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Ergodicity of a collective random walk on a circle

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Abstract

We discuss conditions for unique ergodicity of a collective random walk on a continuous circle. Individual particles in this collective motion perform independent (and different in general) random walks conditioned by the assumption that the particles cannot overrun each other. Additionally to sufficient conditions for the unique ergodicity we discover a new and unexpected way for its violation due to excessively large local jumps. Necessary and sufficient conditions for the unique ergodicity of the deterministic version of this system are obtained as well. Technically our approach is based on the interlacing property of the spin function which describes states of pairs of particles in coupled processes under study.

1 Introduction

We consider a collective random walk of a configuration consisting of n particles on a unit continuous circle. Each particle without interactions with others performs an independent random walk and the interaction between particles consists in the prohibition for particles to overrun each other. The i-th particle in the configuration at time $t \in \mathbb{Z}_+$ is characterized by the position of its center $x_i^t \in S := [0, 1)$, the radius $r_i \geq 0$ of the ball (representing the particle), and the distribution of jumps P_i (i.e. the particle makes a jump equal to a random value ξ distributed according to P_i). In general our theory covers both positive and negative jumps, but to simplify presentation we discuss in the Introduction only the case of nonnegative jumps, i.e. $P_i([0, 1]) = 1$, leaving the general case to Section 4.

The collective random walk under consideration is a close relative to exclusion processes introduced by Frank Spitzer [9] and studied in a number of publications. One of the most prominent and detailed review of statistical properties of such processes considered on a lattice and in continuous time can be found in [7] (see further references therein and [1, 2, 6, 8] for more recent results).

We say that a particle configuration $x^t := \{x_1^t, \ldots, x_n^t\}$ is *admissible* if the open balls corresponding to the particles in the configuration do not intersect (see Fig. 1), i.e. it satisfies the inequality:

$$x_i^t + r_i + r_{i+1} \leq x_{i+1}^t \qquad \forall i.$$

The set of all admissible configurations we denote by X. Here and in the sequel (if the exception is not explicitly mentioned) arithmetic operations with metric elements

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$(x_i, v_i, r_i$, etc.) are taken modulo 1, the comparison between them is performed according to the *clockwise order* as elements of the unit circle, and the indices are taken modulo n, i.e. $x^t_{n+1} \equiv x^t_1, x^t_0 \equiv x^t_n$.

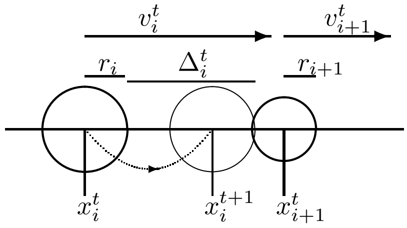

Figure 1: Exclusion process in continuum

Finally the local dynamics (see Fig. 1) of an individual particle is defined by the relation

$$x^{t+1}_i := \min\{x^t_i + u^t_i v^t_i,\ x^t_{i+1} - r_i - r_{i+1}\}, \tag{1.1}$$

where the random variable (velocity) v^t_i is chosen according to the distribution P_i, and the collection of multiplicators $u^t := \{u^t_i\}_i$ with $u^t_i \in \{0, 1\}$ represents the updating rule (see below). The random variables $\{v^t_i\}_{i,t}$ are assumed to be mutually independent.

The moment of time when the i-th particle is stopped by the $i + 1$-th particle (i.e. $x^t_i + u^t_i v^t_i > x^t_{i+1} - r_i - r_{i+1}$) will be referred as the moment of *interaction* between these particles. Note that a homogeneous version of systems of this type (when r_i and P_i do not depend on i) was introduced and studied in [4].

Depending on the updating rule u^t discrete time processes under consideration may be classified into two types: with *parallel* and *sequential* updating. In the former case all particles are trying to move simultaneously which leads to an arbitrary number of simultaneous interactions. In the later case at each moment of time only one particle is chosen to jump according to a certain rule (e.g. by a random choice) and thus at most a single interaction may take place. In terms of u^t the parallel updating means that $u^t_i \equiv 1$ for all i, t. The sequential updating may be realized in a number of ways and we shall consider the following two scenarios:
(a) random sequential updating: at time t the only positive entry in u^t is chosen at random according to a given distribution $q := \{q_1, \ldots, q_n\}$ with $\prod_i q_i > 0$.
(b) deterministic sequential updating: we start by choosing a certain index i starting from which the particles are updated clockwise one at a time until we reach i. Then we repeat the procedure, etc.

If the type of the sequential updating is not specified explicitly we mean that it is either (a) or (b). Surprisingly conditions leading to the unique ergodicity in both these cases coincide. The sequential updating in a sense is equivalent to continuous time collective random walk in which a random alarm clock is attached to each particle and the particle moves only when the clock rings.

The local dynamics (1.1) together with a specific updating rule define a finite dimensional Markov chain in the phase space of admissible configurations. This local dynamics uniquely defines the dynamics of *gaps*

$$\Delta^t_i := (x^t_{i+1} - r_{i+1}) - (x^t_i + r_i)$$

between the particles. Naturally $\Delta^t := \{\Delta^t_i\}_i$ is *admissible* if $\Delta^t_i \geq 0$ and $\sum_i(\Delta^t_i + 2r_i) = 1$.

In terms of gaps the local dynamics of particles may be rewritten as

$$x^{t+1}_i := x^t_i + \min\{u^t_i v^t_i,\ \Delta^t_i\},$$

while the actual dynamics of gaps is described by the relation

$$\Delta_i^{t+1} := \Delta_i^t - \min\{u_i^t v_i^t,\ \Delta_i^t\} + \min\{u_{i+1}^t v_{i+1}^t,\ \Delta_{i+1}^t\}. \tag{1.2}$$

This shows that the dynamics of gaps Δ^t is a Markov chain as well.

Standard arguments about the compactness of the phase space and the continuity of the corresponding Markov operators imply the existence of invariant measures of the processes under study. Question about the uniqueness of the invariant measure is much more delicate. Our main results (Theorems 1,2) give sufficient conditions for the uniqueness of the invariant measure (i.e. for the unique ergodicity) in the true random setting when the distributions P_i are nontrivial. An unexpected counterexample in the case of parallel updating related to excessively large local velocities is constructed in Proposition 1. Despite very week assumptions made in these theorems they cannot be applied in the pure deterministic setting when each distribution P_i is supported by a single constant local velocity $v_i > 0$. Nevertheless we show that this deterministic process still might be uniquely ergodic (albeit due to different reasons) and give necessary and sufficient conditions for this (Theorem 3). Comparing to our earlier note [5], sufficient conditions for the unique ergodicity are formulated in very different terms and became much weaker especially in the general non totally asymmetric case. Technically the main improvement is that instead of using the principle of "isolated interactions" the present approach is based on the interlacing property of the spin function (see Section 4, Lemma 3), which describes states of pairs of particles in statically coupled processes under study.

Let us discuss possible obstacles for the unique ergodicity in the simplest setting. Assume for a moment that instead of a system on the continuous circle we deal with a finite discrete time lattice system with L sites and periodic boundary conditions, i.e. $x_i^t, v_i^t \in \frac{1}{L}\mathbb{Z},\ r_i = 1/(2L)$. Assume also that each particle jumps with probability $p \in (0,1)$ to its nearest right neighboring site if it is not occupied or stays put otherwise. In other words we consider the simple discrete time totally asymmetric exclusion process with parallel updates. This Markov chain has an important property that the probability to reach one state from another in finite time is positive, which implies unique ergodicity of this process. On the other hand, a simple modification of this process allowing longer particle jumps, e.g. by 2 sites instead of one with even L, breaks down this property. Nevertheless as we shall show in this case and in much more complicated case of the continuous circle simple and not especially restrictive assumptions on the particle jumps guarantee the ergodicity of the dynamics of gaps. In some situations (see below) conditions for the unique ergodicity of the original system and of the dynamics of gaps coincide but typically this is not the case.

