consider a nonnegative function $f \in C^2(\mathbb{R}^d)$ such that $f(x) = |x|^\gamma$ for $|x| \geq 1$, and let $\bar{k}(x) := \sup_{z \in \mathbb{R}^d} k(x, z)$. Since the second derivatives of f are bounded in \mathbb{R}^d , and k is also bounded, it follows that

$$\left| \int_{|z| \leq 1} \mathfrak{d}f(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| \leq \kappa_1 \bar{k}(x)$$

for some constant κ_1 which depends on the bound of the trace of the Hessian of f . Following the same steps as in the proof of (3.6), and using the fact that k is bounded in $\mathbb{R}^d \times \mathbb{R}^d$, we obtain

$$\left| \int_{|z| > 1} (|x+z|^\gamma - |x|^\gamma) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| \leq \kappa_2 \bar{k}(x) (1 + |x|^{\gamma-\alpha}) \quad \text{if } |x| > 1, \quad (5.3)$$

for some constant $\kappa_2 > 0$. Since also,

$$\left| \int_{\mathbb{R}^d} \mathbf{1}_{B_1}(x+z) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| \leq \kappa_3 \bar{k}(x) (|x| - 1)^{-\alpha} \quad \text{for } |x| > 2, \quad (5.4)$$

for some constant κ_3 , it follows by the above that

$$\left| \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| \leq \kappa_4 \bar{k}(x) (1 + |x|^{\gamma-\alpha}) \quad \forall x \in \mathbb{R}^d, \quad (5.5)$$

for some constant κ_4 . Therefore by the hypothesis and (5.5), it follows that $\mathcal{I}f(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$. \square

Lemma 5.1. *Let X be the Markov process associated with a generator $\mathcal{I} \in \mathfrak{L}_\alpha(\lambda)$, and suppose that \mathcal{I} satisfies the Lyapunov stability hypothesis (5.1) and the growth condition in (3.5). Then for any $x \in \mathcal{K}^c$ we have*

$$\mathbb{E}_x[\tau(\mathcal{K}^c)] \leq \frac{2}{\varepsilon} (\mathcal{V}(x) + (\inf \mathcal{V})^-).$$

Proof. Let $R_0 > 0$ be such that $\mathcal{K} \subset B_{R_0}$. We choose a cut-off function χ which equals 1 on B_{R_1} , with $R_1 > 2R_0$, vanishes outside of B_{R_1+1} , and $\|\chi\|_\infty = 1$. Then $f := \chi\mathcal{V}$ is in $C_b^2(\mathbb{R}^d)$. Clearly if $|x| \leq R_0$ and $|x+z| \geq R_1$, then $|z| > R_0$, and thus $|x+z| \leq 2|z|$. Therefore, for large enough R_1 , we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}^d} (f(x+z) - \mathcal{V}(x+z)) \frac{k(x, z)}{|z|^{d+\alpha}} dz \right| &\leq 2 \int_{\{|x+z| \geq R_1\}} |\mathcal{V}(x+z)| \frac{k(x, z)}{|z|^{d+\alpha}} dz \\ &\leq 2^{d+\alpha+1} \lambda_{B_{R_0}} \int_{\{|x+z| \geq R_1\}} |\mathcal{V}(x+z)| \frac{1}{|x+z|^{d+\alpha}} dz \\ &\leq \frac{\varepsilon}{2} \quad \forall x \in B_{R_0}. \end{aligned}$$

Hence, for all R_1 large enough, we have

$$\mathcal{I}f(x) \leq -\frac{\varepsilon}{2} \quad \forall x \in B_{R_0} \setminus \mathcal{K}.$$

Let $\tilde{\tau}_R = \tau(\mathcal{K}^c) \wedge \tau(B_R)$. Then applying Itô's formula we obtain

$$\mathbb{E}_x[\mathcal{V}(X_{\tilde{\tau}_{R_0}})] - \mathcal{V}(x) \leq -\frac{\varepsilon}{2} \mathbb{E}_x[\tilde{\tau}_{R_0}] \quad \forall x \in B_{R_0} \setminus \mathcal{K},$$

implying that

$$\mathbb{E}_x[\tilde{\tau}_{R_0}] \leq \frac{2}{\varepsilon} (\mathcal{V}(x) + (\inf \mathcal{V})^-). \quad (5.6)$$

By the growth condition and Lemma 3.3, $\tau(B_R) \rightarrow \infty$ as $R \rightarrow \infty$ with probability 1. Hence the result follows by applying Fatou's lemma to (5.6). \square

5.1. Existence of invariant probability measures. Recall that a Markov process is said to be positive recurrent if for any compact set G with positive Lebesgue measure it holds that $\mathbb{E}_x[\tau(G^c)] < \infty$ for any $x \in \mathbb{R}^d$. We have the following theorem.

Theorem 5.1. *If $\mathcal{I} \in \mathfrak{L}_\alpha(\lambda)$ satisfies the Lyapunov stability hypothesis, and the growth condition in (3.5), then the associated Markov process is positive recurrent.*

Proof. First we note that if the Lyapunov condition is satisfied for some compact set \mathcal{K} , then it is also satisfied for any compact set containing \mathcal{K} . Hence we may assume that \mathcal{K} is a closed ball centered at origin. Let D be an open ball with center at origin and containing \mathcal{K} . We define

$$\hat{\tau}_1 := \inf \{t \geq 0 : X_t \notin D\}, \quad \hat{\tau}_2 := \inf \{t > \tau : X_t \in \mathcal{K}\}.$$

Therefore for $X_0 = x \in \mathcal{K}$, $\hat{\tau}_2$ denotes the first return time to \mathcal{K} after hitting D^c . First we prove that

$$\sup_{x \in \mathcal{K}} \mathbb{E}_x[\hat{\tau}_2] < \infty. \quad (5.7)$$

By Lemma 5.1 we have $\mathbb{E}_x[\tau(\mathcal{K}^c)] \leq \frac{2}{\varepsilon} [\mathcal{V}(x) + (\inf \mathcal{V})^-]$ for $x \in \mathcal{K}^c$. By Lemma 3.1 we have $\sup_{x \in \mathcal{K}} \mathbb{E}_x[\hat{\tau}_1] < \infty$. Let $\mathcal{P}_{\hat{\tau}_1}(x, \cdot)$ denote the exit distribution of the process X starting from $x \in \mathcal{K}$. In order to prove (5.7) it suffices to show that

$$\sup_{x \in \mathcal{K}} \int_{D^c} (\mathcal{V}(y) + (\inf \mathcal{V})^-) \mathcal{P}_{\hat{\tau}_1}(x, dy) < \infty,$$

and since \mathcal{V} is locally bounded it is enough that

$$\sup_{x \in \mathcal{K}} \int_{B_R^c} (\mathcal{V}(y) + (\inf \mathcal{V})^-) \mathcal{P}_{\hat{\tau}_1}(x, dy) < \infty \quad (5.8)$$

for some ball B_R . To accomplish this we choose R large enough so that

$$\frac{|x - z|}{|z|} > \frac{1}{2} \quad \text{for } |z| \geq R, x \in D.$$

Then, for any Borel set $A \subset B_R^c$, by Proposition 2.1 we have that

$$\begin{aligned} \mathbb{P}_x(X_{\hat{\tau}_1 \wedge t} \in A) &= \mathbb{E}_x \left[\sum_{s \leq \hat{\tau}_1 \wedge t} \mathbf{1}_{\{X_s \in D, X_s \in A\}} \right] \\ &= \mathbb{E}_x \left[\int_0^{\hat{\tau}_1 \wedge t} \mathbf{1}_{\{X_s \in D\}} \int_A \frac{k(X_s, z - X_s)}{|X_s - z|^{d+\alpha}} dz ds \right] \\ &\leq 2^{d+\alpha} \lambda_D \mathbb{E}_x \left[\int_0^{\hat{\tau}_1 \wedge t} \int_A \frac{1}{|z|^{d+\alpha}} dz ds \right] \end{aligned}$$

$$= 2^{d+\alpha} \lambda_D \mathbb{E}_x[\hat{\tau}_1 \wedge t] \mu(A),$$

where μ is the σ -finite measure on \mathbb{R}_*^d with density $\frac{1}{|z|^{d+\alpha}}$. Thus letting $t \rightarrow \infty$ we obtain

$$\mathcal{P}_{\hat{\tau}_1}(x, A) \leq 2^{d+\alpha} \lambda_D \left(\sup_{x \in \mathcal{K}} \mathbb{E}_x[\hat{\tau}_1] \right) \mu(A).$$

Therefore, using a standard approximation argument, we deduce that for any nonnegative function g it holds that

$$\int_{B_R^c} g(y) \mathcal{P}_{\hat{\tau}_1}(x, dy) \leq \tilde{\kappa} \int_{B_R^c} g(y) \mu(dy)$$

for some constant $\tilde{\kappa}$. This proves (5.8) since \mathcal{V} is integrable on B_R^c with respect to μ and $\mu(B_R^c) < \infty$.

Next we prove that the Markov process is positive recurrent. We need to show that for any compact set G with positive Lebesgue measure, $\mathbb{E}_x[\tau(G^c)] < \infty$ for any $x \in \mathbb{R}^d$. Given a compact G and $x \in G^c$ we choose a closed ball \mathcal{K} , which satisfies the Lyapunov condition relative to \mathcal{V} , and such that $G \cup \{x\} \subset \mathcal{K}$. Let D be an open ball containing \mathcal{K} . We define a sequence of stopping times $\{\hat{\tau}_k, k = 0, 1, \dots\}$ as follows:

$$\begin{aligned} \hat{\tau}_0 &= 0 \\ \hat{\tau}_{2n+1} &= \inf\{t > \hat{\tau}_{2n} : X_t \notin D\}, \\ \hat{\tau}_{2n+2} &= \inf\{t > \hat{\tau}_{2n+1} : X_t \in \mathcal{K}\}, \quad n = 0, 1, \dots \end{aligned}$$

Using the strong Markov property and (5.8), we obtain $\mathbb{E}_x[\hat{\tau}_n] < \infty$ for all $n \in \mathbb{N}$. From Lemma 3.5 there exist positive constants t and r such that

$$\sup_{x \in \mathcal{K}} \mathbb{P}_x(\tau(D) < t) \leq \sup_{x \in \mathcal{K}} \mathbb{P}_x(\tau(B_r(x)) < t) \leq \frac{1}{4}.$$

Therefore, using a similar argument as in Lemma 4.2, we can find a constant $\delta > 0$ such that

$$\inf_{x \in \mathcal{K}} \mathbb{P}_x(\tau(G^c) < \tau(D)) > \delta.$$

Hence

$$p := \sup_{x \in \mathcal{K}} \mathbb{P}_x(\tau(D) < \tau(G^c)) \leq 1 - \delta < 1.$$

Thus by the strong Markov property we obtain

$$\mathbb{P}_x(\tau(G^c) > \hat{\tau}_{2n}) \leq p \mathbb{P}_x(\tau(G^c) > \hat{\tau}_{2n-2}) \leq \dots \leq p^n \quad \forall x \in \mathcal{K}.$$

