Simple characters and coefficient systems on the building

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Abstract. Let F be a non-archimedean local field and G be the group $\mathrm{GL}(N, F)$ for some integer $N \geq 2$. Let π be a smooth complex representation of G lying in the Bernstein block $\mathcal{B}(\pi)$ of some simple type in the sense of Bushnell and Kutzko. Refining the approach of the second author and U. Stuhler, we canonically attach to π a subset X_π of the Bruhat-Tits building X of G, as well as a G-equivariant coefficient system $\mathcal{C}[\pi]$ on X_π. Roughly speaking the coefficient system is obtained by taking isotypic components of π according to some representations constructed from the Bushnell and Kutzko type of π. We conjecture that when π has central character, the augmented chain complex associate to $\mathcal{C}(\pi)$ is a projective resolution of π in the category $\mathcal{B}(\pi)$. Moreover we reduce this conjecture to a technical lemma of representation theoretic nature. We prove this lemma when π is an irreducible discrete series of G. We then attach to any irreducible discrete series π of G an explicit pseudo-coefficient f_π and obtain a Lefschetz type formula for the value of the Harish-Chandra character of π at a regular elliptic element. In contrast to that obtained by U. Stuhler and the second author, this formula allows explicit character value computations.

Résumé. Soient F un corps local non archimédien et G le groupe $\mathrm{GL}(N, F)$, pour un entier $N \geq 2$. Soit π une représentation lisse complexe de G appartenant au block de Bernstein $\mathcal{B}(\pi)$ d'un type simple au sens de Bushnell et Kutzko. En affinant l'approche que proposent le second auteur et U. Stuhler, nous attachons canoniquement à π un sous-ensemble X_π de l'immeuble de Bruhat-Tits X de G, ainsi qu'un système de coefficients G-équivariant $\mathcal{C}[\pi]$ sur X_π. Grossièrement parlant, le système de coefficients est construit en prenant des composantes isotypiques de π selon des représentations construites à partir du type de Bushnell et Kutzko de π. Nous conjecturons que lorsque π possède un caractère central, le complexe de chaînes augmenté associé à $\mathcal{C}(\pi)$ est une résolution de π dans la catégorie $\mathcal{B}(\pi)$. De plus nous réduisons cette conjecture à un lemme technique en théorie des représentations. Nous démontrons ce lemme lorsque π est une représentation irréductible de la série discrète de G. Nous attachons ensuite à toute représentation irréductible π de la série discrète de G un pseudo-coefficient explicite f_π et obtenons une formule de type Lefschetz pour la valeur du caractère de Harish-Chandra de π en un élément elliptique régulier. Contrairement celle obtenue par U. Stuhler et le second auteur, notre formule permet des calculs explicites.

Introduction

Let F be a non-archimedean local field and, for some integer $N \geq 2$, let G denote the locally compact group $\mathrm{GL}(N, F)$ and X its Bruhat-Tits building The aim of this work is to refine the construction of [SS] (also see [SS2]) to attach to certain representations of G new equivariant coefficient systems on the Bruhat-Tits building. These representations belong to the Bernstein blocks of the category of smooth complex representations of G corresponding to *simple types* in the sense of Bushnell and Kutzko [BK1]. Let $(\pi, \mathcal{V})$ be a smooth complex representation of G. In [SS] an equivariant coefficient system $\mathcal{C}(\pi)$ is constructed by attaching to each simplex σ of X the space of vectors fixed by a certain congruence subgroup of level e of the parahoric subgroup of G fixing σ. Here the integer e is such that $\mathcal{V}$ is generated as a G-module by its vectors fixed by the principal congruence subgroup of level e of some maximal compact subgroup of G. In [SS] it is proved that the augmented chain complex $C_\bullet(X, \mathcal{C}(\pi)) \longrightarrow \mathcal{V}$ of X with coefficients in $\mathcal{C}(\pi)$ is exact. If one moreover assumes that $(\pi, \mathcal{V})$ admits a central character χ, then $C_\bullet(X, \mathcal{C}(\pi)) \longrightarrow \mathcal{V}$ is a projective resolution of $(\pi, \mathcal{V})$ in the category of smooth representations of G with central character χ. In [Br], the first author gave another proof of this fact for Iwahori-spherical representations. In [SS2], the second author and U. Stuhler draw some important consequences concerning the harmonic analysis on G as well as the homological algebra of the category of smooth representations of G. Among other things they prove that these projective resolutions give rise to pseudo-coefficients for discrete series representations (generalizing the pseudo-coefficient constructed by Kottwitz in [Kot] for the Steinberg representation) as well as a Lefschetz type character formula for the Harish-Chandra character of any smoooth representation. Note that if the construction of [SS] is restricted to the group G, [SS2] gives a generalization to any connected reductive F-group $\mathbb{G}$ and most of its results are valid without restriction on G (but sometimes F is assumed to have characteritic 0, and $\mathbb{G}(F)$ to have compact center).

If the construction and results of [SS], [SS2] have important theoretic consequences, they do not allow explicit calculations. Indeed in general the coefficient system $C(\pi)$ cannot be explicitely computed (except may be in the *level* 0 *case*, but this is nowhere written). Indeed the only explicit way to be given an irreducible smooth representation of G is to specify its Bushnell and Kutzko type. This is why it is natural to seek for a refinement of [SS] based on Bushnell and Kutzko theory.

In this paper, for technical reasons, we restrict to representations belonging to Bernstein blocks of G attached to simple types. These Bernstein blocks are exactly those containing discrete series representations. We fix a simple type (J, λ) and denote by $\mathcal{R}_\lambda(G)$ the category of smooth representations of J that are generated by their λ-isotypic component. We fix a smooth representation

$(\pi, \mathcal{V})$ of G lying in $\mathcal{R}_\lambda(G)$. To the datum (J, λ), in a non canonical way, one may associate a field extension E/F of degree dividing N whose multiplicative group $E^\times$ is embedded in G. The centralizer G_E of $E^\times$ in G is isomorphic to $\mathrm{GL}(N/[E:F], E)$. Using a result of the first author and B. Lemaire [BL], we may view the Bruhat-Tits building X_E of G_E as being embedded in X in a G_E-equivariant way. We show that *hidden* in the properties of *Heisenberg representations* constructed in [BK1]§(5.1) and in the *mobility* of simple characters established in *loc. cit.* §(3.6), there is a *geometric structure* allowing to attach to π a G_E-equivariant coefficient system $\mathcal{C}_E[\pi]$ on the first barycentric subdivision $\mathrm{sd}(X_E)$ of X_E. More precisely, in a non canonical way, we attach to (J, λ) a collection of pairs $(J^1(\sigma, \tau), \eta(\sigma, \tau))_{\sigma \subset \tau}$, where σ and τ run over the simplices of X_E satisfying $\sigma \subset \tau$. Here $J^1(\sigma, \tau)$ is some compact open subgroup of G and $\eta(\sigma, \tau)$ a Heisenberg representation of $J^1(\sigma, \tau)$ as considered in *loc. cit.* (5.1.14) (but Bushnell and Kutzko do not use this language nor this notation). Moreover the collection $(J^1(\sigma, \tau), \eta(\sigma, \tau))_{\sigma \subset \tau}$ is G_E-equivariant. Exploiting the compatibility relations among the various $\eta(\sigma, \tau)$ proved in *loc. cit.* §(5.1), and by taking isotypic components of $\mathcal{V}$ according to the Heisenberg representations $\eta(\sigma, \tau)$, we construct our equivariant coefficient system $\mathcal{C}_E[\pi]$.

We then show that the subset $X[E]$ of X obtained by taking the union of the $g.X_E$, where g runs over G has the structure of a G_E-simplicial complex containing X_E as a subcomplex. We naturally attach to $\mathcal{C}_E[\pi]$ a G-equivariant coefficient system $\mathcal{C}[\pi]$ on the first barycentric subdivision of $X[E]$ and show that it actually derives from a coefficient complex on $X[E]$, still denoted by $\mathcal{C}[\pi]$. We prove that the simplicial complex $X[E]$ and the coefficient system $\mathcal{C}[\pi]$ are actually independent of any choice made in their construction: these are objects canonically attached to π. Moreover the support X_π of $\mathcal{C}[\pi]$ maybe explicitely determined. In [BK1]§5, the Hecke algebra of (J, λ) is described using a non canonical unramified field extension L/E. It gives rise to a general linear group $G_L \subset G_E \subset G$, to a Bruhat-Tits building $X_L \subset X_E \subset X$ and to a simplicial complex

$$X[L] = \bigcup_{g \in G} g.X_L \subset X[E] \ .$$

Then the support of $\mathcal{C}[\pi]$ is $X_\pi = X[L]$.

We then consider the augmented chain complex of X_π with coefficients in $\mathcal{C}[\pi]$:

$$(*) \qquad C_\bullet(X_\pi, \mathcal{C}[\pi]) \longrightarrow \mathcal{V} \ .$$

We show that this complex lies in the category $\mathcal{R}_\lambda(G)$. We cannot in general prove its exactness that we consider as a conjecture. However we propose a strategy to tackle this exactness that generalizes the approach that the first author uses in [Br]. Indeed if $(\pi, \mathcal{V})$ has level 0 then $X[L] = X$ and the coefficient system $\mathcal{C}[\pi]$ coincides with that constructed in [SS]. In [Br], for Iwahori-spherical representations (they have level 0), one proves the exactness of $(*)$ using type theory and an argument of geometric nature.

Let us explain how this generalized approach works. Let $\mathcal{H}(G)$ be the Hecke algebra of locally constant complex functions with compact support on G. It is equipped with the convolution product $\star$ coming from a fixed Haar measure on G. Let e_λ be the idempotent of $\mathcal{H}(G)$ attached to λ so that for any smooth complex representation $\mathcal{W}$ of G, $e_\lambda \star \mathcal{W} = \mathcal{W}^\lambda$ is the λ-isotypic component of $\mathcal{W}$. One basic fact of type theory is that the functor

$$\mathcal{R}_\lambda(G) \longrightarrow e_\lambda \star \mathcal{H}(G) \star e_\lambda - \mathrm{Mod}\ ,\ \mathcal{W} \longrightarrow \mathcal{W}^\lambda$$

induces an equivalence of categories. It follows that in order to prove the exactness of $(*)$, we are reduced to proving the exactness of the chain complex $(**)$ in $e_\lambda.\mathcal{H}(G).e_\lambda$-Mod obtained from $(*)$ by applying the functor $\mathcal{W} \longrightarrow \mathcal{W}^\lambda$:

$$(**) \qquad C_\bullet(X_\pi, \mathcal{C}[\pi])^\lambda \longrightarrow \mathcal{V}^\lambda\ .$$

In fact we shall not work with the type λ, but with an equivalent type λ' defining the same Bernstein block; to make things simpler we ignore this difficulty in the introduction. Then generalizing [Br] we prove that modulo a conjectural technical hypothesis (Conjecture (X.4.1)), as a complex of $\mathbb{C}$-vector spaces, $(**)$ is canonically isomorphic to the augmented chain complex of a certain apartment $\mathcal{A}_L$ of X_L with *constant* coefficients in $\mathcal{V}^\lambda$. Of course $\mathcal{A}_L$ being a finite dimensional euclidean space, it is a contractible topological space, and its augmented chain complex with constant coefficients in any abelian group is exact.

We prove Conjecture (X.4.1), whence the exactness of $(*)$, when the representation π belongs to the discrete series of G. Indeed in that case we are able to entirely compute the coefficient system $\mathcal{C}[\pi]$ by using some technical lemmas proved by the second author and Zink in [SZ]. We actually prove that there exists a G-equivariant collection of pairs $(G_\sigma, \lambda_\sigma)$ such that the coefficient system is given by $\mathcal{C}[\pi]_\sigma = \mathcal{V}^{\lambda_\sigma}$ (isotypic component), where σ runs over the simplices of X_π, G_σ denotes the stabilizer of σ in G, and λ_σ is an irreducible smooth representation of G_σ. Moreover for any simplex σ of $X[L]$, the restriction of λ_σ to the maximal compact subgroup of G_σ only depends on (J, λ) but not on π.

Closely following [SS2], we attach to the coefficient system $\mathcal{C}[\pi]$ an Euler-Poincaré function f_{EP}^π on G and prove that it is a pseudo-coefficient of π. This pseudo-ceofficient should be very close to that constructed in [Br2] by the first author using an entirely different approach (but also based on Bushnell and Kutzko type theory), however the comparison has to be done. In contrast with that of [Br2], the pseudo coefficient f_{EP}^π is given by a formula adapted to explicit computations. In particular by computing certain orbital integrals, we derive a Lefschetz type character formula for the value of the Harish-Chandra character Θ_π of π at a regular elliptic element γ of G. This formula takes the form:

$$(***) \qquad \Theta_\pi(\gamma) = \mathrm{Tr}\ (\gamma\ ,\ \mathrm{EP}\, H^*(X_\pi^\gamma, \mathcal{C}[\pi])\ )$$

where EP $H^*(X_\pi^\gamma, \mathcal{C}[\pi])$ denotes the homology Euler-Poincaré module of the restriction of $\mathcal{C}(\pi)$ to the fixed point set X_π^γ of γ in X_π. We cannot expect to make formula $(***)$ entirely explicit. Indeed if γ is an element of G there is no known easy description of the fixed point set X^γ. Nevertheless when the elliptic regular element γ is *minimal over* F in the sense of Bushnell and Kutzko, then X_π^γ is either empty or reduced to a point. In that case the Lefschetz formula for $\Theta_\pi(\gamma)$ takes a striking simple form and allows explicit computations. In particular, in that case we recover the two character formulas obtained in [Br2]. However our approach gives a much more general result.

The paper is organized as follows. In section I we establish some crucial properties of the embedding $X_E \longrightarrow X$, where E/F is a field extension such that $E^\times$ embeds in G. In sections II and III we review the main properties of simple characters and of their endo-classes. The construction of the G_E-equivariant coefficient system $\mathcal{C}_E[\pi]$ on X_E is given in section IV and its extension $\mathcal{C}[\pi]$ to a G-equivariant coefficient system is done in sections V and VI. The canonicity of the coefficient complex $\mathcal{C}[\pi]$ is studied in sections VII and VIII. To state this result the right language is that of *endo-classes* (Propositions (VII.2) and (VIII.1.2)). The support of $\mathcal{C}[\pi]$ is described in Proposition (VIII.2.6). In section IX we prove that the chain complex attached to $\mathcal{C}[\pi]$ actually lies in the Bernstein block of π (Proposition (IX.2)). In section X we reduce the acyclicity of the augmented chain complex attached to $\mathcal{C}[\pi]$ to a technical lemma (Conjecture (X.4.1)). For an irreducible discrete series representation, the conjecture is proved in section XI (Theorem XI.2.7). The last section XII is devoted to applications. We first construct an explicit pseudo-coefficient for any irreducible discrete series representations (Theorem (XII.2.3)) and then derive an explicit character formula for the Harish-Chandra character of such a representation (Theorem (XII.3.2)). For elliptic minimal element the formula simplifies a lot (Proposition (XII.4.4)) and give a new proof of formulas already obtained in [Br2].

We shall assume that the reader is familiar with the formalism of [BK1]. Indeed this work may be somehow viewed as a geometric reformulation of Bushnell and Kutzko's construction of the discrete series of G.

We want to thank Shaun Stevens for his help. Proposition (XI.1.2) and its proof are due to him as well as the proof of Lemma (X.4.4).

This work has a long story. Both authors started to collaborate as the first one was in post-doctoral stay in Muenster in 2000/2001. Results from sections I to IX where obtained already in 2004.

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I. Field extensions and centralizers.

I.1 Vector spaces and orders.

If K is a non-archimedean local field we shall denote by $\mathfrak{o}_K$ its ring of integers and by $\mathfrak{p}_K$ the maximal ideal of $\mathfrak{o}_K$. Once for all we fix such a field F.

Let E/F be a finite field extension and V a finite dimensional E-vector space. Then V is naturally an F-vector space. We write $A = \mathrm{End}_F V$, $G = \mathrm{Aut}_F V$, $B = \mathrm{End}_E V$ and $G_E = \mathrm{Aut}_E V$. We have a natural inclusion of F-algebras $B \subseteq A$ and the group G_E is naturally a subgroup of G. As an F-algebra E embeds canonically in A and its centralizer is B. Similarly, the left action of E on V allows us to see $E^\times$ as a subgroup of G; its centralizer is G_E.