The paper is organized as follows. In Section 2 we formulate our main results. Section 3 is dedicated to technical constructions allowing the analysis of synchronization type phenomena and proofs of the main results in the totally asymmetric case. An alternative construction based on the dynamics of gaps only is discussed here as well. In Section 4 we deal with a more general situation when the local velocities are taking both signs (and hence particles may move toward each other), discuss briefly the lattice version of the collective random walk and a more strict version of the particles conflicts resolution. The last Section is dedicated to the analysis how unique ergodicity may take place in the pure deterministic collective walk.

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2 Main results

We start with the totally asymmetric setting, i.e. $v_i^t \ge 0$ for all i, t.

Theorem 1 *(Ergodicity) Let the distributions of jumps v_i^t satisfy the following non-degeneracy condition*

$$P(v_i^t > v_j^t) > 0 \ \ \forall i \neq j, \ \forall t. \tag{2.1}$$

Then the Markov process $\Delta^t := \{\Delta_i^t\}$ with the sequential updating is uniquely ergodic, namely for each admissible initial Δ^0 the distributions of the random variables Δ^t converges as $t \to \infty$ in Cesaro means to a limit which does not depend on Δ^0. In the case of the parallel updating the same claim holds if additionally

$$P(v_i^t < \varepsilon) > 0 \ \ \forall \varepsilon > 0, \ \forall i, t. \tag{2.2}$$

The non-degeneracy condition (2.1) is a rather distant generalization of the simplest law $P_i := (1-p)\delta_0 + p\delta_\sigma$, when a particle makes a jump of length $\sigma > 0$ with the probability $p > 0$ or stays put otherwise. The condition (2.2) implies that in the parallel updating scenario not all particles interact all the time. See also the discussion of the necessity of this condition after the proof of Theorem 1 in Section 3. The following result shows that the presence of non-interacting particles is "almost" necessary for the unique ergodicity under parallel updating.

Proposition 1 *Let there exist a collection of positive values $v^0 := \{v_i^0\}$ such that* $\mathrm{Var(v^0) := \sum_i |v_i^0 - v_{i+1}^0|} > 0$ *and*

$$\sum_i v_i^0 > 1 - 2\sum_i r_i. \tag{2.3}$$

Then the Markov process Δ^t with parallel updating and $P(v_i^t \ge v_i^0) = 1$ for all i, t has infinitely many ergodic invariant measures.

Remark 2 Using the static coupling construction developed in Section 3 one can show that each two x^t-invariant measures coincide up to a spatial shift, which together with the results of Theorem 1 implies the unique ergodicity of the original process under factorization upon spatial shifts.

Now we formulate sufficient conditions of the unique ergodicity in the general non totally asymmetric case when the local velocities may take both positive and negative values.

Theorem 2 *Let the distributions of jumps v_i^t satisfy one of the following non-degeneracy conditions*

$$P(v_i^t > v_j^t) > 0 \quad \forall i \neq j, \ \forall t, \tag{2.4}$$

$$P(v_i^t < v_j^t) > 0 \quad \forall i \neq j, \ \forall t. \tag{2.5}$$

Then the Markov process $\Delta^t := \{\Delta_i^t\}$ with sequential updating is uniquely ergodic. The same claim holds true in the case of parallel updating if additionally at least for one index i we have

$$P(v_i^t v_{i+1}^t < 0) > 0. \tag{2.6}$$

Observe that if the condition (2.6) is violated we are back to the totally asymmetric case.

Very weak sufficient conditions of the unique ergodicity formulated in Theorems 1,2 always include a version of a non-degeneracy assumption which gives an impression that in the deterministic setting (when the jump distributions P_i are concentrated at a single point) the unique ergodicity is excluded. The following result addresses this question and shows that the corresponding deterministic dynamical system may posses a single nontrivial invariant measure.

Theorem 3 *Let $P(v_i^t = v_i) = 1$ for some constant positive local velocities $v := \{v_i\}_{i=1}^n$ and $u_i^t \equiv 1$. Denote $v_{\min} := \min_i\{v_i\}$ and $\alpha := 1 - 2\sum_{i=1}^n r_i - nv_{\min}$. Then the dynamical system defined by the relation (1.1) is uniquely ergodic if and only if*
(a) $v_{\min}$ is achieved at a single index $i_{\min}$,
and one of the following assumptions holds true:
(b) $\alpha \ge 0$ and $v_{\min}$ is irrational,
(c) $\alpha < 0$ and $\alpha + v_{\min}$ is irrational.

The proofs of Theorems 1,2 use a technical result which (especially due to its deterministic nature) is of interest by itself. Assume that the local particle velocities v_i^t are given for all i, t. We say that the particle process satisfies the *chain-interacting* property if for any initial configuration x^0 subsequent particles will interact in finite time (see a more detailed version of this assumption in Section 3) .

Theorem 4 *(Synchronization) Let the chain-interacting property hold. Then*
(a) in the case of sequential updating for any initial admissible configurations $x^0, \acute{x}^0$ the processes $x^t, \acute{x}^t$ are getting synchronized with time, namely

$$\sum_i^n |\Delta_i^t - \acute{\Delta}_i^t| \stackrel{t\to\infty}{\longrightarrow} 0.$$

(b) in the case of parallel updating the same claim holds if additionally for infinitely many moments of time t there exists $j = j(t)$ such that either $v_j^t < 0 < v_{j+1}^t$ or $v_j^t = 0$.
The chain-interacting property holds in the case of sequential updating if

$$\inf_{i,t}\{v_i^t\} > \frac{1}{n}(1 - 2\sum_i r_i) > 0, \tag{2.7}$$

while in the case of parallel updating it is enough to assume that

$$\forall t_0 \ge 0,\ \forall i \in \{1, \dots, n\}\ \ \exists \tau_i < \infty : \quad \sum_{t=t_0}^{t_0+\tau_i} (v_i^t - v_{i+1}^t) > 1. \tag{2.8}$$

3 Synchronization phenomenon and proofs in the totally asymmetric setting

Define a *static coupling* between the processes $x^t, \acute{x}^t$ satisfying the relation (1.1) as a pair-process $(x, \acute{x})^t$ in which all random choices related to particles with equal indices coincide, i.e. $v_i^t = \acute{v}_i^t$ for all i, t. In the statically coupled processes we say that the i-th pair of particles *interacts* with the $(i+1)$-th one if at least one of the i-th particles in these processes is doing so.

3.1 Construction in terms of particle's positions

Consider a "lifting" of the process x^t acting on the circle S to the real line $\mathbb{R}$ defined by the relation

$$R(x,t,i) := \begin{cases} 0 & \text{if } t=0, i=1 \\ \sum_{j=1}^{i-1} \Delta_j^0 & \text{if } t=0, i>1 \\ R(x,t-1,i) + \min\{u_i^{t-1} v_i^{t-1}, \Delta_i^{t-1}\} & \text{if } t>0. \end{cases} \tag{3.1}$$

Rewriting the definition of a gap in terms of the lifting map we have

$$\Delta_i^t = \begin{cases} R(x,t,i+1) - R(x,t,i) & \text{if } 1 \leq i < n \\ R(x,t,1) + 1 - R(x,t,n) & \text{if } i = n \end{cases},$$

and the total distance covered by the i-th particle during the time from 0 to t is equal to $R(x,t,i) - R(x,0,i)$.