This implies $\mathbb{P}_x(\tau(G^c) < \infty) = 1$. Hence, for $x \in \mathcal{K}$, we obtain

$$\begin{aligned} \mathbb{E}_x[\tau(G^c)] &\leq \sum_{n=1}^{\infty} \mathbb{E}_x[\hat{\tau}_{2n} \mathbf{1}_{\{\hat{\tau}_{2n-2} < \tau(G^c) \leq \hat{\tau}_{2n}\}}] \\ &= \sum_{n=1}^{\infty} \sum_{l=1}^n \mathbb{E}_x[(\hat{\tau}_{2l} - \hat{\tau}_{2l-2}) \mathbf{1}_{\{\hat{\tau}_{2n-2} < \tau(G^c) \leq \hat{\tau}_{2n}\}}] \\ &= \sum_{l=1}^{\infty} \sum_{n=l}^{\infty} \mathbb{E}_x[(\hat{\tau}_{2l} - \hat{\tau}_{2l-2}) \mathbf{1}_{\{\hat{\tau}_{2n-2} < \tau(G^c) \leq \hat{\tau}_{2n}\}}] \\ &= \sum_{l=1}^{\infty} \mathbb{E}_x[(\hat{\tau}_{2l} - \hat{\tau}_{2l-2}) \mathbf{1}_{\{\hat{\tau}_{2l-2} < \tau(G^c)\}}] \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{l=1}^{\infty} p^{l-1} \sup_{x \in \mathcal{K}} \mathbb{E}_x[\hat{\tau}_2] \\
&= \frac{1}{1-p} \sup_{x \in \mathcal{K}} \mathbb{E}_x[\hat{\tau}_2] < \infty.
\end{aligned}$$

Since also $\mathbb{E}_x[\tau(\mathcal{K}^c)] < \infty$ for all $x \in \mathbb{R}^d$, this completes the proof. \square

Theorem 5.2. *Let X be a Markov process associated with a generator $\mathcal{I} \in \mathfrak{L}_\alpha^{\text{sym}}(\lambda) \cup \mathfrak{J}_\alpha(\theta, \lambda)$, and suppose that the Lyapunov stability hypothesis (5.1) and the growth condition in (3.5) hold. Then X has an invariant probability measure.*

Proof. The proof is based on Has'minskiĭ's construction. Let \mathcal{K} , D , $\hat{\tau}_1$, and $\hat{\tau}_2$ be as in the proof of Theorem 5.1. Let \hat{X} be a Markov process on \mathcal{K} with transition kernel given by

$$\hat{\mathbb{P}}_x(dy) = \mathbb{P}_x(X_{\hat{\tau}_2} \in dy).$$

Let f be any bounded, nonnegative measurable function on D . Define $Q_f(x) = \mathbb{E}_x[f(X_{\hat{\tau}_2})]$. We claim that Q_f is harmonic in D . Indeed if we define $\tilde{f}(x) = \mathbb{E}_x[f(X_{\tau(\mathcal{K}^c)})]$ for $x \in D^c$, then by the strong Markov property we obtain $Q_f(x) = \mathbb{E}_x[\tilde{f}(X_{\hat{\tau}_1})]$, and the claim follows. By Theorem 4.1 there exists a positive constant C_H , independent of f , satisfying

$$Q_f(x) \leq C_H Q_f(y) \quad \forall x, y \in \mathcal{K}. \quad (5.9)$$

We note that $Q_{\mathbf{1}_{\mathcal{K}}} \equiv 1$. Let $Q(x, A) := Q_{\mathbf{1}_A}(x)$, for $A \subset \mathcal{K}$. For any pair of probability measures μ and μ' on \mathcal{K} , we claim that

$$\left\| \int_{\mathcal{K}} (\mu(dx) - \mu'(dx)) Q(x, \cdot) \right\|_{\text{TV}} \leq \frac{C_H - 1}{C_H} \|\mu - \mu'\|_{\text{TV}}. \quad (5.10)$$

This implies that the map $\mu \rightarrow \int_{\mathcal{K}} Q(x, \cdot) \mu(dx)$ is a contraction and hence it has a unique fixed point $\hat{\mu}$ satisfying $\hat{\mu}(A) = \int_{\mathcal{K}} Q(x, A) \hat{\mu}(dx)$ for any Borel set $A \subset \mathcal{K}$. In fact, $\hat{\mu}$ is the invariant probability measure of the Markov chain \hat{X} . Next we prove (5.10). Given any two probability measure μ, μ' on \mathcal{K} , we can find subsets F and G of \mathcal{K} such that

$$\begin{aligned}
\left\| \int_{\mathcal{K}} (\mu(dx) - \mu'(dx)) Q(x, \cdot) \right\|_{\text{TV}} &= 2 \int_{\mathcal{K}} (\mu(dx) - \mu'(dx)) Q(x, F), \\
\|\mu - \mu'\|_{\text{TV}} &= 2(\mu - \mu')(G).
\end{aligned}$$

In fact, the restriction of $(\mu - \mu')$ to G is a nonnegative measure and its restriction to G^c it is non-positive measure. By (5.9), we have

$$\inf_{x \in G^c} Q(x, F) \geq \sup_{x \in G} Q(x, F) \quad (5.11)$$

Hence, using (5.11), we obtain

$$\begin{aligned}
\left\| \int_{\mathcal{K}} (\mu(dx) - \mu'(dx)) Q(x, \cdot) \right\|_{\text{TV}} &= 2 \int_G (\mu(dx) - \mu'(dx)) Q(x, F) + 2 \int_{G^c} (\mu(dx) - \mu'(dx)) Q(x, F) \\
&\leq 2(\mu - \mu')(G) \sup_{x \in G} Q(x, F) + 2(\mu - \mu')(G^c) \inf_{x \in G^c} Q(x, F) \\
&\leq 2(\mu - \mu')(G) \sup_{x \in G} Q(x, F) - \frac{2}{C_H} (\mu - \mu')(G) \sup_{x \in G} Q(x, F) \\
&\leq (1 - C_H^{-1}) \|\mu - \mu'\|_{\text{TV}}.
\end{aligned}$$

This proves (5.10).

We define a probability measure ν on \mathbb{R}^d as follows.

$$\int_{\mathbb{R}^d} f(x) \nu(dx) = \frac{\int_{\mathcal{K}} \mathbb{E}_x \left[\int_0^{\hat{\tau}_2} f(X_s) ds \right] \hat{\mu}(dx)}{\int_{\mathcal{K}} \mathbb{E}_x [\hat{\tau}_2] \hat{\mu}(dx)}, \quad f \in C_b(\mathbb{R}^d).$$

It is straight forward to verify that ν is an invariant probability measure of X (see for example, [3, Theorem 2.6.9]). \square

Remark 5.1. If $k(\cdot, \cdot) = 1$ and the drift b belongs to certain Kato class, in particular bounded, (see [14]) then the transition probability has a continuous density, and therefore any invariant probability measure has a continuous density. Since any two distinct ergodic measures are mutually singular, this implies the uniqueness of the invariant probability measure. As shown later in Proposition 5.3 open sets have strictly positive mass under any invariant measure.

The following result is fairly standard.

Proposition 5.2. *Let $\mathcal{I} \in \mathcal{L}_\alpha$, and $\mathcal{V} \in C^2(\mathbb{R}^d)$ be a nonnegative function satisfying $\mathcal{V}(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, and $\mathcal{I}\mathcal{V} \leq 0$ outside some compact set \mathcal{K} . Let ν be an invariant probability measure of the Markov process associated with the generator \mathcal{I} . Then*

$$\int_{\mathbb{R}^d} |\mathcal{I}\mathcal{V}(x)| \nu(dx) \leq 2 \int_{\mathcal{K}} |\mathcal{I}\mathcal{V}(x)| \nu(dx).$$

Proof. Let $\varphi_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a smooth non-decreasing, concave, function such that

$$\varphi_n(x) = \begin{cases} x & \text{for } x \leq n, \\ n + 1/2 & \text{for } x \geq n + 1. \end{cases}$$

Due to concavity we have $\varphi_n(x) \leq |x|$ for all $x \in \mathbb{R}_+$. Then $\mathcal{V}_n(x) := \varphi_n(\mathcal{V}(x))$ is in $C_b^2(\mathbb{R}^d)$ and it also follows that $\mathcal{I}\mathcal{V}_n(x) \rightarrow \mathcal{I}\mathcal{V}(x)$ as $n \rightarrow \infty$. Since ν is an invariant probability measure, it holds that

$$\int_{\mathbb{R}^d} \mathcal{I}\mathcal{V}_n(x) \nu(dx) = 0. \quad (5.12)$$

By concavity, $\varphi_n(y) \leq \varphi_n(x) + (y - x) \cdot \varphi'_n(x)$ for all $x, y \in \mathbb{R}_+$. Hence

$$\begin{aligned} \mathcal{I}\mathcal{V}_n(x) &= \int_{\mathbb{R}^d} \mathfrak{D}\mathcal{V}_n(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz + \varphi'_n(\mathcal{V}(x)) b(x) \cdot \nabla \mathcal{V}(x) \\ &\leq \int_{\mathbb{R}^d} \varphi'_n(\mathcal{V}(x)) \mathfrak{D}\mathcal{V}(x; z) \frac{k(x, z)}{|z|^{d+\alpha}} dz + \varphi'_n(\mathcal{V}(x)) b(x) \cdot \nabla \mathcal{V}(x) \\ &= \varphi'_n(\mathcal{V}(x)) \mathcal{I}\mathcal{V}(x), \end{aligned}$$

which is negative for $x \in \mathcal{K}^c$. Therefore using (5.12) we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{I}\mathcal{V}_n(x)| \nu(dx) &= \int_{\mathcal{K}} |\mathcal{I}\mathcal{V}_n(x)| \nu(dx) - \int_{\mathcal{K}^c} \mathcal{I}\mathcal{V}_n(x) \nu(dx) \\ &= \int_{\mathcal{K}} |\mathcal{I}\mathcal{V}_n(x)| \nu(dx) + \int_{\mathcal{K}} \mathcal{I}\mathcal{V}_n(x) \nu(dx) \\ &\leq 2 \int_{\mathcal{K}} |\mathcal{I}\mathcal{V}_n(x)| \nu(dx). \end{aligned} \quad (5.13)$$

On the other hand, with $A_n := \{y \in \mathbb{R}^d : \mathcal{V}(y) \geq n\}$, and provided $\mathcal{V}(x) < n$, we have

$$|\mathcal{I}\mathcal{V}_n(x)| \leq |\mathcal{I}\mathcal{V}(x)| + \int_{x+z \in A_n} |\mathcal{V}(x+z) - \mathcal{V}_n(x+z)| \frac{k(x, z)}{|z|^{d+\alpha}} dz$$

$$\leq |\mathcal{I}\mathcal{V}(x)| + \int_{x+z \in A_n} |\mathcal{V}(x+z)| \frac{k(x,z)}{|z|^{d+\alpha}} dz.$$

This together with (5.2) imply that there exists a constant κ such that

$$|\mathcal{I}\mathcal{V}_n(x)| \leq \kappa + |\mathcal{I}\mathcal{V}(x)| \quad \forall x \in \mathcal{K},$$

and all large enough n . Therefore, letting $n \rightarrow \infty$ and using Fatou's lemma for the term on the left hand side of (5.13), and the dominated convergence theorem for the term on the right hand side, we obtain the result. \square

5.2. A class of operators with variable order kernels. It is quite evident from Theorem 5.2 that the Harnack inequality plays a crucial role in the analysis. Therefore one might wish to establish positive recurrence for an operator with a variable order kernel, and deploy the Harnack inequality from [10] to prove a similar result as in Theorem 5.2.

