Towards a Characterization of Leaf Powers by Clique Arrangements

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Abstract

The class $\mathcal{L}_k$ of k-leaf powers consists of graphs $G = (V, E)$ that have a k-leaf root, that is, a tree T with leaf set V, where $xy \in E$, if and only if the T-distance between x and y is at most k. Structure and linear time recognition algorithms have been found for 2-, 3-, 4-, and, to some extent, 5-leaf powers, and it is known that the union of all k-leaf powers, that is, the graph class $\mathcal{L} = \bigcup_{k=2}^{\infty} \mathcal{L}_k$, forms a proper subclass of strongly chordal graphs. Despite from that, no essential progress has been made lately.

In this paper, we use the new notion of clique arrangements to suggest that leaf powers are a natural special case of strongly chordal graphs. The clique arrangement $\mathcal{A}(G)$ of a chordal graph G is a directed graph that represents the intersections between maximal cliques of G by nodes and the mutual inclusion of these vertex subsets by arcs. Recently, strongly chordal graphs have been characterized as the graphs that have a clique arrangement without bad k-cycles for $k \geq 3$. We show that the clique arrangement of every graph of $\mathcal{L}$ is free of bad 2-cycles. The question whether this characterizes the class $\mathcal{L}$ exactly remains open.

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1 Introduction

Leaf powers are a family of graph classes that has been introduced by Nishimura et al. [19] to model the problem of reconstructing phylogenetic trees. In particular, a given finite simple graph $G = (V, E)$ is called the k-leaf power of a tree T for some $k \geq 2$, if V is the set of leaves in T and any two distinct vertices $x, y \in V$ are adjacent, that is $xy \in E$, if and only if the distance of x and y in T is at most k. For all $k \geq 2$, the class of graphs that are a k-leaf power of some tree, is simply called k-leaf powers and denoted by $\mathcal{L}_k$. The general problem, from a graph theoretic point of view, is to structurally characterize $\mathcal{L}_k$ for all fixed $k \geq 2$ and to provide efficient recognition algorithms.

Obviously, a graph G is a 2-leaf power, if and only if it is the disjoint union of cliques, that is, G does not contain a chordless path of length 2. Dom et al. [13, 14] prove that 3-leaf powers are exactly the graphs that do not contain an induced bull, dart, or gem. Brandstädt et al. [4] contribute to the characterization of 3-leaf powers by showing that they are exactly the graphs that result from substituting cliques into the nodes of a tree. Moreover, they give a linear time algorithm to recognize 3-leaf powers building on their characterization. A characterization of 4-leaf powers in terms of forbidden subgraphs is yet unknown. However, basic 4-leaf powers, the 4-leaf powers without true twins, are characterized by eight forbidden subgraphs [20]. The structure of basic 4-leaf powers has further been analyzed by

Brandstädt et al. [8], who provide a nice characterization of the two-connected components of basic 4-leaf powers that leads to a linear time recognition algorithm even for 4-leaf powers. For 5-leaf powers, a polynomial time recognition was given in [12]. However, no structural characterization is known, even for basic 5-leaf powers. Only for distance-hereditary basic 5-leaf powers a characterization in terms of 34 forbidden induced subgraphs has been discovered [6]. Except from the result in [10] that $\mathcal{L}_k \subseteq \mathcal{L}_{k+1}$ is not true for every k, there have not been any more essential advances in determining the structure of k-leaf powers for $k \geq 5$ since 2007. Instead, research has focused on generalizations of leaf powers [5, 9], which also turned into dead ends, very soon.

On the other hand, if we push k to infinity, then it turns out that not every graph is a k-leaf power for some $k \geq 2$. In particular, a k-leaf power is, by definition, the subgraph of the k-th power of a tree T induced by the leaves of T. Since trees are sun-free chordal and as taking powers and induced subgraphs do not destroy this property, it follows trivially that every k-leaf power, despite the value of k, is strongly chordal [15]. But even not every strongly chordal graph is a k-leaf power for some $k \geq 2$. In fact, we are aware of exactly one counter example, which has been found by Bibelnieks et al. [1] and is shown as G_7 in Figure 1. Insofar, it is reasonable to ask for a precise characterization of the graphs that are not a k-leaf power for any $k \geq 2$. This problem can equivalently be formulated as to describe the graphs in the class $\mathcal{L} = \bigcup_{k=2}^{\infty} \mathcal{L}_k$, which we call leaf powers, for short.

Interestingly, Brandstädt et al. [3] show that $\mathcal{L}$ coincides with the class of fixed tolerance NeST (neighborhood subtree tolerance) graphs, a well-known graph class with an absolutely different motivation given by Bibelnieks et al. [1]. Naturally, characterizations and an efficient recognition algorithms for this class are also open questions today. However, by Brandstädt et al. [2, 3], it is know that $\mathcal{L}$ is a superclass of ptolemaic graphs, that is, gem-free chordal graphs [17], and even a superclass of directed rooted path graphs, introduced by Gavril [16].

Recently, we introduced the clique arrangement in [18], a new data structure that is especially valuable for the analysis of strongly chordal graphs. The clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ of a chordal graph G is a directed acyclic graph that has certain vertex subsets of G as a node set and describes the mutual inclusion of these sets by arcs. In particular, for every set $C_1, C_2, \ldots$ of maximal cliques of G there is a node in $\mathcal{X}$ for $X = C_1 \cap C_2 \cap \ldots$ and two nodes $X, Z \in \mathcal{X}$ are joined by an arc $XZ \in \mathcal{E}$, if $X \subset Z$ and there is no $Y \in \mathcal{X}$ with $X \subset Y \subset Z$. In [18], we give a new characterization of strongly chordal graphs in terms of a forbidden cyclic substructure in the clique arrangement, called bad k-cycles for $k \geq 3$, and we show how to construct the clique arrangement of a strongly chordal graph in nearly linear time.

It is known that the clique arrangements of ptolemaic graphs are even directed trees [21]. Since all ptolemaic graphs are leaf powers and all leaf powers are strongly chordal, it appears likely that the degree of acyclicity in clique arrangements of leaf powers is between forbidden bad k-cycles, $k \geq 3$, and the complete absence of cycles.

This paper describes a cyclic substructure that is forbidden in the clique arrangement of leaf powers. For convenience, we call these substructures bad 2-cycles, although they are not the obvious continuation of the concept of bad k-cycles for $k \geq 3$. As the main result of this paper, we show that bad 2-cycles occur in $\mathcal{A}(G)$, if and only if G contains at least one of seven induced subgraphs $G_1, \ldots, G_7$ depicted in Figure 1.

We leave it as an open question, if these seven graphs are sufficient to characterize $\mathcal{L}$ in terms of forbidden subgraphs. However, we conjecture that this is the case. This would imply a polynomial time recognition algorithm for $\mathcal{L}$, by using the possibility of efficiently recognizing strongly chordal graphs and checking the containment of a finite number of forbidden induced subgraphs.

2 Preliminaries

We refer to several graph classes which are not explicitly defined due to space limitations. For a comprehensive survey on graph classes we would like to refer to [7].

Throughout this paper, all graphs $G = (V, E)$ are simple, without loops and, with the exception of clique arrangements, undirected. We usually denote the vertex set by V and the edge set by E, where the edges are also called arcs in a directed graph. We write $x{-}y$, respectively $x{\to}y$ in the directed case, for $xy \in E$ and $x|y$ for $xy \notin E$. For all vertices $x \in V$ in an undirected graph, we let $N(x) = \{y \mid xy \in E\}$ denote the *open neighborhood* and $N[x] = N(x) \cup \{x\}$ the *closed neighborhood* of x in G. In a directed graph, $N_o(x) = \{y \mid xy \in E\}$ denotes the set of neighbors that are reachable from x by a single arc and $N_i(x) = \{y \mid yx \in E\}$ are the neighbors that reach x by a single arc. If $|N_i(x)| = 0$ then x is a *source* and if $|N_o(x)| = 0$ then x is a *sink*.

An *independent set* in G is a set of mutually nonadjacent vertices. A *clique* $C \subseteq V$ is a set of mutually adjacent vertices and C is called maximal, if there is no clique C' with $C \subset C'$. The set of all maximal cliques of G is denoted by $\mathcal{C}(G)$.

A (simple) path in a graph G is a sequence $x_1, x_2 \ldots, x_k$ of non-repeating vertices in G, such that $x_i x_{i+1} \in E$ for all $i \in \{1, \ldots, k-1\}$. If E is clear from the context, then we denote the path by $x_1{-}x_2{-}\ldots{-}x_k$ in an undirected graph. In a directed graph, $x_1{\to}x_2{\to}\ldots{\to}x_k$ specifies a directed path and we say that x_1 *reaches* x_k. The distance $d_G(x, y)$ between two vertices x, y of an (un-) directed graph G is the minimum number of edges in an (un-) directed path starting in x and ending in y. If the edge $x_k x_1$ is additionally present in E, then we talk of a (simple) cycle in G, and as for paths, an undirected cycle is denoted by $x_1{-}x_2{-}\ldots{-}x_k{-}x_1$. An undirected cycle is called *induced k-cycle C_k*, if G contains $x_i x_j$, if and only if $j = i + 1$ or $i = k$ and $j = 1$.

A tree T is an undirected connected acyclic graph, that is, for all pairs x, y of vertices there exists a path $x{-}\ldots{-}y$, and T is free of cycles. Directed graphs are acyclic, if they are free of directed cycles.