Let $\mathrm{Her}(A)$ (resp. $\mathrm{Her}(B)$) denote the set of hereditary $\mathfrak{o}_F$-orders in A (resp. hereditary $\mathfrak{o}_E$-orders in B). These sets are posets (for inclusion) and G and G_E respectively act on them by conjugation. We have a natural map j_{order}: $\mathrm{Her}(B) \longrightarrow \mathrm{Her}(A)$, defined as follows. If $\mathfrak{B}$ is in $\mathrm{Her}(B)$, it is the stabilizer in B of an $\mathfrak{o}_E$-lattice chain $\mathcal{L}$ in V; this lattice chain may be seen as an $\mathfrak{o}_F$-lattice chain in V and $j_{\mathrm{order}}(\mathfrak{B})$ is the attached order in A. We shall use the notation $j_{\mathrm{order}}(\mathfrak{B}) = \mathfrak{A}(\mathfrak{B})$. The map j_{order} is G_E-equivariant and, by [BK1] (1.2.1), its image consists of those orders in $\mathrm{Her}(A)$ that are stabilized by $E^\times$.

I.2 Buildings.

We keep the notation as in (I.1). Let X (resp. X_E) denote the semisimple affine building of G (resp. G_E). The following fact will be crucial for our construction.

(I.2.1) Theorem. ([BL] Theorem 1.1). *There exists a unique affine and G_E-equivariant map*

$$j_E : X_E \longrightarrow X.$$

It induces a bijection $X_E \longrightarrow X^{E^\times}$.

We are going to give a more precise version of this theorem. Recall that the building X is triangulated in a canonical way: it is the geometric realization of a G-simplicial complex that we still denote by X. Let $F(X)$ be the set of simplices of X. It is a poset for inclusion and is equipped with an action of G via poset isomorphisms. It is a standard result (compare [BT] Cor. 2.15) that we have an anti-isomorphism of posets, compatible with the G-actions:

$$\begin{array}{rcl} \mathrm{Her}(A)^{\mathrm{opp}} & \longrightarrow & F(X) \\ \mathfrak{A} & \longmapsto & F(\mathfrak{A}) \end{array}$$

where $F(\mathfrak{A})$ is the unique simplex stabilized by the normalizer of $\mathfrak{A}$ in G. Similarly, we have an anti-isomorphism of posets, compatible with the G_E-actions:

$$\begin{array}{rcl} \mathrm{Her}(B)^{\mathrm{opp}} & \longrightarrow & F(X_E) \\ \mathfrak{B} & \longmapsto & F(\mathfrak{B}) \end{array}$$

where the notation is obvious. We write j_{simp} for the morphism of G_E-posets $F(X_E) \longrightarrow F(X)$ obtained from j_{order} through the two previous isomorphisms.

Let $\text{sd}(X)$ (resp. $\text{sd}(X_E)$) be the first barycentric subdivision of X (resp. of X_E). This is the flag complex attached to the poset $F(X)$ (resp. $F(X_E)$). Since j_{simp} is increasing, it induces a G_E-equivariant simplicial map $\text{sd}(X_E) \longrightarrow \text{sd}(X)$.

(I.2.2) Proposition. *The map j_{simp}: $\text{sd}(X_E) \longrightarrow \text{sd}(X)$ induces j_E on the geometric realizations.*

Proof. Let us denote by j_{sd} the map $X_E \longrightarrow X$ induced by j_{simp} on the geometric realizations (constructed with standard affine simplices). By construction j_{sd} is affine and G_E-equivariant. By unicity in Theorem (I.2.1), it must coincide with j_E.

In the sequel we shall use the language of hereditary orders instead of simplices. In particular a q-simplex σ in $\text{sd}(X)$ is a strictly decreasing sequence of orders $\sigma = (\mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \ldots \supset \mathfrak{A}_q)$. The map $j_E = j_{\text{simp}}$ is then given by

$$j_E(\mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \ldots \supset \mathfrak{B}_q) = (\mathfrak{A}(\mathfrak{B}_0) \supset \mathfrak{A}(\mathfrak{B}_1) \supset \ldots \supset \mathfrak{A}(\mathfrak{B}_q)).$$

We shall also see $\text{sd}(X_E)$ as being embedded in $\text{sd}(X)$: j_E is now an inclusion.

The map j_E enjoys another property that is not proved in [BL]. Recall that X_E and X have invariant metrics which are unique up to a > 0 factor. Since G (resp. G_E) acts transitively on the apartments of X (resp. of X_E) fixing a metric on X (resp. on X_E) amounts to fixing it on one of its apartments.

(I.2.3) Proposition. *There exist normalizations of metrics on X_E and X such that the map j_E is an isometry.*

Proof. By invariance it suffices to prove that the restriction of j_E to some apartement $\mathcal{A}_E$ of X_E is an isometry. By [BL](5.1), $j_E(\mathcal{A}_E)$ is contained in an apartment $\mathcal{A}$ of X_E. Set $n = \text{Dim}_E(V)$ and consider $\mathbb{R}^n$ and $\mathbb{R}^{n/[E:F]}$ equipped with their standard euclidean structures. Then by the proof of Lemma (4.1) of [BL], one may choose the apartment $\mathcal{A}$ and metrics on X_E and X such that :

– $\mathcal{A}$ identifies to the orthogonal of $(1, 1, ..., 1)$ in $\mathbb{R}^n$

– $\mathcal{A}_E$ identifies to the orthogonal of $(1, 1, .., 1)$ in $\mathbb{R}^{n/[E:F]}$

– the map j_E is given by the restriction of the following linear map:

$$J \ : \ \mathbb{R}^{n/[E:F]} \longrightarrow \mathbb{R}^n \ , \ (x_1, ..., x_{n/[E:F]}) \mapsto (x_i/e + \mu_j)_{i=1,...,n/[E:F],\ j=1,...,[E:F]}$$

where e is the ramification index of E/F and the μ_i are some real constants. It is clear that up to a scalar J is an isometry. Our result follows.

I.3 Some properties of the embedding $\mathrm{sd}(X_E) \longrightarrow \mathrm{sd}(X)$.

We keep the notation as in (I.1) and (I.2). We need first some more notation and facts on orders. If $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A, then its multiplicative group is a compact open subgroup of G that we denote by $U(\mathfrak{A})$ (this is indeed a parahoric subgroup of G). Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$. Then the quotient $\mathfrak{A}/\mathfrak{P}$ is a semisimple $\mathbb{F}$-algebra, where $\mathbb{F}$ is the residue field of F. In particular the multiplicative group $(\mathfrak{A}/\mathfrak{P})^\times$ is the group of $\mathbb{F}$-points of a product of general linear groups defined over $\mathbb{F}$. The subgroup $U^1(\mathfrak{A}) = 1+\mathfrak{P}$ of 1-units is a normal subgroup of $U(\mathfrak{A})$ and the quotient canonically identifies with $(\mathfrak{A}/\mathfrak{P})^\times$.

For $\mathfrak{B}$ a hereditary order in B, the symbol $\mathcal{N}(\mathfrak{B})$ denotes the normalizer of $\mathfrak{B}$ in G_E, while if $\mathfrak{A}$ is a hereditary order in A, $\mathcal{N}(\mathfrak{A})$ denotes the normalizer of $\mathfrak{A}$ in G.

(I.3.1) Lemma. *For any hereditary order $\mathfrak{B}$ in B, we have*

$$\mathcal{N}(\mathfrak{A}(\mathfrak{B})) = \mathcal{N}(\mathfrak{B})U(\mathfrak{A}(\mathfrak{B})).$$

Proof. Let $(L_k)_{k\in\mathbb{Z}}$ be an $\mathfrak{o}_E$-lattice chain in V defining $\mathfrak{B}$. Let $v_{\mathfrak{A}(\mathfrak{B})} : A \longrightarrow \mathbb{Z}$ be the the valuation map given by

$$v_{\mathfrak{A}}(a) = m \text{ iff } a \in \mathfrak{P}^m \backslash \mathfrak{P}^{m+1} \ , \ m \in \mathbb{Z}$$

where $\mathfrak{P}$ is the radical of $\mathfrak{A}(\mathfrak{B})$. Write $v_{\mathfrak{B}}$ for the similar map $B \longrightarrow \mathbb{Z}$ defined by the powers of the radical of $\mathfrak{B}$. From [BK1]§1, we have

(I.3.2) $(v_{\mathfrak{A}})_{|B} = v_{\mathfrak{B}}$ and $\mathcal{N}(\mathfrak{A}(\mathfrak{B})) \cap G_E = \mathcal{N}(\mathfrak{B})$.

Let $t\mathbb{Z}$, $t > 0$, be the image of the group homomorphism

$$v_{\mathfrak{A}} \ : \mathcal{N}(\mathfrak{A}(\mathfrak{B})) \longrightarrow \mathbb{Z}.$$

Then $\mathcal{N}(\mathfrak{A}(\mathfrak{B})) = z^{\mathbb{Z}}U(\mathfrak{A}(\mathfrak{B}))$ for any z in $\mathcal{N}(\mathfrak{A}(\mathfrak{B}))$ with $\mathfrak{A}$-valuation t. A similar statement holds for $\mathcal{N}(\mathfrak{B})$. Now from [BF] one knows that t is the smallest positive period of the map $k \mapsto \dim_{\mathbb{F}} L_k/L_{k+1}$. So t is also the smallest positive period of $k \mapsto \dim_{\mathbb{F}_E} L_k/L_{k+1}$, where $\mathbb{F}_E$ is the residue class field of E. Together with (I.3.2) this implies that we can actually choose z in $\mathcal{N}(\mathfrak{B})$ and the result follows.

(I.3.3) Lemma. *Let $\sigma = (\mathfrak{B}_0^\sigma \supset \ldots \supset \mathfrak{B}_q^\sigma)$ and $\tau = (\mathfrak{B}_0^\tau \supset \ldots \supset \mathfrak{B}_q^\tau)$ be two q-simplices in $\mathrm{sd}(X_E)$. Assume that $\sigma = g\tau$ for some $g \in G$. Then there exists g_E in G_E such that $\sigma = g_E\tau$. In particular any g as above can be written $g = g_E g_\tau$ with $g_E \in G_E$ and $g_\tau \in \mathrm{Stab}_G(\tau)$.*

Proof. First we need to recall the classification of conjugacy classes of hereditary orders in A (cf. [BF] or [Rei]). Let $\mathfrak{A}$ be such an order and let $(L_k)_{k\in\mathbb{Z}}$ be a lattice chain in V defining $\mathfrak{A}$. To $\mathfrak{A}$ we attach the sequence of integers $d(\mathfrak{A})_k = \dim_{\mathbb{F}} L_k/L_{k+1}$, $k \in \mathbb{Z}$. Then two hereditary orders $\mathfrak{A}_1$, $\mathfrak{A}_2$ are conjugate if and only if the sequences $d(\mathfrak{A}_1)$ and $d(\mathfrak{A}_2)$ coincide up to a translation of the indexing. We use the notation d_E for the sequences attached to hereditary orders in B. If $\mathfrak{B}$ is such an order, attached to an $\mathfrak{o}_E$-lattice chain $(L_k)_{k\in\mathbb{Z}}$ in V, we have:

$$d(\mathfrak{A}(\mathfrak{B}))_k = [\mathbb{F}_E : \mathbb{F}] d_E(\mathfrak{B})_k \ , \ k \in \mathbb{Z}.$$

We deduce:

(I.3.4) *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be hereditary orders in B. Then they are G_E-conjugate if and only if the orders $\mathfrak{A}(\mathfrak{B}_1)$ and $\mathfrak{A}(\mathfrak{B}_2)$ are G-conjugate. In other words Lemma (I.3.3) holds when $q = 0$.*

Now let us turn to the general case. By using (I.3.4), we may replace τ by a conjugate under G_E so that $\mathfrak{B}_q^\sigma = \mathfrak{B}_q^\tau =: \mathfrak{B}_q$. By assumption there exists a $g \in G$ such that $\mathfrak{A}(\mathfrak{B}_i^\sigma) = \mathfrak{A}(\mathfrak{B}_i^\tau)^g$ for $i = 0, \ldots, q$. In particular $\mathfrak{A}(\mathfrak{B}_q) = \mathfrak{A}(\mathfrak{B}_q)^g$, and, thanks to (I.3.1), we may, by replacing τ by a G_E-conjugate, assume that

$$\sigma = g\tau \quad , \quad \mathfrak{B}_q^\sigma = \mathfrak{B}_q^\tau = \mathfrak{B}_q, \text{ and } g \in U(\mathfrak{A}(\mathfrak{B}_q)).$$

But then $g \in U(\mathfrak{A}(\mathfrak{B}_i))$ for any $i = 0, \ldots, q$ which means that g fixes τ, i.e., that $\sigma = \tau$.

It is not possible to characterize the image of $\mathrm{sd}(X_E)$ using numerical invariants attached to simplices. But we are going to give a criterion for a simplex of $\mathrm{sd}(X)$ to belong to:

$$X(E) := \bigcup_{g\in G} g\mathrm{sd}(X_E) \ .$$

Here $\mathrm{sd}(X_E)$ is of course seen as being embedded in $\mathrm{sd}(X)$.

Let $(L_k)_{k\in\mathbb{Z}}$ be an $\mathfrak{o}_F$-lattice chain in V and $\mathfrak{A}$ be the attached order in A. Write $e = e(\mathfrak{A})$ for the period of $\mathfrak{A}$. The sequence of positive integers defined by $d(\mathfrak{A})_k = \dim_{\mathbb{F}} L_k/L_{k+1}$ is e-periodic and we have the partition:

$$n = \dim V = d(\mathfrak{A})_0 + \ldots + d(\mathfrak{A})_{e-1} \ .$$

We denote by $p(\mathfrak{A})$ the least positive period of $(d(\mathfrak{A})_k)_{k\in\mathbb{Z}}$. We can rephrase [BK2] Prop. (1.2) as follows.

(I.3.5) Proposition. *The order $\mathfrak{A}$ has a conjugate normalized by $E^\times$ if and only if the following assertions hold:*

i) $f(E/F)$ divides $d(\mathfrak{A})_k$ for all $k \in \mathbb{Z}$;

ii) $e(E/F)$ divides $e(\mathfrak{A})/p(\mathfrak{A})$.

In other words the vertices of $\mathrm{sd}(X)$ *which are in* $X(E)$ *are exactly those vertices which correspond to hereditary orders* $\mathfrak{A}$ *satisfying conditions (i) and (ii).*

We remark that the simplicial complex $X(E)$ is not simply connected in general. For instance take $G = \mathrm{GL}(4, F)$ and E/F quadratic unramified. Then $\mathrm{sd}(X_E)$ is the building of $G_E = \mathrm{GL}(2, F)$ which is 1-dimensional. Using the criterion of (I.3.5), we get that any vertex of X belongs to $X(E)$. On the other hand the barycenter of an edge in X attached to a 2-periodical $\mathfrak{o}_F$-lattice chain $(L_k)_{k\in\mathbf{Z}}$ in V lies in $X(E)$ if and only if $\dim_{\mathbb{F}}(L_k/L_{k+1}) = 2$ for all k. Any given chamber of X therefore has exactly two opposite edges σ_0 and σ_1 that lie in $X(E)$. If we consider all chambers in an apartment of X which contain σ_0 then the corresponding edges opposite to σ_0 form a cycle in $X(E)$.

II. Simple characters and their endo-classes

Here we recall some basic facts about simple characters. References are to be found in [BK1] and [BH]. We continue to use the notation of (I).

II.1 Simple pairs and their realizations.

Recall that a simple pair $[0, \beta]$ ([BH](1.5)) is a finite field extension E/F, equipped with a generator β (i.e. $E = F(\beta)$) and satisfying the following conditions:

(SP1) $\beta \notin \mathfrak{o}_E$,

(SP2) $k_o(\beta, \mathfrak{A}(E)) < 0$ (cf. [BK1]§1).

For each finite dimensional E-vector space V, and for each $\mathfrak{B} \in \mathrm{Her}(B)$, we have a simple stratum $[\mathfrak{A}(\mathfrak{B}), n_{\mathfrak{B}}, 0, \beta]$ in A, called a *realization* of $[0, \beta]$ ([BH] p. 133). Here $n_{\mathfrak{B}}$ is the valuation of $\beta \in A$ with respect to $\mathfrak{A}(\mathfrak{B})$.

Attached to $[\mathfrak{A}(\mathfrak{B}), n_{\mathfrak{B}}, 0, \beta]$ (so to $[0, \beta]$, V and $\mathfrak{B}$), we have the following data:

– Two *open compact subgroups* of G: $U^1(\mathfrak{B}) \subseteq H^1(\mathfrak{B}) \subseteq J^1(\mathfrak{B}) \subseteq U^1(\mathfrak{A}(\mathfrak{B}))$; they are both normalized by $\mathcal{N}(\mathfrak{B})$.

