

Noncommutative covering projections and K -homology

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Abstract

If X is a topological space then there is a natural homomorphism $\pi_1(X) \rightarrow K_1(X)$ from a fundamental group to a K_1 -homology group. Covering projections depend of fundamental group. So K_1 -homology groups are interrelated with covering projections. This article is concerned with a noncommutative analogue of this interrelationship.

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1 Introduction

It is known that $K_1(S^1) \approx \mathbb{Z}$. If x is a generator of $K(S^1)$ than there is a natural homomorphism $\varphi_K : \pi_1(X) \rightarrow K_1(X)$ given by

$$[f] \mapsto K_1(f)(x) \quad (1)$$

where f is a representative of $[f] \in \pi_1(X)$. This homomorphism does not depend on a basepoint because $K_1(X)$ is an abelian group . So the basepoint is omitted. Let $K_{11}(X) \subset K_1(X)$ be the image of φ_K . Then $K_{11}(X)$ is a homotopical invariant.

Example 1.1. We have a natural isomorphism $\varphi_K : \pi_1(S^1) \rightarrow K_1(S^1)$. From $\pi_1(S^1) = \mathbb{Z}$ it follows that there is a n -listed covering projection $f_n : S^1 \rightarrow S^1$ for any $n \in \mathbb{N}$.

Example 1.2. Let $f : S^1 \rightarrow S^1$ be an n listed covering projection, C_f is the mapping cone [12] of f . Then $\pi_1(C_f) \approx K_1(C_f) \approx \mathbb{Z}_n$ and there is a natural isomorphism $\varphi_K : \pi_1(C_f) \rightarrow K_1(C_f)$. There is n - listed universal covering projection $f_n : \widehat{C}_f \rightarrow C_f$.

Finitely listed covering projections depend of fundamental group. Any epimorphism $\pi_1(X) \rightarrow \mathbb{Z}$ (resp. $\pi_1(X) \rightarrow \mathbb{Z}_n$) corresponds to the infinite sequence of finitely listed covering projections (resp. an n - listed covering projection). If $\varphi : \pi_1(X) \rightarrow G$ is an epimorphism ($G \approx \mathbb{Z}$ or $G \approx \mathbb{Z}_n$) such that $\ker \varphi_K \subset \ker \varphi$ then there is an algebraic construction of these covering projections which is described in this article. A noncommutative analogue of $K_{11}(X)$ is discussed.

This article assumes elementary knowledge of following subjects

1. Algebraic topology [12].
2. C^* - algebras and K -theory [1], [4], [9], [10].

Following notation is used.

Symbol	Meaning
A^+	Unitization of C^* - algebra A
A_+	A positive cone of C^* - algebra A
A^G	Algebra of G invariants, i.e. $A^G = \{a \in A \mid ga = a, \forall g \in G\}$
\hat{A}	Spectrum of C^* - algebra A with the hull-kernel topology (or Jacobson topology)
$\text{Aut}(A)$	Group $*$ - automorphisms of C^* algebra A
$B(H)$	Algebra of bounded operators on Hilbert space H
$B_\infty = B_\infty(\{z \in \mathbb{C} \mid z = 1\})$	Algebra of Borel measured functions on the $\{z \in \mathbb{C} \mid z = 1\}$ set.
\mathbb{C} (resp. \mathbb{R})	Field of complex (resp. real) numbers
\mathbb{C}^*	$\{z \in \mathbb{C} \mid z = 1\}$
$C(X)$	C^* - algebra of continuous complex valued functions on topological space X
$C^b(X)$	C^* - algebra of bounded continuous complex valued functions on topological space X
H	Hilbert space
$I = [0, 1] \subset \mathbb{R}$	Closed unit interval
$G_{tors} \subset G$	The torsion subgroup of an abelian group
$\mathcal{K}(H)$ or \mathcal{K}	Algebra of compact operators on Hilbert space H
$\mathbb{M}_n(A)$	The $n \times n$ matrix algebra over C^* - algebra A
$\text{Map}(X, Y)$	The set of maps from X to Y
$M(A)$	A multiplier algebra of C^* -algebra A
$M^s(A) = M(A \otimes \mathcal{K})$	Stable multiplier algebra of C^* - algebra A
\mathbb{N}	Monoid of natural numbers
$Q(A) = M(A)/A$	Outer multiplier algebra of C^* - algebra A
$Q^s(A) = (M(A \otimes \mathcal{K}))/ (A \otimes \mathcal{K})$	Stable outer multiplier algebra of C^* - algebra A
\mathbb{Q}	Field of rational numbers
$\text{sp}(a)$	Spectrum of element of C^* -algebra $a \in A$
$U(H) \subset \mathcal{B}(H)$	Group of unitary operators on Hilbert space H
$U(A) \subset A$	Group of unitary operators of algebra A
\mathbb{Z}	Ring of integers
\mathbb{Z}_m	Ring of integers modulo m
Ω	Natural contravariant functor from category of commutative C^* - algebras, to category of Hausdorff spaces