A vertex subset $U = \{x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}\} \subseteq V$ induces a *k-sun* in G, if $X = \{x_0, \ldots, x_{k-1}\}$ is a clique and $Y = \{y_0, \ldots, y_{k-1}\}$ is an independent set and for every edge $x_i y_j$ between X and Y, either $i = j$ or $i + 1 = j$, where the indices are counted modulo k. By definition, a graph is *chordal*, if and only if it does not contain induced k-cycles for all $k \geq 4$, and by Farber [15] a graph is *strongly chordal*, if and only if it does not contain induced k-suns for all $k \geq 3$.

Beside the many useful properties of (strongly) chordal graphs, see for example [7], this paper uses in particular the following two properties, that are folklore but nevertheless have been shown in [18]:

▶ **Lemma 1.** *If G is a chordal graph and C_1, C_2 are maximal cliques of G, then there is a vertex $x \in C_1 \setminus C_2$ such that $x|y$ for all $y \in C_2 \setminus C_1$.*

▶ **Lemma 2.** *If G is a strongly chordal graph and $\mathcal{C}$ any nonempty subset of $\mathcal{C}(G)$, then there are two maximal cliques $C_1, C_2 \in \mathcal{C}$ such that $\bigcap_{C \in \mathcal{C}} C = C_1 \cap C_2$.*

A strongly chordal graph $G = (V, E)$ is the k-leaf power of a tree T for $k \geq 2$, if V is the set of leaves in T and for all $x, y \in V$ there exists $xy \in E$, if and only if $d_T(x, y) \leq k$. The tree T is called a k-leaf root of G, in this case. Notice that k-leaf roots are not necessarily unique for given k-leaf powers. For all $k \geq 2$, the class $\mathcal{L}_k$ consists of all graphs that are a k-leaf power for some tree and $\mathcal{L} = \bigcup_{k=2}^{\infty} \mathcal{L}_k$ is the class of leaf powers.

The clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ of a chordal graph G, as introduced in [18], is a directed acyclic graph with node set

$$\mathcal{X} = \left\{ X \;\middle|\; X = \bigcap_{C \in \mathcal{C}} C \text{ with } \mathcal{C} \subseteq \mathcal{C}(G) \text{ and } X \neq \emptyset \right\},$$

that contains exactly all intersections of the maximal cliques of G, and arc set

$$\mathcal{E} = \{ XZ \mid X, Z \in \mathcal{X} \text{ with } X \subset Z \text{ and } \nexists Y \in \mathcal{X} : X \subset Y \subset Z \}$$

that describes their mutual inclusion. Clearly, the set of sinks in $\mathcal{A}(G)$ corresponds exactly to $\mathcal{C}(G)$.

The following simple facts for clique arrangements are also introduced in [18]:

▶ **Lemma 3** (Nevries and Rosenke [18]). *If $X \in \mathcal{X}$ is a node in the clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ of a chordal graph G and if $\{Y_1, \dots, Y_\ell\} = N_o(X)$, then $X = Y_1 \cap \ldots \cap Y_\ell$. Moreover, if $C_1, \dots, C_k$ are the sinks of $\mathcal{A}(G)$ that are reached from X by directed paths, then $X = C_1 \cap \ldots \cap C_k$.*

▶ **Lemma 4** (Nevries and Rosenke [18]). *If $Y_1, \dots, Y_k \in \mathcal{X}$ are nodes in the clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ of a chordal graph G such that their intersection $X = Y_1 \cap \ldots \cap Y_k$ is not empty, then $X \in \mathcal{X}$.*

Although $\mathcal{A}(G)$ is acyclic by definition, we call the following structure a cycle in $\mathcal{A}(G)$ for the lack of a better term. For any $k \in \mathbb{N}$, a *k-cycle* of $\mathcal{A}(G)$ is a set of nodes $S_0, \dots, S_{k-1}, T_0, \dots, T_{k-1}$ such that for all $i \in \{0, \dots, k-1\}$ there is a directed path from S_i to T_i and a directed path from S_i to T_{i-1} (counted modulo k). The nodes $S_0, \dots, S_{k-1}$ are called *starters* of the cycle and the nodes $T_0, \dots, T_{k-1}$ are called *terminals* of the cycle. Note that by definition, $S_i \subseteq T_i \cap T_{i-1}$ for all $i \in \{0, \dots, k-1\}$. In [18], we call a k-cycle *bad*, if $k \geq 3$ and for all $i, j \in \{0, \dots, k-1\}$ there is a directed path from S_i to T_j, if only if $j \in \{i, i-1\}$ (counted modulo k).

▶ **Theorem 5** (Nevries and Rosenke [18]). *Let $G = (V, E)$ be a chordal graph and $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ be the clique arrangement of G. Then G is strongly chordal, if and only if $\mathcal{A}(G)$ is free of bad k-cycles for all $k \geq 3$.*

In this paper we apply two other properties of clique arrangements for strongly chordal graphs:

▶ **Lemma 6** (Proof in Section 6). *Let G be a strongly chordal graph with clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ and let $X, Y, Z \in \mathcal{X}$ be three distinct nodes such that $X = Y \cap Z$. There are sinks $C_1, C_2 \in \mathcal{X}$ such that C_1 is reachable from Y and C_2 is reachable from Z and $X = C_1 \cap C_2$.*

▶ **Lemma 7** (Proof in Section 6). *Let $G = (V, E)$ be a chordal graph with clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$ that occurs as an induced subgraph of a chordal graph $G' = (V', E')$ with clique arrangement $\mathcal{A}(G') = (\mathcal{X}', \mathcal{E}')$, that is, $G = G'[V]$. There exists a function $\phi : \mathcal{X} \to \mathcal{X}'$ that fulfills the following two conditions for all $X, Y \in \mathcal{X}$:*

1. *$X = Y \Leftrightarrow \phi(X) = \phi(Y)$, and*
2. *$\mathcal{A}(G)$ has a directed path from X to Y, if and only if $\mathcal{A}(G')$ has a directed path from $\phi(X)$ to $\phi(Y)$.*

3 Forbidden Induced Subgraphs

Bibelnieks et al. [1] are the first to find a strongly chordal graph, namely G_7, that is not in $\mathcal{L}$ and, consequently, show that the classes are not equivalent. In fact, they were looking for a strongly chordal graph that is not a fixed tolerance NeST graph, but by Brandstädt et al. [3], we know that $\mathcal{L}$ and this class are equal. Since then, it has been conjectured that G_7 is the smallest forbidden induced subgraph of leaf powers.

To show that G_7 is not in $\mathcal{L}$, Bibelnieks et al. [1] use a lemma of Broin et al. [11]. The basic idea of the proof of this lemma is to show for certain pairs of edges x_1y_1 and x_2y_2 in G that the path between x_1 and y_1 is disjoint from the path between x_2 and y_2 in every leaf root of G. In particular, this happens, if vertices a, b exist in G with $x_1, y_1 \in N(a) \setminus N[b]$ and $x_2, y_2 \in N(b) \setminus N[a]$. The graph G_7 has a cycle $x_0 - y_{00} - y_{10} - x_1 - y_{11} - y_{01} - x_0$, where the condition is fulfilled for many pairs of edges in the cycle. It follows that every leaf root of G_7 would have a cycle, which is a contradiction.

In this section, we want to show that there are at least six other strongly chordal graphs $G_1, \dots, G_6$ that are not in $\mathcal{L}$. Interestingly, every of these six graphs is smaller than G_7. For our proof, we generalize the argument of Bibelnieks et al. [1] for pairs of edges x_1y_1 and x_2y_2 that correspond to disjoint paths in leaf roots. The following Lemma provides three corresponding conditions:

▶ **Lemma 8.** *Let $G = (V, E)$ be a k-leaf power of a tree T for some $k \geq 2$ and let x_1y_1 and x_2y_2 be two edges of G on distinct vertices $x_1, y_1, x_2, y_2 \in V$. The paths $x_1 - \ldots - y_1$ and $x_2 - \ldots - y_2$ in T are disjoint, that is, do not share any node, if at least one of the following conditions holds:*

1. *At most one of the edges $x_1x_2, x_1y_2, y_1x_2, y_1y_2$ is in E.*
2. *There is a vertex $a \in V$ such that $x_1, y_1 \in N(a)$, and $x_2, y_2 \notin N[a]$, and $N(x_1) \cap \{x_2, y_2\} \leq 1$, and $N(y_1) \cap \{x_2, y_2\} \leq 1$.*
3. *There are distinct vertices $a, b \in V$ such that $x_1, y_1 \in N(a) \setminus N[b]$, and $x_2, y_2 \in N(b) \setminus N[a]$.*

Proof.