– A finite set of *simple characters* $\mathcal{C}(\mathfrak{B}) = \mathcal{C}(\mathfrak{A}(\mathfrak{B}), 0, \beta)$ of $H^1(\mathfrak{B})$; each character in $\mathcal{C}(\mathfrak{B})$ having a G-intertwining given by $J^1(\mathfrak{B})G_E J^1(\mathfrak{B})$.

– Moreover, for each $\theta \in \mathcal{C}(\mathfrak{B})$, there exists (up to isomorphism) a unique irreducible representation $\eta(\theta)$ of $J^1(\mathfrak{B})$ such that $\eta(\theta)_{|H^1(\mathfrak{B})}$ contains θ. The intertwining of $\eta(\theta)$ is again $J^1(\mathfrak{B})G_E J^1(\mathfrak{B})$ and the representation $\mathrm{Ind}_{H^1(\mathfrak{B})}^{J^1(\mathfrak{B})}\theta$ is a multiple of $\eta(\theta)$.

In addition we need the *degenerate* simple characters ([BK1] p. 184). To have a uniform notation we in this case set $E := F$ and $B := A$; for any $\mathfrak{B} \in \mathrm{Her}(B) =$

Her(A) we let $H^1(\mathfrak{B}) := J^1(\mathfrak{B}) := U^1(\mathfrak{B})$ and let $\mathcal{C}(\mathfrak{B})$ denote the one element set consisting of the trivial representation $\mathbf{1}_{H^1(\mathfrak{B})}$ of $H^1(\mathfrak{B})$.

If we need to keep track of the E-vector space V we some times write $H^1(V,\mathfrak{B})$, $J^1(V,\mathfrak{B}), \mathcal{C}(V,\mathfrak{B})$ instead of $H^1(\mathfrak{B}), J^1(\mathfrak{B}), \mathcal{C}(\mathfrak{B})$, respectively.

II.2 Potential simple characters (cf. [BH] §8).

Let $[0,\beta]$ be a simple pair and V_1, V_2 be two finite dimensional E-vector spaces. Write $B_i = \mathrm{End}_E V_i$, $A_i = \mathrm{End}_F V_i$, $i = 1,2$. For $i = 1,2$, fix a hereditary order $\mathfrak{B}_i \in \mathrm{Her}(B_i)$. Recall ([BK1] (3.6)) that we have a canonical bijection (called a transfer map):

$$\tau_{\mathfrak{B}_1,\mathfrak{B}_2,\beta} : \mathcal{C}(V_1,\mathfrak{B}_1) \longrightarrow \mathcal{C}(V_2,\mathfrak{B}_2).$$

These transfer maps satisfy the properties:

$$\tau_{\mathfrak{B}_1,\mathfrak{B}_2,\beta} = \tau^{-1}_{\mathfrak{B}_2,\mathfrak{B}_1,\beta}\ ,\ \tau_{\mathfrak{B}_1,\mathfrak{B}_3,\beta} = \tau_{\mathfrak{B}_1,\mathfrak{B}_2,\beta} \circ \tau_{\mathfrak{B}_2,\mathfrak{B}_3,\beta}.$$

Write $\mathcal{R}[\prime,\beta] = \bigcup \mathcal{C}(\mathcal{V},\mathfrak{B})$, where $\mathcal{C}(V,\mathfrak{B})$ runs over the sets of simple characters attached to all possible realizations of $[0,\beta]$. We say that $\theta_1,\ \theta_2 \in \mathcal{R}[\prime,\beta]$, attached to $(V_i,\mathfrak{B}_i)$, $i = 1,2$ are *equivalent* if $\theta_2 = \tau_{\mathfrak{B}_1,\mathfrak{B}_2,\beta}\theta_1$. This is indeed an equivalence relation and the equivalence classes are called *potential simple characters* (or *ps-characters*) supported by $[0,\beta]$.

In addition we let all possible degenerate simple characters form a single class which will be called the *degenerate ps-character*.

Remarks. (i) To be given a ps-characters amounts to fixing some $\theta \in \mathcal{C}(V,\mathfrak{B})$ in some realization.

(ii) A ps-character Θ may be seen as a *function* of the pairs $(V,\mathfrak{B})$: to $(V,\mathfrak{B})$ we attach the simple character $\theta \in \Theta$ that lies in $\mathcal{C}(V,\mathfrak{B})$. We shall also say that $\Theta(V,\mathfrak{B})$ is a *realization of* Θ *associated to* $(V,\mathfrak{B})$.

II.3 Endo-classes of ps-characters (cf. [BH]§8).

Let Θ_i, for $i = 1,2$, be two ps-characters. Then each Θ_i is either supported by a simple pair $[0,\beta_i]$ (with $E_i := F(\beta_i)$) or is degenerate (with $E := F$). We say that two realizations $\Theta_1(V_1,\mathfrak{B}_1)$ and $\Theta_2(V_2,\mathfrak{B}_2)$ are *simultaneous* if $[E_1 : F] = [E_2 : F]$ and if the F-vector spaces V_1 and V_2 are the same.

(II.3.1) Definition ([BH](8.6)). *Two ps-characters* Θ_1 *and* Θ_2 *are called endo-equivalent, denoted* $\Theta_1 \simeq \Theta_2$*, if there exist simultaneous realizations* $\Theta_1(V_1,\mathfrak{B}_1)$ *and* $\Theta_2(V_2,\mathfrak{B}_2)$ *that intertwine in* $\mathrm{Aut}_F V$*, where* $V = V_1 = V_2$*. We shall summarize this condition by saying that* Θ_1 *and* Θ_2 *intertwine in some simultaneous realization.*

The following proposition shows that $\simeq$ is indeed an equivalence relation.

(II.3.2) Proposition (cf. [BK1](3.6) and [BH] pp. 154-157). (i) *If Θ is a ps-character then any pair of simultaneous realizations of Θ intertwine.*

(ii) *If Θ_1 and Θ_2 are ps-characters, they intertwine in some simultaneous realization if and only if they intertwine in any simultaneous realization.*

A class for $\simeq$ is called an *endo-class of ps-characters.*

We shall need the following two facts.

(II.3.3) Proposition ([BH] (8.11)). *Let $\mathbf{\Theta}$ be an endo-class of non-degenerate ps-characters and $\Theta \in \mathbf{\Theta}$ supported by $[0, \beta]$. Then the following integers only depend on $\mathbf{\Theta}$: $k_o(\beta, \mathfrak{A}(E))$, $v_E(\beta)$, $e(E/F)$ (ramification index) and $f(E/F)$ (inertial degree).*

(II.3.4) Proposition ([BK1] (3.5.11)). *Let $\mathbf{\Theta}$ be an endo-class of ps-characters and Θ_1, $\Theta_2 \in \mathbf{\Theta}$. Let $\theta_1 = \Theta_1(V_1, \mathfrak{B}_1)$ and $\theta_2 = \Theta_2(V_2, \mathfrak{B}_2)$ be simultaneous realizations. Assume that $\mathfrak{A}(\mathfrak{B}_1) = \mathfrak{A}(\mathfrak{B}_2) =: \mathfrak{A}$. Then there exists $x \in U(\mathfrak{A})$ such that $\theta_2 = \theta_1^x$.*

III. Ps-characters and pairs of orders

III.1 Extensions to mixed groups.

We fix a simple pair $[0, \beta]$, a ps-character Θ supported by $[0, \beta]$, as well as a finite dimensional E-vector space V. We keep the notation as in (I) and (II).

The ps-character Θ gives rise to a function θ; it maps an order $\mathfrak{B} \in \mathrm{Her}(B)$ to the simple character $\theta(\mathfrak{B}) = \Theta(V, \mathfrak{B})$ of $\mathcal{C}(\mathfrak{B})$. For each $\mathfrak{B} \in \mathrm{Her}(B)$, let $\eta(\mathfrak{B}) = \eta(V, \mathfrak{B})$ be the Heisenberg representation of $J^1(\mathfrak{B})$ which contains $\theta(\mathfrak{B})$ when restricted to $H^1(\mathfrak{B})$.

For each pair of hereditary orders $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ in $\mathrm{Her}(B)$, we have $U(\mathfrak{B}_1) \subseteq U(\mathfrak{B}_2)$ and $U^1(\mathfrak{B}_2) \subseteq U^1(\mathfrak{B}_1)$. Since $U^1(\mathfrak{B}_1) \subseteq U(\mathfrak{B}_2)$ and $U(\mathfrak{B}_2)$ normalizes $J^1(\mathfrak{B}_2)$, one may form the group

$$J^1(\mathfrak{B}_1, \mathfrak{B}_2) := U^1(\mathfrak{B}_1) J^1(\mathfrak{B}_2).$$

(III.1.1) Proposition ([BK1](5.1.14-16), (5.1.18), (5.1.19)). *There exists a unique family of irreducible representations $\{(J^1(\mathfrak{B}_1, \mathfrak{B}_2), \eta(\mathfrak{B}_1, \mathfrak{B}_2))\}_{\mathfrak{B}_1 \subseteq \mathfrak{B}_2}$ (determined up to isomorphism) which extends $\{\eta(\mathfrak{B})\}_{\mathfrak{B}}$ in the following sense:*

(i) $\eta(\mathfrak{B}, \mathfrak{B}) = \eta(\mathfrak{B})$ for any $\mathfrak{B}$ in $\mathrm{Her}(B)$;

(ii) $\eta(\mathfrak{B}_1, \mathfrak{B}_2)_{|J^1(\mathfrak{B}_2)} \simeq \eta(\mathfrak{B}_2)$ for all $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ in $\mathrm{Her}(B)$;

(iii) the following induced representations are irreducible and equivalent:

$$\mathrm{Ind}_{J^1(\mathfrak{B}_1)}^{U^1(\mathfrak{A}(\mathfrak{B}_1))}\eta(\mathfrak{B}_1) \simeq \mathrm{Ind}_{J^1(\mathfrak{B}_1,\mathfrak{B}_2)}^{U^1(\mathfrak{A}(\mathfrak{B}_1))}\eta(\mathfrak{B}_1,\mathfrak{B}_2)\ .$$

Moreover we have:

(iv) The compatibility condition: $\eta(\mathfrak{B}_1,\mathfrak{B}_3)_{|J^1(\mathfrak{B}_2,\mathfrak{B}_3)} = \eta(\mathfrak{B}_2,\mathfrak{B}_3)$*, for any triple* $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}_3$ *in* $\mathrm{Her}(B)$*;*

(v) the intertwining formula:

$$\mathcal{I}_G(\eta(\mathfrak{B}_1,\mathfrak{B}_2)) = J^1(\mathfrak{B}_2)G_E J^1(\mathfrak{B}_2)\ .$$

Note that the representation

$$\eta(\mathfrak{A}(\mathfrak{B})) := \eta(V,\mathfrak{A}(\mathfrak{B})) := \mathrm{Ind}_{J^1(\mathfrak{B})}^{U^1(\mathfrak{A}(\mathfrak{B}))}\eta(\mathfrak{B})$$

is irreducible for all $\mathfrak{B}$. Its intertwining is given by

$$\mathcal{I}_G(\eta(\mathfrak{A}(\mathfrak{B})) = U^1(\mathfrak{A}(\mathfrak{B}))G_E U^1(\mathfrak{A}(\mathfrak{B}))\ .$$

(III.1.2) Proposition. *For all g in G_E and $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$, we have* $[J^1(\mathfrak{B}_1,\mathfrak{B}_2)]^g = J^1(\mathfrak{B}_1^g,\mathfrak{B}_2^g)$ *and the representations* $\eta(\mathfrak{B}_1,\mathfrak{B}_2)^g$ *and* $\eta(\mathfrak{B}_1^g,\mathfrak{B}_2^g)$ *are isomorphic.*

First we need the following result. Let V' denote the F-vector space V equipped with a possibly different E-vector space structure. We then find an element $x \in G$ such that $B' := \mathrm{End}_E V'$ satisfies $B' = B^x := xBx^{-1}$ and hence $\mathrm{Her}(B') = \{\mathfrak{B}^x : \mathfrak{b} \in \mathrm{Her}(B)\}$.

(III.1.3) Lemma. *a) For any $\mathfrak{B} \in \mathrm{Her}(B)$ we have:*

i) $H^1(V',\mathfrak{B}^x) = H^1(V,\mathfrak{B})^x$ *and* $J^1(V',\mathfrak{B}^x) = J^1(V,\mathfrak{B})^x$.

ii) If $\theta \in \mathcal{C}(V,\mathfrak{B})$*, then* $\theta^x \in \mathcal{C}(V',\mathfrak{B}^x)$.

b) For $g \in G_E$ *and* $\mathfrak{B} \in \mathrm{Her}(B)$ *we have* $\theta(\mathfrak{B})^g = \theta(\mathfrak{B}^g)$.

Proof. The point a) follows immediately from the inductive definition of simple characters and groups (cf. [BK1] §3).

We need to recall the characterization of the transfer maps $\tau_{\mathfrak{B}_1,\mathfrak{B}_2,\beta}$ for a pair of orders $\mathfrak{B}_i$, $i = 1,2$, in $\mathrm{Her}(B)$ ([BK1](3.6)): If $\theta_i \in \mathcal{C}(\mathfrak{B}_i)$, $i = 1,2$, then $\theta_2 = \tau_{\mathfrak{B}_1,\mathfrak{B}_2,\beta}\theta_1$ if and only if $1 \in G_E$ intertwine θ_1 and θ_2.

Consider the two characters $\theta(\mathfrak{B}^g)$ and $\theta(\mathfrak{B})^g$ of $H^1(\mathfrak{B}^g)$. Since g intertwines $\theta(\mathfrak{B})$, we must have $\theta(\mathfrak{B})^g_{|H^1(\mathfrak{B})\cap H^1(\mathfrak{B}^g)} = \theta(\mathfrak{B})_{|H^1(\mathfrak{B})\cap H^1(\mathfrak{B}^g)}$. So $\theta(\mathfrak{B})^g \in \mathcal{C}(\mathfrak{B}^g)$ coincides with $\tau_{\mathfrak{B},\mathfrak{B}^g,\beta}(\theta(\mathfrak{B}))$, that is with $\theta(\mathfrak{B}^g)$ by definition of Θ.

Turning to the proof of Proposition (III.1.2) the G_E-equivariance of the family $\{(J^1(\mathfrak{B}_1,\mathfrak{B}_2),\eta(\mathfrak{B}_1,\mathfrak{B}_2))\}_{\mathfrak{B}_1\subseteq\mathfrak{B}_2}$ follows now from that of $\{\theta(\mathfrak{B})\}_{\mathfrak{B}\in\mathrm{Her}(B)}$ by a unicity argument.

III.2 Extensions to 1-units of orders.

We now quote some properties of the representations $\eta(\mathfrak{A}(\mathfrak{B}))$. In the following we abbreviate $\mathfrak{A}(\mathfrak{B}_*) = \mathfrak{A}_*$ for any subscript "$*$". Let $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ be hereditary orders in B.

We first note that

$$J^1(\mathfrak{B}_1,\mathfrak{B}_2) \subseteq U^1(\mathfrak{B}_1)U^1(\mathfrak{A}_2) \subseteq U^1(\mathfrak{A}_1)\ .$$

So we can consider the irreducible representation

$$\eta(\mathfrak{A}_1,\mathfrak{A}_2) := \mathrm{Ind}_{J^1(\mathfrak{B}_1,\mathfrak{B}_2)}^{U^1(\mathfrak{B}_1)U^1(\mathfrak{A}_2)}\eta(\mathfrak{B}_1,\mathfrak{B}_2)\ .$$

(III.2.1) Proposition *a) The representation* $\eta(\mathfrak{A}_1,\mathfrak{A}_2)$ *satisfies*

(i) $\eta(\mathfrak{A}_1,\mathfrak{A}_2)_{|U^1(\mathfrak{A}_2)} \simeq \eta(\mathfrak{A}_2)$;

(ii) $\mathrm{Ind}_{U^1(\mathfrak{B}_1)U^1(\mathfrak{A}_2)}^{U^1(\mathfrak{A}_1)}\eta(\mathfrak{A}_1,\mathfrak{A}_2) \simeq \eta(\mathfrak{A}_1)$.

b) Moreover for any triple of hereditary orders $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}_3$, *we have*

$$\eta(\mathfrak{A}_1,\mathfrak{A}_3)_{|U^1(\mathfrak{B}_2)U^1(\mathfrak{A}_3)} = \eta(\mathfrak{A}_2,\mathfrak{A}_3)\ .$$

Proof. Assertion a) (ii) is a consequence of Proposition (III.1.1)(iii). We must prove b). By Mackey's restriction formula and since the double quotient

$$U^1(\mathfrak{B}_2)U^1(\mathfrak{A}_3)\backslash U^1(\mathfrak{B}_1)U^1(\mathfrak{A}_3)/U^1(\mathfrak{B}_1)J^1(\mathfrak{B}_3)$$

is reduced to one element, we get that the restriction in b) is

$$\mathrm{Ind}_{U^1(\mathfrak{B}_1)J^1(\mathfrak{B}_3)\cap U^1(\mathfrak{B}_2)U^1(\mathfrak{A}_3)}^{U^1(\mathfrak{B}_2)U^1(\mathfrak{A}_3)}\eta(\mathfrak{B}_1,\mathfrak{B}_3) = \mathrm{Ind}_{U^1(\mathfrak{B}_2)J^1(\mathfrak{B}_3)}^{U^1(\mathfrak{B}_2)U^1(\mathfrak{A}_3)}\eta(\mathfrak{B}_1,\mathfrak{B}_3)\ .$$

Now the result follows from Proposition (III.1.1)(iv).