2 Galois extensions of C^* - algebras and noncommutative covering projections

2.1 General theory

2.1. *Galois extensions.* Let G be a finite group, a G -Galois extensions can be regarded as particular case of Hopf-Galois extensions [8], where Hopf algebra is a commutative alge-

bra $C(G)$. Let A be a C^* -algebra, let $G \subset \text{Aut}(A)$ be a finite group of $*$ - automorphisms. Let ${}_A\mathcal{M}^G$ be a category of G -equivariant modules. There is a pair of adjoint functors (F, U) given by

$$F = A \otimes_{A^G} - : {}_{A^G}M \rightarrow {}_A\mathcal{M}^G; \quad (2)$$

$$U = (-)^G : {}_A\mathcal{M}^G \rightarrow {}_{A^G}M. \quad (3)$$

The unit and counit of the adjunction (F, U) are given by the formulas

$$\eta_N : N \rightarrow (A \otimes_{A^G} N)^G, \eta_N(n) = 1 \otimes n;$$

$$\varepsilon_M : A \otimes_{A^G} M^G \rightarrow M, \varepsilon_M(a \otimes m) = am.$$

Consider a following map

$$\text{can} : A \otimes_{A^G} A \rightarrow \text{Map}(G, A) \quad (4)$$

given by

$$a_1 \otimes a_2 \mapsto (g \mapsto a_1(ga_2)), (a_1, a_2 \in A, g \in G).$$

The can is a ${}_A\mathcal{M}^G$ morphism.

Theorem 2.2. [3] Let A be an algebra, let G be a finite group which acts on A , (F, U) functors given by (2), (3). Consider the following statements:

1. (F, U) is a pair of inverse equivalences;
2. (F, U) is a pair of inverse equivalences and $A \in {}_{A^G}\mathcal{M}$ is flat;
3. The can is an isomorphism and $A \in {}_{A^G}\mathcal{M}$ is faithfully flat.

These the three conditions are equivalent.

Definition 2.3. If conditions of theorem 2.2 are hold, then A is said to be *left faithfully flat G -Galois extension*

Remark 2.4. Theorem 2.2 is an adapted to finite groups version of theorem from [3].

In case of commutative C^* -algebras definition 2.3 supplies finitely listed covering projections of topological spaces. However I think that above definition is not quite good analogue of noncommutative covering projections. Noncommutative algebras contains inner automorphisms. Inner automorphisms are rather gauge transformations [6] than geometrical ones. So I think that inner automorphisms should be excluded. Importance of outer automorphisms was noted by Miyashita [7]. It is reasonably take to account outer automorphisms only. I have set more strong condition.

Definition 2.5. [11] Let A be C^* - algebra. A $*$ - automorphism α is said to be *generalized inner* if is obtained by conjugating with unitaries from multiplier algebra $M(A)$.

Definition 2.6. [11] Let A be C^* - algebra. A $*$ - automorphism α is said to be *partly inner* if its restriction to some non-zero α - invariant two-sided ideal is generalized inner. We call automorphism *purely outer* if it is not partly inner.

Instead definitions 2.5, 2.6 following definitions are being used.

Definition 2.7. Let $\alpha \in \text{Aut}(A)$ be an automorphism. A representation $\rho : A \rightarrow B(H)$ is said to be α - *invariant* if a representation ρ_α given by

$$\rho_\alpha(a) = \rho(\alpha(a)) \quad (5)$$

is unitary equivalent to ρ .

Definition 2.8. Automorphism $\alpha \in \text{Aut}(A)$ is said to be *strictly outer* if for any α - invariant representation $\rho : A \rightarrow B(H)$, automorphism ρ_α is not a generalized inner automorphism.

Definition 2.9. Let A be a C^* - algebra and $G \subset \text{Aut}(A)$ be a finite subgroup of $*$ - automorphisms. An injective $*$ - homomorphism $f : A^G \rightarrow A$ is said to be a *noncommutative finite covering projection* (or *noncommutative G - covering projection*) if f satisfies following conditions:

1. A is a finitely generated equivariant projective left and right A^G Hilbert C^* -module.
2. If $\alpha \in G$ then α is strictly outer.
3. f is a left faithfully flat G -Galois extension.

The G is said to be *covering transformation group* of f . Denote by $G(B|A)$ covering transformation group of covering projection $A \rightarrow B$.

2.10. Irreducible representations of noncommutative covering projections. Let $f : A^G \rightarrow A$ be a noncommutative G - covering projection. Let $\rho : A \rightarrow B(H)$ be an irreducible representation. Let $g \in G$ and $\rho_g : A \rightarrow B(H)$ be such that

$$\rho_g(a) = \rho(ga).$$

So it is an action of G on \hat{A} such that

$$g \mapsto (\rho \mapsto \rho_g); \forall g \in G, \forall \rho \in \hat{A}. \quad (6)$$

Let us enumerate elements of G by integers, i. e. $g_1, \dots, g_n \in G$, $n = |G|$ and define action of $\sigma : G \times \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ such that $\sigma(g, i) = j \Leftrightarrow g_j = gg_i$. Let $\rho_\oplus = \oplus_{g \in G} \rho_g : A \rightarrow B(H^n)$ be such that

$$\rho_\oplus(a)(h_1, \dots, h_n) = (\rho(g_1 a)h_1, \dots, \rho(g_n a)h_n). \quad (7)$$

Let us define such linear action of G on H^n that

$$g(h_1, \dots, h_n) = (h_{\sigma(g^{-1}, 1)}, \dots, h_{\sigma(g^{-1}, n)}). \quad (8)$$