Theorem 5.3. *Let $\pi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a nonnegative measurable function satisfying the following properties, for $1 < \alpha' < \alpha < 2$:*

- (a) *There exists a constant $c_1 > 0$ such that $\mathbf{1}_{\{|z|>1\}}\pi(x,z) \leq \frac{c_1}{|z|^{d+\alpha}}$ for all $x \in \mathbb{R}^d$;*
- (b) *There exists a constant $c_2 > 0$ such that*

$$\pi(x, z-x) \leq c_2 \pi(y, y-z), \quad \text{whenever } |z-x| \wedge |z-y| \geq 1, |x-y| \leq 1;$$
- (c) *For each $R > 0$ there exists $q_R > 0$ such that*

$$\frac{q_R^{-1}}{|z|^{d+\alpha'}} \leq \pi(x, z) \leq \frac{q_R}{|z|^{d+\alpha}} \quad \forall x \in \mathbb{R}^d, \forall z \in B_R;$$

- (d) *For each $R > 0$ there exists $R_1 > 0$, $\sigma \in (1, 2)$, and $\kappa_\sigma = \kappa_\sigma(R, R_1) > 0$ such that*

$$\frac{\kappa_\sigma^{-1}}{|z|^{d+\sigma}} \leq \pi(x, z) \leq \frac{\kappa_\sigma}{|z|^{d+\sigma}} \quad \forall x \in B_R, \forall z \in B_{R_1}^c;$$

- (e) *There exists $\mathcal{V} \in C^2(\mathbb{R}^d)$ that is bounded from below in \mathbb{R}^d , a compact set $\mathcal{K} \subset \mathbb{R}^d$ and a constant $\varepsilon > 0$, such that*

$$\int_{\mathbb{R}^d} \mathfrak{d}\mathcal{V}(x; z) \pi(x, z) dz < -\varepsilon \quad \forall x \in \mathcal{K}^c.$$

Then the Markov process associated with the above kernel has an invariant probability measure.

The first three assumptions guarantee the Harnack property for associated harmonic functions [10]. Then the conclusion of Theorem 5.3 follows by using an argument similar to the one used in the proof of Theorem 5.2.

Next we present an example of a kernel π that satisfies the conditions in Theorem 5.3. We accomplish this by adding a non-symmetric bump function to a symmetric kernel.

Example 5.1. Let $\varphi : \mathbb{R}^d \rightarrow [0, 1]$ be a smooth function such that

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{1}{2}, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

Define for $1 < \alpha' < \beta' < \alpha < 2$,

$$\gamma(x, z) := \varphi\left(2\frac{x+z}{1+|x|}\right)(1 - \varphi(4x))(\alpha' - \beta'),$$

and let

$$\tilde{\pi}(x, z) := \frac{1}{|z|^{d+\beta'+\gamma(x,z)}},$$

$$\pi(x, z) := \frac{1}{|z|^{d+\alpha}} + \tilde{\pi}(x, z).$$

We prove that π satisfies the conditions of Theorem 5.3. Let us also mention that there exists a unique solution to the martingale problem corresponding to the kernel π [26, 27]. We only show that conditions (b) and (e) hold. It is straightforward to verify (a), (c) and (d).

Note that $\alpha' - \beta' \leq \gamma(x, z) \leq 0$ for all x, z . Let $x, y, z \in \mathbb{R}^d$ such that $|x - z| \wedge |y - z| \geq 1$ and $|x - y| \leq 1$. Then $|z - y| \leq 1 + |z - x|$. By a simple calculation we obtain

$$\begin{aligned} \tilde{\pi}(x, z - x) &\leq \left(1 + \frac{1}{|z - x|}\right)^{d+\beta'+\gamma(x, z-x)} \frac{1}{|z - y|^{d+\beta'+\gamma(x, z-x)}} \\ &\leq 2^{d+\beta'} \frac{1}{|z - y|^{d+\beta'+\gamma(y, z-y)}} |z - y|^{-\beta'(x, z-x)+\gamma(y, z-y)}. \end{aligned}$$

Hence it is enough to show that

$$|z - y|^{-\gamma(x, z-x)+\gamma(y, z-y)} < \varrho \quad (5.14)$$

for some constant ϱ which does not depend on x, y and z . Note that if $|x| \leq 2$, which implies that $|y| \leq 3$, then for $|z| \geq 4$ we have $\gamma(x, z - x) = 0 = \gamma(y, z - y)$. Therefore for $|x| \leq 2$, it holds that

$$|z - y|^{-\gamma(x, z-x)+\gamma(y, z-y)} \leq 7^{\beta'-\alpha'}. \quad (5.15)$$

Suppose that $|x| \geq 2$. Then $|y| \geq 1$. Since we only need to consider the case where $\gamma(x, z - x) \neq \gamma(y, z - y)$ we restrict our attention to $z \in \mathbb{R}^d$ such that $|z| \leq 2(1 + |x|)$. We obtain

$$\begin{aligned} \log(|z - y|)(-\gamma(x, z - x) + \gamma(y, z - y)) &\leq \log(3(1 + |x|)) \|\varphi'\|_\infty \frac{2|z|(\beta' - \alpha')}{(1 + |x|)(1 + |y|)} \\ &\leq \log(3(1 + |x|)) \|\varphi'\|_\infty \frac{4(1 + |x|)(\beta' - \alpha')}{(1 + |x|)|x|}. \end{aligned} \quad (5.16)$$

Since the term on the right hand side of (5.16) is bounded in \mathbb{R}^d , the bound in (5.14) follows by (5.15)–(5.16).

Next we prove the Lyapunov property. We fix a constant $\eta \in (\alpha', \beta')$, and choose some function $\mathcal{V} \in C^2(\mathbb{R}^d)$ such that $\mathcal{V}(x) = |x|^\eta$ for $|x| > 1$. Since $\tilde{\pi}(x, z) \leq \frac{1}{|z|^{d+\alpha'}}$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}_*^d$, it follows that

$$x \mapsto \left| \int_{|z| \leq 1} \partial \mathcal{V}(x; z) \tilde{\pi}(x, z) dz \right|$$

is bounded by some constant on \mathbb{R}^d . By (5.5),

$$\left| \int_{\mathbb{R}^d} \partial \mathcal{V}(x; z) \pi(x, z) dz \right| \leq c_0 (1 + |x|^{\eta-\alpha'}) \quad \forall x \in \mathbb{R}^d,$$

for some constant c_0 . Therefore, in view of (5.4), it is enough to show that for $|x| \geq 4$, there exist positive constants c_1 and c_2 such that

$$\int_{|z| > 1} (|x + z|^\eta - |x|^\eta) \tilde{\pi}(x, z) dz \leq c_1 - c_2 |x|^{\eta-\alpha'}. \quad (5.17)$$

By the definition of γ it holds that

$$\tilde{\pi}(x, z) = \frac{1}{|z|^{d+\beta'}}, \quad \text{if } |x + z| \geq \frac{3}{4}|x|, \text{ and } |x| \geq 2, \quad (5.18)$$

while

$$\tilde{\pi}(x, z) = \frac{1}{|z|^{d+\alpha'}}, \quad \text{if } |x + z| \leq \frac{|x|}{4}. \quad (5.19)$$

Suppose that $|x| > 2$. Since $|x+z| \leq \frac{|x|}{4}$ implies that $\frac{3}{4}|x| \leq |z| \leq \frac{5}{4}|x|$, we obtain by (5.19) that

$$\begin{aligned} \int_{|x+z| \leq \frac{|x|}{4}, |z| > 1} (|x+z|^\eta - |x|^\eta) \tilde{\pi}(x, z) dz &\leq - \int_{|x+z| \leq \frac{|x|}{4}} \left(1 - \frac{1}{4^\eta}\right) |x|^\eta \left(\frac{4}{5}\right)^{d+\alpha'} \frac{1}{|x|^{d+\alpha'}} dz \\ &\leq - \left(1 - \frac{1}{4^\eta}\right) \left(\frac{4}{5}\right)^{d+\alpha'} |x|^{\eta-\alpha'} \int_{|x+z| \leq \frac{|x|}{4}} \frac{dz}{|x|^d} \\ &\leq -m_1 |x|^{\eta-\alpha'}, \quad \text{if } |x| > 2, \end{aligned} \quad (5.20)$$

for some constant $m_1 > 0$, where we use the fact that the integral in the second inequality is independent of x due to rotational invariance. Also, $|x+z| \leq \frac{3}{4}|x|$ implies $\frac{1}{4}|x| \leq |z| \leq \frac{7}{4}|x|$, and in a similar manner, using (5.18), we obtain

$$\begin{aligned} \int_{|x+z| \leq \frac{3|x|}{4}, |z| > 1} (|x+z|^\eta - |x|^\eta) \frac{1}{|x|^{d+\beta'}} dz &\geq - \int_{\frac{1}{4}|x| \leq |z| \leq \frac{7}{4}|x|} |x|^\eta 4^{d+\beta'} \frac{1}{|x|^{d+\beta'}} dz \\ &\geq -m_2 |x|^{\eta-\beta'}, \quad \text{if } |x| > 2, \end{aligned} \quad (5.21)$$

for some constant $m_2 > 0$. Let $A_1 := \{z : \frac{1}{4}|x| \leq |x+z| \leq \frac{3}{4}|x|\}$. Since η is positive, we have

$$\int_{\{|z| \geq 1\} \cap A_1} (|x+z|^\eta - |x|^\eta) \tilde{\pi}(x, z) dz \leq 0.$$

Thus, combining this observation with (5.3) and (5.21), we obtain

$$\begin{aligned} \int_{|x+z| > \frac{|x|}{4}, |z| > 1} (|x+z|^\eta - |x|^\eta) \tilde{\pi}(x, z) dz &\leq \int_{|x+z| > \frac{3}{4}|x|, |z| > 1} (|x+z|^\eta - |x|^\eta) \frac{1}{|z|^{d+\beta'}} dz \\ &= \int_{|z| > 1} (|x+z|^\eta - |x|^\eta) \frac{1}{|z|^{d+\beta'}} dz \\ &\quad - \int_{|x+z| \leq \frac{3|x|}{4}, |z| > 1} (|x+z|^\eta - |x|^\eta) \frac{1}{|x|^{d+\beta'}} dz \\ &\leq m_3 (1 + |x|^{\eta-\beta'}) \end{aligned} \quad (5.22)$$

for some constant $m_3 > 0$. Combining (5.20) and (5.22), we obtain

$$\int_{|z| > 1} (|x+z|^\eta - |x|^\eta) \tilde{\pi}(x, z) dz \leq m_3 (1 + |x|^{\eta-\beta'}) - m_1 |x|^{\eta-\alpha'}, \quad \text{if } |x| > 2. \quad (5.23)$$

Therefore, (5.17) follows by (5.23), and the Lyapunov property holds.