1. Assume that the two paths are not disjoint. Then T contains (not necessarily distinct) nodes s and t such that (i) the path $x_1 - \ldots - y_1$ consists of three subpaths, firstly $x_1 - \ldots - s$, secondly $s - \ldots - t$, and thirdly $t - \ldots - y_1$ and (ii) the path $x_2 - \ldots - y_2$ consists of three subpaths, too, without loss of generality, the first is $x_2 - \ldots - s$ and the last is $t - \ldots - y_2$. Hence, the path between s and t is the intersection between the two paths. Because $x_1 - y_1$ and $x_2 - y_2$ in G we get the following inequations by definition:

$$d_T(x_1, y_1) = d_T(x_1, s) + d_T(s, t) + d_T(t, y_1) \leq k \text{ and} \tag{1}$$
$$d_T(x_2, y_2) = d_T(x_2, s) + d_T(s, t) + d_T(t, y_2) \leq k. \tag{2}$$

As at most one of the edges $x_1x_2, x_1y_2, y_1x_1, y_1y_2$ is in E, we know that at least one of $x_1y_2, y_1x_2 \notin E$ and $x_1x_2, y_1y_2 \notin E$ is true. If $x_1|y_2$ and $y_1|x_2$, then we get

$$d_T(x_1, y_2) = d_T(x_1, s) + d_T(s, t) + d_T(t, y_2) > k \text{ and} \tag{3}$$
$$d_T(y_1, x_2) = d_T(y_1, t) + d_T(t, s) + d_T(s, x_2) > k \tag{4}$$

such that combining (1) and (3) yields $d_T(t, y_1) < d_T(t, y_2)$ and combining (2) and (4) yields $d_T(t, y_2) < d_T(t, y_1)$, a contradiction. Otherwise, if $x_1|x_2$ and $y_1|y_2$, we get the inequations

$$d_T(x_1, x_2) = d_T(x_1, s) + d_T(s, x_2) > k \text{ and} \tag{5}$$
$$d_T(y_1, y_2) = d_T(y_1, t) + d_T(t, y_2) > k \tag{6}$$

such that combining equation (1) and (5) yields $d(x_2, s) > d_T(s,t) + d_T(t, y_1)$. Putting this estimate of $d_T(x_2, s)$ into (2) yields $d_T(y_1, t) + d_T(t, y_2) + 2d_T(s, t) < k$. By (6) we can conclude that $2d_T(s,t) < 0$, which is a contradiction to the preconditions.

2. As the edges ax_1 and x_2y_2 are joined in G by at most one edge, x_1x_2 or x_1y_2, it follows from 1. that $a - \ldots - x_1$ is disjoint from $x_2 - \ldots - y_2$ in T. Analogously, the edges ay_1 and x_2y_2 are joined by at most one edge in G, either y_1x_2 or y_1y_2. Hence, in T, the path $a - \ldots - y_1$ is disjoint from $x_2 - \ldots - y_2$, too. Because T is a tree, it follows that the nodes on $x_1 - \ldots - y_1$ are a subset of the combined nodes of the paths $a - \ldots - x_1$ and $a - \ldots - y_1$. Consequently, there is no node that simultaneously belongs to $x_1 - \ldots - y_1$ and $x_2 - \ldots - y_2$.

3. If $a - b$ then $z_1 | z_2$ for all $z_1 \in \{x_1, y_1\}$ and $z_2 \in \{x_2, y_2\}$. Otherwise, $z_1 - a - b - z_2 - z_1$ is an induced C_4 in G. Hence, in this case $x_1 | x_2$, $x_1 | y_2$, $y_1 | x_2$ and $y_1 | y_2$ and we are done.

If $a | b$, then for all $z_1 \in \{x_1, y_1\}$ and $z_2 \in \{x_2, y_2\}$, the edges $a - z_1$ and $b - z_2$ are joined at most by the edge $z_1 - z_2$ in G. This means by 1. that $a - \ldots z_1$ is disjoint from $b - \ldots - z_2$ in T. Again, as T is a tree, it follows that the nodes on $x_1 - \ldots - y_1$ are a subset of the accumulated nodes on $a - \ldots - x_1$ and $a - \ldots - y_1$ and, similarly, the nodes on $x_2 - \ldots - y_2$ are a subset of the nodes on $b - \ldots - x_2$ and $b - \ldots - y_2$. Consequently, there cannot be a node that simultaneously belongs to $x_1 - \ldots - y_1$ and $x_2 - \ldots - y_2$. ◀

Based on this more general concept, we can find a cycle $x_0 - y_{00} - y_{10} - x_1 - y_{11} - y_{01} - x_0$ in every graph from $G_1, \ldots, G_7$ such that many pairs of edges in the cycle fulfill at least one of the three conditions. The following theorem states that this is never compatible with the existence of a leaf root.

▶ **Theorem 9** (Proof in Section 6). *The graphs $G_1, \ldots, G_7$ are not in $\mathcal{L}$.*

This implies that $G_1, \ldots, G_7$ are forbidden induced subgraphs for $\mathcal{L}$. In the following section, we analyze the clique arrangement of these seven graphs and show that they share one particular cyclic property, related to bad k-cycles.

4 Forbidden Cycles in Leaf Power Clique Arrangements

As shown in [18], strongly chordal graphs can be characterized by forbidden bad k-cycles in their clique arrangements, where $k \geq 3$. But by Theorem 9, this does not fully capture the cyclic structure that is forbidden in leaf powers. In this section, we show that there are certain kinds of 2-cycles which may not occur as a subgraph in the clique arrangement of a leaf power. In particular, we call a 2-cycle *bad*, if for all $i, j \in \{0, 1\}$ there is a directed path from starter S_i to terminal T_j that does not contain a node X which fulfills $S_0 \cup S_1 \subseteq X \subseteq T_0 \cap T_1$. The following theorem provides the main argument of this paper:

▶ **Theorem 10.** *Let $G = (V, E)$ be a strongly chordal graph with clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$. The graph $\mathcal{A}(G)$ contains a bad 2-cycle, if and only if G contains one of the graphs $G_1, \ldots, G_7$ as an induced subgraph.*

Proof. The proof starts by showing the first direction, that is, if $\mathcal{A}(G)$ contains a bad 2-cycle, then G contains one of the graphs $G_1, \ldots, G_7$ as an induced subgraph. Among the bad 2-cycles of $\mathcal{A}(G)$ we select a cycle with starters S_0, S_1 and terminals T_0, T_1 that primarily minimizes the summed cardinalities of the terminals $|T_0| + |T_1|$ and secondarily maximizes the summed cardinalities of the starters $|S_0| + |S_1|$. Because T_0 and T_1 have a non-empty intersection, which contains at least $S_0 \cup S_1$, Lemma 4 provides a node $T = T_0 \cap T_1$.

In the following we provide a number of claims to support our arguments. The proofs of all these claims are found in Section 6. We start by shaping the bad 2-cycle:

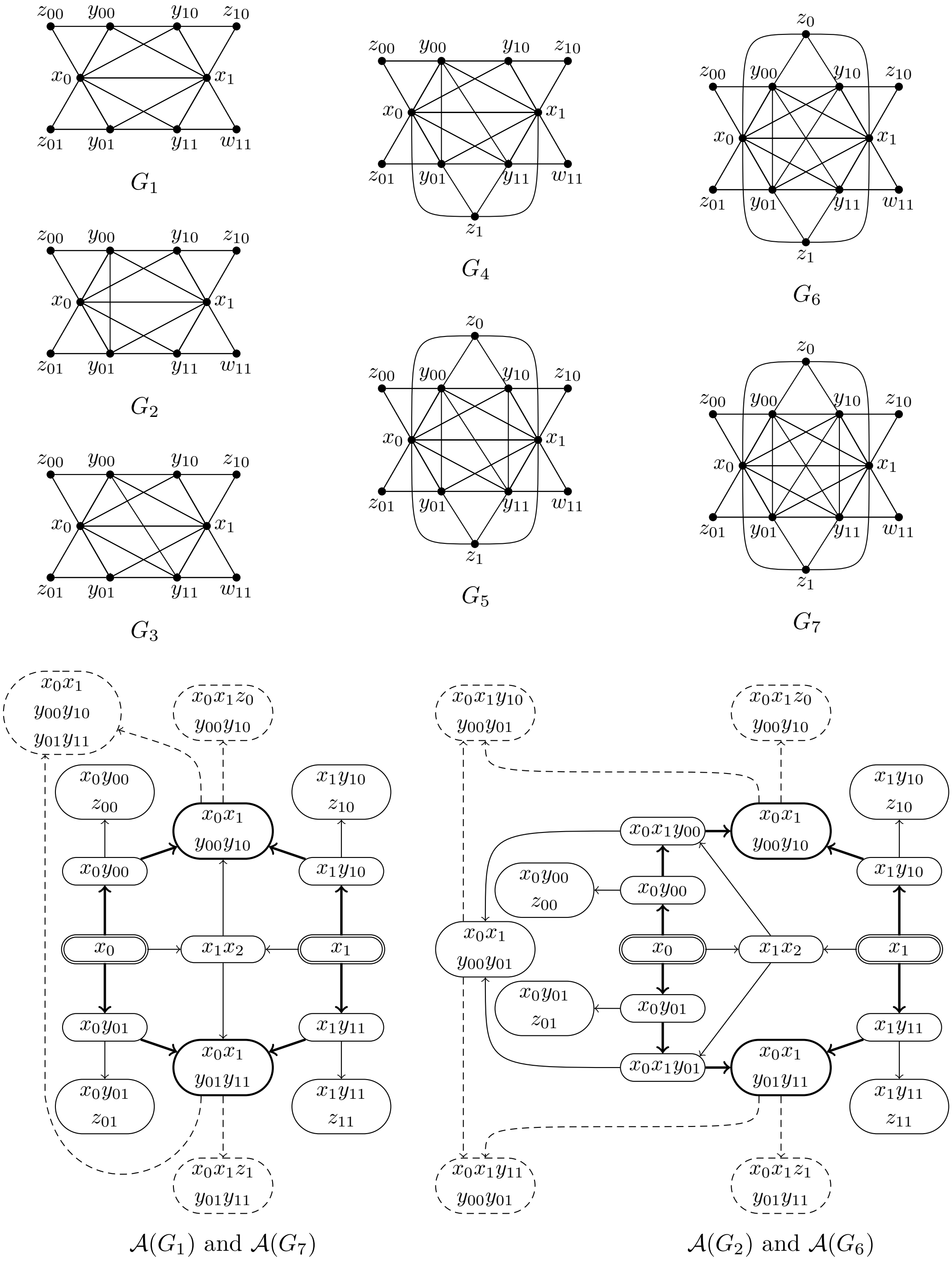

Figure 1 The graphs $G_1, \ldots, G_7$. The bottom left figure displays $\mathcal{A}(G_7)$ and, without dashed nodes and arcs, it shows $\mathcal{A}(G_1)$. Analogously, the bottom right figure presents $\mathcal{A}(G_6)$ or, without the dashed parts, $\mathcal{A}(G_2)$. Bold arcs emphasize the bad 2-cycle, where starters are double framed and terminals bold framed.