From (7), (8) it follows that

$$g(ah) = (ga)(gh); \forall a \in A, \forall g \in G, \forall h \in H^n,$$

i.e. $H^n \in_A \mathcal{M}^G$. Equivariant representation ρ_\oplus defines representation $\eta : A^G \rightarrow B(K)$. $K = (H^n)^G$. If η is not an irreducible then there is a nontrivial A^G - submodule $N \subsetneq K$. From ${}_A \mathcal{M}^G \approx_{A^G} \mathcal{M}$ it follows that $A \otimes_{A^G} N \subsetneq H^n$ is a nontrivial A - submodule. If we identify H with first summand of H^n then $(A \otimes_{A^G} K) \cap H \subsetneq H$ is a nontrivial A - submodule. This fact contradicts with that ρ is irreducible. So η is an irreducible representation. In result we have a natural map

$$\hat{f} : \hat{A} \rightarrow \widehat{A^G}, (\rho \mapsto \eta) \quad (9)$$

and

$$\widehat{A^G} \approx \hat{A}/G. \quad (10)$$

2.2 Covering projection of C^* -algebras with continuous trace

Definition 2.11. [10] A positive element in C^* - algebra A is *abelian* if subalgebra $xAx \subset A$ is commutative.

Proposition 2.12. [10] A positive element x in C^* - algebra A is *abelian* if $\dim \pi(x) \leq 1$ for every irreducible representation $\pi : A \rightarrow B(H)$ of A .

2.13. Let A be a C^* - algebra. For each $x \in A_+$ the (canonical) trace $\text{Tr}(\pi(x))$ of $\pi(x)$ depends only on the equivalence class of an irreducible representation $\pi : A \rightarrow B(H)$, so that we may define a function $\hat{x} : \hat{A} \rightarrow [0, \infty]$ by $\hat{x}(t) = \text{Tr}(\pi(x))$ whenever $\pi \in t$. From Proposition 4.4.9 [10] it follows that \hat{x} is lower semicontinuous function on a in Jacobson topology.

Definition 2.14. [10] We say that element $x \in A_+$ has *continuous trace* if $\hat{x} \in C^b(\hat{A})$. We say that A is a C^* - algebra *with continuous trace* if set of elements with continuous trace is dense in A_+ . We say that a C^* - algebra A is of type I if each non-zero quotient of A contains non-zero abelian element. If A is even generated (as C^* - algebra) by its abelian elements we say that it is of type I_0 .

Theorem 2.15. (Theorem 5.6 [10]) For each C^* - algebra A there is a dense hereditary ideal $K(A)$, which is minimal among dense ideals.

Proposition 2.16. [10] Let A be a C^* - algebra with continuous trace Then

1. A is of type I_0 ;
2. \hat{A} is a locally compact Hausdorff space;
3. For each $t \in \hat{A}$ there is an abelian element $x \in A$ such that $\hat{x} \in K(\hat{A})$ and $\hat{x}(t) = 1$.

The last condition is sufficient for A to have continuous trace.

Remark 2.17. From [5], Proposition 10, II.9 it follows that a continuous trace C^* -algebra is always a CCR-algebra, a C^* -algebra where for every irreducible representation $\pi : A \rightarrow B(H)$ and for every element $x \in A$, $\pi(x)$ is a compact operator, i.e. $\pi(A) = \mathcal{K}(H)$.

Lemma 2.18. *Let $A^G \rightarrow A$ be a noncommutative covering projection such that A is a CCR-algebra. Then G acts freely on \hat{A} .*

Proof. Suppose that G does not act freely on \hat{A} . Then there are $x \in \hat{A}$ and $g \in G$ such that $gt = t$ ($t \in \hat{A}$). By definition 2.9 g should be strictly outer. Let $\rho : A \rightarrow \mathcal{B}(H)$ be representative of x . Then ρ_g is also representative of x . So ρ is unitary equivalent to ρ_g , i. e. there is unitary $U \in U(H)$ such that $\rho_g(a) = U\rho(a)U^*$ ($\forall a \in A$). According to 2.17 $\rho(A) = \mathcal{K}(H)$, $\rho(M(A)) = B(H)$, $\rho(U(M(A))) = U(H)$. So it is $u \in M(A)$ such that $\rho(u) = U$ and we have $\rho_g(a) = \rho(u)\rho(a)\rho(u^*)$. It means that g is inner with respect to ρ , so action of g is not strictly outer. This contradiction proves the lemma. \square

Lemma 2.19. [10] *Let G be a finite group and $f : A^G \rightarrow A$ is a G -covering projection. If A^G is a continuous trace C^* -algebra then A is also a continuous trace C^* -algebra.*

Proof. From 2.10 it follows that for any irreducible representation $\rho : A \rightarrow \mathcal{B}(H)$ there is a irreducible representation $\eta : A^G \rightarrow \mathcal{B}(H)$ such that

$$\rho|_{A^G} = \eta \tag{11}$$

Let $x \in A^G$ be an abelian element of A^G . From 2.12 it follows that $\dim \eta(x) \leq 1$ for any irreducible representation $\eta : A^G \rightarrow \mathcal{B}(H)$. From (11) it follows that $\dim \rho(x) \leq 1$ for any irreducible representation $\rho : A \rightarrow \mathcal{B}(H)$. So any abelian element of A^G is also an abelian element of A . Let $t \in \hat{A}$ and $s = \hat{f}(t) \in \hat{A}^G$ where \hat{f} is defined by (9). From 2.12 it follows that there is an abelian element $x \in A^G$ such that $\hat{x} \in K(\hat{A}^G)$ and $\hat{x}(s) = 1$. However x is a abelian element of A , $\hat{x} \in K(A)$ and $\hat{x}(t) = \hat{x}(s) = 1$. From 2.16 it follows that A is a continuous trace C^* -algebra. \square