Proposition 5.3. *Let D be any bounded open set in \mathbb{R}^d and X be a Markov process associated with either $\mathcal{I} \in \mathfrak{L}_\alpha$, or a generator with kernel π as in Theorem 5.3. Suppose that for any compact set K and any open set G , it holds that $\sup_{x \in K} \mathbb{P}_x(\tau(G^c) > T) \rightarrow 0$ as $T \rightarrow \infty$. Then for any invariant probability measure ν of X we have $\nu(D) > 0$.*

Proof. We argue by contradiction. Suppose $\nu(D) = 0$. Let $x_0 \in D$ and $r \in (0, 1)$ be such that $B_{2r}(x_0) \subset D$. By Lemma 3.5 and Remark 3.2 (see also [10, Proposition 3.1]), we have

$$\sup_{x \in B_r(x_0)} \mathbb{P}_x(\tau(B_r(x)) \leq t) \leq \kappa t, \quad t > 0,$$

for some constant κ which depends on r . Therefore there exists $t_0 > 0$ such that

$$\inf_{x \in B_r(x_0)} \mathbb{P}_x(\tau(B_r(x)) \geq t_0) \geq \frac{1}{2}.$$

Let K be a compact set satisfying $\nu(K) > \frac{1}{2}$. By the hypothesis there exists $T_0 > 0$ such that $\sup_{x \in K} \mathbb{P}_x(\tau(B_r^c(x_0)) > T) \leq 1/2$ for all $T \geq T_0$. Hence

$$\begin{aligned}
 0 = \nu(D) &\geq \frac{1}{T_0 + t_0} \int_0^{T_0+t_0} \int_{\mathbb{R}^d} \nu(dx) P(t, x; B_{2r}(x_0)) dt \\
 &= \frac{1}{T_0 + t_0} \int_{\mathbb{R}^d} \nu(dx) \mathbb{E}_x \left[\int_0^{T_0+t_0} \mathbf{1}_{\{B_{2r}(x_0)\}}(X_s) dt \right] \\
 &\geq \frac{1}{T_0 + t_0} \int_K \nu(dx) \mathbb{E}_x \left[\mathbf{1}_{\{\tau(B_r^c(x_0)) \leq T_0\}} \mathbb{E}_{X_{\tau(B_r^c(x_0))}} \left[\mathbf{1}_{\{\tau(B_{2r}(x_0)) \geq t_0\}} \right. \right. \\
 &\qquad \qquad \qquad \left. \left. \int_{\tau(B_r(x_0))}^{T_0+t_0} \mathbf{1}_{\{B_{2r}(x_0)\}}(X_s) dt \right] \right] \\
 &\geq \frac{1}{T_0 + t_0} \nu(K) \inf_{x \in K} \mathbb{P}_x(\tau(B_r^c(x_0)) \leq T_0) \inf_{x \in B_r(x_0)} \mathbb{P}_x(\tau(B_{2r}(x_0)) \geq t_0) t_0 \\
 &\geq \frac{1}{T_0 + t_0} \frac{\nu(K)}{2} \inf_{x \in B_r(x_0)} \mathbb{P}_x(\tau(B_r(x)) \geq t_0) t_0 \\
 &\geq \frac{t_0}{T_0 + t_0} \frac{\nu(K)}{4} > 0.
 \end{aligned}$$

But this is a contradiction. Hence $\nu(D) > 0$. □

5.3. Mean recurrence times for weakly Hölder continuous kernels. This section is devoted to the characterization of the mean hitting time of bounded open sets for Markov processes with generators studied in Section 3.2. The results hold for any bounded domain D with C^2 boundary, but for simplicity we state them for the unit ball centered at 0. As introduced earlier, we use the notation $B \equiv B_1$.

For nondegenerate continuous diffusions, it is well known that if some bounded domain D is positive recurrent with respect to some point $x \in \bar{D}^c$, then the process is positive recurrent and its generator satisfies the Lyapunov stability hypothesis in (5.1) [3, Lemma 3.3.4]. In Theorem 5.4 we show that the same property holds for the class of operators $\mathfrak{I}_\alpha(\beta, \theta, \lambda)$.

Theorem 5.4. *Let $\mathcal{I} \in \mathfrak{I}_\alpha(\beta, \theta, \lambda)$. We assume that \mathcal{I} satisfies the growth condition in (3.5). Moreover, we assume that $\mathbb{E}_x[\tau(B^c)] < \infty$ for some x in \bar{B}^c . Then $u(x) := \mathbb{E}_x[\tau(B^c)]$ is a viscosity solution to*

$$\begin{aligned}
 \mathcal{I}u &= -1 \quad \text{in } \bar{B}^c, \\
 u &= 0 \quad \text{in } \bar{B}.
 \end{aligned}$$

In order to prove Theorem 5.4 we need the following two lemmas.

Lemma 5.2. *Let $\mathcal{I} \in \mathfrak{I}_\alpha(\beta, \theta, \lambda)$, and G a bounded open set containing \bar{B} . Then there exist positive constants r_0 and M_0 depending only on G such that*

$$\int_{\bar{B}^c(x)} \mathbb{E}_z[\tau(B^c)] \frac{1}{|z|^{d+\alpha}} dz < \frac{M_0}{r^\alpha} \mathbb{E}_x[\tau(B^c)]$$

for every $r < \text{dist}(x, B) \wedge r_0$, and for all $x \in G \setminus \bar{B}$, such that $\mathbb{E}_x[\tau(B^c)] < \infty$.

Proof. Let $\check{\tau} := \tau(B^c)$, and $\hat{\tau}_r := \tau(B_r(x))$. We select r_0 as in Lemma 4.1, and without loss of generality we assume $r_0 \leq 1$. We have

$$\mathbb{E}_x \left[\mathbf{1}_{\{\hat{\tau}_r < \check{\tau}\}} \mathbb{E}_{X_{\hat{\tau}_r}}[\check{\tau}] \right] \leq \mathbb{E}_x[\check{\tau}]. \quad (5.24)$$

By Definition 3.3 we have

$$k(y, z) \geq \lambda_G^{-1} > 0 \quad \forall y \in B_{r_0}(x).$$

Let $A \subset \bar{B}_r^c(x) \cap \bar{B}^c$ be any Borel set. Using Proposition 2.1, we have

$$\begin{aligned} \mathbb{P}_x(X_{\hat{\tau}_r \wedge t} \in A) &= \mathbb{E}_x \left[\sum_{s \leq \hat{\tau}_r \wedge t} \mathbf{1}_{\{X_s \in B_r(x), X_s \in A\}} \right] \\ &= \mathbb{E}_x \left[\int_0^{\hat{\tau}_r \wedge t} \mathbf{1}_{\{X_s \in B_r(x)\}} \int_A \pi(X_s, z - X_s) dz ds \right] \\ &\geq \lambda_G^{-1} \mathbb{E}_x \left[\int_0^{\hat{\tau}_r \wedge t} \int_A \frac{1}{|z|^{d+\alpha}} dz ds \right] \\ &\geq \lambda_G^{-1} \mathbb{E}_x[\hat{\tau}_r \wedge t] \int_A \frac{1}{|z|^{d+\alpha}} dz. \end{aligned}$$

Letting $t \rightarrow \infty$, we obtain

$$\mathbb{P}_x(X_{\hat{\tau}_r} \in A) \geq \lambda_G^{-1} \mathbb{E}_x[\hat{\tau}_r] \int_A \frac{1}{|z|^{d+\alpha}} dz. \quad (5.25)$$

By Lemma 4.1 it holds that $\mathbb{E}_x[\hat{\tau}_r] > \kappa_1 r^\alpha$ for some positive constant κ_1 which depends on G . Hence combining (5.24) and (5.25) we obtain

$$\begin{aligned} \lambda_G^{-1} \kappa_1 r^\alpha \int_{\bar{B}^c(x)} \mathbb{E}_z[\check{\tau}] \frac{1}{|z|^{d+\alpha}} dz &\leq \mathbb{E}_x[\mathbf{1}_{\{X_{\hat{\tau}_r} \in \bar{B}^c\}} \mathbb{E}_{X_{\hat{\tau}_r}}[\check{\tau}]] \\ &\leq \mathbb{E}_x[\check{\tau}], \end{aligned}$$

where the first inequality follows by the standard approximation technique using step functions. This completes the proof. \square

Lemma 5.2 of course implies that if $\mathbb{E}_x[\tau(B^c)] < \infty$ at some point $x \in \bar{B}^c$ then $\mathbb{E}_x[\tau(B^c)]$ is finite a.e.- x . We can express the bound in Lemma 5.2 without reference to Lemma 4.1 as

$$\int_{\bar{B}^c(x)} \mathbb{E}_z[\tau(B^c)] \frac{1}{|z|^{d+\alpha}} dz \leq \lambda_G \frac{\mathbb{E}_x[\tau(B^c)]}{\mathbb{E}_x[\bar{B}^c]}.$$

Now let x' be any point such that $\text{dist}(x', x) \wedge \text{dist}(x', B) = 2r$. We obtain

$$\frac{\omega(r)}{|2r|^{d+\alpha}} \inf_{z \in B_r(x')} \mathbb{E}_z[\tau(B^c)] \leq \frac{M_0}{r^\alpha} \mathbb{E}_x[\tau(B^c)].$$

Therefore for some $y \in B_r(x')$, we have $\mathbb{E}_y[\tau(B^c)] < C_1 \mathbb{E}_x[\tau(B^c)]$. Applying Lemma 5.2 once more we obtain

$$\int_{\mathbb{R}^d} \mathbb{E}_{x+z}[\tau(B^c)] \frac{1}{(1+|z|)^{d+\alpha}} dz \leq C_0 \mathbb{E}_x[\tau(B^c)],$$

with the constant C_0 depending only on $\text{dist}(x, B)$ and the parameter λ , i.e., the local bounds on k . We introduce the following notation.

Definition 5.2. We say that $v \in L^1(\mathbb{R}^d, s)$ if

$$\int_{\mathbb{R}^d} \frac{|v(z)|}{(1+|z|)^{d+\alpha}} dz < \infty.$$

Thus we have the following.

Corollary 5.1. *If $\mathbb{E}_{x_0}[\tau(B^c)] < \infty$ for some $x_0 \in \bar{B}^c$, then the function $u(x) := \mathbb{E}_x[\tau(B^c)]$ is in $L^1(\mathbb{R}^d, s)$.*

In what follows, without loss of generality we assume that $\beta < s$. Then, by Theorem 3.1, $u_n(x) := \mathbb{E}_x[\tau(B_n \cap \bar{B}^c)]$ is the unique solution in $C^{\alpha+\beta}(B_n \setminus \bar{B}) \cap C(\bar{B}_n \setminus B)$ of

$$\begin{aligned} \mathcal{I}u_n &= -1 \quad \text{in } B_n \cap \bar{B}^c, \\ u_n &= 0 \quad \text{in } B_n^c \cup B. \end{aligned} \tag{5.26}$$

The following lemma provides a uniform barrier on the solutions u_n near B .