Let the processes $x^t, \acute{x}^t$ be statically coupled. For each i we associate to the i-th particle a new random variable

$$s_i^t := R(x,t,i) - R(\acute{x},t,i),$$

to which we refer as a *spin*. It is useful to think about the pair of points $R(x,t,i), R(\acute{x},t,i)$ as a *dumbbell* whose disks centers lye on two parallel straight lines. Then the spin s_i^t describes the state of this dumbbell.

As we shall see, after the interaction between the i-th and $i+1$-th particles in one of the processes x^t and $\acute{x}^t$ (or in both of them) the corresponding spins become closer to each other in comparison to the situation just before the interaction (see Fig. 2,3). On the other hand, this might lead to the increase of the distinction with the spins of neighboring $(i-1)$-th and $(i+1)$-th particles, i.e. either $|s_{i-1}^t - s_i^t|$ or $|s_i^t - s_{i+1}^t|$ may grow with time t. Nevertheless in the worst case the amount to which one of the distinctions was enlarged cannot be greater than the amount to which another distinction became smaller. The idea of our approach is to show the under dynamics the *variation* of the collection of spins s^t, defined as

$$\mathrm{Var(s^t) := \sum_{i=1}^{n} |s_i^t - s_{i+1}^t|},$$

is a non-increasing function of the variable t and converges to zero monotonically with time.

Lemma 3 *(Interlacing) Let at time t one (or both) of the i-th particles in the processes* $x^t, \acute{x}^t$ *interact with the* $(i+1)$*-th one. Then the interlacing property*

$$\min\{s_i^t, s_{i+1}^t\} \leq s_i^{t+1} \leq \max\{s_i^t, s_{i+1}^t\} \tag{3.2}$$

takes place. Additionally, if $s_i^t \neq s_{i+1}^t$ *then*

$$|s_i^t - s_{i+1}^t| > |s_i^{t+1} - s_{i+1}^t|. \tag{3.3}$$

Proof. Consider the interaction at time t of the i-th pair of particles with the $(i+1)$-th pair and assume that

$$\Delta_i^t < v_i^t \leq \acute{\Delta}_i^t. \tag{3.4}$$

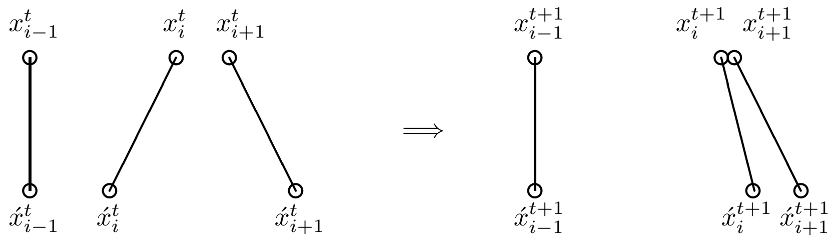

Figure 2: Isolated interaction of the i-th pair with the $(i+1)$-th one. Here we use x_i^t instead of $R(x,t,i)$ to simplify the presentation.

This situation is depicted in Fig. 2. By definition of the spin, (3.4) implies that $s_i^t > s_{i+1}^t$.

We have $s_i^t = R(x,t,i) - R(\acute{x},t,i)$ and

$$\begin{aligned} R(x,t+1,i) &= \begin{cases} R(x,t,i+1) - r_i - r_{i+1} < R(x,t,i) + v_i^t & \text{if } 1 \le i < n \\ R(x,t,1) + 1 - r_n - r_1 < R(x,t,n) + v_n^t & \text{if } i = n \end{cases} \\ R(\acute{x},t+1,i) &= R(\acute{x},t,i) + v_i^t \le R(\acute{x},t,i+1) - r_i - r_{i+1}. \end{aligned}$$

Therefore

$$\min\{s_i^t, s_{i+1}^t\} = s_{i+1}^t \le s_i^{t+1} < s_i^t = \max\{s_i^t, s_{i+1}^t\},$$

which additionally implies (3.3).

The situation $\Delta_i^t \ge v_i^t > \acute{\Delta}_i^t$ is considered similarly exchanging the roles played by the processes x^t and $\acute{x}^t$. It remains to study the case

$$v_i^t > \max\{\Delta_i^t, \acute{\Delta}_i^t\}.$$

This inequality means that after the interaction $s_i^{t+1} = s_i^t$, which implies (3.2), and if additionally $s_i^t \ne s_{i+1}^t$ we get (3.3) as well. □

Lemma 4 *The variation of the spin function* $\mathrm{Var}(\mathrm{s}^\mathrm{t})$ *does not increase under dynamics.*

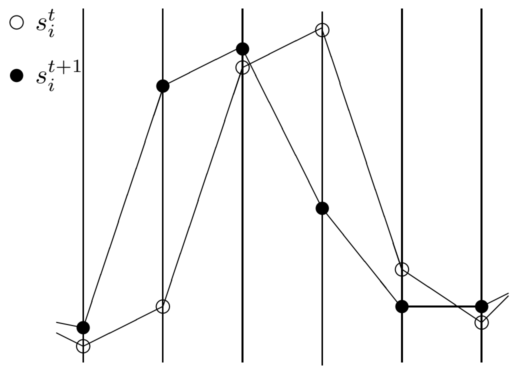

Figure 3: Change of the spin function under dynamics. s_i^t/s_i^{t+1} are marked by open/closed circles.

Proof. We say that $[j,k] := \{j, j+1, \dots, k\}$ is the interval of *positive monotonicity* of the collection $s := \{s_i\}$ considered as a function of the variable i if $s_j \le s_{j+1} \le \dots \le s_k$, and the interval of *negative monotonicity* if $s_j \ge s_{j+1} \ge \dots \ge s_k$.

By the property (3.2) for each locally maximal interval of positive monotonicity $[i, j]$ of the function s^{t+1} we have

$$\min\{s^t_i, s^t_{i+1}\} \le s^{t+1}_i \le s^{t+1}_j \le \max\{s^t_j, s^t_{j+1}\}. \tag{3.5}$$

Similarly for each locally maximal interval of negave monotonicity $[i, j]$

$$\max\{s^t_i, s^t_{i+1}\} \ge s^{t+1}_i \ge s^{t+1}_j \ge \min\{s^t_j, s^t_{j+1}\}. \tag{3.6}$$

Consider two consecutive locally maximal intervals of positive monotonicity $[i^+_k, j^+_k]$ and $[i^+_{k+1}, j^+_{k+1}]$ of the function s^{t+1}. Then by (3.5)

$$\mathrm{ind}\max\{s^t_{j^+_k}, s^t_{j^+_k+1}\} \le \mathrm{ind}\min\{s^t_{i^+_{k+1}+1}, s^t_{i^+_{k+1}}\}, \tag{3.7}$$

where

$$\mathrm{ind}\max\{s_i, s_j\} := \begin{cases} i & \text{if } s_i \ge s_j \\ j & \text{otherwise} \end{cases}, \quad \mathrm{ind}\min\{s_i, s_j\} := \begin{cases} i & \text{if } s_i \le s_j \\ j & \text{otherwise} \end{cases}.$$

The main difficulty in the analysis of the change of variation of the spin function is that the intervals of monotonicity of the functions s^t and s^{t+1} need not coincide and even may be very different from each other (see, e.g. Fig 3). Additionally some individual slopes of the function s^{t+1} might be much larger than the corresponding slopes of the function s^t.

