An Erdős-Ko-Rado theorem for permutations with fixed number of cycles

Cheng Yeaw Ku *   Kok Bin Wong †

October 9, 2018

Abstract

Let S_n denote the set of permutations of $[n] = \{1, 2, \dots, n\}$. For a positive integer k, define $S_{n,k}$ to be the set of all permutations of $[n]$ with exactly k disjoint cycles, i.e.,

$$S_{n,k} = \{\pi \in S_n : \pi = c_1 c_2 \cdots c_k\},$$

where $c_1, c_2, \dots, c_k$ are disjoint cycles. The size of $S_{n,k}$ is given by $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)$, where $s(n,k)$ is the Stirling number of the first kind. A family $\mathcal{A} \subseteq S_{n,k}$ is said to be *t-intersecting* if any two elements of $\mathcal{A}$ have at least t common cycles. In this paper, we show that, given any positive integers k, t with $k \geq t+1$, there exists an integer $n_0 = n_0(k,t)$, such that for all $n \geq n_0$, if $\mathcal{A} \subseteq S_{n,k}$ is t-intersecting, then

$$|\mathcal{A}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix},$$

with equality if and only if $\mathcal{A}$ is the stabiliser of t fixed points.

KEYWORDS: t-intersecting family, Erdős-Ko-Rado, permutations

1 Introduction

Let $[n] = \{1, \dots, n\}$, and let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A family $\mathcal{A}$ of subsets of $[n]$ is *t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in \mathcal{A}$. One of the most beautiful results in extremal combinatorics is the Erdős-Ko-Rado theorem.

Theorem 1.1 (Erdős, Ko, and Rado [13], Frankl [14], Wilson [37]). *Suppose $\mathcal{A} \subseteq \binom{[n]}{k}$ is t-intersecting and $n > 2k - t$. Then for $n \geq (k-t+1)(t+1)$, we have*

$$|\mathcal{A}| \leq \binom{n-t}{k-t}.$$

Moreover, if $n > (k-t+1)(t+1)$ then equality holds if and only if $\mathcal{A} = \{A \in \binom{[n]}{k} : T \subseteq A\}$ for some t-set T.

*Department of Mathematics, National University of Singapore, Singapore 117543. E-mail: matkcy@nus.edu.sg

†Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia. E-mail: kbwong@um.edu.my.

Later, Ahlswede and Khachatrian [1] extended the Erdős-Ko-Rado theorem by determining the structure of all t-intersecting set systems of maximum size for all possible n (see also [3, 15, 22, 28, 32, 34, 35] for some related results). There have been many recent results showing that a version of the Erdős-Ko-Rado theorem holds for combinatorial objects other than set systems (see [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 29, 30, 31, 33, 36, 38]). Most notably is the results of Ellis, Friedgut and Pilpel [12] who showed that for sufficiently large n depending on t, a t-intersecting family $\mathcal{A}$ of permutations has size at most $(n-t)!$, with equality if and only if $\mathcal{A}$ is a coset of the stabilizer of t points, thus settling an old conjecture of Deza and Frankl in the affirmative. The proof uses spectral methods and representations of the symmetric group.

Let S_n denote the set of permutations of $[n]$. For a positive integer k, define $S_{n,k}$ to be the set of all permutations of $[n]$ with exactly k disjoint cycles, i.e.,

$$S_{n,k} = \{\pi \in S_n : \pi = c_1 c_2 \cdots c_k\},$$

where $c_1, c_2, \dots, c_k$ are disjoint cycles. It is well known that the size of $S_{n,k}$ is given by $\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n-k} s(n,k)$, where $s(n,k)$ is the *Stirling number of the first kind.*

We shall use the following notations:

(a) $N(c) = \{a_1, a_2, \dots, a_l\}$ for a cycle $c = (a_1, a_2, \dots, a_l)$;

(b) $M(\pi) = \{c_1, c_2, \dots, c_k\}$ for a $\pi = c_1 c_2 \dots c_k \in S_{n,k}$;

A family $\mathcal{A} \subseteq S_{n,k}$ is said to be *t-intersecting* if any two elements of $\mathcal{A}$ have at least t common cycles, i.e., $|M(\pi_1) \cap M(\pi_2)| \geq t$ for all $\pi_1, \pi_2 \in \mathcal{A}$.