Lemma 5.3. *Let $\mathcal{I} \in \mathcal{I}_\alpha(\beta, \theta, \lambda)$, and*

$$\tilde{\tau}_n := \tau(B_n \cap \bar{B}^c), \quad n \in \mathbb{N}.$$

Then, provided that $\sup_{x \in F} \mathbb{E}_x[\tau(B^c)] < \infty$ for all compact sets $F \subset \bar{B}^c$, there exists a continuous, nonnegative radial function φ that vanishes on B , and satisfies, for some $\eta > 0$,

$$\mathbb{E}_x[\tilde{\tau}_n] \leq \varphi(x) \quad \forall x \in B_{1+\eta} \setminus B, \quad \forall n > 1.$$

Proof. The proof relies on the construction of barrier. Let $\hat{k}(x, z) = k(x, z) - k(x, 0)$. By Lemma 6.2, for $q \in (\alpha-1/2, \alpha/2)$, there exists a constant $c_0 > 0$ such that for $\varphi_q(x) := [(1 - |x|)^+]^q$ we have

$$(-\Delta)^{\alpha/2} \varphi_q(x) > c_0 (1 - |x|)^{q-\alpha} \quad \forall x \in B.$$

We recall the Kelvin transform from [31]. Define $\hat{\varphi}(x) = |x|^{\alpha-d} \varphi_q(x^*)$ where $x^* := \frac{x}{|x|^2}$. Then by [31, Proposition A.1] there exists a positive constant c_1 such that

$$(-\Delta)^{\alpha/2} \hat{\varphi}(x) > c_1 (|x| - 1)^{q-\alpha} \quad \forall x \in B_2 \setminus \bar{B}.$$

We restrict $\hat{\varphi}$ outside a large compact set so that it is bounded on \mathbb{R}^d . By $\hat{\mathcal{I}}$ we denote the operator

$$\hat{\mathcal{I}}f(x) = b(x) \cdot \nabla f(x) + \int_{\mathbb{R}^d} \mathfrak{d}f(x; z) \frac{\hat{k}(x, z)}{|z|^{d+\alpha}} dz.$$

It is clear that $|\nabla \hat{\varphi}(x)| \leq c_2 (|x| - 1)^{q-1}$ for all $|x| \in (1, 2)$, for some constant c_2 . Also, using the fact that $\hat{\varphi}$ is Hölder continuous of exponent q and (3.10) we obtain

$$\left| \int_{\mathbb{R}^d} \mathfrak{d}\hat{\varphi}(x; z) \frac{\hat{k}(x, z)}{|z|^{d+\alpha}} dz \right| \leq c_3 (|x| - 1)^{q+\theta-\alpha} \quad \forall x \in B_2 \setminus \bar{B},$$

for some constant c_3 . Hence

$$|\hat{\mathcal{I}}\hat{\varphi}(x)| \leq c_4 (|x| - 1)^{(q-1) \wedge (q+\theta-\alpha)}, \quad \text{for } x \in B_2 \setminus \bar{B},$$

for some constant c_4 . Since $\theta > 0$, $\alpha > 1$, and $\mathcal{I} = \hat{\mathcal{I}} - k(x, 0)(-\Delta)^{\alpha/2}$, it follows that we can find η small enough such that

$$\mathcal{I}\hat{\varphi}(x) < -4, \quad \text{for } x \in B_{1+\eta} \setminus \bar{B}.$$

Let K be a compact set containing $B_{1+\eta}$. We define

$$\tilde{\varphi}(x) = \hat{\varphi}(x) \mathbf{1}_K(x) + \mathbb{E}_x[\tau(B^c)] \mathbf{1}_{K^c}(x).$$

Since the hypotheses of Lemma 5.2 are met, we conclude that $\mathbf{1}_{K^c}(x) \mathbb{E}_x[\tau(B^c)]$ is integrable with respect to the kernel π . For $x \in B_{1+\eta} \setminus \bar{B}$, we obtain

$$\begin{aligned} \mathcal{I}\tilde{\varphi}(x) &< -4 + \int_{\mathbb{R}^d} (\mathbb{E}_{x+z}[\tau(B^c)] - \hat{\varphi}(x+z)) \mathbf{1}_{K^c}(x+z) \pi(x, z) dz \\ &= -4 + \int_{K^c} \mathbb{E}_z[\tau(B^c)] \frac{\pi(x, z-x)}{\pi(x, z)} \pi(x, z) dz - \int_{\mathbb{R}^d} \hat{\varphi}(x+z) \mathbf{1}_{K^c}(x+z) \pi(x, z) dz. \end{aligned}$$

Since the kernel is comparable to $|z|^{-d-\alpha}$ on any compact set, we may choose K large enough and use Lemma 5.2 to obtain

$$\mathcal{I}\tilde{\varphi}(x) < -2 \quad \forall x \in B_{1+\eta} \setminus \bar{B}.$$

Let

$$\psi(x) := \left(1 \vee \sup_{z \in K \setminus B_{1+\eta}} \mathbb{E}_z[\tau(B^c)]\right) \left(1 \vee \sup_{z \in K \setminus B_{1+\eta}} \frac{1}{\tilde{\varphi}(z)}\right) \tilde{\varphi}(x).$$

Then, $\mathcal{I}\psi < -2$ on $B_{1+\eta} \setminus \bar{B}$, while $\psi \geq u_n$ on $B_{1+\eta}^c \cup B$. Therefore, by the comparison principle, $u_n \leq \psi$ on $B_{1+\eta} \setminus \bar{B}$ for all $n \in \mathbb{N}$ and the proof is complete. \square

Proof of Theorem 5.4. Consider the sequence of solutions $\{u_n\}$ defined in (5.26). First we note that $u_n(x) \leq \mathbb{E}_x[\tau(B^c)]$ for all x . Clearly $u_{n+1} - u_n$ is bounded, nonnegative and harmonic in $B_n \setminus \bar{B}$. By Theorem 4.1 the operator \mathcal{I} has the Harnack property. Therefore

$$\sup_{x \in F} \sum_{n \geq 1} (u_{n+1}(x) - u_n(x)) < \infty$$

for any compact subset F in \bar{B}^c . Hence Lemma 3.3 combined with Fatou's lemma implies that $\sup_{x \in F} \mathbb{E}_x[\tau(B^c)] < \infty$ for any compact set $F \subset \bar{B}^c$.

We write

$$u_n = u_1 + \sum_{m=1}^{n-1} (u_{m+1}(x) - u_m(x)),$$

and use the Harnack property once more to conclude that $u_n \nearrow u$ uniformly over compact subsets of \bar{B}^c . Since $u \leq \varphi$ in a neighborhood of ∂B by Lemma 5.3, and φ vanishes on ∂B , it follows that $u \in C(\mathbb{R}^d)$. That u is a viscosity solution follows from the fact that $u_n \rightarrow u$ uniformly over compacta as $n \rightarrow \infty$ and Lemma 5.2. \square

6. THE DIRICHLET PROBLEM FOR WEAKLY HÖLDER CONTINUOUS KERNELS

This section is devoted to the study of the Dirichlet problem

$$\begin{aligned} \mathcal{I}u(x) &= f(x) \quad \text{in } D, \\ u &= 0 \quad \text{in } D^c, \end{aligned} \tag{6.1}$$

where $\mathcal{I} \in \mathfrak{I}_\alpha(\beta, \theta, \lambda)$, f is Hölder continuous with exponent β , and D is a bounded open set with a C^2 boundary. In this section, it is convenient to use $s \equiv \frac{\alpha}{2}$ as the parameter reflecting the order of the kernel. Throughout this section, we assume $s > 1/2$.

Recall the definition of weighted Hölder norms in Section 1.1. We start with the following lemma.

Lemma 6.1. *Let D be a C^2 bounded domain in \mathbb{R}^d , and $r \in (0, s]$. Suppose $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ and the constants $\beta \in (0, 1)$, $\theta \in (0, (2s-1) \wedge \beta)$, and $\lambda_D > 0$ satisfy parts (c) and (d) of Definition 3.3. We define*

$$\begin{aligned} \tilde{k}(x, z) &:= c(d, 2s) \left(\frac{k(x, z)}{k(x, 0)} - 1 \right), \\ \mathcal{H}[v](x) &:= \int_{\mathbb{R}^d} \mathfrak{d}v(x; z) \frac{\tilde{k}(x, z)}{|z|^{d+2s}} dz, \end{aligned}$$

where $c(d, 2s) = c(d, \alpha)$ is the normalization constant of the fractional Laplacian.

Suppose that either of the following assumptions hold:

- (i) $\beta \leq r$.

(ii) $\beta \in (r, 1)$ and $\frac{\tilde{k}(x, z)}{|z|^\theta}$ is bounded on $(x, z) \in D \times \mathbb{R}^d$, or, equivalently, it satisfies

$$|k(x, z) - k(x, 0)| \leq \tilde{\lambda}_D |z|^\theta \quad \forall x \in D, \forall z \in \mathbb{R}^d, \quad (6.2)$$

for some positive constant $\tilde{\lambda}_D$.

Then, if $v \in \mathcal{C}_{2s-\theta}^{(-r)}(D)$, we have

$$\llbracket \mathcal{H}[v] \rrbracket_{0;D}^{(2s-r-\theta)} \leq M_0 \|v\|_{2s-\theta;D}^{(-r)},$$

and if $v \in \mathcal{C}_{2s+\beta-\theta}^{(-r)}(D)$, it holds that $\mathcal{H}[v] \in \mathcal{C}_\beta^{(2s-r-\theta)}(D)$, and

$$\|\mathcal{H}[v]\|_{\beta;D}^{(2s-r-\theta)} \leq M_1 \|v\|_{2s+\beta-\theta;D}^{(-r)} \quad (6.3)$$

for some constants M_0 and M_1 which depend only on d, s, β, r , and D .

Moreover, over a set of parameters of the form $\{(r, \beta) : r \in (\varepsilon, 1), \beta \in (0, 1)\}$, constants M_0 and M_1 can be selected which do not depend on β or r , but only on $\varepsilon > 0$.

Proof. Let $x \in D$, and define $R = \frac{d_x}{4}$. We suppose that $R < 1$. It is clear that \tilde{k} satisfies (3.10), and that it is Hölder continuous. Abusing the notation, we'll use the same symbol λ_D as a constant in the estimates. We have,

$$|\mathfrak{d}v(x; z)| \leq |z|^{2s-\theta} R^{r+\theta-2s} \|v\|_{2s-\theta;D}^{(-r)} \quad \forall z \in B_R. \quad (6.4)$$

Also, since $|z| \geq R$ on B_R^c , we obtain

$$\begin{aligned} |\mathfrak{d}v(x; z)| &\leq \left(|z|^r \|v\|_{r;D}^{(-r)} + |z| R^{r-1} \|v\|_{1;D}^{(-r)} \right) \mathbf{1}_{\{|z| \leq 1\}} + 2 \|v\|_{C(D)} \mathbf{1}_{\{|z| > 1\}} \\ &\leq (|z| \wedge 1)^{2s-\theta} R^{r+\theta-2s} \left(\|v\|_{r;D}^{(-r)} + \|v\|_{1;D}^{(-r)} \right) + 2 \|v\|_{C(D)} \mathbf{1}_{\{|z| > 1\}} \end{aligned} \quad (6.5)$$

for all $z \in B_R^c$. Integrating, using (3.10), and (6.4)–(6.5), as well as the Hölder interpolation inequalities, we obtain

$$|\mathcal{H}[v](x)| \leq c_1 (4d_x)^{r+\theta-2s} \|v\|_{2s-\theta;D}^{(-r)} \quad \forall x \in D,$$

for some constant c_1 . Therefore, for some constant M_0 , we have

$$\llbracket \mathcal{H}[v] \rrbracket_{0;D}^{(2s-r-\theta)} \leq M_0 \|v\|_{2s-\theta;D}^{(-r)}. \quad (6.6)$$