Proposition 2.20. [2] *If a topological group G acts properly on a topological space then orbit space X/G is Hausdorff. If also G is Hausdorff, then X is Hausdorff.*

Theorem 2.21. *Let $f : A^G \rightarrow A$ be a noncommutative finite covering projection and A^G is a continuous trace algebra. Then $\hat{A} \rightarrow \hat{A}/G$ is a (topological) covering projection.*

Proof. From lemma 2.19 it follows that A is a continuous trace algebra. From 2.16 it follows that a space \hat{A} is Hausdorff. From 2.18 it follows that G acts freely on \hat{A} . From (10) it follows that $\widehat{A^G} \approx \hat{A}/G$. It is known [12] that if a finite group G acts freely on Hausdorff space X then $X \rightarrow X/G$ is a covering projection. \square

Remark 2.22. From theorem 2.21 it follows that finite covering projections of commutative algebras are just covering projections of their character spaces. If A^G is a commutative C^* -algebra then $\dim \pi(A^G) = 1$ for all irreducible $\pi : A \rightarrow \mathcal{B}(H)$. If $f : A^G \rightarrow A$ is noncommutative G covering projection and A^G is commutative then A^G is continuous trace algebra $\Omega(A^G) \approx \hat{A}^G$. From 2.19 it follows that A is also a continuous trace C^* -algebra. If $\rho : A \rightarrow \mathcal{B}(H)$ then $\rho(A) = \mathcal{K}(H)$. Let us recall construction from 2.10. Let us enumerate elements of G by integers, i. e. $g_1, \dots, g_n \in G$, $n = |G|$ and define action of σ :

$G \times \{i, \dots, n\} \rightarrow \{i, \dots, n\}$ such that $\sigma(g, i) = j \Leftrightarrow g_j = gg_i$. Let $\rho_{\oplus} = \bigoplus_{g \in G} \rho_g : A \rightarrow \mathcal{B}(H^n)$ be such that

$$\rho_{\oplus}(a)(h_1, \dots, h_n) = (\rho(g_1 a)h_1, \dots, \rho(g_n a)h_n). \quad (12)$$

Let us define such linear action of G on H^n that

$$g(h_1, \dots, h_n) = (h_{\sigma(g^{-1}, 1)}, \dots, h_{\sigma(g^{-1}, n)}). \quad (13)$$

From (7), (13) it follows that

$$g(ah) = (ga)(gh); \forall a \in A, \forall g \in G, \forall h \in H^n,$$

i.e. $H^n \in_A \mathcal{M}^G$. Representation ρ_{\oplus} defines representation $\eta : A^G \rightarrow \mathcal{B}(K)$. $K = (H^n)^G$. From 2.10 η is irreducible representation and since A^G is commutative it follows that $\dim K = 1$. From (13) it follows that $\dim H = 1$. Thus the dimension of any irreducible representation of A equals to 1. It means that any irreducible representation is commutative. From this fact it follows that A is a commutative C^* -algebra $\hat{A} = \Omega(A)$ and $\Omega(f) : \Omega(A) \rightarrow \Omega(A^G)$ is a (topological) covering projection.

2.3 Covering projections of noncommutative torus

2.23. A noncommutative torus [13] A_{θ} is C^* -norm completion of algebra generated by two unitary elements u, v which satisfy following conditions

$$uu^* = u^*u = vv^* = v^*v = 1;$$

$$uv = e^{2\pi i \theta} vu,$$

where $\theta \in \mathbb{R}$. If $\theta = 0$ then $A_{\theta} = A_0$ is commutative algebra of continuous functions on commutative torus $C(S^1 \times S^1)$. There is a trace τ_0 on A_{θ} such that $\tau_0(\sum_{-\infty < i < \infty, -\infty < j < \infty} a_{ij} u^i v^j) = a_{00}$. C^* -norm of A_{θ} is defined by following way $\|a\| = \sqrt{\tau_0(a^*a)}$. Let us consider $*$ -homomorphism $f : A_{\theta} \rightarrow A_{\theta'}$, where $A_{\theta'}$ is generated by unitary elements u' and v' . Homomorphism f is defined by following way:

$$u \mapsto u'^m;$$

$$v \mapsto v'^n;$$

It is clear that

$$\theta' = \frac{\theta + k}{mn}; \quad (k = 0, \dots, mn - 1). \quad (14)$$

Lemma 2.24. Above $*$ -homomorphism $A_{\theta} \rightarrow A_{\theta'}$ is a noncommutative covering projection.