Let $[i^+_k, j^+_k]$ and $[i^-_k, j^-_k]$ be locally maximal intervals of positive and negative monotonicity of the function s respectively, and let $[i_k, j_k]$ be any collection of non-intersecting intervals. (We say that integer intervals do not intersect if they have at most one common point.) Then by the triangle inequality

$$\begin{aligned} \mathrm{Var(s)} &= \sum_k \Big((-s_{i^+_k} + s_{j^+_k}) + (s_{i^-_k} - s_{j^-_k}) \Big) \\ &= 2\sum_k (-s_{i^+_k} + s_{j^+_k}) \ge 2\sum_k (-s_{i_k} + s_{j_k}). \end{aligned} \tag{3.8}$$

Therefore, combining (3.5) and (3.1) and using (3.7) we obtain

$$\begin{aligned} \mathrm{Var(s^{t+1})} &= 2\sum_k (-s^{t+1}_{i^+_k} + s^{t+1}_{j^+_k}) \\ &\le 2\sum_k \Big(-\min\{s^t_{i^+_k+1}, s^t_{i^+_k}\} + \max\{s^t_{j^+_k}, s^t_{j^+_k+1}\} \Big) \\ &\le \mathrm{Var(s^t)}, \end{aligned}$$

which gives the desired inequality. □

Since the result of Lemma 4 seems to have an independent interest giving a comparison between the variation of two interlacing collections of points s^t and s^{t+1}, we describe also a sketch of an alternative proof of this result based on the induction on the number of points n. The base of induction – the case $n = 2$ is trivial. Indeed, assume for definiteness that $s^t_1 < s^t_2$. Then

$$s^t_1 \le s^{t+1}_1 \le s^t_2, \qquad s^t_1 \le s^{t+1}_2 \le s^t_2.$$

Therefore $\mathrm{Var}(\mathrm{s^t}) = 2|\mathrm{s_1^t} - \mathrm{s_2^t}| \geq 2|\mathrm{s_1^{t+1}} - \mathrm{s_2^{t+1}}| = \mathrm{Var}(\mathrm{s^{t+1}})$.

It remains to show that the case of general $n > 2$ can be reduced to the case of a smaller number of points. There are two possibilities: there exists at least one interval of monotonicity J of the function s^{t+1} of length greater than 1, or all locally maximal intervals of monotonicity of the function s^{t+1} are short of length 1. In the first case we may remove one of the particles in the middle of the interval J. This will preserve the interlacing conditions (3.2), but not change the variation of s^{t+1} without the removed point, while the variation of s^t may only decrease. Thus we get the reduction to the smaller number of particles. In the second case we have only short intervals of monotonicity of the function s^{t+1} and the types (increasing or decreasing) of corresponding intervals of s^t are either opposite, or between two opposite type pairs of intervals there is a single pair of intervals with the same types of monotonicity. This situation is depicted in Fig. 4. The analysis of the case of short monotonicity intervals is straightforward if one recalls the inequality (3.7).

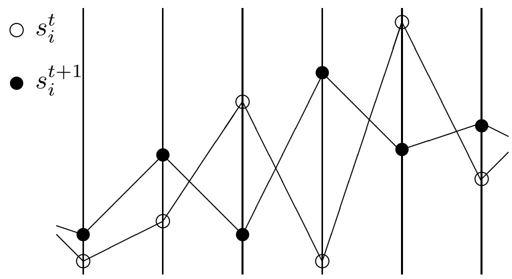

Figure 4: Oscillation of the spin functions. s_i^t/s_i^{t+1} are marked by open/closed circles.

Now a nontrivial point is to show that under assumptions made above the variation $\mathrm{Var}(\mathrm{s^t})$ vanishes with time a.s. To this end one needs to have some additional control over particle's interactions.

We say that the particles with indices $i \leq j$ are *clockwise chain-interacting* if for any initial configuration x^0 a.s. there exists a sequence of (random) moments of interaction $t_i \leq t_{i+1} \leq \ldots \leq t_{j-1} < \infty$ between the corresponding particles. In other words, for each $k \in \{0, 1, \ldots, j-i-1\}$ at time t_{i+k}

$$v_{i+k}^{t_{i+k}} > \Delta_{i+k}^{t_{i+k}}. \tag{3.9}$$

Similarly one defines the *anti clockwise chain-interaction* when $i > j$. If for any pair of indices i, j the (anti) clockwise chain-interacting property holds we say that the process satisfies the *chain-interacting property.*

Lemma 5 *Let $x^t, \acute{x}^t$ be statically coupled copies of the same particle process with sequential updating satisfying the chain-interaction property and let* $\mathrm{Var}(\mathrm{s^0}) > 0$. *Then* $\exists \tau = \tau(x^0, \acute{x}^0) < \infty : \quad \mathrm{Var}(\mathrm{s}^{\tau+1}) < \mathrm{Var}(\mathrm{s}^\tau)$.

Proof. By Lemma 3

$$\mathrm{Var}(\mathrm{s^{t+1}}) \leq \mathrm{Var}(\mathrm{s^t}) \quad \forall \mathrm{t} \geq 0$$

and we need to show only that for some t this inequality becomes strict. Assume from the contrary, that this is not the case, i.e.

$$\mathrm{Var}(\mathrm{s^{t+1}}) = \mathrm{Var}(\mathrm{s^t}) \quad \forall \mathrm{t} \geq 0. \tag{3.10}$$

By the definition of the sequential updating at the moments of time t_i the particle's interactions are isolated in the sense that the particles neighboring to the interacting ones make no interactions with other particles.

Let us show that if there exists an index i such that

$$(s_{i-1}^t - s_i^t)(s_i^t - s_{i+1}^t) < 0, \quad s_i^{t+1} \neq s_i^t, \tag{3.11}$$

then the variation strictly decreases at time t (i.e. $\mathrm{Var}(\mathrm{s}^{\mathrm{t}+1}) < \mathrm{Var}(\mathrm{s}^{\mathrm{t}})$) . The condition (3.11) means that s_i^t as a function on i is non-monotone at i and changes its value at this index at time t. This situation is depicted in Fig. 5.

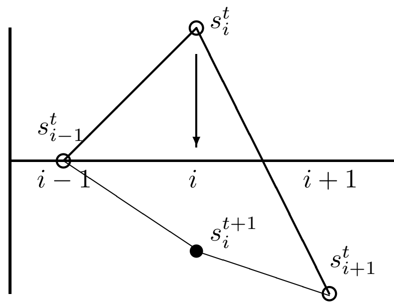

Figure 5: Change of the i-th spin during the isolated interaction.

By the interlacing property (3.2) if

$$\min\{s_{i-1}^t, s_{i+1}^t\} \leq s_i^t \leq \max\{s_{i-1}^t, s_{i+1}^t\}$$

then $\mathrm{Var}(\mathrm{s}^{\mathrm{t}+1}) = \mathrm{Var}(\mathrm{s}^{\mathrm{t}})$, while the violation of this inequality leads by (3.3) to $\mathrm{Var}(\mathrm{s}^{\mathrm{t}+1}) < \mathrm{Var}(\mathrm{s}^{\mathrm{t}})$. Therefore if the index i is the position of a local extremum of the function s^t, then the variation becomes strictly smaller (see Fig. 5).

Now by the chain-interacting property the interaction occurs eventually between all neighboring particles and thus the preservation of the variation implies that the spin function considered as a function on the index variable is monotone (otherwise by the argument above the equality (3.10) cannot hold). Finally the observation that the spin function is spatially periodic (is defined on a circle) shows that this function cannot be monotone, unless its variation vanishes (i.e. all spins are equal to each other). We came to a contradiction. □

Corollary 6 *Let $x^t, \acute{x}^t$ be statically coupled copies of the same particle process with sequential updating satisfying the chain-interaction property. Then* $\mathrm{Var}(\mathrm{s}^{\mathrm{t}}) \stackrel{\mathrm{t}\to\infty}{\longrightarrow} 0$.