III.3 The degenerate case.

The constructions of (III.1) and (III.2) trivially extend to the case where Θ is the degenerate ps-character. Indeed, in that case, we set $E = F$ and for all pairs of orders $\mathfrak{B}_1 \subset \mathfrak{B}_2$ in $\mathrm{Her}(B)$, we set:

– $J^1(\mathfrak{B}_1, \mathfrak{B}_2) = U^1(\mathfrak{B}_1)U^1(\mathfrak{B}_2) = U^1(\mathfrak{A}_1)U^1(\mathfrak{A}_2) = U^1(\mathfrak{A}_1)$;

– $\eta(\mathfrak{B}_1, \mathfrak{B}_2) = \eta(\mathfrak{A}_1, \mathfrak{A}_2) = \mathbf{1}_{U^1(\mathfrak{A}_1)}$.

IV. The coefficient system on $\mathrm{sd}(X_E)$

As in the previous section, we fix a ps-character Θ. It is either degenerate ($E := F$) or supported by a simple pair $[0, \beta]$ ($E := F(\beta)$). We also fix a finite dimensional E-vector space V.

Let $\mathcal{V}$ be a smooth complex representation of $G = \mathrm{Aut}_F V$. In a first step, we are going to construct a G_E-equivariant coefficient system $\mathcal{C}_o(\mathcal{V}) = \mathcal{C}_o(\Theta, V, \mathcal{V})$ on $\mathrm{sd}(X_E)$. We shall first construct this coefficient system on the stars of the vertices of X_E and then extend it to any simplex.

We call a simplex $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ *semistandard* if it belongs to the star of some vertex in X_E, that is if $\mathfrak{B}_0$ is a maximal order.

(IV.1) Definition. *i) For any semistandard simplex $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ of* $\mathrm{sd}(X_E)$*, we set*

$$\mathcal{V}(\sigma) = \mathcal{V}^{\eta(\mathfrak{B}_q, \mathfrak{B}_0)} \ ,$$

the $\eta(\mathfrak{B}_q, \mathfrak{B}_0)$-isotypic component of $\mathcal{V}$.

ii) For an arbitrary simplex σ of $\mathrm{sd}(X_E)$*, we set*

$$\mathcal{V}(\sigma) = \sum_{\tau \ \text{semistandard}, \tau \supseteq \sigma} \mathcal{V}(\tau) \ .$$

(IV.2) Proposition. *i) The previous definition is consistent.*

ii) For any pair of simplices σ, τ of $\mathrm{sd}(X_E)$ *with $\sigma \subseteq \tau$, we have $\mathcal{V}(\tau) \subseteq \mathcal{V}(\sigma)$.*

Proof. We only need to prove the second assertion in the case of semistandard simplices σ, τ. Suppose therefore that $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ is semistandard. The stars of two distinct vertices being disjoint, the simplex τ must then have the form $\tau = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_r)$ containing $(\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ as a subflag. By (III.1.1)(iv), we have $\eta(\mathfrak{B}_r, \mathfrak{B}_0) | J^1(\mathfrak{B}_q, \mathfrak{B}_0) = \eta(\mathfrak{B}_q, \mathfrak{B}_0)$. So $\mathcal{V}^{\eta(\mathfrak{B}_r, \mathfrak{B}_0)} \subseteq \mathcal{V}^{\eta(\mathfrak{B}_q, \mathfrak{B}_0)}$ and the result follows.

By taking inclusions as transition maps, the family $\mathcal{C}_o(\mathcal{V}) = (\mathcal{V}(\sigma))_\sigma$, σ running over the simplices of $\mathrm{sd}(X_E)$, is then a coefficient system of $\mathbb{C}$-vector spaces over $\mathrm{sd}(X_E)$.

(IV.3) Proposition. *For the obvious action of G_E on the $\mathcal{V}(\sigma)$, σ simplex of* $\mathrm{sd}(X_E)$, *$\mathcal{C}_o(\mathcal{V})$ is naturally endowed with a structure of G_E-equivariant coefficient system.*

Proof. We must prove that $g\mathcal{V}(\sigma) = \mathcal{V}(g\sigma)$, for all $g \in G_E$ and σ simplex of $\mathrm{sd}(X_E)$. Also we may clearly reduce to the case where σ and τ are semi-standard.

Let $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ be semistandard and g be in G_E. Then $g\sigma = (\mathfrak{B}_0^g \supset \ldots \supset \mathfrak{B}_q^g)$ and

$$g\mathcal{V}(\sigma) = g\mathcal{V}^{\eta(\mathfrak{B}_q,\mathfrak{B}_0)} \text{ and } \mathcal{V}(g\sigma) = \mathcal{V}^{\eta(\mathfrak{B}_q^g,\mathfrak{B}_0^g)} \ .$$

By (III.1.2), this last vector space is $\mathcal{V}^{\eta(\mathfrak{B}_q,\mathfrak{B}_0)^g}$. Now our result follows from the following observation. Let (K,ρ) be a smooth irreducible representation of a compact open subgroup K of G and let $g \in G$. Then $g\mathcal{V}^\rho = \mathcal{V}^{\rho^g}$, where ρ^g is the representation of $K^g = gKg^{-1}$ given by $\rho^g(k) = \rho(g^{-1}kg)$.

V. The coefficient system on $\mathrm{sd}(X)$

We keep the notations from the previous sections. As in (I) we see $\mathrm{sd}(X_E)$ as a subcomplex of $\mathrm{sd}(X)$. We are now going to construct a coefficient system $\mathcal{C}(\mathcal{V}) = \mathcal{C}(\Theta, V, \mathcal{V})$ on $\mathrm{sd}(X)$.

For any subscript "$*$", we shall write $\mathfrak{A}_*$ for $\mathfrak{A}(\mathfrak{B}_*)$. In particular, if $(\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ is a flag of orders in $\mathrm{Her}(B)$ then $(\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$, $(\mathfrak{A}(\mathfrak{B}_0) \supset \ldots \supset \mathfrak{A}(\mathfrak{B}_q))$ and $(\mathfrak{A}_0 \supset \ldots \supset \mathfrak{A}_q)$ denote the same object, i.e. a simplex of $\mathrm{sd}(X_E)$ seen as a simplex of $\mathrm{sd}(X)$.

We shall need the two following lemmas.

(V.1) Lemma. *Let (ρ, W) be a smooth irreducible representation of some compact open subgroup $K \subseteq G$ and let $\mathcal{V}^\rho$ denote the ρ-isotypic component of $\mathcal{V}$. Then $\mathcal{V}^\rho$ is invariant under any subgroup of $N_G(K)$ which intertwines ρ.*

Proof. Let $v \in \mathcal{V}^\rho$ and $g \in G$ be an element normalizing K and intertwining ρ. By definition, there exist φ in $\mathrm{Hom}_\rho(W, \mathcal{V})$ and $w \in W$ such that $v = \varphi(w)$. Since $gKg^{-1} = K$ and $\rho^g \simeq \rho$, and since ρ is irreducible, there must exist an intertwining operator $\psi \in \mathrm{Aut}_{\mathbb{C}}(W)$ such that $\rho^g(k) = \psi^{-1} \circ \rho(k) \circ \psi$, for all $k \in K$. It easily follows that $g\varphi\psi^{-1}$ belongs to $\mathrm{Hom}_\rho(W, \mathcal{V})$. So $gv = [g\varphi\psi^{-1}](\psi(w))$ and $gv \in \mathcal{V}^\rho$, as required.

(V.2) Lemma. *Let $H \subseteq K$ be compact open subgroups of G. Let η_H be an irreducible smooth representation of H and assume that $\eta_K := \mathrm{Ind}_H^K \eta_H$ is irreducible as well. Then $\mathcal{V}^{\eta_K} = K\mathcal{V}^{\eta_H}$.*

Proof. Let $\Phi \in \mathrm{Hom}(\eta_K, \mathcal{V})$. We may see η_H as an H-submodule of η_K so that

$$\eta_K = \bigoplus_{k \in K/H} k\eta_H \ .$$

So

$$\Phi(\eta_K) = \sum_{k \in K/H} k\Phi(\eta_H) \ ,$$

with $\Phi(\eta_H)$ contained in $\mathcal{V}^{\eta_H}$ since Φ is H-equivariant. This gives the inclusion $\mathcal{V}^{\eta_K} \subseteq K\mathcal{V}^{\eta_H}$ Conversely, since the smooth representations of H are semisimple, $\mathcal{V}^{\eta_H}$ decomposes into a direct sum

$$\mathcal{V}^{\eta_H} = \bigoplus_{i \in I} \mathcal{V}_i \ ,$$

each $\mathcal{V}_i$ being isomorphic to η_H as an H-module. Now each $K.\mathcal{V}_i \subseteq \mathcal{V}$ is isomorphic to η_K as a K-module and the opposite inclusion follows.

(V.3) Definition. *For σ a semistandard simplex in* $\mathrm{sd}(X_E)$*, we set*

$$\mathcal{V}_\sigma = \sum_{g \in \mathrm{Stab}_G(\sigma)} g\mathcal{V}(\sigma) \subseteq \mathcal{V} \ .$$

(V.4) Lemma. *Let $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ be a semistandard simplex of* $\mathrm{sd}(X_E)$*. Then*

$$\mathrm{Stab}_G(\sigma) = E^\times U(\mathfrak{A}_q) \ .$$

Proof. The group $E^\times U(\mathfrak{A}_q)$ certainly normalizes $(\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q) = (\mathfrak{A}_0 \supset \ldots \supset \mathfrak{A}_q)$ and lies in $\mathrm{Stab}_G(\sigma)$. Conversely if $g \in \mathrm{Stab}_G(\sigma)$, then g normalizes the principal order $\mathfrak{A}_0$ and must lie in its stabilizer which by (I.3.1) is equal to $E^\times U(\mathfrak{A}_0)$. Write $g = \lambda h$, with $\lambda \in E^\times$ and $h \in U(\mathfrak{A}_0)$. Since λ is in $\mathcal{N}(\mathfrak{A}_q) = \mathcal{N}(\mathfrak{B}_q)U(\mathfrak{A}_q)$, so is h. Now h must be in the maximal compact subgroup of $\mathcal{N}(\mathfrak{B}_q)U(\mathfrak{A}_q)$, that is $U(\mathfrak{A}_q)$, and the lemma follows.

(V.5) Proposition. *Let $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ be a semistandard simplex of* $\mathrm{sd}(X_E)$*. Then:*

$$\mathcal{V}_\sigma = \sum_{g \in U(\mathfrak{A}_q)/U(\mathfrak{B}_q)J^1(\mathfrak{B}_0)} g\mathcal{V}(\sigma) = \sum_{g \in U(\mathfrak{A}_q)/U^1(\mathfrak{A}_q)} g\mathcal{V}^{\eta(\mathfrak{A}_q)} \ .$$

Proof. The subgroup $U(\mathfrak{B}_q)J^1(\mathfrak{B}_0)$ normalizes $J^1(\mathfrak{B}_q, \mathfrak{B}_0) = U^1(\mathfrak{B}_q)J^1(\mathfrak{B}_0)$. Moreover it intertwines $\eta(\mathfrak{B}_q, \mathfrak{B}_0)$ by (III.1.1)(v). As a consequence of (V.1), (V.4), and the definition of $\mathcal{V}(\sigma)$ we therefore obtain the first equality in

$$\mathcal{V}_\sigma = \sum_{E^\times U(\mathfrak{A}_q)/U(\mathfrak{B}_q)J^1(\mathfrak{B}_0)} g\mathcal{V}(\sigma) = \sum_{U(\mathfrak{A}_q)/U^1(\mathfrak{B}_q)J^1(\mathfrak{B}_0)} g\mathcal{V}(\sigma) \ .$$

The second one is immediate from the fact that $E^\times$ stabilizes $\mathcal{V}(\sigma)$. Now, using (III.1.1)(iii), we may apply (V.2) with $H = J^1(\mathfrak{B}_q, \mathfrak{B}_0)$, $K = U^1(\mathfrak{A}_q)$, $\eta_H = \eta(\mathfrak{B}_q, \mathfrak{B}_0)$, $\eta_K = \eta(\mathfrak{A}_q)$ to get the second equality in the proposition.

(V.6) Proposition. *Let σ and τ be semistandard simplices of* $\mathrm{sd}(X_E)$ *with $\sigma \subseteq \tau$. Then $\mathcal{V}_\tau \subseteq \mathcal{V}_\sigma$.*

Proof. Write $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ and $\tau = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_r)$, with $\mathfrak{B}_0$ maximal. By (IV.2)(ii), we have $\mathcal{V}(\tau) \subseteq \mathcal{V}(\sigma)$. Moreover $U(\mathfrak{A}_r) \subseteq U(\mathfrak{A}_q)$. Our inclusion follows now from the first equality in (V.5).

(V.7) Proposition. *Let σ and τ be semistandard simplices of* $\mathrm{sd}(X_E)$*, and assume that $\tau = g\sigma$ for some $g \in G$. Then $g\mathcal{V}_\sigma = \mathcal{V}_\tau$.*

Proof. Write $\sigma = (\mathfrak{B}_0^\sigma \supset \ldots \supset \mathfrak{B}_q^\sigma)$ and $\tau = (\mathfrak{B}_0^\tau \supset \ldots \supset \mathfrak{B}_q^\tau)$. Using (I.3.3), we can decompose g as $g_E g_\sigma$, $g_E \in G_E$, $g_\sigma \in \mathrm{Stab}_G(\sigma)$. By construction we have $g_\sigma V_\sigma = V_\sigma$. So $gV_\sigma = g_E V_\sigma$. We get:

$$gV_\sigma = \sum_{g \in U(\mathfrak{A}(\mathfrak{B}_q^\sigma))} g_E g g_E^{-1} g_E V(\sigma) \ .$$

By (IV.3), we have $g_E V(\sigma) = V(\tau)$, and it follows that

$$gV_\sigma = \sum_{h \in U(\mathfrak{A}(\mathfrak{B}_q^\sigma))^{g_E}} hV(\tau) \ .$$

Now the result follows from the G_E-equivariance of the map $\mathfrak{B} \mapsto U(\mathfrak{A}(\mathfrak{B}))$ and from the definition of V_τ.

(V.8) Definition. *A simplex of* $\mathrm{sd}(X)$ *is called* E-semistandard *if it is conjugate to a semistandard simplex of* $\mathrm{sd}(X_E)$*. We define a vector space $\mathcal{V}_\sigma$, for each simplex σ of* $\mathrm{sd}(X)$*, as follows:*

i) If $\sigma = g\tau$, for τ semistandard in $\mathrm{sd}(X_E)$ *and $g \in G$, then $\mathcal{V}_\sigma = g\mathcal{V}_\tau$;*

ii) If σ is an arbitrary simplex of $\mathrm{sd}(X)$*, then*

$$\mathcal{V}_\sigma = \sum_{\tau \ E-\text{semistandard}, \tau \supseteq \sigma} \mathcal{V}_\tau \ .$$

(V.9) Proposition. *i) The previous definition is consistent.*

ii) For any pair of simplices $\sigma \subseteq \tau$ of $\mathrm{sd}(X)$*, we have $\mathcal{V}_\tau \subseteq \mathcal{V}_\sigma$. In particular, by taking inclusions as transition maps, the collection $\mathcal{C}(\mathcal{V}) := (\mathcal{V}_\sigma)_\sigma$ is a coefficient system of $\mathbb{C}$-vector spaces on* $\mathrm{sd}(X)$*.*

iii) For the obvious action of G, the coefficient system $\mathcal{C}(\mathcal{V})$ is equivariant.

Proof. i) By (V.7) the definition of $\mathcal{V}_\sigma$ in (V.8)(i) does not depend on the choice of g.