▶ Claim 1. For all $i, j \in \{0,1\}$ there is a path B_{ij} from S_i to T_j that does not contain a node X with $S_0 \cup S_1 \subseteq X \subseteq T_0 \cap T_1$, in particular B_{ij} does not contain T, such that B_{ij} contains a node P_{ij} with

(1) $S_i \subseteq P_{ij} \subseteq T_j$,
(2) $S_{1-i} \not\subseteq P_{ij}$,
(3) $P_{ij} \not\subseteq T$, and
(4) there exists a sink Q_{ij} in $\mathcal{A}(G)$ with $S_{1-i} \not\subseteq Q_{ij}$ that fulfills $P_{ij} = Q_{ij} \cap T_j$.

In the following we refer to the nodes P_{ij} by the P-nodes and we call Q_{ij} the Q-nodes. The pure existence of the Q-nodes does not directly imply that they are different:

▶ Claim 2. For all $i, j, i', j' \in \{0,1\}$ with $(i,j) \neq (i',j')$, the sinks Q_{ij} and $Q_{i'j'}$ differ.

For the pairwise intersection between the P-nodes, Claim 1 directly implies for all $i, j, j' \in \{0,1\}$ that $P_{ij} \not\subseteq P_{(1-i)j'}$. We can now infer the following two additional statements about the intersections between the P-nodes and the intersections between the Q-nodes:

▶ Claim 3. For all $i \in \{0,1\}$ it is true that $P_{i0} \cap P_{i1} = S_i$.

▶ Claim 4. For all $i, i' \in \{0,1\}$ it is true that $P_{0i} \cap P_{1i'} \subseteq T$ and $Q_{0i} \cap Q_{1i'} \subseteq T$.

We deduce that $P_{ij} \cap P_{i'j'} \subseteq T$ for all $i, j, i', j' \in \{0,1\}$ with $(i,j) \neq (i',j')$. Following the construction of the P-nodes, we also know for all $i, j \in \{0,1\}$ that the set $P'_{ij} = P_{ij} \setminus T$ is not empty.

Using the collected facts about the mentioned nodes on the bad 2-cycle, the next two claims start selecting vertices to construct one of the induced subgraphs $G_1, \ldots, G_7$:

▶ Claim 5. For all $i \in \{0,1\}$, the starter S_i contains a vertex u_i such that $u_i \notin Q_{(1-i)0} \cup Q_{(1-i)1}$.

▶ Claim 6. For all $i, j \in \{0,1\}$, there is a vertex $w_{ij} \in Q_{ij} \setminus P_{ij}$ such that

(1) for all $i', j' \in \{0,1\}$ it is true that $w_{ij} = w_{i'j'} \iff (i,j) = (i',j')$ and
(2) w_{ij} is neither adjacent to $u_{1-i}, w_{i(1-j)}, w_{(1-i)j}), w_{(1-i)(1-j)}$, nor to any vertex in $P'_{i(1-j)}$, in $P'_{(1-i)j}$ or in $P'_{(1-i)(1-j)}$.

Depending on the edges between the six central vertices of $G_1, \ldots, G_7$, there exist up to two additional vertices in $G_4, \ldots, G_7$. This dependency is also visible in the clique arrangement. Consider the sets $V_0 = P_{00} \cup P_{01}$, $V_1 = P_{10} \cup P_{11}$, $D_0 = P_{00} \cup P_{11}$ and $D_1 = P_{01} \cup P_{10}$ and moreover, for all $i, j \in \{0,1\}$ let $C_{ij} = V_i \cup D_j$. If one of the sets $C_{ij}, i, j \in \{0,1\}$ induces a clique in G, then it follows that T_0 or T_1 are proper subsets of maximal cliques in G:

▶ Claim 7. For all $i, j \in \{0,1\}$ and $k = (i + j + 1) \mod 2$, the node T_k is not a sink in $\mathcal{A}(G)$, if C_{ij} is a clique in G.

In such a case, if C_{ij} is a clique, we select an additional vertex from the sink that is reachable from T_k:

▶ Claim 8. For all $i, j \in \{0,1\}$ and $k = (i + j + 1) \mod 2$, if C_{ij} is a clique in G, then there is a sink T'_k which is reachable from T_k and contains a vertex $w_k \in T'_k \setminus (P_{0k} \cup P_{1k} \cup T_{1-k})$ such that

(1) w_k is not one of the vertices $w_{1-k}, w_{00}, w_{01}, w_{10}, w_{11}$,
(2) w_k is neither adjacent to $w_{1-k}, w_{0(1-k)}, w_{1(1-k)}$ nor to any vertex in $T_{1-k} \setminus T$, and
(3) w_k is adjacent to at most one vertex of w_{0k} and w_{1k}.

In the remainder of the proof we select the central vertices v_{ij} from P'_{ij} for all $i, j \in \{0,1\}$ to ultimately induce a forbidden subgraph. But before explaining how to select these four vertices, we briefly summarize the results gathered in the proof so far. By Claim 5, we know that there are vertices u_0, u_1 and, from the construction of the P-nodes in Claim 1, it follows that $\{u_0, u_1, v_{00}, v_{10}\}$ and $\{u_0, u_1, v_{01}, v_{11}\}$ are cliques in G, regardless of the choice of $v_{00}, v_{01}, v_{10}, v_{11}$. Moreover, by Claim 6, there exists an independent set $\{w_{00}, w_{01}, w_{10}, w_{11}\}$ in G such that for all $i, j \in \{0, 1\}$, the vertex w_{ij} is adjacent to u_i and v_{ij} but not to any of the vertices $u_{1-i}, v_{i(1-j)}, v_{(1-i)j}, v_{(1-i)(1-j)}$. Finally, Claim 8 states that certain circumstances imply the existence of two non-adjacent vertices w_0 and w_1 in G that are both adjacent to u_0 and u_1 and such that for all $k \in \{0, 1\}$ it is true that w_k is adjacent to v_{0k} and v_{1k} but not adjacent to $v_{0(1-k)}, v_{1(1-k)}, w_{0(1-k)}$ and $w_{0(1-k)}$. The claim leaves it open, if w_k can be adjacent to either w_{0k} or w_{1k} and, consequently, we cope with this problem during the following vertex selection. These facts are subsequently used without explicit mentioning.

Moreover, in the following vertex selection we write

$$G_i(x_0, x_1, y_{00}, y_{01}, y_{10}, y_{11}, z_{00}, z_{01}, z_{10}, z_{11}, [z_0, z_1])$$

to state that G contains an induced G_i for $i \in \{1, \ldots, 7\}$ on vertices x_0, x_1, y_{00}, y_{01}, y_{10}, y_{11}, z_{00}, z_{01}, z_{10}, z_{11}, optionally including z_0, z_1. Hence, both vertex sets x_0, x_1, y_{00}, y_{10} and x_0, x_1, y_{01}, y_{11} form a clique in G and every z_{ij} is exactly adjacent to x_i, y_{ij}. Depending on i the vertices z_0 and z_1 are present and z_i is exactly adjacent to x_0, x_1, y_{0i}, y_{1i} for $i \in \{0, 1\}$. The adjacency between $y_{00}, y_{01}, y_{10}, y_{11}$ depends on i, too.

To find suitable vertices for the forbidden induced subgraphs, we have to distinguish between three cases:

1. **Assume that at most one of the sets V_0, V_1, D_0, D_1 is a clique in G:** Because of symmetry we just have the following two subcases:
   a. **Assume that at most V_0 is a clique in G:** Because D_0, D_1 are not cliques, we can select vertices $v_{ij} \in P'_{ij}$ for all $i, j \in \{0, 1\}$ such that $v_{00}|v_{11}$ and $v_{01}|v_{10}$. At most one of the edges $v_{00}v_{01}$ or $v_{10}v_{11}$ is present in E because otherwise $v_{00}{-}v_{01}{-}v_{11}{-}v_{10}{-}v_{00}$ is an induced C_4 in G. If $v_{00}|v_{01}$ and $v_{10}|v_{11}$, then

   $$G_1(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}).$$

   Clearly, if V_0 is a clique, then $v_{00}{-}v_{01}$ and $v_{10}|v_{11}$ and then we have

   $$G_2(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}).$$

   b. **Assume that at most D_0 is a clique in G:** Analogously to the previous case, we can select $v_{ij} \in P'_{ij}$ for all $i, j \in \{0, 1\}$ such that $v_{00}|v_{01}$ and $v_{10}|v_{11}$ and again, either $v_{00}{-}v_{11}$ or $v_{01}{-}v_{10}$ as otherwise $v_{00}{-}v_{11}{-}v_{01}{-}v_{10}{-}v_{00}$ is an induced C_4 in G. The case of $v_{00}|v_{01}$ and $v_{10}|v_{11}$ yields an induced G_1 and has already been handled in the first case. If without loss of generality $v_{00}{-}v_{11}$, then