Theorem 1.2. *Given any positive integers k, t with $k \geq t+1$, there exists an integer $n_0 = n_0(k,t)$, such that for all $n \geq n_0$, if $\mathcal{A} \subseteq S_{n,k}$ is t-intersecting, then*

$$|\mathcal{A}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix},$$

with equality if and only if $\mathcal{A}$ is the stabiliser of t fixed points.

2 Stirling number revisited

The unsigned Stirling number $\begin{bmatrix} n \\ k \end{bmatrix}$ satisfies the recurrence relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}, \tag{1}$$

with initial conditions $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$ and $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} = 0$, $n > 0$. Note that $\begin{bmatrix} n \\ n \end{bmatrix} = 1$. By using equation (1) and induction on n,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!. \tag{2}$$

By applying equation (1) repeatedly,

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
&= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + (n-1)(n-2) \begin{bmatrix} n-2 \\ k \end{bmatrix} \\
&= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} + (n-1)(n-2) \begin{bmatrix} n-3 \\ k-1 \end{bmatrix} + (n-1)(n-2)(n-3) \begin{bmatrix} n-3 \\ k \end{bmatrix} \\
&\vdots \\
&= \sum_{r=k-1}^{n-1} \frac{(n-1)!}{r!} \begin{bmatrix} r \\ k-1 \end{bmatrix}. \qquad (3)
\end{aligned}$$

In particular (by equations (2) and (3)),

$$\begin{bmatrix} n \\ 2 \end{bmatrix} = (n-1)! \sum_{r=1}^{n-1} \frac{1}{r}. \qquad (4)$$

By elementary calculus, it is easy to show that for sufficiently large n,

$$\frac{\ln n}{2} < \ln(n-1) + \frac{1}{n-1} \leq \sum_{r=1}^{n-1} \frac{1}{r} \leq \ln(n-1) + 1 < 2 \ln n. \qquad (5)$$

Hence, there are positive constants $\alpha(2)$ and $\beta(2)$ such that

$$\alpha(2)((n-1)!(\ln n)) < \begin{bmatrix} n \\ 2 \end{bmatrix} < \beta(2)((n-1)!(\ln n)), \qquad (6)$$

for all $n \geq 2$.

Again, by elementary calculus, for $m \geq 1$ and sufficiently large n depending on m,

$$\frac{\ln^{m+1} n}{2(m+1)} < \sum_{r=1}^{n-1} \frac{\ln^m r}{r} < \frac{2 \ln^{m+1} n}{m+1}. \qquad (7)$$

The following lemma follows by using equations (3) and (7), and induction on k.

Lemma 2.1. *There are positive constants $\alpha(k)$ and $\beta(k)$ such that*

$$\alpha(k)((n-1)!(\ln^{k-1} n)) < \begin{bmatrix} n \\ k \end{bmatrix} < \beta(k)((n-1)!(\ln^{k-1} n)),$$

for all $n \geq k$.

3 Main results

A family $\mathcal{B} \subseteq S_{n,k}$ is said to be *independent* if $M(\pi_1) \cap M(\pi_2) = \varnothing$ for all $\pi_1, \pi_2 \in \mathcal{B}$.

Lemma 3.1. *Let $\mathcal{A} \subseteq S_{n,k}$ and $k \geq 2$. If a maximal independent subset of $\mathcal{A}$ is of size at most l, then*

$$|\mathcal{A}| \leq kl \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

Proof. Let $\mathcal{B} = \{\pi_1, \pi_2, \ldots, \pi_l\}$ be a maximal independent subset of $\mathcal{A}$ of size l. Let $\pi_i = c_{i1}c_{i2}\ldots c_{ik}$ where $c_{i1}, c_{i2}, \ldots, c_{ik}$ are disjoint cycles, and

$$Q = \bigcup_{i=1}^{l} M(\pi_i).$$

Note that $|Q| = kl$.