To prove that the definition (V.8)(ii) is consistent, we must prove that if σ and τ are E-semistandard simplices of $\mathrm{sd}(X)$ satisfying $\tau \supseteq \sigma$, then $\mathcal{V}_\tau \subseteq \mathcal{V}_\sigma$. Write $\sigma = g\sigma_o$, $\tau = h\tau_o$, with $g, h \in G$ and $\sigma_o = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$, $\tau_o = (\mathfrak{C}_0 \supset \ldots \supset \mathfrak{C}_r)$ semistandard in $\mathrm{sd}(X_E)$. By definition $\mathcal{V}_\sigma = g\mathcal{V}_{\sigma_o}$ and $\mathcal{V}_\tau = h\mathcal{V}_{\tau_o}$.

The hypothesis $\tau \supseteq \sigma$ implies $\mathfrak{B}_i = g^{-1}h\mathfrak{C}_{j(i)}$ for some $j(0) = 0 < \ldots < j(q) \leq r$. By (V.6) and (V.7), we have

$$\mathcal{V}_\tau = h\mathcal{V}_{\tau_o} = h\mathcal{V}_{(\mathfrak{C}_0 \supset \ldots \supset \mathfrak{C}_r)} \subseteq h\mathcal{V}_{(\mathfrak{C}_{j(0)} \supset \ldots \supset \mathfrak{C}_{j(q)})} = hh^{-1}g\mathcal{V}_{\sigma_o} = g\mathcal{V}_{\sigma_o} = \mathcal{V}_\sigma \ .$$

ii) We can obviously reduce to the case where σ and τ are E-semistandard, and the inclusion has just been proved in i).

iii) We must simply prove that $g\mathcal{V}_\sigma = \mathcal{V}_{g\sigma}$, for any simplex σ of $\mathrm{sd}(X)$ and $g \in G$. We may reduce to the case where σ is E-semistandard where the result follows trivially from the definition of $\mathcal{C}(\mathcal{V})$.

By construction the coefficient system $\mathcal{C}(\mathcal{V})$ is supported on

$$X(E) = \bigcup_{g \in G} g\mathrm{sd}(X_E)$$

viewed naturally as a simplicial subcomplex of $\mathrm{sd}(X)$. But $\mathcal{C}(\mathcal{V})$ has the following additional constancy property.

(V.10) Proposition. *Let the vertex $\sigma_o = (\mathfrak{B})$ in $\mathrm{sd}(X_E)$ be the barycenter of a simplex $\widetilde{\sigma}$ of X_E; then $\mathcal{V}_\sigma = \mathcal{V}_{\sigma_o}$ for any simplex σ in $\mathrm{sd}(X_E)$ such that $\sigma_o \subseteq \sigma \subseteq \widetilde{\sigma}$.*

Proof. Put $\mathfrak{A} := \mathfrak{A}(\mathfrak{B})$ and

$$\mathcal{V}_o := \sum_{g \in U(\mathfrak{A})/U^1(\mathfrak{A})} g\mathcal{V}^{\eta(\mathfrak{A})} \ .$$

If σ is semistandard then $\mathcal{V}_\sigma = \mathcal{V}_o$ by (V.5). Consider therefore the case that $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ with $\mathfrak{B}_q = \mathfrak{B}$ is not semistandard, and let τ be any E-semistandard simplex in $\mathrm{sd}(X)$ such that $\tau \supseteq \sigma$. We have to show that $\mathcal{V}_\tau \subseteq \mathcal{V}_o$. Write $\tau = g\tau_o$ with $g \in G$ and $\tau_o = (\mathfrak{C}_0 \supset \ldots \supset \mathfrak{C}_r)$ semistandard in $\mathrm{sd}(X_E)$. By (I.3.3) we may assume that $\tau_0 \supseteq g^{-1}\sigma = \sigma$. We then have $\mathfrak{B}_i = \mathfrak{C}_{j(i)}$ for some $0 \leq j(0) < \ldots < j(q) \leq r$. Since τ_0 is semistandard whereas σ is not the order $\mathfrak{C}_0$ is maximal but $\mathfrak{B}_0$ is not. This means that $0 < j(0)$. It follows that $\tau_1 := (\mathfrak{C}_0 \supset \mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ is a semistandard simplex in $\mathrm{sd}(X_E)$ such that $\tau_0 \supseteq \tau_1 \supseteq \sigma$. Hence $\tau = g\tau_0 \supseteq g\tau_1 \supseteq g\sigma = \sigma$. Since $\tau \supseteq g\tau_1$ both are E-semistandard we know from the proof of (V.9)(i) that $\mathcal{V}_\tau \subseteq \mathcal{V}_{g\tau_1}$. On the other hand, by (I.3.1) we may write $g = hg'$ with $h \in \mathcal{N}(\mathfrak{B}) \subseteq G_E$ and $g' \in U(\mathfrak{A})$. We obtain that $g\tau_1 = h\tau_1$ in fact is semistandard in $\mathrm{sd}(X_E)$. Using (V.5) we conclude that $\mathcal{V}_\tau \subseteq \mathcal{V}_{g\tau_1} = \mathcal{V}_{h\tau_1} = \mathcal{V}_o$.

This result is best expressed in the following way. The simplicial structure of X_E (before subdivision) can be described in terms of the simplicial structure of X as follows: The interiors $\widetilde{\sigma}^{\mathrm{o}}$ of simplices $\widetilde{\sigma}$ of X_E are precisely the (nonempty) subsets of the form $\widetilde{\tau}^{\mathrm{o}} \cap X_E$ for some simplex $\widetilde{\tau}$ of X.

Suppose now that $\widetilde{\sigma}_1, \widetilde{\sigma}_2$ are simplices of X_E such that

$$g(\widetilde{\sigma}_1)^{\mathrm{o}} \cap (\widetilde{\sigma}_2)^{\mathrm{o}} \neq \emptyset \qquad \text{for some } g \in G\ .$$

Write $(\widetilde{\sigma}_i)^{\mathrm{o}} = (\widetilde{\tau}_i)^{\mathrm{o}} \cap X_E$ with simplices $\widetilde{\tau}_i$ of X. Then $g(\widetilde{\tau}_1)^{\mathrm{o}} \cap (\widetilde{\tau}_2)^{\mathrm{o}} \neq \emptyset$ and hence $g\widetilde{\tau}_1 = \widetilde{\tau}_2$ since the G-action on X is simplicial. In particular, g maps the barycenter of $\widetilde{\tau}_1$ into the barycenter of $\widetilde{\tau}_2$. Since both barycenters lie in X_E we conclude from (I.3.3) that there also is an element $g_E \in G_E$ which maps the first barycenter into the second one. It follows that $g_E\widetilde{\sigma}_1 = \widetilde{\sigma}_2$ and $g_E\widetilde{\tau}_1 = \widetilde{\tau}_2$. Hence gg_E^{-1} fixes $\widetilde{\tau}_2$ and, by (I.3.1) applied to its barycenter, can be written $gg_E^{-1} = h_E h$ with $h_E \in G_E$ fixing $\widetilde{\sigma}_2$ and $h \in G$ fixing $\widetilde{\tau}_2$ pointwise. We obtain

$$g(\widetilde{\sigma}_1)^{\mathrm{o}} = gg_E^{-1}(\widetilde{\sigma}_2)^{\mathrm{o}} = h_E h(\widetilde{\sigma}_2)^{\mathrm{o}} = h_E(\widetilde{\sigma}_2)^{\mathrm{o}} = (\widetilde{\sigma}_2)^{\mathrm{o}}\ .$$

¿From this fact one deduces in a straightforward way that $X(E)$ carries a simplicial structure where the simplices are the subsets of the form $g\widetilde{\sigma}$ for $\widetilde{\sigma}$ a simplex of X_E and $g \in G$. We write $X[E]$ for $X(E)$ equipped with this simplicial structure. Similarly as for X_E the interiors of simplices of $X[E]$ are the nonempty intersections $\widetilde{\tau}^{\mathrm{o}} \cap X[E]$ for $\widetilde{\tau}$ running over the simplices of X. The barycentric subdivision of $X[E]$ is $X(E)$.

The Proposition (V.10) (together with G-equivariance) says that $\mathcal{C}(\mathcal{V})$ in an obvious way derives from an equivariant coefficient system $\mathcal{C}[\mathcal{V}] = \mathcal{C}[\Theta, V, \mathcal{V}] := (\mathcal{V}[\sigma])_\sigma$ on $X[E]$ given by

$$\mathcal{V}[\sigma] := \sum_{g \in U(\mathfrak{A})/U^1(\mathfrak{A})} hg\mathcal{V}^{\eta(\mathfrak{A})}$$

for any simplex σ of $X[E]$ where σ is the image under some $h \in G$ of a simplex of X_E with barycenter $(\mathfrak{B})$ and where $\mathfrak{A} := \mathfrak{A}(\mathfrak{B})$.

VI. The degenerate case

We recall that in the degenerate case we have $E = F$, $B = A$, $H^1(\mathfrak{A}) = J^1(\mathfrak{A}) = U^1(\mathfrak{A})$, and $\theta(\mathfrak{A}) = \eta(\mathfrak{A}) = \mathbf{1}$. The coefficient system $\mathcal{C}(\mathcal{V}) = (\mathcal{V}_\sigma)_\sigma$ on X

associated with a smooth complex representation $\mathcal{V}$ of G is given by the fixed vectors

$$\mathcal{V}_\sigma = \mathcal{V}^{U^1(\mathfrak{A})} \qquad \text{for } \sigma = F(\mathfrak{A}).$$

This is precisely one of the coefficient systems considered in [SS] (namely the one corresponding to "level" $n = 1$). From loc. cit. we therefore have the following result.

(VI.1) Theorem. *The oriented chain complex of $\mathcal{C}(\mathcal{V})$ is an exact resolution in the category of all smooth complex G-representations of the subrepresentation of $\mathcal{V}$ generated (as a G-representation) by $\mathcal{V}^{U^1(\mathfrak{A}_0)}$ for some vertex $F(\mathfrak{A}_0)$. Moreover, if the center of G acts on $\mathcal{V}$ through a character χ then this resolution is a projective resolution in the category of smooth G-representations with central character χ.*

VII. Dependence on the endo-class

We fix a finite dimensional F-vector space V. Let Θ_i, for $i = 1, 2$, be two ps-characters with simultaneous realizations in $A = \mathrm{Aut}_F V$. So for each i we are in one of the following cases:

1) The ps-character Θ_i is supported by a simple pair $[0, \beta_i]$ and V is an E_i-vector space, where $E_i = F(\beta_i)$, and as such will be denoted by V_i. Following previous notations we have the centralizer B_i of E_i in A, the centralizer G_{E_i} of $E_i^\times$ in $G = A^\times$, the affine building X_{E_i} of G_{E_i}, and $X[E_i] = \bigcup_{g\in G} gX_{E_i}$ equipped with the simplicial structure defined in (V).

2) The degenerate case.

(VII.1) Lemma. *If Θ_1 and Θ_2 are endo-equivalent then $X[E_1] = X[E_2]$ as sets and simplicial complexes.*

Proof. It suffices to prove that the barycentric subdivisions $X(E_1)$ and $X(E_2)$ coincide as simplicial subcomplexes of $\mathrm{sd}(X)$. Being the fixed points sets of groups acting simplicially on $\mathrm{sd}(X)$ they are full subcomplexes. Hence they coincide if they have the same vertices. For $i = 1, 2$, a vertex $(\mathfrak{A})$ of $\mathrm{sd}(X)$ lies in $X(E_i)$ if and only if the order $\mathfrak{A} \in \mathrm{Her}(A)$ satisfies the numerical criterion of (I.3.5). But by (II.3.3), since Θ_1 and Θ_2 are endo-equivalent, we have $f(E_1/F) = f(E_2/F)$, $e(E_1/F) = e(E_2/F)$ and the numerical criteria for $X(E_1)$ and $X(E_2)$ are the same.

(VII.2) Proposition. *Let $\mathcal{V}$ be a smooth representation of G. If Θ_1 and Θ_2 are endo-equivalent, then the two coefficient systems $\mathcal{C}[\Theta_i, V_i, \mathcal{V}]$ coincide.*

Proof. If Θ_1 and Θ_2 are endo-equivalent, then they are both supported by simple pairs or are both degenerate. In this latter case the coefficient systems in question coincide for trivial reasons. So we may assume in the following that Θ_i, for $i = 1, 2$, is supported by a simple stratum $[0, \beta_i]$ and use the notations of case 1). Recall that the ps-character Θ_i gives rise to the simple character valued function $\theta_i := \Theta_i(V_i, .)$ on $\mathrm{Her}(B_i)$. We write $\eta_i(V_i, .)$ for the representations corresponding to θ_i which were introduced in (III.1).

We now fix a simplex σ of $X[E_1] = X[E_2]$ and write $\sigma = h_i\sigma_i$ with $h_i \in G$ and σ_i a simplex of X_{E_i}. Let $(\mathfrak{B}_i)$, for $\mathfrak{B}_i \in \mathrm{Her}(B_i)$, be the barycenter of σ_i and put $\mathfrak{A}_i := \mathfrak{A}(\mathfrak{B}_i) \in \mathrm{Her}(A)$. We have to show that the identity

$$h_1(\sum_{g\in U(\mathfrak{A}_1)/U^1(\mathfrak{A}_1)} g\mathcal{V}^{\eta_1(V_1,\mathfrak{A}_1)}) = h_2(\sum_{g\in U(\mathfrak{A}_2)/U^1(\mathfrak{A}_2)} g\mathcal{V}^{\eta_2(V_2,\mathfrak{A}_2)})$$

holds true. Since $\mathfrak{A}_1^{h_1}$ and $\mathfrak{A}_2^{h_2}$ both correspond to the barycenter of σ they are equal. Hence setting $h := h_1^{-1}h_2$ the above identity can equivalently be written as

$$\sum_{g\in U(\mathfrak{A}_1)/U^1(\mathfrak{A}_1)} g\mathcal{V}^{\eta_1(V_1,\mathfrak{A}_1)} = \sum_{g\in U(\mathfrak{A}_1)/U^1(\mathfrak{A}_1)} gh\mathcal{V}^{\eta_2(V_2,\mathfrak{A}_2)} .$$

It therefore suffices to find an $x \in U(\mathfrak{A}_1)$ such that

$$\eta_1(V_1, \mathfrak{A}_1)^x \cong \eta_2(V_2, \mathfrak{A}_2)^h .$$

For this in turn it certainly is sufficient to show that

$$\theta_1(\mathfrak{B}_1)^x = \theta_2(\mathfrak{B}_2)^h .$$

Let V_2^h be the F-vector space $V_2 = V$ with the new E_2-vector space structure given by $E_2 \hookrightarrow \mathrm{End}_{E_2}V_2 \xrightarrow{h.h^{-1}} \mathrm{End}_F V$. By (III.1.3)(a) we have $\Theta_2(V_2, \mathfrak{B}_2)^h \in \mathcal{C}(V_2^h, \mathfrak{B}_2^h)$. Hence there is a unique ps-character Θ_2^h supported by $[0, \beta_2]$ such that $\Theta_2^h(V_2^h, \mathfrak{B}_2^h) = \Theta_2(V_2, \mathfrak{B}_2)^h$. Obviously $\Theta_2(V_2, \mathfrak{B}_2)$ and $\Theta_2^h(V_2^h, \mathfrak{B}_2^h)$ are simultaneous realizations which intertwine in G. Therefore Θ_2 and Θ_2^h and hence Θ_1 and Θ_2^h are endo-equivalent. Since $\mathfrak{A}(\mathfrak{B}_2^h) = \mathfrak{A}_2^h = \mathfrak{A}_1 = \mathfrak{A}(\mathfrak{B}_1)$ we may apply (II.3.4) to Θ_1 and Θ_2^h and obtain an $x \in U(\mathfrak{A}_1)$ such that

$$\Theta_1(V_1, \mathfrak{B}_1)^x = \Theta_2^h(V_2^h, \mathfrak{B}_2^h) = \Theta_2(V_2, \mathfrak{B}_2)^h .$$

VIII. On the support of $\mathcal{C}(\Theta, V, \mathcal{V})$.

We fix a simple type (J, λ) in the sense of [BK1](5.5.10). Recall that this means one of the following two possibilities.