Next consider two points $x, y \in D$. If $|x - y| \geq 4d_{xy}$, then (6.6) provides a suitable estimate. Indeed, if $x, y \in D$ are such $4d_{xy} \leq |x - y|$, then, for any r we have

$$\begin{aligned} d_{xy}^{2s-r-\theta} d_{xy}^\beta \frac{|\mathcal{H}[v](x) - \mathcal{H}[v](y)|}{|x - y|^\beta} &\leq \frac{1}{4^\beta} d_{xy}^{2s-r-\theta} |\mathcal{H}[v](x) - \mathcal{H}[v](y)| \\ &\leq \frac{1}{4^\beta} d_x^{2s-r-\theta} |\mathcal{H}[v](x)| + \frac{1}{4^\beta} d_y^{2s-r-\theta} |\mathcal{H}[v](y)| \\ &\leq \frac{2M_0}{4^\beta} \|v\|_{2s-\theta;D}^{(-r)}. \end{aligned}$$

So it suffices to consider the case $|x - y| < 4d_{xy}$. Therefore, we may suppose that x is as above and that $y \in B_R(x)$. Then $d_{xy} \leq 4R$. With $\tilde{\pi}(x, z) := \frac{\tilde{k}(x, z)}{|z|^{d+2s}}$, we write

$$\begin{aligned} F(x, y; z) &:= \mathfrak{d}v(x; z) \tilde{\pi}(x, z) - \mathfrak{d}v(y; z) \tilde{\pi}(y, z) \\ &= F_1(x, y; z) + F_2(x, y; z), \end{aligned}$$

with

$$F_1(x, y; z) := \left(\mathfrak{d}v(x; z) + \mathfrak{d}v(y; z) \right) \frac{\tilde{\pi}(x, z) - \tilde{\pi}(y, z)}{2},$$

$$F_2(x, y; z) := \left(\mathfrak{d}v(x; z) - \mathfrak{d}v(y; z) \right) \frac{\tilde{\pi}(x, z) + \tilde{\pi}(y, z)}{2}.$$

We modify the estimate in (6.4), and write

$$|\mathfrak{d}v(x; z) + \mathfrak{d}v(y; z)| \leq 2|z|^{\gamma_0} R^{r-\gamma_0} [v]_{\gamma_0; D}^{(-r)}, \quad \text{if } z \in B_R,$$

with $\gamma_0 = (2s + \beta - \theta) \wedge (s + 1)$, and

$$|\mathfrak{d}v(x; z) + \mathfrak{d}v(y; z)| \leq 2 \left(|z|^r [v]_{r; D}^{(-r)} + |z| R^{r-1} [v]_{1; D}^{(-r)} \right) \mathbf{1}_{\{|z| \leq 1\}} + 4\|v\|_{C(D)} \mathbf{1}_{\{|z| > 1\}},$$

if $z \in B_R^c$. We use the Hölder continuity of $x \mapsto \tilde{k}(x, \cdot)$ to obtain

$$\int_{\mathbb{R}^d} F_1(x, y; z) dz \leq c_2 R^{r-2s} |x - y|^\beta \|v\|_{\gamma_0; D}^{(-r)}$$

for some constant c_2 . We write this as

$$R^{2s-r-\theta} R^\beta \frac{\int_{\mathbb{R}^d} F_1(x, y; z) dz}{|x - y|^\beta} \leq R^{2s-r-\beta} R^\beta \frac{\int_{\mathbb{R}^d} F_1(x, y; z) dz}{|x - y|^\beta}$$

$$\leq c_2 \|v\|_{\gamma_0; D}^{(-r)}. \quad (6.7)$$

For F_2 , we use

$$\mathfrak{d}v(x; z) = z \cdot \int_0^1 (\nabla v(x + tz) - \nabla v(x)) dt,$$

combined with the following fact: If $f \in C^\gamma(B)$ for $\gamma \in (0, 1]$ and $x, y, x + z, y + z$ are points in B and $\delta \in (0, \gamma)$, then adopting the notation $\Delta f_x(z) := f(x + z) - f(x)$, we obtain by Young's inequality, that

$$\frac{|\Delta f_x(z) - \Delta f_y(z)|}{|z|^{\gamma-\delta} |x - y|^\delta} \leq \frac{\gamma - \delta}{\gamma} \frac{|\Delta f_x(z)| + |\Delta f_y(z)|}{|z|^\gamma} + \frac{\delta}{\gamma} \frac{|\Delta f_{x+z}(y - x)| + |\Delta f_x(y - x)|}{|x - y|^\gamma}$$

$$\leq 2[f]_{\gamma; B}.$$

The same inequality also holds for $\gamma \in (1, 2)$ and $\delta \in (\gamma - 1, 1)$. For this we use

$$\frac{|\Delta f_x(z) - \Delta f_y(z)|}{|z|^{\gamma-\delta} |x - y|^\delta} \leq \frac{1 - \delta}{2 - \gamma} \frac{|z| \left| \int_0^1 (\nabla f(x + tz) - \nabla f(y + tz)) dt \right|}{|x - y|^{\gamma-1} |z|}$$

$$+ \frac{1 + \delta - \gamma}{2 - \gamma} \frac{|x - y| \left| \int_0^1 (\nabla f(y + z + t(x - y)) - \nabla f(y + t(x - y))) dt \right|}{|z|^{\gamma-1} |x - y|}$$

Therefore, in either of the cases (i) or (ii) we obtain,

$$|\nabla v(x + tz) - \nabla v(x) - \nabla v(y + tz) + \nabla v(y)| \leq 2|tz|^{2s-\theta-1} |x - y|^\beta [\nabla v]_{2s-\theta-1+\beta; B_{2R}(x)}$$

for $t \in [0, 1]$, and

$$|\mathfrak{d}v(x; z) - \mathfrak{d}v(y; z)| \leq \frac{2}{2s - \theta} |z|^{2s-\theta} |x - y|^\beta R^{r+\theta-\beta-2s} [v]_{2s+\beta-\theta; D}^{(-r)} \quad \forall z \in B_R. \quad (6.8)$$

Concerning the integration on B_R^c , we use

$$|v(x) - v(y) - z \cdot (\nabla v(x) - \nabla v(y)) \mathbf{1}_{\{|z| \leq 1\}}|$$

$$\begin{aligned}
 &\leq |x - y|^{\beta \vee r} d_{xy}^{r - \beta \vee r} [v]_{\beta \vee r; D}^{(-r)} + (|z| \wedge 1) |x - y|^\beta d_{xy}^{r - \beta - 1} [v]_{1 + \beta; D}^{(-r)} \\
 &\leq c_3 (|z| \wedge 1)^{2s - \theta} |x - y|^\beta R^{r + \theta - \beta - 2s} \|v\|_{1 + \beta; D}^{(-r)} \quad \forall z \in B_R^c, \tag{6.9}
 \end{aligned}$$

for some constant c_3 , and

$$|v(x + z) - v(y + z)| \leq |x - y|^{\beta \vee r} (d_{x+z} \wedge d_{y+z})^{r - \beta \vee r} [v]_{\beta \vee r; D}^{(-r)} \quad \forall z \in B_R^c. \tag{6.10}$$

Integrating the terms on the right hand side of (6.8)–(6.9) is straightforward. Doing so, and using the fact that $1 + \beta < 2s + \beta - \theta$, one obtains the desired estimate.

Concerning the integral of $|v(x + z) - v(y + z)|$ on B_R^c , we distinguish between the cases (i) and (ii). Let $\tilde{\pi}(z) := \frac{|\tilde{\pi}(x, z) + \tilde{\pi}(y, z)|}{2}$. In case (i) we have

$$\begin{aligned}
 &\int_{B_R^c} |v(x + z) - v(y + z)| \tilde{\pi}(z) dz \\
 &\leq |x - y|^r [v]_{r; D}^{(-r)} \int_{B_R^c} \tilde{\pi}(z) dz \\
 &\leq |x - y|^\beta R^{r - \beta} R^{\theta - 2s} [v]_{r; D}^{(-r)} \int_{\mathbb{R}^d} (|z| \wedge \text{diam}(D))^{2s - \theta} \tilde{\pi}(z) dz, \tag{6.11}
 \end{aligned}$$

where we use the fact that $|z| > R$ on B_R^c . In case (ii) the integral is estimated over disjoint sets. We define

$$\mathcal{Z}_{xy}(a) := \{z \in \mathbb{R}^d : d_{x+z} \wedge d_{y+z} < a\} \quad \text{for } a \in (0, R).$$

Since $d_{x+z} \wedge d_{y+z} \in [R, \text{diam}(D)]$ for $x \in \mathcal{Z}_{xy}^c(R)$, integration is straightforward, after replacing $(d_{x+z} \wedge d_{y+z})^{r - \beta}$ in (6.10) with $R^{r - \beta}$. Thus, similarly to (6.11), we obtain

$$\begin{aligned}
 &\int_{B_R^c \cap \mathcal{Z}_{xy}^c(R)} |v(x + z) - v(y + z)| \tilde{\pi}(z) dz \\
 &\leq |x - y|^\beta R^{r - \beta} [v]_{\beta; D}^{(-r)} \int_{B_R^c \cap \mathcal{Z}_{xy}^c(R)} \tilde{\pi}(z) dz \\
 &\leq |x - y|^\beta R^{r + \theta - \beta - 2s} [v]_{\beta; D}^{(-r)} \int_{\mathbb{R}^d} (|z| \wedge \text{diam}(D))^{2s - \theta} \tilde{\pi}(z) dz. \tag{6.12}
 \end{aligned}$$

Since $\mathcal{Z}_{xy}(R) \subset B_R^c$, it remains to compute the integral on $\mathcal{Z}_{xy}(R)$. Recall the definition of D_ε in (3.11). We also define for $\varepsilon > 0$,

$$\tilde{D}(\varepsilon) = \{z \in D : \text{dist}(z, \partial D) \geq \varepsilon\}.$$

In other words $\tilde{D}(\varepsilon) = (D^c)_\varepsilon^c$. We'll make use of the following simple fact: There exists a constant C_0 , such that for all $x \in D$ and positive constants R and ε which satisfy $0 < \varepsilon \leq R$ and $d_x \geq 3R$, it holds that

$$\int_{x+z \in D_\varepsilon \setminus \tilde{D}(\varepsilon)} \frac{dz}{|z|^d} \leq \frac{C_0 \varepsilon}{R}. \tag{6.13}$$

Observe that the support of $|v(x + z) - v(y + z)|$ in $\mathcal{Z}_{xy}(R)$ is contained in the disjoint union of the sets

$$\tilde{\mathcal{Z}}_{xy}(R) := \{z \in \mathcal{Z}_{xy}(R) : d_{x+z} \wedge d_{y+z} > 0\},$$

and

$$\hat{\mathcal{Z}}_{xy} := \{z \in \mathbb{R}^d : x + z \in D_{|x-y|} \setminus D \text{ or } y + z \in D_{|x-y|} \setminus D\}.$$

We also have the bound $|v(x+z) - v(y+z)| \leq |x-y|^r [v]_{r;D}^{(-r)}$ for $z \in \widehat{\mathcal{Z}}_{xy}$. Therefore, using (6.13), we obtain

$$\begin{aligned} \int_{\widehat{\mathcal{Z}}_{xy}} |v(x+z) - v(y+z)| \tilde{\pi}(z) dz &\leq |x-y|^r [v]_{r;D}^{(-r)} R^{\theta-2s} \int_{\widehat{\mathcal{Z}}_{xy}} |z|^{2s-\theta} \tilde{\pi}(z) dz \\ &\leq |x-y|^r [v]_{r;D}^{(-r)} R^{\theta-2s} \int_{\widehat{\mathcal{Z}}_{xy}} \frac{dz}{|z|^d} \\ &\leq 2\tilde{\lambda}_D C_0 |x-y|^{r+1} [v]_{r;D}^{(-r)} R^{\theta-2s} R^{-1} \\ &\leq 2\tilde{\lambda}_D C_0 |x-y|^\beta R^{r+\theta-\beta-2s} [v]_{r;D}^{(-r)}. \end{aligned} \quad (6.14)$$