Indeed, the monotonicity on t (by Lemma 4) of the nonnegative function $\mathrm{Var}(\mathrm{s}^{\mathrm{t}})$ implies its convergence to a limit, which in turn (by Lemma 5) cannot differ from zero.

Lemma 7 $\mathrm{Var}(\mathrm{s}^{\mathrm{t}}) \stackrel{\mathrm{t}\to\infty}{\longrightarrow} 0$ *implies* $\sum_i^n |\Delta_i^t - \acute{\Delta}_i^t| \stackrel{t\to\infty}{\longrightarrow} 0$.

Proof. By the definition of the spin function

$$\Delta_i^t - \acute{\Delta}_i^t = s_{i+1}^t - s_i^t,$$

which implies the claim. □

Lemma 8 *Let the updating be sequential and let the jump distributions satisfy the non-degeneracy condition (2.1). Then a.s. in the process x^t each pair of particles is either clockwise or anti clockwise chain-interacting.*

Proof. The condition (2.1) implies that a.s. each particle interacts with one (or both) of its neighbors in finite time. Therefore since the total number of particles is finite it follows that a.s. after a finite time each particle will chain-interact with each other. □

Proof of Theorem 1. We start by checking the sequential updating case. By Lemma 8 the chain-interaction property is satisfied. Therefore by Corollary 6 the functional $\mathrm{Var}(\mathrm{s}^\mathrm{t})$ vanishes with time. Thus the coupling time is finite and hence (see e.g. [3] for a suitable version of the corresponding statement) we get the desired claim.

In the parallel updating case the situation is somewhat more complicated. By Lemma 4 the variation of the spin function cannot increase under dynamics and we only need to demonstrate that it strictly decreases with positive probability. To this end we make use of the condition (2.2), according to which with positive probability not all pairs of particles interact simultaneously.

Assume that at time t the i-th pair of particles in the statically coupled processes $x^t, \acute{x}^t$ does not interact with the $(i+1)$-th pair. Then we can rewrite the parallel updating as n deterministic sequential ones with the given local velocities defined in the parallel updating starting from the index $i+1$.

Remark that in the absence of non-interacting particles the reduction to the sequential updating cannot be done since otherwise the position of the $(i+1)$-th particle will be different at the moment of the sequential updating of the i-th particle, which will change its position at time $t+1$. This is the crucial observation in the proof of Proposition 1 below.

Once we made the reduction to the sequential updating, the results proven in that case imply that the variation of the spin function vanishes with time and hence we get the unique ergodicity of the corresponding gap process for the parallel updating as well. □

It is worth noting that in order to guarantee that not all particles interact simultaneously all the time it is enough to make an assumption

$$P\left(\sum_i v_i^t < 1 - 2\sum_i r_i\right) > 0, \tag{3.12}$$

which is much weaker than (2.2). Unfortunately to make the simultaneous reduction of both coupled processes from parallel to sequential updating we need to find not only a single non-interacting particle but a non-interacting dumbbell – a pair of particles which do not interact with their right neighbors. Let us show that under the condition (3.12) this might not work. Choose a sequence $0 < a_0 < a_1 < \ldots < 1/2$ and let $n = 2$, $r_1 = r_2 = 0$. Consider initial configurations of gaps $\Delta_1^0 := a_0, \Delta_2^0 := 1 - a_0, \acute{\Delta}_1^0 := 1 - a_0, \acute{\Delta}_2^0 := a_0$. To simplify the argument we consider only the deterministic setting, choosing the following deterministic sequence of local velocities $\{v_1^t, v_2^t\} = \{a_t, a_t\}$. Then under dynamics the gaps in the process x^t are equal to $\{a_t, 1 - a_t\}$, while for the process $\acute{x}^t$ they are equal to $\{1 - a_t, a_t\}$. Thus the assumption (3.12) holds true for any moment of time, only one of the particles in each process does not interact with its right neighbor, but in both 1st and 2nd pairs (dumbbells) one of the particles makes the interaction with its right neighbor.

Proof of Proposition 1. Set $a := \frac{1}{n}\left(\sum_i v_i^0 - 1 + 2\sum_i r_i\right)$ and choose some $0 < b \ll a$. The value a is positive by (2.3). Consider a configuration of n gaps $\tilde{\Delta} := \{\max\{v_i^0 - a/2 + b, 0\}\}$. To construct an admissible configuration Δ we normalize $\tilde{\Delta}$ as follows:

$$\Delta := \left\{ \frac{\tilde{\Delta}_i\ (1 - 2\sum_j r_j)}{\sum_j \tilde{\Delta}_j} \right\}.$$

By the choice of the parameters a, b for each i we have $v_i^0 > \Delta_i$.

Therefore the application of the dynamics (1.2) to Δ is equivalent to the cyclic right shift: $\Delta_i \to \Delta_{i+1}$ for all $1 \le i < n$ and $\Delta_n \to \Delta_1$. Thus the configuration Δ gives rise to an ergodic invariant measure (uniformly distributed on a finite set of points) of the process under consideration. Noting that choosing different values of the 2nd parameter b we are getting different invariant measures we get the claim. □

Proof of Theorem 4 in the case of non negative local velocities. In the sequential updating case the claim about unique ergodicity follows from Corollary 6, while in the parallel updating case we need additionally the condition that for infinitely many moments of time some particles remain put to use it instead of the similar probabilistic assumption (2.2).

Therefore we need only to check sufficient conditions for the chain-interacting property. In the sequential updating case condition (2.7) implies that during at most $(1 - 2\sum_i r_i)/\min_i\{v_i^t\} < n$ iterations each particle will interact with its nearest right neighbor, which implies the property under question. In the parallel updating case the condition (2.8) plays the same role but does not give explicit estimate of the interaction time. □

3.2 Construction in terms of the gap process

During the discussion of an earlier version of this work a question whether it is possible to prove the unique ergodicity using the dynamics of gaps only was posed. Here we give a positive answer to this question. Note that despite a certain simplification of arguments here we are loosing important information about the original particle process and its geometric interpretation. Therefore we prefer to discuss both approaches rather than to choose only one of them.

Recall that the dynamics of gaps is defined by the relation (1.2). Similarly to the particle processes we say that two processes of gaps (with the same number of elements) Δ^t and $\acute{\Delta}^t$ are *statically coupled* if $v_i^t = \acute{v}_i^t$ for all i, t. Define a functional

$$V(\Delta^t, \acute{\Delta}^t) := \sum_{i=1}^{n} |\Delta_i^t - \acute{\Delta}_i^t|.$$

Lemma 9 *For a pair of statically coupled processes of gaps Δ^t and $\acute{\Delta}^t$ with sequential updating we have*

$$V(\Delta^{t+1}, \acute{\Delta}^{t+1}) \le V(\Delta^t, \acute{\Delta}^t) \quad \forall t. \tag{3.13}$$

Proof. In terms of gaps the interaction between the i-th and $(i+1)$-th pair of particles at time t in the processes $x^t, \acute{x}^t$ takes place if and only if

$$v_i^t > \min\{\Delta_i^t, \acute{\Delta}_i^t\}. \tag{3.14}$$

There are 3 possibilities

$$\acute{\Delta}_i^t \geq v_i^t > \Delta_i^t, \tag{3.15}$$

$$\Delta_i^t \geq v_i^t > \acute{\Delta}_i^t, \tag{3.16}$$

$$v_i^t > \max\{\Delta_i^t, \acute{\Delta}_i^t\}. \tag{3.17}$$

We start with the case (3.15). Then

$$\begin{aligned} \Delta_{i-1}^{t+1} &= \Delta_{i-1}^t + \Delta_i^t, \quad \Delta_i^{t+1} = 0, \\ \acute{\Delta}_{i-1}^{t+1} &= \acute{\Delta}_{i-1}^t + v_i^t, \quad \acute{\Delta}_i^{t+1} = \acute{\Delta}_i^t - v_i^t. \end{aligned}$$