Let

$$\mathcal{A}_{ij} = \{\pi \in \mathcal{A} \ : \ c_{ij} \in M(\pi)\}.$$

Let $\pi \in \mathcal{A} \setminus \mathcal{B}$. By the maximality of $\mathcal{B}$, $M(\pi) \cap Q \neq \varnothing$. So, $c_{ij} \in M(\pi)$ for some i, j, and $\pi \in \mathcal{A}_{ij}$. Hence,

$$\mathcal{A} = \bigcup_{i,j} \mathcal{A}_{ij}.$$

The lemma follows by noting that

$$|A_{ij}| \leq \begin{bmatrix} n - |N(c_{ij})| \\ k-1 \end{bmatrix} \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

□

Let $\mathcal{A} \subseteq S_{n,k}$ and $c_1, \ldots, c_t$ by cycles such that $N(c_i) \cap N(c_j) = \varnothing$ for $i \neq j$. Let $T = \{c_1, \ldots, c_t\}$. We set

$$\mathcal{A}(T) = \{\pi \in \mathcal{A} \ : \ T \subseteq M(\pi)\}.$$

Now, for each element $\pi \in \mathcal{A}(T)$, we remove all the cycles $c_1, c_2, \ldots, c_t$ from π and denote the resulting set by $\mathcal{A}^*(T)$. Let $P = \bigcup_{i=1}^{t} N(c_i)$. Note that $|\mathcal{A}(T)| = |\mathcal{A}^*(T)|$ and $\mathcal{A}^*(T) \subseteq S_{n-|P|,k-t}$. Here, $S_{n-|P|,k-t}$ is the set of all permutations of $[n] \setminus P$ with exactly $k - t$ disjoint cycles.

Lemma 3.2. *Let $\mathcal{A} \subseteq S_{n,k}$ be maximal t-intersecting and $k \geq t + 1$. Let $T = \{c_1, \ldots, c_t\}$ with $N(c_i) \cap N(c_j) = \varnothing$ for $i \neq j$. If $\mathcal{A}^*(T)$ has an independent set of size at least $k + 1$, then*

$$\mathcal{A} = \{\pi \in S_{n,k} \ : \ T \subseteq M(\pi)\}.$$

Proof. Let $\{\pi_1, \ldots, \pi_{k+1}\}$ be an independent subset of $\mathcal{A}^*(T)$ of size $k + 1$. For $l = 1, 2, \ldots, k + 1$, let

$$\pi_l = c_1 \ldots c_t d_{l,t+1} \ldots d_{l,k},$$

where $c_1, \ldots, c_t, d_{l,t+1}, \ldots, d_{l,k}$ are disjoint cycles.

Suppose there is a $\pi \in \mathcal{A}$ such that $c_{i_0} \notin M(\pi)$ for a fixed i_0. Since $\mathcal{A}$ is t-intersecting,

$$|M(\pi) \cap M(\pi_l)| \geq t.$$

Therefore there is a j_l $(t+1 \le j_l \le k)$ with $d_{l,j_l} \in M(\pi)$. Since all the d_{l,j_l} are distinct, $k = |M(\pi)| \ge k+1$, a contradiction. Hence,

$$\mathcal{A} \subseteq \{\pi \in S_{n,k} \ : \ T \subseteq M(\pi)\}.$$

By the maximality of $\mathcal{A}$, the lemma follows. □

Proof of Theorem 1.2. We may assume that $\mathcal{A}$ is maximal t-intersecting.