In order to evaluate the integral over $\widetilde{\mathcal{Z}}_{xy}(R)$, we define

$$G(z) := \frac{|v(x+z) - v(y+z)|}{|x-y|^\beta [v]_{\beta;D}^{(-r)}}.$$

By (6.10) we have

$$\{z \in \widetilde{\mathcal{Z}}_{xy}(R) : G(z) > h\} \subset \{z \in \mathbb{R}^d : x+z \in \widetilde{D}^c(h^{\frac{-1}{\beta-r}})\} \cup \{z \in \mathbb{R}^d : y+z \in \widetilde{D}^c(h^{\frac{-1}{\beta-r}})\}.$$

Therefore, by (6.13), we obtain

$$\begin{aligned} \tilde{\pi}(\{z \in \widetilde{\mathcal{Z}}_{xy}(R) : G(z) > h\}) &\leq 2R^{\theta-2s} \int_{\widehat{\mathcal{Z}}_{xy}} |z|^{2s-\theta} \tilde{\pi}(z) dz \\ &\leq 2\tilde{\lambda}_D C_0 R^{\theta-2s-1} h^{\frac{-1}{\beta-r}}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\widetilde{\mathcal{Z}}_{xy}(R)} G(z) \tilde{\pi}(z) dz &= \int_0^\infty \tilde{\pi}(\{z \in \widetilde{\mathcal{Z}}_{xy}(R) : G(z) > h\}) dh \\ &\leq 2\tilde{\lambda}_D C_0 R^{\theta-2s-1} \int_{R^{r-\beta}}^\infty h^{\frac{-1}{\beta-r}} dh \\ &\leq \frac{2(\beta-r)}{1+r-\beta} \tilde{\lambda}_D C_0 R^{\theta-2s-1} R^{1+r-\beta}. \end{aligned} \quad (6.15)$$

Thus, combining (6.8)–(6.9) with (6.11) in case (i), or with (6.12), (6.14) and (6.15) in case (ii), and using the Hölder interpolation inequalities, we obtain

$$R^{2s-r-\theta} R^\beta \frac{\int_{\mathbb{R}^d} F_2(x, y; z) dz}{|x-y|^\beta} \leq c_4 [v]_{2s+\beta-\theta;D}^{(-r)} \quad (6.16)$$

for some constant c_4 .

Therefore, by (6.6), (6.7) and (6.16) we obtain (6.3), and the proof is complete. \square

Remark 6.1. It is evident from the proof of Lemma 6.1 that the assumption in (6.2) may be replaced by the following: There exists a constant M_D , such that for all $x \in D$ and positive constants R and ε which satisfy $0 < \varepsilon \leq R$ and $d_x \geq 3R$, it holds that

$$\int_{x+z \in D_\varepsilon \setminus \widetilde{D}(\varepsilon)} \frac{\tilde{k}(x, z)}{|z|^{d-\theta}} dz \leq M_D \frac{\varepsilon}{R}.$$

The same applies to Theorems 3.1 and 6.1.

In order to proceed, we need certain properties of solutions of $(-\Delta)^s = f$ in a bounded domain D , and $u = 0$ on D^c , with f not necessarily in $L^\infty(D)$. We start with exhibiting a suitable supersolution.

Lemma 6.2 (Supersolution). *For any $q \in (s - 1/2, s)$ there exists a constant $c_0 > 0$ and a radial continuous function φ such that*

$$\begin{cases} (-\Delta)^s \varphi(x) \geq d_x^{q-2s}, & \text{in } B_4 \setminus \bar{B}_1, \\ \varphi = 0 & \text{in } B_1, \\ 0 \leq \varphi \leq c_0(|x| - 1)^q & \text{in } B_4 \setminus B_1, \\ 1 \leq \varphi \leq c_0 & \text{in } \mathbb{R}^d \setminus B_4. \end{cases}$$

Proof. In view of the Kelvin transform [31, Proposition A.1] it is enough to prove the following: for $q \in (s - 1/2, s)$, and with $\psi(x) := [(1 - |x|)^+]^q$, we have

$$(-\Delta)^s \psi(x) \geq c_1 (1 - |x|)^{q-2s}, \quad \text{for all } x \in B_1, \quad (6.17)$$

for some positive constant c_1 . To prove (6.17) we let $x_0 \in B_1$. Due to the rotational symmetry we may assume $x_0 = re_1$ for some $r \in (0, 1)$. Denote $z = (z_1, \dots, z_d)$. Then

$$\begin{aligned} -(-\Delta)^s \psi(x_0) &= c(d, \alpha) \int_{\mathbb{R}^d} (\psi(x_0 + z) - \psi(x_0)) \frac{1}{|z|^{d+2s}} dz \\ &= c(d, \alpha) \int_{\mathbb{R}^d} \left([(1 - |re_1 + z|)^+]^q - (1 - r)^q \right) \frac{1}{|z|^{d+2s}} dz \\ &\leq c(d, \alpha) \int_{\mathbb{R}^d} \left([(1 - |re_1 + z_1|)^+]^q - (1 - r)^q \right) \frac{1}{|z|^{d+2s}} dz \\ &= c_2 \int_{\mathbb{R}} \left([(1 - |r + z|)^+]^q - (1 - r)^q \right) \frac{1}{|z|^{1+2s}} dz \\ &\leq c_2 \int_{\mathbb{R}} \left([(1 - r - z)^+]^q - (1 - r)^q \right) \frac{1}{|z|^{1+2s}} dz \\ &= c_2 (1 - r)^{q-2s} \int_{\mathbb{R}} \left([(1 - z)^+]^q - 1 \right) \frac{1}{|z|^{1+2s}} dz \end{aligned}$$

for some constant c_2 , where in the first inequality we use the fact that $(1 - |z|)^+ \leq (1 - |z_1|)^+$ and in the second inequality we use $1 - |z| \leq 1 - z$. Define

$$\begin{aligned} A(q) &:= \int_{\mathbb{R}} \left([(1 - z)^+]^q - 1 \right) \frac{1}{|z|^{1+2s}} dz = \int_0^\infty \frac{z^q - 1}{|1 - z|^{1+2s}} dz - \int_{-\infty}^0 \frac{1}{|1 - z|^{1+2s}} dz, \\ B(q) &:= \int_0^\infty \frac{z^q - 1}{|1 - z|^{1+2s}} dz. \end{aligned}$$

We need to show that $A(q) < 0$ for q close to s . It is known that $A(s) = 0$ [31, Proposition 3.1]. Therefore it is enough to show that $B(q)$ is strictly increasing for $q \in (s - 1/2, s)$. We have

$$\begin{aligned} B(q) &= \int_0^1 \frac{z^q - 1}{|1 - z|^{1+2s}} dz + \int_1^\infty \frac{z^q - 1}{|1 - z|^{1+2s}} dz \\ &= \int_0^1 \frac{(z^q - 1)(1 - z^{2s-1-q})}{|1 - z|^{1+2s}} dz. \end{aligned}$$

Therefore, for $q \in (s - 1/2, s)$, we obtain

$$\frac{dB(q)}{dq} = \int_0^1 \frac{(z^q - z^{2s-1-q}) \log z}{|1-z|^{1+2s}} dz > 0,$$

where we use the fact that $\log z \leq 0$ in $[0, 1]$. This completes the proof. \square

Lemma 6.3. *Let f be a continuous function in D satisfying $\sup_{x \in D} d_x^\delta |f(x)| < \infty$ for some $\delta < s$. Then there exists a viscosity solution $u \in C(\mathbb{R}^d)$ to*

$$\begin{aligned} (-\Delta)^s u(x) &= -f(x) \quad \text{in } D, \\ u &= 0 \quad \text{in } D^c. \end{aligned}$$

Also, for every $q < s$ we have

$$|u(x)| \leq C_1 [f]_{0;D}^{(\delta)} d_x^q \quad \forall x \in \bar{D}, \quad (6.18a)$$

$$\|u\|_{C^q(\bar{D})} \leq C_1 \sup_{x \in D} d_x^\delta |f(x)|, \quad (6.18b)$$

for some constant C_1 that depends only on s, δ, q and the domain D . Moreover, since $u = 0$ on D^c , it follows that the Hölder norm of u on \mathbb{R}^d is bounded by the same constant.

Proof. Existence of a continuous viscosity solution follows from Lemma 6.2 and Perron's method, since we can always choose q close enough to s in Lemma 6.2 so as to satisfy $2s - q > \delta$, and obtain a bound on the solution u . From the barrier there exists a compact set $K_1 \subset D$ such that

$$|u(x)| \leq \kappa_1 \left(\sup_{x \in K_1} |u(x)| + [f]_{0;D}^{(\delta)} \right) d_x^q \quad \forall x \in K_1^c, \quad (6.19)$$

where the constant κ_1 depends only on K_1 and D . Also, using the same argument as in Lemma 3.2, we can show that for any compact $K_2 \subset D$, there exists a constant κ_2 , depending on D , and satisfying

$$\sup_{x \in K_2} |u(x)| \leq \kappa_2 \left(\sup_{x \in K_2} |f(x)| + \sup_{x \in D \setminus K_2} |u(x)| \right). \quad (6.20)$$

We choose K_2 and $K_1 \subset K_2$ such that $\sup_{x \in K_2^c \cap D} d_x^q < \frac{1}{2\kappa_1 \kappa_2}$. Then from (6.19)–(6.20) we obtain

$$\sup_{x \in K_2} |u(x)| \leq \kappa_3 [f]_{0;D}^{(\delta)} \quad (6.21)$$

for some constant κ_3 . Hence the bound in (6.18a) follows by combining (6.19) and (6.21).

The estimate in (6.18b) is easily obtained by following the argument in the proof of [31, Proposition 1.1]. \square

Our main result in this section is the following.

Theorem 6.1. *Let $\mathcal{I} \in \mathfrak{I}_{2s}(\beta, \theta, \lambda)$, f be locally Hölder continuous with exponent β , and D be a bounded domain with a C^2 boundary. We assume that neither β , nor $2s + \beta$ are integers, and that either $\beta < s$, or that $\beta \geq s$ and*

$$|k(x, z) - k(x, 0)| \leq \tilde{\lambda}_D |z|^\theta \quad \forall x \in D, \forall z \in \mathbb{R}^d,$$

for some positive constant $\tilde{\lambda}_D$. Then the Dirichlet problem in (6.1) has a unique solution in $C_{\text{loc}}^{2s+\beta}(D) \cap C(\bar{D})$. Moreover, for any $r < s$, we have the estimate

$$\|u\|_{2s+\beta;D}^{(-r)} \leq C_0 \|f\|_{C^\beta(\bar{D})}$$

for some constant C_0 that depends only on d, β, r, s and the domain D .