Thus

$$\begin{aligned} |\Delta_{i-1}^{t+1} - \acute{\Delta}_{i-1}^{t+1}| + |\Delta_i^{t+1} - \acute{\Delta}_i^{t+1}| &= |\Delta_{i-1}^t + \Delta_i^t - \acute{\Delta}_{i-1}^t - v_i^t| + \acute{\Delta}_i^t - v_i^t \\ &= |(\Delta_{i-1}^t - \acute{\Delta}_{i-1}^t) + (\Delta_i^t - v_i^t)| + \acute{\Delta}_i^t - v_i^t \\ &\leq |\Delta_{i-1}^t - \acute{\Delta}_{i-1}^t| + v_i^t - \Delta_i^t + \acute{\Delta}_i^t - v_i^t \\ &= |\Delta_{i-1}^t - \acute{\Delta}_{i-1}^t| - \Delta_i^t + \acute{\Delta}_i^t \\ &\leq |\Delta_{i-1}^t - \acute{\Delta}_{i-1}^t| + |\Delta_i^t - \acute{\Delta}_i^t|. \end{aligned}$$

Here the inequality in the 3nd line follows from the triangle inequality. Note that this inequality becomes equality if and only if $\Delta_{i-1}^t \leq \acute{\Delta}_{i-1}^t$. Together with the assumption (3.15) this implies that two consecutive gaps in the process Δ^t is less or equal to the corresponding gaps in the process $\acute{\Delta}^t$.

Similarly in the case (3.16) we get

$$|\Delta_{i-1}^{t+1} - \acute{\Delta}_{i-1}^{t+1}| + |\Delta_i^{t+1} - \acute{\Delta}_i^{t+1}| \leq |\Delta_{i-1}^t - \acute{\Delta}_{i-1}^t| + |\Delta_i^t - \acute{\Delta}_i^t|$$

and the inequality takes place if and only if two consecutive gaps in the process Δ^t is larger or equal to the corresponding gaps in the process $\acute{\Delta}^t$.

In the remaining case (3.17) the calculation is even simpler:

$$\Delta_{i-1}^{t+1} = \Delta_{i-1}^t + \Delta_i^t, \quad \Delta_i^{t+1} = 0,$$

$$\acute{\Delta}_{i-1}^{t+1} = \acute{\Delta}_{i-1}^t + \acute{\Delta}_i^t, \quad \acute{\Delta}_i^{t+1} = 0.$$

Therefore

$$\begin{aligned} |\Delta_{i-1}^{t+1} - \acute{\Delta}_{i-1}^{t+1}| + |\Delta_i^{t+1} - \acute{\Delta}_i^{t+1}| &= |(\Delta_{i-1}^t + \Delta_i^t) - (\acute{\Delta}_{i-1}^t + \acute{\Delta}_i^t)| + 0 \\ &\leq |\Delta_{i-1}^t - \acute{\Delta}_{i-1}^t| + |\Delta_i^t - \acute{\Delta}_i^t|. \end{aligned}$$

Again the equality takes place if and only if two consecutive gaps in the process Δ^t are both larger or both smaller than the corresponding gaps in the process $\acute{\Delta}^t$.

Observe now that, by the assumption that the process has the sequential updating, during the interaction of the i-th particle only the $(i-1)$-th and the i-th gaps may change. Thus the claim (3.13) follows. □

Using this result instead of Lemma 3 and the functional $V(\Delta^t, \acute{\Delta}^t)$ instead of the variation of the spin function one can follow arguments of the previous Section to prove Theorem 1.

4 Local velocities of both signs and other generalizations

4.1 General (non totally asymmetric) collective random walks

So far to simplify the setting we assumed that all particles move in the same direction, i.e. the local velocities v_i^t have the same (positive) sign. The presence of particles moving in opposite directions leads to a significant modification of the violation of the admissibility condition for local velocities. Now one needs to take into account not only the position of the succeeding particle, but also its velocity, as well as the corresponding quantities related to the preceding particle. In this more general case the i-th local velocity does not break the admissibility condition if and only if

$$\begin{aligned}\max\{x_{i-1}^t, x_{i-1}^t + u_{i-1}^t v_{i-1}^t\} + r_{i-1} &\le \min\{x_i^t, x_i^t + u_i^t v_i^t\} - r_i \\ \max\{x_i^t, x_i^t + u_i^t v_i^t\} + r_i &\le \min\{x_{i+1}^t, x_{i+1}^t + u_{i+1}^t v_{i+1}^t\} - r_{i+1}.\end{aligned}$$

If for some $i \in \{1, 2, \ldots, n\}$ and $j = i \pm 1$ the corresponding inequality is not satisfied we say that there is a *conflict* between the i-th particle and the j-th one and one needs to resolve it. In terms of gaps Δ_i^t the inequalities above may be rewritten as follows:

$$\Delta_j^t \ge \max\{u_j^t v_j^t, \ -u_{j+1}^t v_{j+1}^t, \ u_j^t v_j^t - u_{j+1}^t v_{j+1}^t\}, \quad j \in \{i-1, i\} \tag{4.1}$$

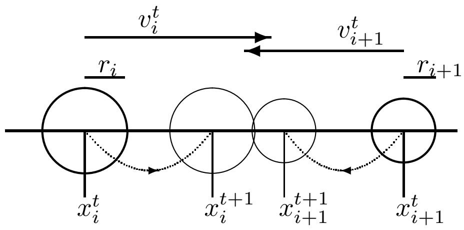

Figure 6: Velocities of both signs

In distinction to the case of particles moving in the same direction the resolution of the conflict between particles is not uniquely defined: to resolve a conflict between two mutually conflicting particles moving simultaneously in opposite directions (see Fig. 6) one needs to specify the positions of the particles after the conflict. This can be done in a number of ways and we shall consider a *natural* resolution of the conflict allowing each particle to move with the corresponding velocity as far as possible imitating a continuous time motion. Namely in the case of the mutual conflict between the i-th and the $(i+1)$-th particles, i.e. $v_i^t > 0 > v_{i+1}^t$, the natural resolution of the conflict leads to

$$x_i^{t+1} := x_i^t + \frac{\Delta_i^t v_i^t}{v_i^t - v_{i+1}^t}, \qquad x_{i+1}^{t+1} := x_{i+1}^t + \frac{\Delta_{i+1}^t v_{i+1}^t}{v_i^t - v_{i+1}^t}.$$

The difference between the condition (2.1) in the formulation of Theorem 1 and the conditions (2.4, 2.5) in Theorem 2 is that the latter are able to deal with particles jumping in both directions. To show that this is indeed necessary, consider an example with $n = 4$ particles whose velocity distributions satisfy the relations

$$P_1((0,1]) = P_2([-1,0)) = P_3((0,1]) = P_4([-1,0)) = 1.$$

In this example the particles will eventually meet in pairs (1+2 and 3+4), and the pairs will stay put in two random places:

$$\Delta_1^t, \Delta_3^t \stackrel{t\to\infty}{\longrightarrow} 0, \quad \text{while} \quad \sum_i^4 \Delta_i^t \equiv 1.$$

In the simplest case when the distributions of jumps P_i do not depend on the index i it is enough to assume that this common distribution is not supported by a single point.