Proof. We need check conditions of definition 2.9. $A_{\theta'}$ is a free A_{θ} module generated by monomials $u'^i v'^j$ ($i = 0, \dots, m - 1; j = 0, \dots, n - 1$), so it is projective finitely generated A_{θ} -module. Commutative C^* -subalgebras $C(u') \subset A_{\theta'}$ and $C(v') \subset A_{\theta'}$ generated by u' and v' respectively are isomorphic to algebra $C(S^1)$, where S^1 is one dimensional circle. There

are induced by f^* -homomorphisms $C(S^1) = C(u) \rightarrow C(u') = C(S^1)$, $C(S^1) = C(v) \rightarrow C(v') = C(S^1)$. These $*$ -homomorphisms induces m and n listed covering projections respectively. Covering groups of these covering projections are $G_1 \approx \mathbb{Z}_m$ and $G_2 \approx \mathbb{Z}_n$ respectively. Generators of these groups are presented below:

$$u' \mapsto e^{\frac{2\pi i}{m}} u'; \quad (15)$$

$$v' \mapsto e^{\frac{2\pi i}{n}} v'. \quad (16)$$

Equations (15), (16) define action of $G = \mathbb{Z}_m \times \mathbb{Z}_n$ on $A_{\theta'}$ and $A_{\theta} = A_{\theta'}^G$. Inner automorphisms of $A_{\theta'}$ are given by

$$v' \mapsto u'^p v' u'^{*p} = e^{\frac{2\pi i p \theta}{mn}} v'.$$

$$u' \mapsto v'^q u' v'^{*q} = e^{\frac{2\pi i q \theta}{mn}} u'.$$

These inner automorphisms do not coincide with automorphisms given by (15), (16). Let us show that $\text{can} : A_{\theta'} \otimes_{A_{\theta}} A_{\theta'} \rightarrow \text{Map}(G, A_{\theta'})$ is an isomorphism in ${}_{A_{\theta}}\mathcal{M}^G$ category. This fact follows from the set theoretic bijectivity of the can . Homomorphisms of commutative algebras $C(u) \rightarrow C(u')$, $C(v) \rightarrow C(v')$ correspond to covering projection, it follows that there are elements $x_i \in C(u')$ ($i = 1, \dots, r$), $y_j \in C(v')$ ($j = 1, \dots, s$) such that

$$\sum_{1 \leq i \leq r} x_i^2 = 1_{C(u')}; \quad (17)$$

$$\sum_{1 \leq i \leq r} x_i(g_1 x_i) = 0; g_1 \in G_1; \quad (18)$$

$$\sum_{1 \leq j \leq s} y_j^2 = 1_{C(v')}; \quad (19)$$

$$\sum_{1 \leq j \leq s} y_j(g_2 y_j) = 0; g_2 \in G_2, \quad (20)$$

where g_1 and g_2 are nontrivial elements of \mathbb{Z}_m and \mathbb{Z}_n .

Let $a_k, b_k \in A_{\theta}$ be such that

$$a_k = y_j x_i,$$

$$b_k = x_i y_j,$$

where $k = 1, \dots, rs$.

From (17)- (20) it follows that

$$\sum_{1 \leq k \leq rs} a_k b_k = 1_{A_{\theta'}},$$

$$\sum_{1 \leq k \leq rs} a_k (g b_k) = 0,$$

where $g \in G = \mathbb{Z}_m \times \mathbb{Z}_n$ is a nontrivial element. If $\varphi \in \text{Map}(G, A_{\theta})$ is such that $g_i \mapsto c_i$ ($i = 1, \dots, mn$) then

$$\varphi = \text{can} \left(\sum_{i=1}^{mn} \sum_{k=1}^{rs} a_k \otimes g_i^{-1} b_k c_i \right). \quad (21)$$

So can is a surjective map. Let us show that can is injective. $A_{\theta'}$ is a free left A_{θ} module, because any element $a \in A_{\theta'}$ has following unique representation

$$a = \sum_{r=0, s=0}^{m-1, n-1} a_{rs} u^r v^s \quad (a_{rs} \in A_{\theta}). \quad (22)$$

From (22) it follows that any element $x \in A_{\theta'} \otimes_{A_{\theta}} A_{\theta'}$ has following unique representation

$$x = \sum_{r=0, s=0}^{m-1, n-1} a_{rs} \otimes u^r v^s \quad (a_{rs} \in A_{\theta'}). \quad (23)$$

Let us prove that can maps above sum of linearly independent elements of $A_{\theta'} \otimes_{A_{\theta}} A_{\theta'}$ to sum of linearly independent elements of $\text{Map}(\mathbb{Z}_m \times \mathbb{Z}_n, A_{\theta'})$. Really if

$$\varphi = \text{can}(a \otimes u^r v^s) \quad (24)$$

and $(p, q) \in \mathbb{Z}_m \times \mathbb{Z}_n$ then

$$\varphi((p, q)) = \varphi((0, 0)) e^{\frac{2\pi i p r}{m}} e^{\frac{2\pi i q s}{n}}. \quad (25)$$

i.e. linearly independent elements of (23) correspond to different representations of $G = \mathbb{Z}_m \times \mathbb{Z}_n$, but different representations are linearly independent. So can is injective. \square