   $$G_3(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}).$$

2. **Assume that exactly two of the sets V_0, V_1, D_0, D_1 are cliques in G:** If V_0 and V_1 are cliques but not D_0 and D_1, then vertices $v_{ij} \in P'_{ij}$ exist for all $i, j \in \{0, 1\}$ such that $v_{00}|v_{11}$ and $v_{01}|v_{10}$ and consequently, $v_{00}{-}v_{01}{-}v_{11}{-}v_{10}{-}v_{00}$ is an induced C_4. Analogously, D_0 and D_1 being the cliques implies $v_{00}{-}v_{11}{-}v_{01}{-}v_{10}{-}v_{00}$ as an induced C_4. Because of this and symmetry, we have only one remaining case, namely V_0 and D_0 are the cliques and this implies that C_{00} is a clique.
   Next we show that there exist vertices $v_{ij} \in P'_{ij}$ for all $i, j \in \{0, 1\}$ such that $v_{01}|v_{10}$ and $v_{10}|v_{11}$. For that purpose, assume that every vertex in P'_{10}, that is adjacent to some vertex in $P'_{(1-k)1}$ for $k \in \{0, 1\}$, is also adjacent to all vertices in P'_{k1}. Then, as V_1 and D_1 are not cliques, there are vertices $x \neq y \in P'_{10}$ such that there is $x' \in P'_{01}$ and $y' \in P'_{11}$

with $x|x'$ and $y|y'$. By our assumption, it follows that $x{-}y'$ and $x'{-}y$ and hence, there is $x{-}y{-}x'{-}y'{-}x$, an induced C_4 in G. Consequently, the assumption was wrong and we can select the vertices such that $v_{01}|v_{10}$ and $v_{10}|v_{11}$.
Because C_{00} is a clique, it follows by Claim 8 that w_1 exists, and if w_1 is neither adjacent to w_{01} nor to w_{11}, then

$$G_4(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}, w_1).$$

Otherwise, if w_1 is adjacent to w_{10}, then we get

$$G_3(u_0, u_1, v_{00}, w_1, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}),$$

and if $w_1{-}w_{11}$, then

$$G_2(u_0, u_1, v_{00}, v_{01}, v_{10}, w_1, w_{00}, w_{01}, w_{10}, w_{11}).$$

3. **Assume that at least three of the sets V_0, V_1, D_0, D_1 are cliques in G:** In this case, we select any vertex $v_{ij} \in P'_{ij}$ for all $i, j \in \{0, 1\}$. As at least one of the sets $C_{00} = V_0 \cup D_0$ or $C_{11} = V_1 \cup D_1$ is a clique, it follows from Claim 8 that w_1 exists. Analogously, $C_{01} = V_0 \cup D_1$ or $C_{10} = V_1 \cup D_0$ is a clique and thus, w_0 exists.
Assume first that w_0 and w_1 are completely disjoint from $w_{00}, w_{01}, w_{10}, w_{11}$. By symmetry we just have to consider the cases of (i) $v_{01}|v_{10}$, which leads to

$$G_5(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}, w_0, w_1),$$

(ii) $v_{10}|v_{11}$, which yields

$$G_6(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}, w_0, w_1),$$

and (iii) $v_{00}, v_{01}, v_{10}, v_{11}$ are a clique where

$$G_7(u_0, u_1, v_{00}, v_{01}, v_{10}, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}, w_0, w_1).$$

Finally, we have to check all the cases where w_0 or w_1 are adjacent to one of the vertices $w_{00}, w_{01}, w_{10}, w_{11}$. Because of symmetry we can simply assume that w_0 is adjacent to w_{10}.
Assume that w_1 is neither adjacent to w_{01} nor to w_{11}. Then (iv) $v_{00}{-}v_{01}$ and $v_{00}|v_{11}$ implies

$$G_2(u_0, u_1, v_{00}, v_{01}, w_0, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}),$$

(v) $v_{00}|v_{01}$ and $v_{00}{-}v_{11}$ yields

$$G_3(u_0, u_1, v_{00}, v_{01}, w_0, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}),$$

and (vi) $v_{00}{-}v_{01}$ and $v_{00}{-}v_{11}$ gives

$$G_4(u_0, u_1, v_{00}, v_{01}, w_0, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}, w_1).$$

If $w_1{-}w_{11}$, then (vii) $v_{00}|v_{01}$ implies

$$G_1(u_0, u_1, v_{00}, v_{01}, w_0, w_1, w_{00}, w_{01}, w_{10}, w_{11}),$$

and (viii) $v_{00}{-}v_{01}$ yields

$$G_2(u_0, u_1, v_{00}, v_{01}, w_0, w_1, w_{00}, w_{01}, w_{10}, w_{11}).$$

Moreover, if $w_1{-}w_{01}$, then (ix) $v_{00}|v_{11}$ results in

$$G_1(u_0, u_1, v_{00}, w_1, w_0, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}),$$

and (x) $v_{00}{-}v_{11}$ provides

$$G_3(u_0, u_1, v_{00}, w_1, w_0, v_{11}, w_{00}, w_{01}, w_{10}, w_{11}).$$

The following shows the converse direction, that is, if G contains one of $G_1, \dots, G_7$ as an induced subgraph, then $\mathcal{A}(G)$ has a bad 2-cycle.

We basically use Lemma 7. The clique arrangement of all graphs $G_1, \dots, G_7$ contains a bad 2-cycle with starters $S_0 = \{x_0\}$, $S_1 = \{x_1\}$ and terminals $T_0 = \{x_0, x_1, y_{00}, y_{10}\}$, $T_1 = \{x_0, x_1, y_{01}, y_{11}\}$. Moreover, there are nodes $P_{ij} = \{x_i, y_{ij}\}$, $Q_{ij} = \{x_i, y_{ij}, z_{ij}\}$ such that $S_i \to \dots \to P_{ij} \to \dots \to T_j$ and $P_{ij} \to \dots \to Q_{ij}$ for all $i, j \in \{0, 1\}$. If G contains an induced subgraph $G_1, \dots, G_7$, then there is a function ϕ, that maps these nodes to some nodes of the clique arrangement $\mathcal{A}(G)$ such that $\phi(S_i) \to \dots \to \phi(P_{ij}) \to \dots \to \phi(T_j)$ and $\phi(P_{ij}) \to \dots \to \phi(Q_{ij})$ for all $i, j \in \{0, 1\}$.

Assume that at least one of these four paths in $\mathcal{A}(G)$, say $\phi(S_0) \to \dots \to \phi(T_0)$, contains a node X with $\phi(S_0) \cup \phi(S_1) \subseteq X \subseteq \phi(T_0) \cap \phi(T_1)$. If X is situated on the subpath $\phi(S_0) \to \dots \to \phi(P_{00})$, then it follows that $X \subset Q_{00}$ and, hence, $x_1{-}z_{00}$, a contradiction.

Hence, X is on the subpath $\phi(P_{00}) \to \dots \to \phi(T_0)$. Here, $\phi(P_{00})$ is a subset of $X \subseteq \phi(T_0) \cap \phi(T_1)$ and thus, also a subset of $\phi(T_1)$. This means that $y_{00} \in \phi(T_1)$, which implies $y_{00}{-}y_{01}$ and $y_{00}{-}y_{11}$. Consequently, we are in the case were the induced subgraph in G is one of $G_4, \dots, G_7$. The clique arrangement of all these graphs contains a sink $T_1' = \{x_0, x_1, y_{01}, y_{11}, z_1\}$ that is reached from T_1. In $\mathcal{A}(G)$, we have $\phi(T_1) \to \dots \to \phi(T_1')$, thus, $\phi(P_{00}) \subset \phi(T_1')$, which finally means that $y_{00}{-}z_1$, a contradiction.

Hence, X does not exist and $\mathcal{A}(G)$ contains a bad 2-cycle with starters $\phi(S_0), \phi(S_1)$ and terminals $\phi(T_0), \phi(T_1)$. ◀

The main theorem, presented in this section, and Theorem 9 lead to the following conclusion:

▶ **Corollary 11.** *Let $G = (V, E)$ be a graph in $\mathcal{L}$ with clique arrangement $\mathcal{A}(G) = (\mathcal{X}, \mathcal{E})$. The graph $\mathcal{A}(G)$ does not contain a bad 2-cycle.*

Hence, leaf powers fit naturally into the hierarchy of chordal graphs, right between strongly chordal graphs, which have clique arrangements without bad k-cycles for $k \geq 3$, and ptolemaic graphs, whose clique arrangements are entirely free of cycles.

5 Conclusion and Future Directions

In this paper, we were able to indicate that leaf powers $\mathcal{L}$ are a natural subclass of strongly chordal graphs by showing that their clique arrangements are not only free of bad k-cycles for $k \geq 3$ but also for $k = 2$. Moreover, we proved that the clique arrangement of a strongly chordal graph G comprises a bad 2-cycle, if and only if G contains at least one of $G_1, \dots, G_7$ as an induced subgraph. This means that, beside the forbidden induced subgraphs of strongly chordal graphs, that is, the family of suns, this finite number of graphs describe a cyclic composition of cliques that is not realizable by a k-leaf root for any $k \geq 2$.