Proof of Theorem 2. Observe that the constructions developed during the analysis of the totally asymmetric setting remain valid in the general case as well. The only difference is that additionally to the interaction of a given particle with its right nearest neighbor one needs to take into account the interactions with the left nearest neighbor when the particle's local velocity becomes negative. Fortunately only one of these interactions may take place at a given moment of time.

To apply the machinery developed in Section 3 we need to check that in the case of the mutual conflict the interlacing property holds. All definitions made in Section 3 remain valid except for the change in the last line of the definition of the lifting (3.1), where the term $\min\{u_i^{t-1}v_i^{t-1}, \Delta_i^{t-1}\}$ should be changed to the actual distance covered by the i-th particle during its jump at time $(t-1)$.

Let the processes $x^t, \acute{x}^t$ be statically coupled and the mutual conflict between at least one of the i-th particles takes place. We restrict ourselves only to the situation $\Delta_i^t \le \acute{\Delta}_i^t$ (and hence $s_i^t \ge s_{i+1}^t$) since the the analysis of the alternative situation is completely similar. There are two possibilities.

(a) $\Delta_i^t < v_i^t - v_{i+1}^t \le \acute{\Delta}_i^t$. Then denoting $\ell := \frac{\Delta_i^t v_i^t}{v_i^t - v_{i+1}^t} < v_i^t$ we get

$$\begin{aligned} R(x,t+1,i) &= R(x,t,i) + \ell, \qquad & R(x,t+1,i+1) &= R(x,t,i+1) - (\Delta_i^t - \ell), \\ R(\acute{x},t+1,i) &= R(\acute{x},t,i) + v_i^t, \qquad & R(x,t+1,i+1) &= R(x,t,i+1) + v_{i+1}^t. \end{aligned}$$

Therefore

$$\begin{aligned} s_i^{t+1} &= s_i^t + (\ell - v_i^t) < s_i^t = \max\{s_i^t, s_{i+1}^t\}, \\ s_{i+1}^{t+1} &= s_{i+1}^t + (\Delta_i^t - \ell + v_{i+1}^t) > s_{i+1}^t = \min\{s_i^t, s_{i+1}^t\} \end{aligned}$$

since $\ell - v_i^t < 0$ and $\Delta_i^t - \ell + v_{i+1}^t < 0$.

The observation that $\Delta_i^{t+1} = 0 < \acute{\Delta}_i^{t+1}$ implies $s_i^{t+1} > s_{i+1}^{t+1}$, which finishes the analysis of this possibility.

(b) $v_i^t - v_{i+1}^t > \max\{\Delta_i^t, \acute{\Delta}_i^t\}$. Denoting $\acute{\ell} := \frac{\acute{\Delta}_i^t v_i^t}{v_i^t - v_{i+1}^t} < v_i^t$ we get

$$\begin{aligned} R(x,t+1,i) &= R(x,t,i) + \ell, \qquad & R(x,t+1,i+1) &= R(x,t,i+1) - (\Delta_i^t - \ell), \\ R(\acute{x},t+1,i) &= R(\acute{x},t,i) + \acute{\ell}, \qquad & R(\acute{x},t+1,i+1) &= R(\acute{x},t,i+1) - (\acute{\Delta}_i^t - \acute{\ell}). \end{aligned}$$

Therefore

$$\begin{aligned} s_i^{t+1} &= s_i^t + (\ell - \acute{\ell}) \le s_i^t, \\ s_{i+1}^{t+1} &= s_{i+1}^t - (\Delta_i^t - \ell) + (\acute{\Delta}_i^t - \acute{\ell}) \\ &= s_{i+1}^t - (\Delta_i^t - \acute{\Delta}_i^t)(1 - \frac{v_i^t}{v_i^t - v_{i+1}^t}) \ge s_{i+1}^t \end{aligned}$$

since $\ell \leq \acute{\ell}$, $\Delta_i^t \leq \acute{\Delta}_i^t$ and $\frac{v_i^t}{v_i^t - v_{i+1}^t} < 1$. Eventually we obtain

$$\min\{s_i^t, s_{i+1}^t\} \leq s_{i+1}^{t+1} = s_i^{t+1} \leq s_i^t = \max\{s_i^t, s_{i+1}^t\}.$$

Additionally a close look to the calculations above shows that if $s_i^t \neq s_{i+1}^t$ then

$$|s_i^t - s_{i+1}^t| > |s_i^{t+1} - s_{i+1}^{t+1}|,$$

which gives the analog of the inequality (3.3).

It remains to prove that under the assumptions of Theorem 2 the chain interacting property holds true. Assume first that the process x^t has sequential updating. Then each of the non-degeneracy conditions obviously implies the chain-interacting property (in one of the directions). Namely the condition (2.4) implies the clockwise chain interaction, while the condition (2.5) implies the anti clockwise chain interaction. Therefore the claim follows from the same arguments as in the proof of the totally asymmetric setting.

The situation with the parallel updating is slightly more subtle. The point is that additionally to the absence of interactions between the nearest particles (used in Section 3) we get an additional way to make the reduction from parallel to consecutive updating. Indeed, if at time t two consecutive particles have opposite local velocities then there is at least another pair of consecutive particles satisfying this property. Moreover among such pairs there is at least one, say j and $j+1$ such that $v_j^t < 0 < v_{j+1}^t$. Therefore these two particles do not interact and hence in this situation one can make the reduction to the sequential updating, which starts at the index $j+1$ and goes up to the index j.

Using the above trick we make the reduction to the sequential updating if the event described in the assumption (2.2) takes place. The remaining part follows the same arguments as in the proof of Theorem 1. □

Proof of Theorem 4 in the case of local velocities taking both signs. Additionally to the already proven part related to the local velocities of the same sign, we use the condition that for infinitely many moments of time t there exists $j = j(t)$ such that $v_j^t < 0 < v_{j+1}^t$ in order to make the reduction from the parallel updating to the sequential one at these moments of time. After this reduction one applies the same arguments as in the probabilistic setting. □

4.2 Strict exclusion

In all our previous constructions we have considered only those rules resolving particle conflicts allowing the particles to move as far as possible according to their local velocities. On the other hand, as we already mentioned the collective random walk under consideration is a generalization of the simple exclusion process, where a particle moves to the neighboring site only if the latter is not occupied by another particle. From this point of view it seems to be natural to consider a similar conflict resolution rule. Namely we say that the particle process x^t satisfies the *strict exclusion* rule if in the case of a conflict the corresponding particle stays put, i.e.

$$\text{if} \quad v_i^t > \Delta_i^t \quad \text{or} \quad -v_i^t > \Delta_{i-1}^t \quad \text{then} \quad x_i^{t+1} = x_i^t.$$

It turns out that this "natural" conflict resolution rule typically leads to a non-ergodic behavior.

Proposition 10 *Let $P_i([-\varepsilon, \varepsilon]) = 0$ for some $\varepsilon > 0$ and all i. Then for n large enough the process Δ^t is non-ergodic.*

Proof. Let $n > 1/\varepsilon$. Consider a configuration $x := \{x_i\}_{i=1}^n$ such that $\max_i \Delta_i < \varepsilon$. Then under the assumptions of the Theorem, due to the strict exclusion interaction rule, all particles in the configuration x stay put under dynamics. Hence the Dirac measure δ_x supported by the configuration x is invariant under dynamics as well as the Dirac measure supported by the sequence of the corresponding gaps Δ_i. Passing from the configuration x to close enough configuration $\acute{x}$ such that $\acute{\Delta}_i < \varepsilon$ we are getting infinitely many different fixed points of the process x^t having different configurations of gaps. This proves the non-ergodicity of the process Δ^t. □

In distinction to the non-strict exclusion case here it is much more difficult to give sufficient conditions of the unique ergodicity. At present we only can formulate the following hypothesis.