Remark 2.25. Let $\theta \in \mathbb{R}$ be irrational number, $m, n \in \mathbb{N}$, $mn > 1$, $\theta' = \theta/mn$, $\theta'' = (\theta + k)/mn$ ($k \neq 0 \pmod{mn}$). Let $u, v \in A_{\theta}$, $u', v' \in A_{\theta'}$, $u'', v'' \in A_{\theta''}$ be unitary generators, $f' : A_{\theta} \rightarrow A_{\theta'}$ (resp. $f'' : A_{\theta} \rightarrow A_{\theta''}$) be $*$ -homomorphism $u \mapsto u'^m$, $v \mapsto v'^n$ (resp. $u \mapsto u''^m$, $v \mapsto v''^n$). We have $A_{\theta'} \not\cong A_{\theta''}$. So this noncommutative covering projections are not isomorphic. However these covering projections can be regarded as equivalent because they are Morita equivalent. Let $U, V \in \mathbb{M}_{N=mn}(\mathbb{C})$ be unitary matrices such that

$$UV = e^{\frac{2\pi i k}{nm}} VU.$$

There is following G equivariant isomorphism $A_{\theta'} \otimes \mathbb{M}_N(\mathbb{C}) \cong A_{\theta''} \otimes \mathbb{M}_N(\mathbb{C})$

$$u' \otimes 1 \rightarrow u'' \otimes U; \quad v' \otimes 1 \rightarrow v'' \otimes V.$$

This isomorphism is also $A_{\theta} - A_{\theta}$ bimodule isomorphism. From $\mathcal{K} \otimes \mathbb{M}_N(\mathbb{C}) \cong \mathcal{K}$ it follows that there exist isomorphism $A_{\theta'} \otimes \mathcal{K} \cong A_{\theta''} \otimes \mathcal{K}$ and there is following commutative diagram

$$\begin{array}{ccc} A_{\theta'} \otimes \mathcal{K} & \xrightarrow{\cong} & A_{\theta''} \otimes \mathcal{K} \\ & \searrow & \swarrow \\ & A_{\theta} \otimes \mathcal{K} & \end{array}$$

I find that good theory of noncommutative covering projections should be invariant with respect to Morita equivalence. This theory can replace C^* -algebras with their stabilizations (recall that the stabilization of a C^* algebra A is a C^* -algebra $A \otimes \mathcal{K}$).

3 Covering projections and K -homology

3.1 Extensions of C^* -algebras generated by unitary elements

Definition 3.1. Let A be a C^* -algebra, $A \rightarrow B(H)$ is a faithful representation, $u \in U(A^+)$, $v \in U(B(H))$, is such that $v^n = u$ and $v^i \notin U(A^+)$, ($i = 1, \dots, n-1$). A *generated by v extension* is a minimal subalgebra of $B(H)$ which contains following operators:

1. $v^i a$; ($a \in A$, $i = 0, \dots, n-1$)
2. av^i .

Denote by $A\{v\}$ a generated by v extension.

Remark 3.2. Sometimes a $*$ -homomorphism $A \rightarrow A\{v\}$ is a noncommutative covering projection but it is not always true. If the homomorphism is a covering projection then there is a relationship between the covering projection and K -homology.

Lemma 3.3. Let A be a C^* -algebra, $A \rightarrow B(H)$ is a faithful representation, $u \in U(A^+)$ is an unitary element such that $\text{sp}(u) = \mathbb{C}^* = \{z \in \mathbb{C} \mid |z| = 1\}$, $\zeta, \eta \in B_\infty(\text{sp}(u))$ are Borel measured functions such that $\zeta(z)^n = \eta(z)^n = z$ ($\forall z \in \text{sp}(u)$). Then there is an isomorphism

$$A\{\zeta(u)\} \otimes \mathcal{K} \rightarrow A\{\eta(u)\} \otimes \mathcal{K} \quad (26)$$

which is a left A -module isomorphism. The isomorphism is given by

$$\zeta(u) \otimes x \mapsto \eta(u) \otimes \zeta \eta^{-1}(u)x; \quad (x \in \mathcal{K}). \quad (27)$$

Proof. Follows from the equality $\zeta(u) = \zeta \eta^{-1}(\eta(u))$. □

Remark 3.4. See remark 2.25.

Definition 3.5. A n^{th} root of identity map is a Borel-measurable function $\phi \in B_\infty(\mathbb{C}^*)$ such that

$$(\phi(z))^n = z \quad (\forall z \in U(\mathbb{C}^*)). \quad (28)$$

Lemma 3.6. Let A be a C^* -algebra, $u \in U((A \otimes \mathcal{K})^+)$ is such that $[u] \neq 0 \in K_1(A)$ then $\text{sp}(u) = \mathbb{C}^* = \{z \in \mathbb{C} \mid |z| = 1\}$.

Proof. $\text{sp}(u) \subset \mathbb{C}^*$ since u is an unitary. Suppose $z_0 \in \mathbb{C}$ be such that $z_0 \notin \text{sp}(u)$ and $z_1 = -z_0$. Let $\varphi : \text{sp}(u) \times [0, 1] \rightarrow \mathbb{C}^*$ be such that

$$\varphi(z_1 e^{i\phi}, t) = z_1 e^{i(1-t)\phi}; \quad \phi \in (-\pi, \pi), \quad t \in [0, 1].$$

There is a homotopy $u_t = \varphi(u, t) \in U((A \otimes \mathcal{K})^+)$ such that $u_0 = u$, $u_1 = z_1$. From $[z_1] = 0 \in K_1(A)$ it follows that $[u] = 0 \in K_1(A)$. So there is a contradiction which proves this lemma. □