It remains for future work to find a complete characterization of $\mathcal{L}$ in terms of forbidden subgraphs. During our deep analysis of leaf powers we have considered a huge variety of graphs and their clique arrangements. We have not a single example of a graph G that has a clique arrangement $\mathcal{A}(G)$ without bad k-cycles for $k \geq 2$, where a corresponding leaf root of G is unknown. Therefore, we conjecture that a strongly chordal graph G has a k-leaf root for some $k \geq 2$, if and only if $\mathcal{A}(G)$ is free of bad 2-cycles. If this was true, a polynomial time recognition algorithm is straight found by the efficient recognition of strongly chordal graphs and the possibility to check for a finite number of induced subgraphs in polynomial time.

Answering this question implies the challenge of constructing leaf roots from bad-cycle-free clique arrangements. This turns out to be sophisticated, especially if the clique arrangement has 2-cycles that are not bad.

6 Technical Proofs

The Proof of Lemma 6

Proof. First of all, we know that $Y \not\subseteq Z$ and $Z \not\subseteq Y$ as otherwise X, Y, Z are not distinct nodes. Let $\mathcal{X}_Y$ be the set of sinks reachable from Y and $\mathcal{X}_Z$ be the set of sinks reachable from Z. If Y and Z are sinks themselves then $\mathcal{X}_Y = \{Y\}$ and $\mathcal{X}_Z = \{Z\}$ and trivially we set $C_1 = Y$ and $C_2 = Z$.

If just one of the nodes, Y or Z, is a sink, without loss of generality, $\mathcal{X}_Y = \{Y\}$, then Lemma 2 implies that there are sinks $A, B \in \mathcal{X}_Z$ such that $A \neq B$ and $Z = A \cap B$. Clearly, $A \neq Y$ and $B \neq Y$ as otherwise $Z \subseteq Y$. By Lemma 4, $A' = A \cap Y$ and $B' = B \cap Y$ are nodes in $\mathcal{X}$ because both contain X. It is straight forward that $A' \neq B'$ and moreover, $A' \neq Z$ and $B' \neq Z$ as otherwise $Z \subseteq Y$. We have a 3-cycle with starters A', B', Z and terminals A, B, Y. By Theorem 5, the cycle is not bad and as $Z \not\subseteq Y$ it follows that $A' \subseteq B$ or $B' \subseteq A$. If $A' \subseteq B$, then $A' \subseteq A$ and $Z = A \cap B$ imply that $A' \subseteq Z$. From $A' \subseteq Y$ and $X = Y \cap Z$ we obtain $A' \subseteq X$ and, as $X \subseteq A'$, we have $X = A'$. Hence, in this case $X = A \cap Z$ and we set $C_1 = A$ and $C_2 = Y$. In an analogous fashion we get $X = B \cap Y$, if $B' \subseteq A$ and then we set $C_1 = B$ and $C_2 = Y$.

If none of the nodes Y, Z is a sink, then Lemma 2 implies that there are sinks $A \neq B \in \mathcal{X}_Y$ and $C \neq D \in \mathcal{X}_Z$ such that $Y = A \cap B$ and $Z = C \cap D$. If Y reaches one of the sinks C and D or Z reaches one of the sinks A and B, without loss of generality, Z reaches B, then $Z \subseteq B \cap C$. Notice that Z cannot reach A in this case, as otherwise $Z \subseteq Y$. By Lemma 3, Z is exactly the intersection of all cliques in $\mathcal{X}_Z$, which includes B. By definition, $Z = C \cap D$ and, hence, we also have $Z = B \cap C \cap D$. As $B \cap C \cap D \subseteq B \cap C$, we conclude that $Z = B \cap C$. The node $A' = A \cap C$ exists by Lemma 4 because it contains X as a subset. Clearly, if $Y = A'$ or $Z = A'$ then $A' = A \cap B \cap C$. Otherwise, we have a 3-cycle with starters Y, Z, A' and terminals A, B, C. By Theorem 5, the cycle is not bad and as $Y \subseteq C$ or $Z \subseteq A$ implies $Y \subseteq Z$ or $Z \subseteq Y$, we have $A' \subseteq A \cap B \cap C$, again. Because $Y = A \cap B$ and $Z = B \cap C$, this implies that $A' \subseteq Y$ and $A' \subseteq Z$, and because $X = Y \cap Z$, we have $A' \subseteq X$ and thus, $X = A'$. Hence, $X = A \cap C$ and we set $C_1 = A$ and $C_2 = C$.

If neither Y reaches the sinks C or D nor Z reaches the sinks A or B, then we consider the nodes $A' = A \cap C$ and $B' = B \cap D$, which exist by Lemma 4, as they all contain X as a subset. Clearly, if A' or B' coincides with Y, then Y reaches one of the sinks C or D and analogously, A' and B' are not Z. Moreover, $A' = B'$ implies that A' is a subset of Y and Z and, by that, a subset of X, which means that $X = A' = A \cap C$ and that $C_1 = A$ and $C_2 = C$. Otherwise, as $Y \neq Z$, we get a 4-cycle with starters A', B', Y, Z and terminals A, B, C, D. Theorem 5 states that the cycle is not bad and as $Y \not\subseteq C \cup D$ and $Z \not\subseteq A \cup B$, it must be true, without loss of generality, that $A' \subseteq B$ and hence, we get that $A' \subseteq Y$ and that there is a 3-cycle with starters A', B', Z and terminals B, C, D. This cycle is not bad, either, and hence, either $A' \subseteq D$ or $B' \subseteq C$. If $A' \subseteq D$, then $A' \subseteq Z$, which implies $A' \subseteq X$ and thus, $X = A' = A \cap C$ and then $C_1 = A$ and $C_2 = C$.

The case $B' \subseteq C$ implies that $B' \subseteq Z$ and then we consider the node $C' = A \cap D$, which exists because it contains X as a subset. If $B' = C'$, B' is contained in $X = Y \cap Z$, which means that $X = B' = B \cap D$ and that $C_1 = B$ and $C_2 = D$. Otherwise, we have a 3-cycle with starters B', C', Y and terminals A, B, D. Because the cycle is not bad, it is true that $B' \subseteq A$ or $C' \subseteq B$. If $B' \subseteq A$, then $B' \subseteq Y$, which implies $B' \subseteq X$ and thus, $X = B' = B \cap D$ and then $C_1 = B$ and $C_2 = D$. In the other case, C' is contained in A and B and thus, in Y. Moreover, C' is a subset of $B \cap D$, thus, a subset of B', and, consequently, contained in $C \cap D$, which means that C' is also in Z. Together, this implies that $X = C' = A \cap D$ and that $C_1 = A$ and $C_2 = D$. ◀

The Proof of Lemma 7

Proof. We first fix a function ϕ. For that purpose notice that for every maximal clique C of H, there is at least one maximal clique C' in G such that $C \subseteq C'$ and we define $\phi(C) = C'$. Notice that $C = \phi(X) \cap V$. Moreover, for every node $X \in \mathcal{X}$ that is not a maximal clique, there is the subset $C_1, \ldots, C_k$ of maximal cliques in H that are reached in $\mathcal{A}(H)$ by a directed path from X. As $X = C_1 \cap \ldots \cap C_k$, we define $\phi(X) = X'$ for the node $X' \in \mathcal{X}'$ that fulfills $X' = \phi(C_1) \cap \ldots \cap \phi(C_k)$.

The proof is completed by showing the two declared properties for all $X, Y \in \mathcal{X}$:

1. Since ϕ is a function, $X = Y$ implies $\phi(X) = \phi(Y)$. Conversely, if $\phi(X) = \phi(Y)$ but $X \neq Y$, then there are non-adjacent vertices $x \in X \setminus Y$ and $y \in Y \setminus X$, which are, by definition, both in $\phi(X)$, a contradiction.
2. By definition, there is a directed path from $\phi(X)$ to $\phi(Y)$ in $\mathcal{A}(G')$, if $\phi(X) \subseteq \phi(Y)$. As $X = \phi(X) \cap V$ and $Y = \phi(Y) \cap V$ this implies $X \subseteq Y$, which, by definition, means that there is a directed path from X to Y in $\mathcal{A}(G)$.
   Conversely, let there be a directed path from X to Y in $\mathcal{A}(G)$, thus, let $X \subseteq Y$. This means that in $\mathcal{A}(G)$ the set $C_1, \ldots, C_k$ of maximal cliques reached from Y is a subset of the maximal cliques $C_1, \ldots, C_\ell$ reached from X, hence, $k \leq \ell$. Consequently, $\phi(C_1) \cap \ldots \cap \phi(C_\ell) = \phi(X) \subseteq \phi(Y) = \phi(C_1) \cap \ldots \cap \phi(C_k)$, which implies that there is a directed path from $\phi(X)$ to $\phi(Y)$ in $\mathcal{A}(G')$.

◀

The Proof of Theorem 9

Proof. The proof works basically the same as in [11]. Assume that at least one of the graphs $G_1, \ldots, G_7$ is a k-leaf power of a tree T for some $k \geq 2$ and that $x'_0, x'_1, y'_{00}, y'_{01}, y'_{10}, y'_{11}$ are the parent nodes of the leaves $x_0, x_1, y_{00}, y_{01}, y_{10}, y_{11}$ in T.

Consider for all $i, j \in \{0, 1\}$ the path $P_{ij} = x'_i - \ldots - y'_{ij}$ in T as well as, for all $i \in \{0, 1\}$, the path $P_i = y'_{0i} - \ldots - y'_{1i}$ in T. From Lemma 8 we get that $P_{00} \cap P_{10} = \emptyset$ and $P_{00} \cap P_{11} = \emptyset$. Similarly, Lemma 8 implies that $P_{01} \cap P_{10} = \emptyset$ and $P_{01} \cap P_{11} = \emptyset$. This means that the subtree T_0 of T given by the union $P_{00} \cup P_{01}$ is disjoint from the subtree T_1 of T given by $P_{10} \cup P_{11}$.