(a) There are given a simple pair $[0,\beta]$, an E-vector space V where $E := F(\beta)$, and a *principal* order $\mathfrak{B}_o$ in $\mathrm{Her}(B)$ where $B := \mathrm{End}_E V$. The representation λ of the group $J := J(\mathfrak{B}_o) := J^1(\mathfrak{B}_o)\cdot U(\mathfrak{B}_o)$ is of the form $\kappa\otimes\rho$, where κ is a β-extension to J of a simple character $\theta_o \in \mathcal{C}(V,\mathfrak{B}_o)$ (cf. [BK1](5.2.1)) and ρ is the inflation of an irreducible cuspidal representation of $J/J^1(\mathfrak{B}_o)$ of the following kind. Recall that $J/J^1(\mathfrak{B}_o)$ identifies with $\mathrm{GL}_{n/e}(\mathbb{F}_E)^{\times e}$, where $n := \dim_E V$, e is the period of the $\mathfrak{o}_E$-lattice chain corresponding to $\mathfrak{B}_o$, and $\mathbb{F}_E$ denotes the residue class field of E. Then the condition on ρ is that, as a representation of $\mathrm{GL}_{n/e}(\mathbb{F}_E)^{\times e}$, it is of the form $\rho_o^{\otimes e}$, where ρ_o is an irreducible cuspidal representation of $\mathrm{GL}_{n/e}(\mathbb{F}_E)$. We let Θ_o denote the unique ps-character supported by $[0,\beta]$ such that $\Theta_o(V,\mathfrak{B}_o)=\theta_o$.

(b) We are in the degenerate case. There is given an F-vector space V and a *principal* order $\mathfrak{A}_o$ in $A := \mathrm{End}_F V$. The representation λ of the group $J := J(\mathfrak{A}_o) := U(\mathfrak{A}_o)$ is the inflation of an irreducible cuspidal representation of $U(\mathfrak{A}_o)/U^1(\mathfrak{A}_o)$ of the following kind. We have $U(\mathfrak{A}_o)/U^1(\mathfrak{A}_o) \cong \mathrm{GL}_{n/e}(\mathbb{F}_F)^{\times e}$, where $n := \dim_F V$ and e is the period of the o_F-lattice chain corresponding to $\mathfrak{A}_o$. Then $\lambda \cong \rho_o^{\otimes e}$ for some irreducible cuspidal representation ρ_o of $\mathrm{GL}_{n/e}(\mathbb{F}_F)$. In order to have a notation consistent with the case a), we set $E := F$, $B := A$, $\mathfrak{B}_o := \mathfrak{A}_o$, $\kappa := \mathbf{1}_J$, $\theta_o := \mathbf{1}_{J^1(\mathfrak{A}_o)}$, $\rho := \lambda$, and we let Θ_o denote the corresponding degenerate ps-character.

Let $\mathcal{R}(G)$ denote the category of smooth complex representations of $G := \mathrm{Aut}_F V$. Recall that the full subcategory $\mathcal{R}_{(J,\lambda)}(G)$ whose objects are the representations generated by their λ-isotypic component is stable under the formation of subquotients. It coincides with a Bernstein component of $\mathcal{R}(G)$ attached to a single point in the Bernstein spectrum of G (cf. [BK1] and [BK3](9.3)).

Throughout this section we fix a nonzero representation $\mathcal{V}$ in $\mathcal{R}_{(J,\lambda)}(G)$.

Remark. The coefficient systems $\mathcal{C}_o(\Theta_o,V,\mathcal{V})$ and $\mathcal{C}(\Theta_o,V,\mathcal{V})$ are nonzero.

Proof. Choose a maximal order $\mathfrak{B}$ in $\mathrm{Her}(B)$ containing $\mathfrak{B}_o$ so that $\sigma := (\mathfrak{B}\supset\mathfrak{B}_o)$ is a semistandard simplex of $\mathrm{sd}(X_E)$. It is a consequence of (III.1.1)(iii) that $\mathcal{V}(\sigma)$ and $\mathcal{V}^{\eta(\theta_o)}$ generate the same $U^1(\mathfrak{A}(\mathfrak{B}_o))$-invariant subspace of $\mathcal{V}$. But the latter contains the isotypic component $\mathcal{V}^\lambda$ which is nonzero by assumption.

VIII.1 Endo-classes.

Let Θ' be an arbitrary ps-character which can be realized in an E'-vector space V' which as an F-vector space coincides with V. We assume that Θ' is either degenerate or supported by a simple pair $[0,\beta']$; in particular $E'=F$ or $E'=F(\beta')$. Let $B' := \mathrm{End}_{E'}(V')$ and let $\eta(\ldots)=\eta(V',\ldots)$ denote the various representations attached to Θ' and V' as introduced in (III.1).

(VIII.1.1) Lemma. *Assume that there exists $\mathfrak{C}_0 \in \mathrm{Her}(B')$ such that $\mathcal{V}$ contains the simple character $\Theta'(V',\mathfrak{C}_0)$. Then there exist a $\mathfrak{C}\in\mathrm{Her}(B')$, a β'-extension κ' of the Heisenberg representation $\eta(\mathfrak{C})$ attached to $\Theta'(V',\mathfrak{C})$, and* *an irreducible cuspidal representation ρ' of the $\mathbb{F}_{E'}$-reductive group $U(\mathfrak{C})/U^1(\mathfrak{C})$ such that $\mathcal{V}$ contains the representation $\kappa' \otimes \rho'$ of $J(\mathfrak{C})$.*

Proof. (This fact actually is a consequence of the proof of [BK1](8.1.5), p. 268/269. We shall nevertheless give the argument, for the context of *loc. cit.* is slightly different.)
Take $\mathfrak{C}$ minimal among the orders of $\mathrm{Her}(B')$ such that $\mathcal{V}$ contains $\Theta'(V', \mathfrak{C})$. Then $\mathcal{V}$ must contain the Heisenberg representation $\eta(\mathfrak{C})$ associated to $\Theta'(V', \mathfrak{C})$ and a fortiori an irreducible representation λ' of $J(\mathfrak{C})$ such that $\lambda'_{|J^1(\mathfrak{C})}$ contains $\eta(\mathfrak{C})$. By [BK1](5.2.2) such a representation λ' is of the form $\lambda' = \kappa' \otimes \rho'$, where κ' is a β'-extension of $\eta(\mathfrak{C})$ and ρ' is (the inflation of) an irreducible representation of $J(\mathfrak{C})/J^1(\mathfrak{C}) = U(\mathfrak{C})/U^1(\mathfrak{C})$. It remains to prove that the minimality condition on $\mathfrak{C}$ implies that ρ' is cuspidal. Assume therefore that ρ' is not cuspidal. Then there exists a proper parabolic subgroup $\mathbb{P}$ of $U(\mathfrak{C})/U^1(\mathfrak{C})$, with unipotent radical $\mathbb{U}$, such that $\rho'_{|\mathbf{U}}$ contains the trivial character. There is a uniquely determined hereditary order $\mathfrak{C}_1 \subseteq \mathfrak{C}$ such that $\mathbb{P}$ (resp. $\mathbb{U}$) is the image of $U(\mathfrak{C}_1)$ (resp. $U^1(\mathfrak{C}_1)$) in the quotient $U(\mathfrak{C})/U^1(\mathfrak{C})$. Since $\mathbb{P}$ is proper, the containment $\mathfrak{C}_1 \subset \mathfrak{C}$ is strict. Let $\eta(\mathfrak{C}_1)$ be the Heisenberg representation associated to $\Theta'(V', \mathfrak{C}_1)$. We show that $\mathcal{V}$ contains $\eta(\mathfrak{C}_1)$, hence also $\Theta'(V', \mathfrak{C}_1)$, which contradicts the minimality assumption on $\mathfrak{C}$.

The representation $\mathcal{V}$ contains

$$(\kappa' \otimes \rho')_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})} = \kappa'_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})} \otimes \rho'_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})}$$

which contains

$$\kappa'_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})} \otimes \mathbf{1}_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})} = \kappa'_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})} \ .$$

Hence our claim follows from [BK1](8.1.6) which says that the representations of $U^1(\mathfrak{A}(\mathfrak{C}_1))$ induced by $\eta(\mathfrak{C}_1)$ and $\kappa'_{|U^1(\mathfrak{C}_1)J^1(\mathfrak{C})}$ are irreducible and equivalent to each other.

(VIII.1.2) Proposition. *If the coefficient system $\mathcal{C}(\Theta', V', \mathcal{V})$ is nonzero then the ps-characters Θ' and Θ_o are endo-equivalent and $\mathcal{C}[\Theta', V', \mathcal{V}] = \mathcal{C}[\Theta_o, V, \mathcal{V}]$.*

Proof. The second part of the assertion is a consequence of the first part by (VII.2). If $\mathcal{C}(\Theta', V', \mathcal{V})$ is nonzero then there is a vertex $(\mathfrak{C})$ of $\mathrm{sd}(X_{E'})$ such that $\mathcal{V}^{\eta(\mathfrak{A}(\mathfrak{C}))} \neq 0$. Let v be a nonzero vector in this isotypic component, let $\mathcal{V}_0$ be the G-subrepresentation of $\mathcal{V}$ generated by v, and let $\mathcal{V}_1 \subseteq \mathcal{V}_0$ be the largest G-subrepresentation which does not contain v. Then $\mathcal{V}_0/\mathcal{V}_1$ is an irreducible G-representation in the category $\mathcal{R}_{(J,\lambda)}(G)$. By construction we have $(\mathcal{V}_0/\mathcal{V}_1)^{\eta(\mathfrak{A}(\mathfrak{C}))} \neq 0$ and hence $\mathcal{C}(\Theta', V', \mathcal{V}_0/\mathcal{V}_1) \neq 0$. In order to show that Θ' and Θ_o are endo-equivalent, we may therefore assume in the following without loss of generality that $\mathcal{V}$ is irreducible. By definition if $\mathcal{C}(\Theta', V', \mathcal{V})$ is nonzero then $\mathcal{C}_o(\Theta', V', \mathcal{V})$ is nonzero, too. In particular there exists a semistandard simplex $(\mathfrak{C}_0 \supset \ldots \supset \mathfrak{C}_q)$ of $\mathrm{sd}(X_{E'})$ such that $\mathcal{V}^{\eta(\mathfrak{C}_q,\mathfrak{C}_0)} \neq 0$. Since $\eta(\mathfrak{C}_q,\mathfrak{C}_0)_{|J^1(\mathfrak{C}_0)} \simeq \eta(\mathfrak{C}_0)$, we have $\mathcal{V}^{\eta(\mathfrak{C}_0)} \neq 0$. On the other hand the condition that $\mathcal{V}$ lies in $\mathcal{R}_{(J,\lambda)}(G)$ implies that $\mathcal{V}^{\eta(\theta_o)} \neq 0$. Since $\mathcal{V}$ is irreducible it follows that the representations $(J^1(\mathfrak{B}_o),\eta(\theta_o))$ and $(J^1(\mathfrak{C}_0),\eta(\mathfrak{C}_0))$ intertwine in G. Moreover $\eta(\theta_o)_{|H^1(\mathfrak{B}_o)}$ (resp. $\eta(\mathfrak{C}_0)_{|H^1(\mathfrak{C}_0)}$) is a multiple of $\theta_o = \Theta_o(V,\mathfrak{B}_o)$ (resp. of $\Theta'(V',\mathfrak{C}_0)$); so these simple characters must intertwine as well. Hence, for the endo-equivalence of Θ_o and Θ', it remains to establish the equality $[E:F] = [E':F]$.

Applying (VIII.1.1) we have that $\mathcal{V}$ contains a pair $(J(\mathfrak{C}),\kappa'\otimes\rho')$, where $\mathfrak{C} \in \mathrm{Her}(B')$, κ' is a β'-extension of the Heisenberg representation $\eta(\mathfrak{C})$ attached to $\Theta'(V',\mathfrak{C})$, and ρ' is an irreducible cuspidal representation of $U(\mathfrak{C})/U^1(\mathfrak{C})$. Write

$$U(\mathfrak{C})/U^1(\mathfrak{C}) \simeq \prod_{i=1}^{e'} \mathrm{GL}_{m_i}(\mathbb{F}_{E'}) \ ,$$

where $e' := e(\mathfrak{C}/\mathfrak{o}_{E'})$ and the m_i are integers ≥ 1. Then ρ' writes $\rho'_1 \otimes \cdots \otimes \rho'_{e'}$, where, for $i = 1,\ldots,e'$, ρ'_i is an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{F}_{E'})$. The pair $(J(\mathfrak{C}),\kappa'\otimes\rho')$ is either a simple type or a split type in the sense of [BK1](8.1), according to whether the ρ'_i are equivalent to each other or are not. When it is a split type it has level $(0,0)$ ([BK1](8.1.2)) or level $(-v_{\mathfrak{A}(\mathfrak{C})}(\beta')/e(\mathfrak{A}(\mathfrak{C})/\mathfrak{o}_F),0)$ ([BK1](8.1.4)), according to whether Θ' is degenerate or not.

Assume first that $(J(\mathfrak{C}),\kappa'\otimes\rho')$ is a simple type. Since $\mathcal{V}$ is irreducible, it then is a type for the same Bernstein component $\mathcal{R}_{(J,\lambda)}(G)$ of $\mathcal{R}(G)$. By [BK1](7.3.17) the pairs (J,λ) and $(J(\mathfrak{C}),\kappa'\otimes\rho')$ must be conjugate in G, and in particular $e = e(\mathfrak{B}_o/\mathfrak{o}_E) = e(\mathfrak{C}/\mathfrak{o}_{E'}) = e'$ (cf. the proof of loc.cit.). Setting $n := \dim_E V$ and $n' := \dim_{E'} V$ it follows that

$$GL_{n/e}(\mathbb{F}_E)^{\times e} \cong J/J(\mathfrak{B}_o) \cong J(\mathfrak{C})/J^1(\mathfrak{C}) \cong GL_{n'/e'}(\mathbb{F}_{E'})^{\times e'} = GL_{n'/e}(\mathbb{F}_{E'})^{\times e}$$

and hence that $n = n'$, i.e., that $[E:F] = [E':F]$.

Assume now that $(J(\mathfrak{C}),\kappa'\otimes\rho')$ is a split type. In this case we need to consider the cuspidal support of the irreducible representation $\mathcal{V}$. From the point of view of the simple type (J,λ) in $\mathcal{V}$ we know from [BK1](7.3.12) that the cuspidal support of $\mathcal{V}$ is of the form (M,μ) where $\mu = \mu_1 \otimes \ldots \otimes \mu_e$ is a supercuspidal representation of the Levi subgroup $M = \mathrm{Aut}_F(W)^{\times e}$ corresponding to a decomposition $V = W \oplus \ldots \oplus W$ (e factors) of the E-vector space V. Moreover each supercuspidal representation μ_i contains “the maximal type” $(\widetilde{J},\widetilde{\lambda})$ attached to (J,λ) ([BK1](7.2.18)(iii)). We do not repeat the definition of $(\widetilde{J},\widetilde{\lambda})$ but only recall that its underlying simple pair still is $[0,\beta]$.

On the other hand, from the point of view of the split type $(J(\mathfrak{C}),\kappa'\otimes\rho')$ in $\mathcal{V}$ we deduce from [BK1](8.3.3) and (6.2.2) that the cuspidal support of $\mathcal{V}$ must be of the form (M', μ') where $\mu' = \mu'_1 \otimes \ldots \otimes \mu'_{e'}$ is a supercuspidal representation of the Levi subgroup $M' = \prod_{i=1}^{e'} \mathrm{Aut}_F(W_i)$ corresponding to a decomposition $V = W_1 \oplus \ldots \oplus W_{e'}$ of the E-vector space V. Moreover each supercuspidal representation μ'_i contains a simple type (J_i, λ_i) with underlying simple pair $[0, \beta']$. By unicity of the cuspidal support, the pairs (M, μ) and (M', μ') are conjugate in G. So after conjugation, we may reduce to the case where, e.g., the representation $\mu_1 \cong \mu'_1$ contains two simple types with underlying simple pairs $[0, \beta]$ and $[0, \beta']$, respectively, and we may conclude as in the first case.

Let $\mathrm{Coeff}_G(\mathrm{sd}(X))$ denote the category of G-equivariant coefficient systems on the simplicial complex $\mathrm{sd}(X)$. The Proposition (VIII.1.1) together with the above Remark imply that, given a simple type (J, λ), the functor

$$\begin{aligned} \mathcal{C}_{(J,\lambda)} : \mathcal{R}_{(J,\lambda)}(G) &\longrightarrow \mathrm{Coeff}_G(\mathrm{sd}(X)) \\ \mathcal{V} &\longmapsto \mathcal{C}(\Theta_o, V, \mathcal{V}) \end{aligned}$$

is independent of any additional choices. In order to be able later on to show that this functor in fact is a fully faithful embedding we first have to analyze the support of these coefficient systems.

VIII.2 The support of $\mathcal{C}[\Theta_o, V, \mathcal{V}]$.