Suppose $k = t+1$. Since $\mathcal{A}$ is t-intersecting, there are $\pi_1, \pi_2 \in \mathcal{A}$ such that

$$\pi_1 = c_1 c_2 \dots c_t d_1$$
$$\pi_2 = c_1 c_2 \dots c_t d_2$$

where $c_1, \dots, c_t, d_1$ are disjoint cycles, $d_2 \ne d_1$ and $N(d_2) = N(d_1)$. Suppose there is a $\pi \in \mathcal{A}$ with $c_{i_0} \notin M(\pi)$ for some i_0. Then $d_1, d_2 \in M(\pi)$. But this is impossible as $N(d_1) = N(d_2)$. Hence,

$$\mathcal{A} = \{\pi \in S_{n,k} \ : \ c_i \in M(\pi) \ \text{ for } \ i = 1, 2, \dots, t\},$$

for $\mathcal{A}$ is maximal t-intersecting. Let $P = \bigcup_{i=1}^{t} N(c_i)$. Then

$$|\mathcal{A}| = \begin{bmatrix} n - |P| \\ 1 \end{bmatrix} \le \begin{bmatrix} n - t \\ 1 \end{bmatrix},$$

with equality if and only if $|N(c_i)| = 1$ for $i = 1, 2 \dots, t$, i.e., $\mathcal{A}$ is the stabilizer of at least t fixed points. Now, if $n \ge t+2$, then $\mathcal{A}$ is the stabilizer of t fixed points.

Suppose $k \ge t+2$. Let $\pi_0 = d_1 d_2 \dots d_k \in \mathcal{A}$ be fixed, where $d_1, \dots, d_k$ are disjoint cycles. Then

$$\mathcal{A} = \bigcup_{T \subseteq M(\pi_0), |T| = t} \mathcal{A}(T).$$

Case 1. Suppose that for each $T \subseteq M(\pi_0)$ with $|T| = t$, all independent subsets of $\mathcal{A}^*(T)$ is of size at most k. Then by Lemma 3.1 and equation (1),

$$|\mathcal{A}(T)| = |\mathcal{A}^*| \le k^2 \begin{bmatrix} n - |P| - 1 \\ k - t - 1 \end{bmatrix} \le k^2 \begin{bmatrix} n - t - 1 \\ k - t - 1 \end{bmatrix},$$

where $P = \bigcup_{c \in T} N(c)$. This implies that

$$|\mathcal{A}| \le k^2 \binom{k}{t} \begin{bmatrix} n - t - 1 \\ k - t - 1 \end{bmatrix}.$$

By Lemma 2.1, there is are positive constants α and β such that

$$|\mathcal{A}| < \beta k^2 \binom{k}{t} \left((n-t-2)! (\ln^{k-t-2} n) \right),$$

and

$$\alpha \left((n-t-2)! (\ln^{k-t-1} n) \right) < \begin{bmatrix} n - t - 1 \\ k - t \end{bmatrix}.$$

So, for sufficiently large n, $|\mathcal{A}| < \begin{bmatrix} n-t-1 \\ k-t \end{bmatrix}$, and by equation (1), $|\mathcal{A}| < \begin{bmatrix} n-t \\ k-t \end{bmatrix}$.

Case 2. Suppose that there is a $T_0 \subseteq M(\pi_0)$ with $|T_0| = t$, such that $\mathcal{A}^*(T_0)$ has an independent set of size at least $k+1$. By Lemma 3.2,

$$\mathcal{A} = \{\pi \in S_{n,k} \ : \ T_0 \subseteq M(\pi)\} .$$

Let $P_0 = \bigcup_{c \in T_0} N(c_0)$. Then

$$|\mathcal{A}| = \begin{bmatrix} n-|P_0| \\ k-t \end{bmatrix} \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix},$$

with equality if and only if $|N(c_i)| = 1$ for $i = 1, 2 \ldots, t$, i.e., $\mathcal{A}$ is the stabilizer of at least t fixed points. □

Acknowledgments

This project is supported by the Advanced Fundamental Research Cluster, University of Malaya (UMRG RG238/12AFR).

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