As T is a tree, there is a node z situated on the path connecting the subtrees T_0 and T_1 such that z is on every path $x - \ldots - y$ in T that connects a node x from T_0 and a node y from T_1. In particular, that also means that z is on P_0, if $x = y'_{00}$ and $y = y'_{10}$, and that z

is on P_1, if $x = y'_{01}$ and $y = y'_{11}$. Hence, $P_0 \cap P_1 \neq \emptyset$, as both paths contain z, which is a contradiction to Lemma 8. ◀

The Proofs of Claims in Theorem 10

The claims proved in the following are stated in a general and simple fashion, and they often use indices $i, j \in \{0, 1\}$ for the occurring nodes. However, because the bad 2-cycle is symmetric, the proofs always show the individual statements just for the case $i = j = 0$ without explicit indication.

The Proof of Claim 1

Proof. As mentioned, we show the claim only for $i = j = 0$.

We start by choosing an arbitrary path B_{00} from S_0 to T_0 that does not contain a node X with $S_0 \cup S_1 \subseteq X \subseteq T_0 \cap T_1$, which exists by the definition of bad 2-cycles. Obviously, this implies that the node T is not on B_{00}.

Firstly, there are nodes P, P' on the path $B_{00} = S_0 \to \ldots \to P \to P' \to \ldots \to T_0$ that are joined by an arc $P \to P'$ such that $P \subseteq T$ and $P' \not\subseteq T$ and $P' \neq T_0$, hence, on B_{00}, the node P is the last exit to T. Clearly, we have $S_0 \subseteq T$ and thus, if such arc does not exist, then every node on the path, except T_0 itself, would be a subset of T. Because T is not on $B_{00} = S_0 \to \ldots \to Q \to T_0$, even the predecessor Q of T_0 reaches T by a directed path. Hence, as $T \subset T_0$, there is a directed path $Q \to \ldots \to T \to \ldots \to T_0$ and, consequently, the arc $Q \to T_0$ is transitive, a contradiction.

Next we show that $S_1 \subseteq P'$ implies also that $S_1 \subseteq P$. This can be seen by the use of the intersection node $X = P' \cap T$, which entirely contains S_1 because $S_1 \subset P'$ and $S_1 \subseteq T$. As $P' \not\subseteq T$ and $X \subseteq T$, it follows that X is not equal to the node P'. Moreover, since $P \subseteq P'$ and $P \subseteq T$, it follows that $P \subseteq X$ and hence, there is a path $P \to \ldots \to X \to \ldots \to T$. But X cannot be a node on that path, unless $X = P$, because otherwise $P \to P'$ would be a transitive arc. But $X = P$ implies that $B_{00} = S_0 \to \ldots \to X = P \to \ldots \to T_0$ passes a node that fulfills $S_0 \cup S_1 \subseteq X = P \subseteq T$, which is a contradiction to the selection of the bad 2-cycle. Hence, $S_1 \not\subseteq P'$ must be true.

However, P' is not necessarily the node P_{00} we are looking for. Particularly, it may happen that no sink Q of $\mathcal{A}(G)$ fulfills $Q \cap T_0 = P'$. For that reason, let $Q_1, \ldots, Q_r$ be the sinks reachable from P' by directed paths and let $P'_1 = Q_1 \cap T_0, \ldots, P'_r = Q_r \cap T_0$. Because $P' = P'_1 \cap \ldots \cap P'_r$, Lemma 3 implies that, if $S_1 \subseteq P'_i$ for all $i \in \{1, \ldots, r\}$, then $S_1 \subseteq P'$. Hence, we can select $i \in \{1, \ldots, r\}$ such that $S_1 \not\subseteq P'_i$ and we set $P_{00} = P'_i$ and $Q_{00} = Q_i$.

Of course, it may happen that P_{00} is not on the path B_{00}, but now we have a new path $B' = S_0 \to \ldots \to P' \to \ldots \to P_{00} \to \ldots \to T_0$. We use B' as a replacement for B_{00}, because it is easy to see that it does not contain a node X with $S_0 \cup S_1 \subseteq X \subseteq T$, too. If such a node X was on the subpath $S_0 \to \ldots \to P_{00}$, then $S_1 \subset P_{00}$, and, if it was on the subpath $P_{00} \to \ldots \to T_0$, then $P_{00} \subset T$, which both contradicts the construction of P_{00}.

Finally, as $P_{00} = Q_{00} \cap T_0$, it follows that $S_1 \not\subseteq Q_{00}$, as otherwise $S_1 \subseteq T_0$ implies that $S_1 \subseteq P_{00}$, too. ◀

The Proof of Claim 2

Proof. The case $Q_{00} = Q_{1j'}$ is impossible for all $j' \in \{0, 1\}$, because then $S_1 \subseteq Q_{1j'}$ implies $S_1 \subseteq Q_{00}$, which is forbidden by Claim 1. If we assume that $Q_{00} = Q_{01}$, then we get a 3-cycle with starters P_{00}, P_{01}, S_1 and terminals T_0, T_1, Q_{00}. Certainly, P_{00} is not contained in T_1 as otherwise $P_{00} \subseteq T_0$ implies $P_{00} \subseteq T$, which is forbidden by Claim 1. Similarly, we get that $P_{01} \not\subseteq T_0$. That $S_1 \not\subseteq Q_{00}$ is a direct consequence of Claim 1. Hence, the 3-cycle is bad, a contradiction to Theorem 5. ◀

The Proof of Claim 3

Proof. Let S be the node representing the intersection $P_{00} \cap P_{01}$, which exists by Lemma 4 as $S_0 \subseteq S$. If we assume that $S_0 \neq S$, then we have two paths $B'_{00} = S \rightarrow \ldots \rightarrow P_{00} \rightarrow \ldots \rightarrow T_0$ and $B'_{01} = S \rightarrow \ldots \rightarrow P_{01} \rightarrow \ldots \rightarrow T_1$ and we obtain a 2-cycle with starters S, S_1 and terminals T_0, T_1. We show that this cycle is bad by arguing that none of B'_{00}, B'_{01}, B_{10}, and B_{11} contains a node X that fulfills $S \cup S_1 \subseteq X \subseteq T$. Clearly, the existence of X on one of B_{10}, B_{11}, $P_{00} \rightarrow \ldots \rightarrow T_0$, and $P_{01} \rightarrow \ldots \rightarrow T_1$ contradicts to the choice of B_{00}, B_{01}, B_{10}, and B_{11}.

If X was on the path $S \rightarrow \ldots \rightarrow P_{00}$, then we would get $S_1 \subset P_{00}$, which has been eliminated in Claim 1. Similarly, the path $S \rightarrow \ldots \rightarrow P_{01}$ does not contain X, and hence, the new 2-cycle is bad. But this contradicts to the choice of the primal bad 2-cycle, because, by $S_0 \subset S$, we obtain $|S_0| + |S_1| < |S| + |S_1|$. ◀

The Proof of Claim 4

Proof. If $Q_{00} \cap Q_{1i'} = \emptyset$, then clearly $Q_{00} \cap Q_{1i'} \subseteq T$. Otherwise, let Q be the intersection node for $Q_{00} \cap Q_{1i'}$, which exists by Lemma 4. We get a 3-cycle with starters S_0, S_1, Q and terminals Q_{00}, Q_{10}, T. If $Q \not\subseteq T$ then the 3-cycle is bad, because $S_0 \not\subseteq Q_{10}$ and $S_1 \not\subseteq Q_{00}$ by Claim 1. This contradicts Theorem 5.

Clearly, we have $P_{00} \cap P_{10} \subseteq Q_{00} \cap Q_{10} \subseteq T$ and $P_{00} \cap P_{11} \subseteq Q_{00} \cap Q_{11} \subseteq T$. ◀

The Proof of Claim 5

Proof. If u_0 does not exist, then S_0 is a subset of $Q_{10} \cup Q_{11}$. As S_0 cannot be entirely contained in a single set, Q_{10} or Q_{11}, we find two distinct nodes $X = S_0 \cap Q_{10}$ and $Y = S_0 \cap Q_{11}$ by Lemma 4. The same lemma reveals the existence of a node $Z = Q_{10} \cap Q_{11}$, because Z contains at least as S_1.

We get a 3-cycle with starters X, Y, Z and terminals S_0, Q_{10}, Q_{11}. We show that this cycle is bad by the help of $S_0 = (S_0 \cap Q_{10}) \cup (S_0 \cap Q_{11})$. Firstly, X cannot be a subset of Q_{11}, because otherwise Q_{11}, which already contains $Y = S_0 \cap Q_{11}$, contains also $S_0 \cap Q_{10}$, which would imply that $S_0 \subseteq Q_{11}$. Secondly and similarly, Y cannot be a subset of Q_{10}, because otherwise $S_0 \subseteq Q_{10}$. Finally, by $S_1 \subseteq Z$, it follows that $Z \not\subseteq S_0$. This bad 3-cycle contradicts Theorem 5, hence, the node u_0 exists. ◀

The Proof of Claim 6

Proof. As the Q-nodes represent distinct maximal cliques in G, Lemma 1 allows to select vertices $x \in Q_{00} \setminus Q_{10}$ and $y \in Q_{00} \setminus Q_{11}$ such that x is not adjacent to any vertex in $Q_{10} \setminus Q_{00}$ and y is not adjacent to any vertex in $Q_{11} \setminus Q_{00}$.