As in (IV) and (V) we write $\mathcal{C}_o(\Theta_o, V, \mathcal{V}) = (\mathcal{V}(\sigma))_\sigma$ and $\mathcal{C}[\Theta_o, V, \mathcal{V}] = (\mathcal{V}[\sigma])_\sigma$. To start with we fix a simplex $\sigma_0 = (\mathfrak{B}_{max} \supset \ldots \supset \mathfrak{B}_{min})$ of maximal dimension in $\mathrm{sd}(X_E)$ such that $\mathfrak{B}_{min} \subseteq \mathfrak{B}_o \subseteq \mathfrak{B}_{max}$. Recall ([BK0](5.2.2-5)) that the β-extension κ gives rise to a compatible family of β-extensions $\kappa(\mathfrak{B})$ where $(\mathfrak{B})$ runs over the vertices of σ_0. These $\kappa(\mathfrak{B})$ are characterized as follows:

(a) The induced representations

$$\mathrm{Ind}_{J(\mathfrak{B})}^{U(\mathfrak{B})U^1(\mathfrak{A}(\mathfrak{B}))} \kappa(\mathfrak{B}) \quad \text{and} \quad \mathrm{Ind}_{U(\mathfrak{B})J^1(\mathfrak{B}_{max})}^{U(\mathfrak{B})U^1(\mathfrak{A}(\mathfrak{B}))} \kappa(\mathfrak{B}_{max})$$

are isomorphic (and irreducible);

(b) $\kappa(\mathfrak{B}_o) = \kappa$.

Set $\mathbb{G} = U(\mathfrak{B}_{max})/U^1(\mathfrak{B}_{max}) \simeq \mathrm{GL}_n(\mathbb{F}_E)$. Following [SZ]§5, we define the $\mathbb{G}$-module $\mathcal{V}(\mathfrak{B}_{max}) := \mathrm{Hom}_{J^1(\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}), \mathcal{V})$, using the obvious action of $J(\mathfrak{B}_{max})$ and the canonical identification

$$J(\mathfrak{B}_{max})/J^1(\mathfrak{B}_{max}) \simeq U(\mathfrak{B}_{max})/U^1(\mathfrak{B}_{max}) \ .$$

Recall (loc. cit.) that for $\mathfrak{B}_{min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{max}$ the image of $U(\mathfrak{B})J^1(\mathfrak{B}_{max})$ in $J(\mathfrak{B}_{max})/J^1(\mathfrak{B}_{max})$ is a parabolic subgroup $\mathbb{P}_{\mathfrak{B}}$ of $\mathbb{G}$ whose unipotent radical $\mathbb{U}_{\mathfrak{B}}$ is the image of $U^1(\mathfrak{B})J^1(\mathfrak{B}_{max})$ and whose Levi quotient $\mathbb{L}_{\mathfrak{B}}$ canonically identifies with $U(\mathfrak{B})/U^1(\mathfrak{B})$.

(VIII.2.1) Proposition. *For any* $\mathfrak{B}_{min} \subseteq \mathfrak{B} \subseteq \mathfrak{B}_{max}$ *we have linear isomorphisms*

$$\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}}} \simeq \mathrm{Hom}_{J^1(\mathfrak{B},\mathfrak{B}_{max})}(\eta(\mathfrak{B},\mathfrak{B}_{max}),\mathcal{V}) \simeq \mathrm{Hom}_{J^1(\mathfrak{B})}(\eta(\mathfrak{B}),\mathcal{V}) \ ,$$

where $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}}}$ *denotes the Jacquet module with respect to the parabolic subgroup* $\mathbb{P}_{\mathfrak{B}}$.

Proof. According to the proof of Lemma 2 in [SZ]§5, we have:

$$\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}}} \simeq \mathrm{Hom}_{J^1(\mathfrak{B},\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V}) \simeq \mathrm{Hom}_{J^1(\mathfrak{B})}(\kappa(\mathfrak{B}),\mathcal{V}) \ .$$

So we must simply prove the isomorphisms:

$$\kappa(\mathfrak{B}_{max})_{|J^1(\mathfrak{B},\mathfrak{B}_{max})} \simeq \eta(\mathfrak{B},\mathfrak{B}_{max}) \ , \ \kappa(\mathfrak{B})_{|J^1(\mathfrak{B})} \simeq \eta(\mathfrak{B}) \ .$$

The second one is clear by definition of a β-extension. Write

$$\eta_{max} := \kappa(\mathfrak{B}_{max})_{|J^1(\mathfrak{B},\mathfrak{B}_{max})} \ .$$

Using Mackey's restriction formula, the restrictions to $U^1(\mathfrak{A}(\mathfrak{B}))$ of the isomorphic representations $\mathrm{Ind}_{J(\mathfrak{B})}^{U(\mathfrak{B})U^1(\mathfrak{A}(\mathfrak{B}))}\kappa(\mathfrak{B})$ and $\mathrm{Ind}_{U(\mathfrak{B})J^1(\mathfrak{B}_{max})}^{U(\mathfrak{B})U^1(\mathfrak{A}(\mathfrak{B}))}\kappa(\mathfrak{B}_{max})$ are

$$\mathrm{Ind}_{J^1(\mathfrak{B})}^{U^1(\mathfrak{A}(\mathfrak{B}))}\eta(\mathfrak{B}) \simeq \mathrm{Ind}_{J^1(\mathfrak{B},\mathfrak{B}_{max})}^{U^1(\mathfrak{A}(\mathfrak{B}))}\eta_{max} \ .$$

Moreover by definition of a β-extension $\eta_{max\,|J^1(\mathfrak{B}_{max})} \simeq \eta(\mathfrak{B}_{max})$. So by definition of $\eta(\mathfrak{B},\mathfrak{B}_{max})$ (cf. (III.1.1) and [BK1](5.1.16)), we have $\eta_{max} \simeq \eta(\mathfrak{B},\mathfrak{B}_{max})$, as required.

We shall also need the following fact from.

(VIII.2.2) Proposition ([SZ]§5 Prop. 3). *Any irreducible constituent of the* $\mathbb{G}$*-module* $\mathcal{V}(\mathfrak{B}_{max})$ *has cuspidal support* $(\mathbb{L}_{\mathfrak{B}_o},\rho)$.

As a corollary of the last two propositions we obtain the following result.

(VIII.2.3) Proposition. *Let* $\sigma = (\mathfrak{B}_{max} \supset \ldots \supset \mathfrak{B}_q)$ *be a semistandard simplex contained in* σ_0. *Then* $\mathcal{V}(\sigma) = \mathcal{V}^{\eta(\mathfrak{B}_q,\mathfrak{B}_{max})} \neq 0$ *if and only if* $\mathbb{L}_{\mathfrak{B}_q}$ *contains a Levi subgroup conjugate in* $\mathbb{G}$ *to* $\mathbb{L}_{\mathfrak{B}_o}$. *In other words, if the invariant of the conjugacy class of* $\mathbb{L}_{\mathfrak{B}_q}$ *is the unordered partition* $(n_1,\ldots,n_s)$ *of* n, *we have* $\mathcal{V}(\sigma) \neq 0$ *if and only if* n/e *divides* n_i *for any* $i = 1,\ldots,s$.

As in (I.3.3) we introduce, for any $\mathfrak{B} \in \mathrm{Her}(B)$, the integers

$$d_E(\mathfrak{B})_k := \dim_{\mathbb{F}_E} L_k/L_{k+1}$$

where $(L_k)_{k\in\mathbf{Z}}$ is an o_E-lattice chain in V corresponding to $\mathfrak{B}$. The condition of the proposition can be read off the sequence $[d_E(\mathfrak{B}_q)_k]_{k\in\mathbf{Z}}$ and, of course, only depends on the G_E-conjugacy class of $\mathfrak{B}_q$.

(VIII.2.4) Corollary. *Let $\sigma = (\mathfrak{B}_0 \supset \ldots \supset \mathfrak{B}_q)$ be any simplex of* $\mathrm{sd}(X_E)$. *Then $\mathcal{V}(\sigma) \neq 0$ if and only if n/e divides $d_E(\mathfrak{B}_q)_k$ for any $k \in \mathbb{Z}$.*

Proof. By definition $\mathcal{V}(\sigma) \neq 0$ if and only if their exists a semistandard simplex τ containing σ such that $\mathcal{V}(\tau) \neq 0$. So $\mathcal{V}(\sigma) \neq 0$ if and only if their exists $\mathfrak{B} \in \mathrm{Her(B)}$ such that $\mathfrak{B} \subseteq \mathfrak{B}_q$ and $(n/e)|d_E(\mathfrak{B})_k$ for any $k \in \mathbb{Z}$. But this implies that $(n/e)|d_E(\mathfrak{B}_q)_k$ for any $k \in \mathbb{Z}$, since any $\mathfrak{o}_E$-lattice occurring in a lattice chain defining $\mathfrak{B}_q$ occurs in any lattice chain defining $\mathfrak{B}$.

(VIII.2.5) Corollary. *Let σ be a simplex of $X[E]$; write σ as the image under some $h \in G$ of a simplex of X_E with barycenter* $(\mathfrak{B})$, $\mathfrak{B} \in \mathrm{Her}(B)$. *Then $\mathcal{V}[\sigma] \neq 0$ if and only if $(n/e)|d_E(\mathfrak{B})_k$ for any $k \in \mathbb{Z}$.*

Proof. By G-equivariance, we may assume that $h = 1$. We have

$$\mathcal{V}[\sigma] = \sum_{g\in U(\mathfrak{A})/U^1(\mathfrak{A})} g\mathcal{V}^{\eta(\mathfrak{A})} ,$$

where $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$. So if $\mathfrak{B}_{max}$ is any maximal order in $\mathrm{Her}(B)$ containing $\mathfrak{B}$, by (V.5), we have

$$\mathcal{V}[\sigma] = \sum_{g\in U(\mathfrak{A})} g\mathcal{V}((\mathfrak{B}_{max} \supset \mathfrak{B}))$$

and the result follows easily.

We are now going to describe the support of $\mathcal{C}[\Theta_o, V, \mathcal{V}]$ in terms of an auxiliary building. Thanks to (I.3.5), we find an unramified extension L of E contained in B, of degree n/e, and such that $L^\times$ normalizes $\mathfrak{B}_o$. Write $C := \mathrm{End}_L V \simeq \mathrm{M}(e, L)$ for the centralizer of L in B. From (I.2.1), we have a canonical G_L-equivariant embedding j_L of the semisimple affine building X_L of G_L into X_E. Since L/E is unramified, this embedding is actually simplicial; indeed in that case if $\mathfrak{C} \in \mathrm{Her}(C)$ is maximal (i.e. corresponds to a vertex in X_L) then the corresponding order $\mathfrak{B}(\mathfrak{C}) \in \mathrm{Her}(B)$ is maximal as well. We see X_L as a simplicial subcomplex of X_E and $\mathrm{sd}(X_L)$ as a simplicial subcomplex of X_E. So as in (V), we may consider the simplicial complex $X[L]$; this is a G-invariant simplicial subcomplex of $X[E]$.

(VIII.2.6) Proposition. *For any simplex σ of $X[E]$, we have $\mathcal{V}[\sigma] \neq 0$ if and only if σ lies in $X[L]$.*

Proof. By G-equivariance we may assume that σ actually lies in $\mathrm{sd}(X_E)$. We then must prove that $\mathcal{V}[\sigma] \neq 0$ if and only if $\sigma \in G_E(X_L)$. By the criterion of (I.3.5) (applied to "E/F"="L/E"), this latter condition is equivalent to $f(L/E)|d_E(\mathfrak{B})_k$ for any $k \in \mathbb{Z}$, where $\mathfrak{B}$ is the barycenter of σ. But $f(L/E) = n/e$. So we are done using (VIII.2.5).

We therefore may and will view the functor $\mathcal{C}_{(J,\lambda)}$ introduced at the end of (VIII.1) as a functor

$$\mathcal{C}_{(J,\lambda)} : \mathcal{R}_{(J,\lambda)}(G) \longrightarrow \mathrm{Coeff}_G(X[L]) \ .$$

IX. The chain complex attached to $\mathcal{C}_{(J,\lambda)}(\mathcal{V})$

As in the previous section we fix a simple type (J, λ) in $G = \mathrm{Aut}_F V$ where $\lambda = \kappa \otimes \rho$, with ps-character Θ_o having a realization in V, and a smooth complex representation $\mathcal{V}$ in $\mathcal{R}_{(J,\lambda)}(G)$. We also keep most of the other notations introduced in (VIII). We consider the G-equivariant coefficient system $\mathcal{C} := \mathcal{C}[\Theta_o, V, \mathcal{V}] = (\mathcal{V}[\sigma])_\sigma$ that we see as a coefficient system on the G-invariant simplicial subcomplex $X[L]$ of $X[E]$. This complex has dimension

$$d := \dim X[L] = \dim_F V/[L : F] - 1 = e - 1$$

where e is the divisor of $\dim_E V$ fixed in (VIII). We denote by $X[L]_q$ the set of q-simplices of $X[L]$ for $q = 0, \ldots, d$. The following considerations are copied from [SS].

An ordered q-simplex in $X[L]$ is a sequence $(\sigma_0, \ldots, \sigma_q)$ of vertices such that $\{\sigma_0, \ldots, \sigma_q\}$ is a q-simplex. Two such ordered simplices are called equivalent if they differ by an even permutation of the vertices; the corresponding equivalence classes are called oriented q-simplices and are denoted by $\langle\sigma_0, \ldots, \sigma_q\rangle$. We let $X[L]_{(q)}$ be the set of oriented q-simplices of $X[L]$. The space $C_q^{\mathrm{or}}(X[L], \mathcal{C})$ of oriented q-chains of $X[L]$ with coefficients in $\mathcal{C}$ is the $\mathbb{C}$-vector space of all maps

$$\omega \ : X[L]_{(q)} \longrightarrow \mathcal{V}$$

such that:

– ω has finite support,

– $\omega(\langle\sigma_0, \ldots, \sigma_q\rangle) \in \mathcal{V}[\{\sigma_0, \ldots, \sigma_q\}]$,

– $\omega(\langle\sigma_{\iota(0)}, \ldots, \sigma_{\iota(q)}\rangle) = \mathrm{sgn}(\iota) \cdot \omega(\langle\sigma_0, \ldots, \sigma_q\rangle)$ for any permutation ι.

The group G acts smoothly on $C_q^{\mathrm{or}}(X[L], \mathcal{C})$ via

$$(g\omega)(\langle\sigma_0, \ldots, \sigma_q\rangle) := g(\omega(\langle g^{-1}\sigma_0, \ldots, g^{-1}\sigma_q\rangle)) \ .$$

With respect to the G-equivariant boundary maps

$$\begin{array}{rcl} \partial \ : \ C^{\rm or}_{q+1}(X[L],\mathcal{C}) & \to & C^{\rm or}_q(X[L],\mathcal{C}) \\ \omega & \mapsto & [\langle\sigma_0,\ldots,\sigma_q\rangle \mapsto \sum\limits_{\{\sigma,\sigma_0,\ldots\sigma_q\}\in X[L]_{q+1}} \omega(\langle\sigma,\sigma_0,\ldots,\sigma_q\rangle)] \end{array}$$

we then have the augmented chain complex in $\mathcal{R}(G)$:

$$\text{(IX.1)} \qquad C^{\rm or}_d(X[L],\mathcal{C}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^{\rm or}_0(X[L],\mathcal{C}) \xrightarrow{\epsilon} \mathcal{V}$$

where $\epsilon(\omega) = \sum\limits_{\sigma\in X[L]_{(0)}} \omega(\sigma) \in \mathcal{V}$.