3.2 Universal coefficient theorem

Universal coefficient theorem [1] establishes (in particular) a relationship between K -theory and K -homology. For any C^* -algebra A there is a natural homomorphism

$$\gamma : KK_1(A, \mathbb{C}) \rightarrow \text{Hom}(K_1(A), K_0(\mathbb{C})) \approx \text{Hom}(K_1(A), \mathbb{Z}) \quad (29)$$

which is the adjoint of following pairing

$$KK(\mathbb{C}, A) \otimes KK(A, \mathbb{C}) \rightarrow KK(\mathbb{C}, \mathbb{C}).$$

If $\tau \in KK^1(A, \mathbb{C})$ is represented by extension

$$0 \rightarrow \mathbb{C} \rightarrow D \rightarrow A \rightarrow 0$$

then γ is given as connecting maps ∂ in the associated six-term exact sequence of K theory

$$\begin{array}{ccccc} K_0(\mathbb{C}) & \longrightarrow & K_0(D) & \longrightarrow & K_0(A) \\ & & & & \downarrow \partial \\ \partial \uparrow & & & & \\ & & K_1(D) & \longleftarrow & K_1(\mathbb{C}) \\ & & \longleftarrow & & \\ & & K_1(A) & & \end{array}$$

If $\gamma(\tau) = 0$ for an extension τ then the six-term K -theory exact sequence degenerates into two short exact sequences

$$0 \rightarrow K_i(A) \rightarrow K_i(D) \rightarrow K_i(\mathbb{C}) \rightarrow 0 \quad (i = 0, 1)$$

and thus determines an element $\kappa(\tau) \in \text{Ext}^1(K_*(A), K_*(\mathbb{C}))$. In result we have a sequence of abelian group homomorphisms

$$\text{Ext}^1(K_0(A), K_0(\mathbb{C})) \rightarrow KK^1(A, \mathbb{C}) \rightarrow \text{Hom}(K_1(A), K_0(\mathbb{C}))$$

such that composition of the homomorphisms is trivial. Above sequence can be rewritten by following way

$$\text{Ext}^1(K_0(A), \mathbb{Z}) \rightarrow K^1(A) \rightarrow \text{Hom}(K_1(A), \mathbb{Z}). \quad (30)$$

If G is an abelian group that

$$\text{Ext}^1(G, \mathbb{Z}) = \text{Ext}^1(G_{tors}, \mathbb{Z}),$$

$$\text{Hom}(G, \mathbb{Z}) = \text{Hom}(G/G_{tors}, \mathbb{Z}).$$

From (30) it follows that $K^1(A)$ depends on $K_0(A)_{tors}$ and $K_1(A)/K_1(A)_{tors}$. We say that dependence(30) on $K_0(A)_{tors}$ is a *torsion special case* and dependence (29) of $K^1(A)$ on $K_1(A)/K_1(A)_{tors}$ is a *free special case*.

3.3 Free special case

Example 3.7. The n - listed coverings of example 1.1 can be constructed algebraically. From (30) it follows that $K_1(C(S^1)) \approx \mathbb{Z}$. Let $u \in U(C(S^1))$ is such that $[u] \in K_1(S^1)$ is a generator of $K_1(S^1)$. Let $C(S^1) \rightarrow B(H)$ be a faithful representation and ϕ is an n^{th} root of identity map. If $v = \phi(u) \in B(H)$ then $v^n = u$ and $v \notin C(S^1)$. According to definition 3.1 we have a $*$ -homomorphism $C(S^1) \rightarrow C(S^1)\{v\}$ which corresponds to n listed covering projection of the S^1 .

3.8. General construction. Construction of example 3.7 can be generalized. Let A be a C^* - algebra such that $K^1(A) \approx G \oplus \mathbb{Z}$. From (30) it follows that

$$K_1(A) = G' \oplus \mathbb{Z}[u] \quad (31)$$

where $u \in U((A \otimes \mathcal{K})^+)$. If ϕ is an n^{th} - root of identity map then we have a generated by $\{\phi(u)\}$ extension $A \rightarrow A\{\phi(u)\}$. Sometimes this extension is a noncommutative covering projection.

Example 3.9. Let A_θ be a noncommutative torus, $K_1(A_\theta) \approx \mathbb{Z}^2$ Let $u, v \in U(A)$ be representatives of generators of $K^1(A_\theta)$ a $\text{sp}(u) = \text{sp}(v) = \{z \in \mathbb{C} \mid |z| = 1\}$. Following $*$ -homomorphisms

$$\begin{aligned} A_\theta &\rightarrow A_\theta\{\phi(u)\}, \\ A_\theta &\rightarrow A_\theta\{\phi(v)\} \end{aligned}$$

are particular cases of noncommutative covering projections which are described in subsection 2.3.

Example 3.10. It is known that S^3 is homeomorphic to $SU(2)$, $K_1(C(SU(2))) \approx \mathbb{Z}$ and $K_1(C(SU(2)))$ is generated by unitary $u \in U(C(SU(2)) \otimes \mathbb{M}_2(\mathbb{C}))$. Element u can be regarded as the natural map $SU(2) \rightarrow \mathbb{M}_2(\mathbb{C})$ and $\text{sp}(u) = \{z \in \mathbb{C} \mid |z| = 1\}$. Denote by $A = C(SU(2)) \otimes \mathbb{M}_2(\mathbb{C})$. Let ϕ be a 2^{th} - root of identity map, and $v = \phi(u)$. There is an extension $A \rightarrow A\{v\}$. Both A and $A\{v\}$ are continuous trace algebras. The \mathbb{Z}_2 group acts on $A\{v\}$ such that action of nontrivial element $g \in \mathbb{Z}_2$ is given by

$$gv = -v.$$

Let $\rho : A\{v\} \rightarrow B(H)$ be a irreducible representation. Then $V = \rho(v)$ is a 2×2 unitary matrix. Suppose that ρ is such that by

$$\rho(v) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$\rho_g(v) = \rho(gv) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Above matrices are unitary equivalent, i. e.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