We show that at least one of x and y is not adjacent to all vertices in $(Q_{10} \cup Q_{11}) \setminus Q_{00}$. If $x = y$ we are done. Otherwise, assume that x has a neighbor $x' \in Q_{11} \setminus Q_{00}$ and that y has a neighbor $y' \in Q_{10} \setminus Q_{00}$. As $x|y'$ and $y|x'$ and $x{-}y$, it follows that $x'|y'$ as otherwise G contains $x{-}y{-}y'{-}x'{-}x$ as an induced C_4.

Now consider the vertex u_1, which is at the same time in $Q_{10} \setminus Q_{00}$ and in $Q_{11} \setminus Q_{00}$ according to Claim 5. Hence, according to the choice of x and y, we have $x|u_1$ and $y|u_1$. Moreover, as u_1 and x' are both in $Q_{11} \setminus Q_{00}$ and because u_1 and y' are both in $Q_{10} \setminus Q_{00}$, we get $x'{-}u_1$ and $y'{-}u_1$, which implies that G has $x{-}y{-}y'{-}u_1{-}x'{-}x$ as an induced C_5. This means, our assumption was wrong and we let w_{00} be a vertex in $\{x, y\}$ that has no neighbors in $Q_{10} \setminus Q_{00}$ and in $Q_{11} \setminus Q_{00}$.

First of all, we have already seen that w_{00} is not adjacent to u_1. Therefore, w_{00} is in $Q_{00} \setminus P_{00}$, as every vertex in P_{00} is adjacent to u_1 by $P_{00} \cup \{u_1\} \subseteq T_0$. Moreover, this means that $w_{00} \neq w_{10}$ and $w_{00} \neq w_{11}$ as both, w_{10} and w_{11}, are adjacent to u_1, which follows from $\{w_{10}, u_1\} \subseteq Q_{10}$ and $\{w_{11}, u_1\} \subseteq Q_{11}$. As $w_{10} \in Q_{10} \setminus Q_{00}$ and $w_{11} \in Q_{11} \setminus Q_{00}$, it follows also that w_{00} is not adjacent to w_{10} and w_{11}.

From Claim 4 we know that $Q_{00} \cap Q_{10}$ and $Q_{00} \cap Q_{11}$ are subsets of T. Because $P_{10} = Q_{10} \cap T_0$ and $P_{11} = Q_{11} \cap T_1$, this means also that $Q_{00} \cap P_{10} = Q_{00} \cap Q_{10} \cap T_0 \subseteq T$ and $Q_{00} \cap P_{11} = Q_{00} \cap Q_{11} \cap T_1 \subseteq T$. Hence, from $P'_{10} = P_{10} \setminus T$ and $P'_{11} = P_{11} \setminus T$ it follows already that w_{00} is not adjacent to vertices in P'_{10} or in P'_{11}. It remains to show that $w_{00} \neq w_{01}$, that $w_{00}|w_{01}$ and that w_{00} is not adjacent to any vertex in P'_{01}.

If $W = N(w_{00}) \cap P'_{01}$ is an empty set, then w_{00} is not adjacent to vertices in P'_{01}. Otherwise, if W is not empty, assume that there are vertices $x \in W$ and $y \in P'_{00}$ such that $x|y$. Recall that w_{00} is adjacent to all vertices in P'_{00} including y and not adjacent to u_1. Unlike w_{00}, the vertices x and y are adjacent to u_1, because $\{u_1, y\} \subseteq T_0$ and $\{u_1, x\} \subseteq T_1$. This means that G has $w_{00}{-}x{-}u_1{-}y{-}w_{00}$ as an induced C_4, a contradiction.

Consequently, $W \cup P'_{00}$ is a clique in G and there is a maximal clique of G represented by a sink T' of $\mathcal{A}(G)$ such that $(W \cup P'_{00} \cup \{u_1\}) \subseteq T'$. Because $T = T_0 \cap T_1$, it follows from Lemma 6 that there are distinct sinks T'_0, reachable from T_0, and T'_1, reachable from T_1, such that $T = T'_0 \cap T'_1$. Clearly, $T'_0 \neq T'_1$ and because $u_1 \in T'_0$ and $u_1 \in T'_1$, we have $T' \neq T'_0$ and $T' \neq T'_1$. We let $X = T' \cap T'_0$ and $Y = T' \cap T'_1$ and obtain a 3-cycle with starters X, Y, T and terminals T', T'_0, T'_1. By construction, there are vertices $x \in P'_{00}$ and $y \in P'_{01}$ that are also contained in T'. Because $P'_{00} \subseteq T'_0$ and $P'_{01} \subseteq T'_1$, it follows that $x \in T'_0$ and $y \in T'_1$ and this in turn means that $x \in X$ and $y \in Y$. Consequently, $X \not\subseteq T'_1$, as otherwise $x \in T'_0 \cap T'_1 = T$, which is a contradiction to the construction $P'_{00} = P_{00} \setminus T$. Analogously, we have $Y \not\subseteq T'_0$. Finally, $T \subseteq T'$ implies that all vertices in T, including u_1, are adjacent to w_{00}, which is impossible. This means that the 3-cycle is bad and, hence, W has to be empty and w_{00} is not adjacent to any vertex in P'_{00}.

As w_{01} is adjacent to all vertices in P'_{01}, it follows that $w_{00} \neq w_{01}$. Assume that w_{00} and w_{01} are adjacent and select any vertices $x \in P'_{00}$ and $y \in P'_{01}$. If $x{-}y$, then we obtain $w_{00}{-}x{-}y{-}w_{01}{-}w_{00}$ as an induced C_4 in G, and otherwise, we get $w_{00}{-}x{-}u_1{-}y{-}w_{01}{-}w_{00}$ as an induced C_5 in G. Hence, $w_{00}{-}w_{01}$ cannot be true. ◀

The Proof of Claim 7

Proof. Let R_{00} be a sink of $\mathcal{A}(G)$ that represents one of the maximal cliques of G with $C_{00} \subseteq R_{00}$. Consider the node T'_1 that results from the intersection $R_{00} \cap T_1$, which exists by Lemma 4, as both, R_{00} and T_1, contain $P_{01} \cup P_{11}$. If $T'_1 \subset T_1$, we get a new 2-cycle with starters S_0, S_1 and terminals T_0, T'_1. Assume that there is a node X with $S_0 \cup S_1 \subseteq X \subseteq T_0 \cap T'_1$ on one of B_{00}, B_{10}, $B'_{01} = S_0 \to \ldots \to P_{01} \to \ldots \to T'_1$, and $B'_{11} = S_1 \to \ldots \to P_{11} \to \ldots \to T'_1$. If X is neither on the path $P_{01} \to \ldots \to T'_1$ nor on the path $P_{11} \to \ldots \to T'_1$, then X is located on one of the paths B_{00}, B_{01}, B_{10}, and B_{11}, a contradiction to the choice of these paths. If $P_{01} \to \ldots \to X \to \ldots \to T'_1$, then it follows that $P_{01} \subseteq X \subseteq T_0 \cap T'_1 \subset T$, which is impossible due to the construction of P_{01}. The same holds if X is on the path $P_{11} \to \ldots \to T'_1$. Hence, the new 2-cycle is bad. This is a contradiction to the choice of the primal cycle, because, by $T'_1 \subset T_1$, we have $|T_0| + |T'_1| < |T_0| + |T_1|$. Consequently, T'_1 equals T_1, and thus, $T_1 \subset R_{00}$, which means that T_1 is not a sink in $\mathcal{A}(G)$. ◀

The Proof of Claim 8

Proof. Lemma 6 implies the existence of two distinct sinks T_0', reachable from T_0, and T_1', reachable from T_1, such that $T_0' \cap T_1' = t$. Because C_{00} is a clique, Claim 7 implies that T_1 is not a sink, hence, $T_1' \neq T_1$. From Lemma 1 it follows that $T_1' \setminus T_0'$ contains at least one vertex w_1 that is not adjacent to any vertex in $T_0' \setminus T_1'$.

As C_{00} is a clique, all vertices in P_{01} and in P_{11} are adjacent to all vertices in P_{00}. Consequently, w_1 is not in $P_{01} \cup P_{11}$. If w_0 exists, then it can neither be the same vertex as w_1 nor be adjacent to w_1, because $w_0 \in T_0' \setminus T_1'$. Clearly, by Claim 6, w_1 is not one of the vertices w_{00}, w_{10}, because, unlike w_1, they are adjacent to vertices in $T_0 \setminus T$. Moreover, Claim 6 implies that w_1 is not w_{01}, because, unlike w_{01}, the vertex w_1 is adjacent to all vertices in P_{11}'. Similarly w_1 is not w_{11}.

It remains to show that w_1 is not adjacent to w_{00} and w_{10} and adjacent to at most one vertex w_{10} or w_{11}. If $w_1 - w_{00}$, then we can select any vertex $x \in P_{01}'$ and get $w_1 - u_1 - x - w_{00} - w_1$ as an induced C_4 in G. Analogously, if $w_1 - w_{10}$, then we select $x \in P_{10}'$ to find $w_1 - u_0 - x - w_{10} - w_1$ as induced C_4 in G. Finally, if w_1 is adjacent to w_{01} and w_{11}, then we select $x \in P_{00}'$ and get an induced 3-sun in G with central clique u_0, u_1, w_1 and independent set x, w_{11}, w_{01}. ◀

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