Proof. Consider the case $\beta \geq s$. We write (6.1) as

$$(-\Delta)^s u(x) = \mathcal{T}[u](x) := \frac{c(d, 2s)}{k(x, 0)} (-f(x) + b(x) \cdot \nabla u(x)) + \mathcal{H}[u](x) \quad \text{in } D, \quad (6.22)$$

$$u = 0 \quad \text{in } D^c,$$

and we apply the Leray–Schauder fixed point theorem. Also, without loss of generality, we assume $\theta < 2s - 1$. We choose any $r \in (0, s)$ which satisfies

$$r > \left(s - \frac{\theta}{2}\right) \vee \left(1 - s + \frac{\theta}{2}\right),$$

and let $v \in \mathcal{C}_{2s+\beta-\theta}^{(-r)}(D)$. Then $\mathcal{H}[v] \in \mathcal{C}_{\beta}^{(2s-r-\theta)}(D)$ by Lemma 6.1. Since $\nabla v \in \mathcal{C}_{2s+\beta-\theta-1}^{(1-r)}(D)$ and $(1-r) \wedge (2s-r-\theta) < s$ by hypothesis, then applying Lemma 6.3 we conclude that there exists a solution u to $(-\Delta)^s u = \mathcal{T}[v]$ on D , with $u = 0$ on D^c , such that $u \in \mathcal{C}_0^{(-q)}(D)$ for any $q < s$.

Next we obtain some estimates that are needed in order to apply the Leray–Schauder fixed point theorem. By Lemma 6.1 we obtain

$$\|\mathcal{H}[v]\|_{0;D}^{(2s-r-\theta/2)} = \|\mathcal{H}[v]\|_{0;D}^{(2s-(r-\theta/2)-\theta)} \leq \kappa_1 \|v\|_{2s-\theta;D}^{(-r+\theta/2)},$$

and similarly,

$$\|\mathcal{H}[v]\|_{\beta;D}^{(2s-r-\theta/2)} \leq \kappa_1 \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)}, \quad (6.23)$$

for some constant κ_1 which does not depend on θ or r . Thus, since by hypothesis $2s - r - \theta/2 < s$ and $1 - r + \theta/2 < s$, we obtain by Lemma 6.3 that

$$\|u\|_{C^r(\mathbb{R}^d)} \leq \kappa'_1 \left(\|f\|_{C(\bar{D})} + \|\nabla v\|_{0;D}^{(1-r+\theta/2)} + \|v\|_{2s-\theta;D}^{(-r+\theta/2)} \right) \quad (6.24)$$

for some constant κ'_1 . Also, by Lemma 2.10 in [31], there exists a constant κ_2 , depending only on β, s, r and d , such that

$$\|u\|_{2s+\beta;D}^{(-r)} \leq \kappa_2 \left(\|u\|_{C^r(\mathbb{R}^d)} + \|\mathcal{T}[v]\|_{\beta;D}^{(2s-r)} \right). \quad (6.25)$$

It follows by (6.24)–(6.25) that $v \mapsto u$ is a continuous map from $\mathcal{C}_{2s+\beta-\theta}^{(-r)}$ to itself. Moreover, since $\mathcal{C}_{2s+\beta}^{(-r)}(D)$ is precompact in $\mathcal{C}_{2s+\beta-\theta}^{(-r)}(D)$, it follows that $v \mapsto u$ is compact.

Next we obtain a bound for $\|u\|_{2s+\beta;D}^{(-r)}$. By (6.23) we have

$$\begin{aligned} \|\mathcal{H}[v]\|_{\beta;D}^{(2s-r)} &\leq (\text{diam}(D))^{\theta/2} \|\mathcal{H}[v]\|_{\beta;D}^{(2s-r-\theta/2)} \\ &\leq \kappa_1 (\text{diam}(D))^{\theta/2} \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)}. \end{aligned}$$

Therefore, since also $2s - r > 1 - r + \theta/2$, we obtain

$$\|\mathcal{T}[v]\|_{\beta;D}^{(2s-r)} \leq \kappa_3 \left(\|f\|_{C^\beta(\bar{D})} + [v]_{1;D}^{(-r+\theta/2)} + \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)} \right) \quad (6.26)$$

for some constant κ_3 . By the Hölder interpolation inequalities, for any $\varepsilon > 0$, there exists $\tilde{C}(\varepsilon) > 0$ such that

$$[v]_{1;D}^{(-r+\theta/2)} + \|v\|_{2s+\beta-\theta;D}^{(-r+\theta/2)} \leq \tilde{C}(\varepsilon) [v]_{0;D}^{(-r+\theta/2)} + \varepsilon \|v\|_{2s+\beta;D}^{(-r+\theta/2)}. \quad (6.27)$$

Combining (6.24), (6.25), and (6.26), and then using (6.27) and the inequality

$$[v]_{2s+\beta;D}^{(-r+\theta/2)} \leq (\text{diam}(D))^{\theta/2} \|v\|_{2s+\beta;D}^{(-r+\theta/2)}$$

we obtain

$$\|u\|_{2s+\beta;D}^{(-r)} \leq \kappa_4(\varepsilon) \left(\|f\|_{C^\beta(\bar{D})} + \|v\|_{0;D}^{(-r+\theta/2)} \right) + \varepsilon \|v\|_{2s+\beta;D}^{(-r)}. \quad (6.28)$$

In order to apply the Leray–Schauder fixed point theorem, it suffices to show that the set of solutions $u \in \mathcal{C}_{2s+\beta}^{(-r)}(D)$ of $(-\Delta)^s u(x) = \xi \mathcal{T}[u](x)$, for $\xi \in [0, 1]$, with $u = 0$ on D^c , is bounded in $\mathcal{C}_{2s+\beta}^{(-r)}(D)$. However, from the above calculations, any such solution u satisfies (6.28) with $v \equiv u$. Moreover by Lemma 3.2,

$$\sup_{x \in D} |u(x)| \leq \kappa_5 \sup_{x \in D} |f(x)| \quad (6.29)$$

for some constant κ_5 . We also have that

$$\begin{aligned} \|u\|_{0;D}^{(-r+\theta/2)} &\leq \varepsilon^{-r+\theta/2} \sup_{x \in D, d_x \geq \varepsilon} |u(x)| + \varepsilon^{\theta/2} \sup_{x \in D, d_x < \varepsilon} d_x^{-r} |u(x)| \\ &\leq \varepsilon^{-r+\theta/2} \sup_{x \in D} |u(x)| + \varepsilon^{\theta/2} \|u\|_{0;D}^{(-r)}. \end{aligned} \quad (6.30)$$

Choosing $\varepsilon > 0$ small enough, and using (6.29)–(6.30) on the right hand side of (6.28) with $v \equiv u$, we obtain

$$\|u\|_{2s+\beta;D}^{(-r)} \leq \kappa_6 \|f\|_{C^\beta(\bar{D})} \quad (6.31)$$

for some constant κ_6 . Hence by the Leray–Schauder fixed point theorem the map $v \mapsto u$ given by (6.22) has a fixed point $u \in \mathcal{C}_{2s+\beta}^{(-r)}(D)$, i.e.,

$$(-\Delta)^s u(x) = \mathcal{T}[u](x).$$

Hence, this is a solution to (6.1). Uniqueness is obvious as u is a classical solution. The bound in (6.31) then applies and the proof is complete. The proof in the case $\beta < s$ is completely analogous. \square

Optimal regularity up to the boundary can be obtained under additional hypotheses. The following result is a modest extension of the results in [31, Proposition 1.1].

Corollary 6.1. *Let $\mathcal{I} \in \mathfrak{I}_{2s}(\beta, \theta, \lambda)$ with $\theta > s$, f be locally Hölder continuous with exponent β , and D be a bounded domain with a C^2 boundary. Suppose in addition that $b = 0$ and that k is symmetric, i.e., $k(x, z) = k(x, -z)$. Then the solution of the Dirichlet problem in (6.1) is in $C^s(\mathbb{R}^d)$. Moreover, for any $\beta < s$ we have $u \in \mathcal{C}_{2s+\beta}^{(-s)}(D)$.*

Proof. By Theorem 6.1, the Dirichlet problem in (6.1) has a unique solution in $C_{\text{loc}}^{2s+\rho}(D) \cap C(\bar{D})$, for any $\rho < \beta \wedge s$. Moreover, for any $r < s$, we have the estimate

$$\|u\|_{2s+\rho;D}^{(-r)} \leq C_0 \|f\|_{C^\beta(\bar{D})}.$$

Fix $r = 2s - \theta$. Then

$$\int_{R < |z| < 1} |z|^r \frac{\tilde{k}(x, z)}{|z|^{d+2s}} dz = \int_{R < |z| < 1} |z|^{2s-\theta} \frac{\tilde{k}(x, z)}{|z|^{d+2s}} dz \leq \lambda_D.$$

By (6.5) and the symmetry of the kernel, it follows that

$$\left| \int_{R < |z|} \mathfrak{d}u(x; z) \frac{\tilde{k}(x, z)}{|z|^{d+2s}} dz \right| \leq \kappa_1 \left([u]_{r;D}^{(-r)} + \|u\|_{C(\bar{D})} \right) \quad \forall x \in D,$$

for some constant κ_1 . Combining this with the estimate in Lemma 6.1 we obtain

$$[\mathcal{H}[u]]_{0;D}^{(0)} \leq M_0 \|u\|_{r;D}^{(-r)} < \infty,$$

implying that $\mathcal{H}[u] \in L^\infty(D)$. It then follows by [31, Proposition 1.1] that $u \in C^s(\mathbb{R}^d)$, and that for some constant C depending only on s , we have

$$\|u\|_{C^s(\mathbb{R}^d)} \leq C \|\mathcal{T}[u]\|_{L^\infty(D)}$$

$$\begin{aligned} &\leq C \lambda_D^{-1} c(d, 2s) \left(\|f\|_{L^\infty(D)} + \|\mathcal{H}[u]\|_{L^\infty(D)} \right) \\ &\leq C \lambda_D^{-1} c(d, 2s) \left(\|f\|_{L^\infty(D)} + M_0 \|u\|_{r; D}^{(-r)} \right). \end{aligned}$$

Using the Hölder interpolation inequalities we obtain from the preceding estimate that

$$\|u\|_{C^s(\mathbb{R}^d)} \leq \tilde{C} \|f\|_{L^\infty(D)}$$

for some constant \tilde{C} depending only on s , θ , and λ_D .

Applying Lemma 6.1 once more, we conclude that $\mathcal{H}[u] \in \mathfrak{C}_{\beta'}^{(s)}(D)$ for any $\beta' \leq r$, and that

$$\|\mathcal{H}[u]\|_{\beta'; D}^{(s)} \leq M_1 \|u\|_{2s+\beta'-\theta; D}^{(-r)}.$$

Hence, applying [31, Proposition 1.4], we obtain

$$\|u\|_{2s+\beta'; D}^{(-s)} \leq C_1 \left(\|u\|_{C^s(\mathbb{R}^d)} + \|\mathcal{T}[u]\|_{\beta'; D}^{(s)} \right)$$

for some constant C_1 , and we can repeat this procedure to reach $u \in \mathfrak{C}_{2s+\beta}^{(-s)}(D)$. \square

Acknowledgments. We thank Dennis Kriventsov for helping us in clarifying some points of his paper [28] and for suggesting us the paper [31]. We also thank Héctor Chang Lara and Gonzalo Dávila for their help. The work of Anup Biswas was supported in part by an award from the Simons Foundation (No. 197982 to The University of Texas at Austin) and in part by the Office of Naval Research through the Electric Ship Research and Development Consortium. The work of Ari Arapostathis was supported in part by the Office of Naval Research through the Electric Ship Research and Development Consortium. This research of Luis Caffarelli is supported by an award from NSF.

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