So the representation ρ is unitary equivalent to the ρ_g and action of g is not strictly outer, extension $f : A \rightarrow A\{v\}$ does not satisfy definition 2.9, i.e. $f : A \rightarrow A\{v\}$ is not a noncommutative covering projection. Algebra A does not have nontrivial noncommutative covering projections because

1. A is a continuous trace algebra,
2. $\hat{A} \approx S^3$,
3. $\pi_1(S^3) = 0$, i.e. S^3 does not have nontrivial covering projections.

Remark 3.11. This construction supplies a covering projection if $x \in K_1(X)$ belongs to image of $\pi_1(X) \rightarrow K_1(X)$.

3.4 Torsion special case

Example 3.12. Universal covering from example 1.2 can be constructed algebraically. Let $f : S^1 \rightarrow S^1$ be a n listed covering projection of the circle, C_f is the (topological) mapping cone of f . $C(f) : C(S^1) \rightarrow C(S^1)$ is a corresponding *-homomorphism of C^* -algebras ($u \mapsto u^n$), where $u \in U(C(S^1))$ is such that $[u] \in K_1(C(S^1))$ is a generator. Algebraic mapping cone [1] $C_{C(f)}$ of $C(f)$ corresponds to the topological space C_f . $C_{C(f)}$ is an algebra of continuous maps $f[0,1] \rightarrow U(C)$ such that

$$f(0) = \sum_{k \in \mathbb{Z}} a_k u^{kn}, \quad a_k \in \mathbb{C}.$$

A map $v = (x \mapsto u)$ ($\forall x \in [0,1]$) is such that $v^i \notin M(C(C_f))$ ($i = 1, \dots, n-1$), $v^n \in M(C_{C(f)})$. Homomorphism $C_{C(f)} \rightarrow C_{C(f)}\{v\}$ corresponds to a n -listed covering projection from the example 1.2.

3.13. General construction. Above construction can be generalized. Let A be a C^* -algebra such that $K^1(A) = G \oplus \mathbb{Z}_n$, where G is an abelian group. From (30) it follows that $K_0(A) \approx G' \oplus \mathbb{Z}_n$. Let $Q^s(A) = M(A \otimes \mathcal{K}) / (A \otimes \mathcal{K})$ be the stable multiplier algebra of C^* -algebra A . Then from [1] it follows that $K_1(Q^s(A)) = K_0(A)$. Let $u \in U(Q^s(A))$ be such that $K_1(Q^s(A)) = G' \oplus \mathbb{Z}_n[u]$. Let ϕ be a n^{th} root of identity map such that $\phi(u^n) = u$. Let $p : M(A \otimes \mathcal{K}) \rightarrow M(A \otimes \mathcal{K}) / (A \otimes \mathcal{K})$ be a natural surjective *-homomorphism. It is known [1] that unitary element $v \in U(Q^s)$ can be lifted to an unitary element $v' \in U(M(A \otimes \mathcal{K}))$ (i.e. $v = p(v')$) if and only if $[v] = 0 \in K^1(Q^s(A))$. From $n[u] = [u^n] = 0$ it follows that there is an unitary $w \in U(M(A \otimes \mathcal{K}))$ such that $p(w) = u^n$. Let $M(A \otimes \mathcal{K}) \rightarrow B(H)$ be a faithful representation, then $\phi(w) \in U(B(H))$. If $\phi(w) \in M(A \otimes \mathcal{K})$ then $p(\phi(w)) = u$, however it is impossible because $[u] \neq 0 \in K^1(Q^s(A))$. So $\phi(w) \notin M(A \otimes \mathcal{K})$ and similarly $\phi(w)^i \notin M(A \otimes \mathcal{K})$ ($i = 1, \dots, n-1$). So we have a generated by $\phi(w)$ extension $A \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K})\{\phi(w)\}$ which can be a noncommutative covering projection. Example 3.12 is a particular case of this general construction.

Example 3.14. Let O_n be a Cuntz algebra [1], $K_0(O_n) = \mathbb{Z}_{n-1}$. Construction 3.13 supplies a \mathbb{Z}_{n-1} -Galois extension $f : O_n \otimes \mathcal{K} \rightarrow \tilde{O}_n$. However it is not known if f is strictly outer.

3.5 A noncommutative generalization of $K_{11}(X)$.

Above construction can generalize $K_{11}(X)$ group. Suppose that $K_1(X)$ is group generated by x_1, \dots, x_n . Let $x \in \{x_1, \dots, x_n\}$ be a generator. Construction of 3.8, 3.13 supplies extension of A which is associated with x . The element x is said to be *proper* if the extension is a noncommutative covering projection. Generalization of $K_{11}(X)$ is a generated by proper elements subgroup of $K^1(A)$.

4 Conclusion

The presented here theory supplies algebraic construction of covering projections. These projections are well known for commutative case. Example 3.9 is principally new application of the theory. It is interesting to find other nontrivial examples of this theory.

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