Hypothesis. Let the assumptions (2.1, 2.2) hold true. Then the process of gaps Δ^t with either sequential or parallel updating is uniquely ergodic.

4.3 Lattice exclusion process

Observe that ergodic type results for lattice versions of the problems under consideration being nontrivial as well may be derived from the present results.

In the lattice setting all elements of a configuration x belong to a finite set $\{0, 1/n, 2/n, \ldots, (n-1)/n\}$ for some $n \in \mathbb{Z}_+$ which defines the number of lattice sites. The radius of a ball representing a particle satisfies the condition $nr_i \in \{0, k+1/2\}$ with $k \in \mathbb{Z}_+$, and the jump distribution is supported by the lattice points $P_i(\cup_{j=0}^{n-1}\{j/n\}) = 1$.

Despite an apparent significant difference between the behavior of the lattice processes in the cases when $r_i \equiv 0$ and $r_i > 0$ (in the former case an arbitrary number of particles may share the same lattice site, while in the latter case at most one particle is allowed per lattice site) the ergodicity conditions turn out to be the same.

5 Unique ergodicity in the deterministic setting

Now we address the question of unique ergodicity of the collective walk in the deterministic setting. This means that the jump distributions are supported by single points: $P(v_i^t = v_i) = 1$ for some constant local velocities $v := \{v_i\}_{i=1}^n$.

We start the analysis from the case when the local velocities v_i take both positive and negative values and consider the partition of the set if indices into groups of consecutive indices of three types corresponding to negative, positive and zero velocities, e.g. the configuration of signs $++000--+--$ has 5 groups of all 3 types. Since there are oppositely signed velocities the number of groups is greater than one.

Theorem 5 *The process of gaps Δ^t (with either sequential or parallel updating) is uniquely ergodic if and only if the number of different groups is at most three, and the only group of zero velocity particles (if it exists) consists of a single element which is located after the group of positive and before negative particles (i.e. $+++0--$).*

Proof. If the conditions of Theorem are satisfied each initial configuration of gaps converges in finite time to the configuration having a single positive gap of length $(1-2\sum_i \Delta_i)$, which proves the unique ergodicity. The presence of two zero velocities or more than two groups of signed velocities obviously contradicts to the unique ergodicity. The observation that the wrong location of the unique zero velocity particle (i.e. $++--0$) leads to the presence of the invariant measure supported by two points and sensitively depending on the position of this particle finishes the proof. □

Now we are ready to address a more interesting situation of local velocities of the same (say positive) sign. If the updating is sequential, Theorem 4 gives a sufficient condition of the unique ergodicity

$$v_{\min} := \min_i \{v_i\} > \frac{1}{n}(1 - 2\sum_i r_i).$$

This condition is probably non optimal and one is tempted to weaken it to

$$\sum_i v_i > 1 - 2\sum_i r_i. \tag{5.1}$$

Unfortunately a simple example with two zero velocities $v_1 = v_2 = 0,\ v_3 = v_4 = \ldots = v_n = 1$, which definitely satisfies (5.1), leads to infinitely many invariant measures of the gap process.

In the case of the parallel updating we able to get both necessary and sufficient conditions for the unique ergodicity, formulated in Theorem 3.

Proof of Theorem 3. Assume that the assumption (a) holds true. Since the system is translationally invariant without any loss of generality we may assume that the only minimum is achieved at the index $i_{\min} = n$. The key point to our argument is that for any initial admissible particles configuration x^0 there exists a finite time $t_n = t_n(x^0, v, \{r_i\})$ such that for each $t > t_n$ the particle configuration x^t satisfy the property:

$$\Delta_1^t = \Delta_2^t = \ldots \Delta_{n-1}^t = \beta, \quad \Delta_n^t \geq \beta, \tag{5.2}$$

for a certain $0 < \beta \leq v_n$. If $\beta < v_n$, then $\Delta_n^t = \beta$.

Indeed, if this is the case, starting from the time t_n our dynamical system is a direct product of n identical irrational rotation maps

$$x_i^{t+1} := x_i^t + \beta \mod(1).$$

This direct product system possesses a number of invariant measures, but the property (5.2) defines a unique invariant measure uniformly distributed on the segment given in the coordinates $(x_1, \ldots, x_n)$ by the relations

$$x_1 = x_n - (n-1)\beta,\ x_2 = x_n - (n-2)\beta, \ldots, x_{n-1} = x_n - \beta,$$

provided the number β is irrational.

Let us prove that the property (5.2) holds true. Observe that after at most

$$t_{n-1} := \left(1 - 2\sum_{i=1}^{n} r_i\right) / (v_{n-1} - v_n)$$

iterations the $(n-1)$-th particle will catch with the n-th one and thus for each $t \geq t_{n-1}$ the corresponding gap Δ_{n-1}^t is exactly equal to the length of jump that the n-th particle will perform at time t, in particular $\Delta_{n-1}^t \leq v_n$.

Similarly

$$\forall t \geq t_{n-2} := t_{n-1} + \left(1 - 2\sum_{i=1}^{n} r_i\right) / (v_{n-2} - v_n)$$

the gap Δ_{n-2}^t will match the length of jump that the n-th particle, etc. Eventually after at most

$$t_1 := \left(1 - 2\sum_{i=1}^{n} r_i\right) \left(\frac{1}{v_{n-1} - v_n} + \frac{1}{v_{n-2} - v_n} + \ldots + \frac{1}{v_1 - v_n}\right)$$

iterations all particles will start moving synchronously.

If the assumption (b) holds true, then for for all $t \geq t_1$

$$\Delta_n^t \geq 1 - 2\sum_{i=1}^{n} r_i - (n-1)v_n = \alpha + v_n$$

and hence the n-th particle will stop interacting with others and will perform the pure rotation through the irrational angle v_n, which guarantees the unique ergodicity. Note that the rationality of the rotation leads to the presence of infinitely many invariant measures.

If (c) holds true then $\Delta_n^t = \alpha + v_n < v_n$ for all $t \geq t_1$ and due to the interactions with other particles the n-th one will perform the pure rotation through another irrational angle $\alpha + v_n$.

It remains to check that the violation of the assumption (a) implies the absence of unique ergodicity. If the minimum is achieved at several indices then each of the slowest particles will generate a "train" of faster particles following it in the same manner as it has been shown for the case of the single minimum. If

$$1 - 2\sum_{i=1}^{n} r_i > n v_{\min} \tag{5.3}$$

then using the same argument as in the case of the single minimum one finds a partition of particles into groups following one of the slowest particles. In each group the particles are moving at the same speed as the leading one. By (5.3) each group may be slightly shifted not perturbing its motion and the motion of the other particles. Therefore the system possesses an infinite number of invariant measures, corresponding to the trajectories of the perturbed configurations.

If the inequality (5.3) is violated, then there are infinitely many different particle configurations x^0 such that $\Delta^0 < v_{\min}$ and this property holds for any $t > 0$. Thus under dynamics we get $\Delta_i^{t+1} := \Delta_{i+1}^t$ for all i, t. Therefore the obtained configuration is a periodic point of the process Δ^t, which again implies the presence of a infinite number of invariant measures. □

Note that the "most natural" case of identical particles and equal constant local velocities, i.e. when $r_1 = \ldots = r_n,\ v_1 = \ldots = v_n$, does not satisfy the conditions of unique ergodicity.

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