(IX.2) Proposition. *For all $q = 0,\ldots,d$, the G-module $C^{\rm or}_q(X[L],\mathcal{C})$ lies in $\mathcal{R}_{(J,\lambda)}(G)$. In particular the complex (IX.1) is a chain complex in the category $\mathcal{R}_{(J,\lambda)}(G)$.*

To prepare for the proof we let $\sigma_{\mathfrak{C}}$, for any $\mathfrak{C} \in \mathrm{Her}(C)$, denote the simplex of X_L with barycenter $(\mathfrak{C})$. Moreover let $<\!\sigma_{\mathfrak{C}}\!>$ be a fixed oriented simplex with underlying simplex $\sigma_{\mathfrak{C}}$ and let $\overline{<\!\sigma_{\mathfrak{C}}\!>}$ denote that oriented simplex with the same underlying simplex $\sigma_{\mathfrak{C}}$ but with the reversed orientation (for vertices we have $<\!\sigma_{\mathfrak{C}}\!> = \overline{<\!\sigma_{\mathfrak{C}}\!>}$). The order $\mathfrak{B}_o$ defining $J = J(\mathfrak{B}_o)$ corresponds to a minimal order $\mathfrak{C}_{min}$ of C. We write $\mathfrak{B}_{min} := \mathfrak{B}_o = \mathfrak{B}(\mathfrak{C}_{min})$ and put $\mathfrak{A}_{min} := \mathfrak{A}(\mathfrak{B}_{min})$. We fix a maximal order $\mathfrak{C}_{max}$ of C containing $\mathfrak{C}_{min}$ and put $\mathfrak{B}_{max} := \mathfrak{B}(\mathfrak{C}_{max})$ and $\mathfrak{A}_{max} := \mathfrak{A}(\mathfrak{B}_{max})$. We have $\mathfrak{A}_{min} \subset \mathfrak{A}_{max}$ and $\mathfrak{B}_{min} \subset \mathfrak{B}_{max}$ and since L/E is unramified, $\mathfrak{B}_{max}$ is a maximal order of B. Note that $\mathfrak{B}_{min}$ in general is not a minimal hereditary order of B.

Any simplex in $X[L]$ lies in the G-orbit of a simplex $\sigma_{\mathfrak{C}}$ with $\mathfrak{C}_{min} \subseteq \mathfrak{C} \subseteq \mathfrak{C}_{max}$. Hence

$$C^{\rm or}_q(X[L],\mathcal{C}) = \sum_{\mathfrak{C}_{min}\subseteq\mathfrak{C}\subseteq\mathfrak{C}_{max},\dim\sigma_{\mathfrak{C}}=q} C^{\rm or}_q(\sigma_{\mathfrak{C}},\mathcal{C})$$

where

$$C^{\rm or}_q(\sigma_{\mathfrak{C}},\mathcal{C}) := \{\omega \in C^{\rm or}_q(X[L],\mathcal{C}) : \omega \text{ has support in } G\!<\!\sigma_{\mathfrak{C}}\!> \cup\ G\overline{<\!\sigma_{\mathfrak{C}}\!>}\}$$

and we are reduced to showing that the G-subrepresentations $C^{\rm or}_q(\sigma_{\mathfrak{C}},\mathcal{C})$, for $\mathfrak{C}_{min} \subseteq \mathfrak{C} \subseteq \mathfrak{C}_{max}$, are generated by their λ-isotypic components. In the following we fix such an order $\mathfrak{C}_{min} \subseteq \mathfrak{C} \subseteq \mathfrak{C}_{max}$ and put $\mathfrak{B} := \mathfrak{B}(\mathfrak{C})$ and $\mathfrak{A} := \mathfrak{A}(\mathfrak{B})$. We may embed $\mathcal{V}[\sigma_{\mathfrak{C}}]$ in a $U(\mathfrak{A})$-equivariant way into $C^{\rm or}_q(\sigma_{\mathfrak{C}},\mathcal{C})$ by

$$\begin{array}{rcl} \mathcal{V}[\sigma_{\mathfrak{C}}] & \longrightarrow & C^{\rm or}_q(\sigma_{\mathfrak{C}},\mathcal{C}) \\ v & \longmapsto & \omega_v \end{array}$$

where

$$\omega_v(.) := \begin{cases} +v & \text{if } \; . = <\sigma_{\mathfrak{C}}> \;\; , \\ -v & \text{if } q > 0 \text{ and } \; . = \overline{<\sigma_{\mathfrak{C}}>} \;\; , \\ 0 & \text{otherwise } \; . \end{cases}$$

In the following we view this embedding as an inclusion. Clearly $\mathcal{V}[\sigma_{\mathfrak{C}}]$ generates $C_q^{\mathrm{or}}(\sigma_{\mathfrak{C}}, \mathcal{C})$ as a G-representation. According to (V.10) and (V.5), we have

$$\mathcal{V}[\sigma_{\mathfrak{C}}] = \mathcal{V}_{(\mathfrak{B}_{max} \supset \mathfrak{B})} = \sum_{g \in U(\mathfrak{A})/U(\mathfrak{B})J^1(\mathfrak{B}_{max})} g\mathcal{V}^{\eta(\mathfrak{B}, \mathfrak{B}_{max})} \; .$$

and hence

$$\text{(IX.3)} \qquad C_q^{\mathrm{or}}(\sigma_{\mathfrak{C}}, \mathcal{C}) = \sum_{g \in G/U(\mathfrak{B})J^1(\mathfrak{B}_{max})} g\mathcal{V}^{\eta(\mathfrak{B}, \mathfrak{B}_{max})} \; .$$

Having fixed a compatible family of β-extensions $\kappa(.)$ as in (VIII.2) we in particular have $\kappa(\mathfrak{B}_{max})$ as a representation of $J(\mathfrak{B}_{max}) = J^1(\mathfrak{B}_{max}) \cdot U(\mathfrak{B}_{max})$. We then may form the representation $\lambda_{max} = \kappa(\mathfrak{B}_{max}) \otimes \rho$ of $U(\mathfrak{B}_{min})J^1(\mathfrak{B}_{max})$. Both factors in this tensor product are irreducible, the second factor by assumption and the first factor since it restricts to the irreducible representation $\eta(\mathfrak{B}_{max})$ on $J^1(\mathfrak{B}_{max})$. Therefore, by the argument in the proof of [BK1](5.3.2), the representation λ_{max} is irreducible.

(IX.4) Lemma. *i) A smooth G-representation lies in $\mathcal{R}_{(J,\lambda)}(G)$ if and only if it is generated by its λ_{max}-isotypic component.*

ii) We have $\mathcal{V}^{\eta(\mathfrak{B}_{min}, \mathfrak{B}_{max})} = \mathcal{V}^{\lambda_{max}}$.

Proof. i) According to the proof of [BK1](5.5.13) we have the isomorphism

$$\mathrm{Ind}_{U(\mathfrak{B}_{min})J^1(\mathfrak{B}_{max})}^{U(\mathfrak{B}_{min})U^1(\mathfrak{A}_{min})} \lambda_{max} \simeq \mathrm{Ind}_{J(\mathfrak{B}_{min})}^{U(\mathfrak{B}_{min})U^1(\mathfrak{A}_{min})} \lambda$$

So by Frobenius reciprocity, the $U(\mathfrak{B}_{min})U^1(\mathfrak{A}_{min})$-submodules in a smooth G-representation $\mathcal{W}$ generated by $\mathcal{W}^{\lambda_{max}}$ and $\mathcal{W}^{\lambda}$, respectively, coincide.

ii) According to the proof of (VIII.2.1) we have

$$\kappa(\mathfrak{B}_{max})_{|J^1(\mathfrak{B}_{min}, \mathfrak{B}_{max})} \simeq \eta(\mathfrak{B}_{min}, \mathfrak{B}_{max}) \; .$$

Hence $\lambda_{max\,|J^1(\mathfrak{B}_{min}, \mathfrak{B}_{max})}$ is $\eta(\mathfrak{B}_{min}, \mathfrak{B}_{max})$-isotypic which shows that $\mathcal{V}^{\lambda_{max}} \subseteq \mathcal{V}^{\eta(\mathfrak{B}_{min}, \mathfrak{B}_{max})}$. But it also implies that $\mathcal{V}^{\eta(\mathfrak{B}_{min}, \mathfrak{B}_{max})}$ is the image of

$$\kappa(\mathfrak{B}_{max}) \otimes \mathrm{Hom}_{J^1(\mathfrak{B}_{min}, \mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}), \mathcal{V})$$

under the canonical map into $\mathcal{V}$. For the reverse inclusion $\mathcal{V}^{\eta(\mathfrak{B}_{min},\mathfrak{B}_{max})} \subseteq \mathcal{V}^{\lambda_{max}}$ it therefore suffices to prove that $\mathrm{Hom}_{J^1(\mathfrak{B}_{min},\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V})$ as a $U(\mathfrak{B}_{min})/U^1(\mathfrak{B}_{min})$-module is ρ-isotypic. But by the first formula in the proof of (VIII.2.1) this latter module is the Jacquet module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}_{min}}}$ (notation of (VIII)) of the $U(\mathfrak{B}_{max})/U^1(\mathfrak{B}_{max})$-module $\mathrm{Hom}_{J^1(\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V})$. From (VIII.2.2) we know that the latter representation has cuspidal support $(\mathbb{L}_{\mathfrak{B}_{min}},\rho)$. Since our ρ is of the form $\rho \simeq \rho_o^{\otimes e}$ it follows that the Jacquet module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}_{min}}}$ indeed is ρ-isotypic.

In order to prove that the G-representation in (IX.3) is generated by its λ-isotypic component, it suffices to prove that it is generated by its λ_{max}-isotypic component. Since the right hand version of this representation visibly is generated by $\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{max})}$ and since $\mathcal{V}^{\eta(\mathfrak{B}_{min},\mathfrak{B}_{max})} \subseteq \mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{max})}$ by (III.1.1)(iv), we are finally reduced to establishing the following fact.

(IX.5) Lemma. *$\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{max})}$, as a $U(\mathfrak{B})J^1(\mathfrak{B}_{max})$-module, is generated by $\mathcal{V}^{\eta(\mathfrak{B}_{min},\mathfrak{B}_{max})}$.*

Proof. We first of all note that $\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{max})}$, by (III.1.1)(v) and (V.1), indeed is $U(\mathfrak{B})J^1(\mathfrak{B}_{max})$-invariant. In the proof of (IX.4)(ii) we have seen that $\mathcal{V}^{\eta(\mathfrak{B}_{min},\mathfrak{B}_{max})}$ is the image of

$$\kappa(\mathfrak{B}_{max}) \otimes \mathrm{Hom}_{J^1(\mathfrak{B}_{min},\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V})$$

under the canonical map into $\mathcal{V}$. Analogously $\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{max})}$ is the image of

$$\kappa(\mathfrak{B}_{max}) \otimes \mathrm{Hom}_{J^1(\mathfrak{B},\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V}) ,$$

and this in fact in a $U(\mathfrak{B})J^1(\mathfrak{B}_{max})$-equivariant way since $U(\mathfrak{B})J^1(\mathfrak{B}_{max})$ normalizes $J^1(\mathfrak{B},\mathfrak{B}_{max})$. We therefore are reduced to proving that

$$\mathrm{Hom}_{J^1(\mathfrak{B}_{min},\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V})$$

generates

$$\mathrm{Hom}_{J^1(\mathfrak{B},\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V})$$

as a $U(\mathfrak{B})J^1(\mathfrak{B}_{max})$-module. But in that proof we also have seen (with the notations of (VIII)) that the former is the Jacquet module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}_{min}}}$ and the latter is the Jacquet module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}}}$ of the $\mathbb{G}$-module $\mathcal{V}(\mathfrak{B}_{max}) = \mathrm{Hom}_{J^1(\mathfrak{B}_{max})}(\kappa(\mathfrak{B}_{max}),\mathcal{V})$. Hence we are further reduced to showing that the module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}}}$ for the Levi group $\mathbb{L}_{\mathfrak{B}}$ is generated by its Jacquet module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}_{min}}}$. For this it suffices that all irreducible constituents of the $\mathbb{L}_{\mathfrak{B}}$-module $\mathcal{V}(\mathfrak{B}_{max})^{\mathbf{U}_{\mathfrak{B}}}$ have cuspidal support on $\mathbb{L}_{\mathfrak{B}_{min}}$. Because of the special form of the group $\mathbb{L}_{\mathfrak{B}_{min}}$ this follows from the fact (cf. (VIII.2.2)) that any irreducible constituent of the $\mathbb{G}$-module $\mathcal{V}(\mathfrak{B}_{max})$ has cuspidal support on $\mathbb{L}_{\mathfrak{B}_{min}}$.

This finishes the proof of Proposition (IX.2).

X. Acyclicity of the chain complex: a strategy

In this section we consider the augmented complex (IX.1). We reduce its exactness to a technical hypothesis (conjecture (X.4.1)) that we cannot prove. In the next section we shall prove this hypothesis for irreducible discrete series representation.

X.1 Some lemmas on $\lambda_{\max}$-isotypic components.

As in §IX we fix a simple type (J,λ) in G and a smooth complex representation $\mathcal{V}$ in $\mathcal{R}_{(J,\lambda)}(G)$. We keep the same notation. We abbreviate $J_{\max} = U(\mathfrak{B}_{\min})J^1(\mathfrak{B}_{\max})$ and write Λ for the representation space of $\lambda_{\max}$.

We fix a Haar measure μ on G and let $\mathcal{H}(G)$ denote the (convolution) Hecke algebra of locally constant functions with compact support on G. For $\varphi \in \mathcal{H}(G)$ and $g \in G$, we also define ${}^g\varphi \in \mathcal{H}(G)$ by ${}^g\varphi(x) = \varphi(g^{-1}x)$. We also recall the Schur orthogonality formula: if $(\rho,\mathcal{W})$ is an irreducible representation of a compact subgroup K of G, with contragredient representation $(\check{\rho},\check{\mathcal{W}})$, then

$$\int_K \langle\rho(x^{-1})w,\check{w}\rangle\langle\rho(x)v,\check{v}\rangle dk = \frac{\dim(\rho)}{\mu(K)}\langle w,\check{v}\rangle\langle v,\check{w}\rangle, \qquad v,w\in\mathcal{W},\ \check{v},\check{w}\in\check{\mathcal{W}},$$

where $\langle -,-\rangle : \mathcal{W}\times\check{\mathcal{W}} \longrightarrow \mathbb{C}$ denotes the canonical pairing.

The irreducible representation $\lambda_{\max}$ gives rise to an idempotent $e_{\max}$ of $\mathcal{H}(G)$ defined as follows: it has support $J_{\max}$ and is given by

$$e_{\max}(j) = \mu(J_{\max})^{-1}\dim(\lambda_{\max})\mathrm{Tr}(\lambda_{\max}(j^{-1}))$$

for $j \in J_{\max}$ (cf. [BK1] §4.2) Note that $e_{\max}$ may be considered as an idempotent of the Hecke algebra $\mathcal{H}(J_{\max}) := \{f \in \mathcal{H}(G)\ ;\ \mathrm{Support}(f) \subset J_{\max}\}$. If $(\zeta,\mathcal{U})$ is a smooth representation of G (resp. of $J_{\max}$) then $(\zeta,\mathcal{U})$ extends to a representation of $\mathcal{H}(G)$ (resp. $\mathcal{H}(J_{\max})$) on $\mathcal{U}$, and we then have $\zeta(e_{\max})\star\mathcal{U} = \mathcal{U}^{\lambda_{\max}}$ (the $\lambda_{\max}$-isotypic component of $\mathcal{U}$).

For $x \in G$, we denote by $\lambda_{\max}^x$ the representation of $J_{\max}^x := xJ_{\max}x^{-1}$ in the space Λ given by $\lambda_{\max}^x(xjx^{-1}) = \lambda_{\max}(j)$, $j \in J_{\max}$.

(X.1.1) Proposition. *i) Any non-zero function f in the scalar Hecke algebra $e_{\max}\star\mathcal{H}(G)\star e_{\max}$ has support in the G-intertwining $I_G(\lambda_{\max})$ of $\lambda_{\max}$.*

ii) Let x be an element of G such that $x \notin I_G(\lambda_{\max})$ and let $(\pi, \mathcal{V})$ be a smooth representation of G. Then the linear map $p_x : \mathcal{V}^{\lambda_{\max}} \longrightarrow \mathcal{V}^{\lambda_{\max}}$, given by $p_x(v) = \pi(e_{\max}) \circ \pi(x) \circ \pi(e_{\max}).v$ is zero.

Remark. These facts are certainly well known but we could not find a reference.

Proof. i) Let $\mathcal{H}(G, \lambda_{\max})$ be the Hecke algebra of $\lambda_{\max}$-spherical functions on G ([BK](4.1)). Recall that if $(\check{\lambda}_{\max}, \check{\Lambda})$ denotes the contragredient representation of $(\lambda_{\max}, \Lambda)$, then $\mathcal{H}(G, \lambda_{\max})$ is the convolution algebra of compactly supported functions $\Phi : G \longrightarrow \mathrm{End}_{\mathbb{C}}(\check{\Lambda})$ satisfying :

$$\Phi(j_1 g j_2) = \check{\lambda}_{\max}(j_1) \circ \Phi(g) \circ \check{\lambda}_{\max}(j_2)\ ,\ j_i \in J_{\max}\ ,\ g \in G\ .$$

¿From [BK](4.1.1), any non-zero $\Phi \in \mathcal{H}(G, \lambda_{\max})$ has support in $I_G(\lambda_{\max})$. Moreover by [BK], proposition (4.2.4), we have an algebra isomorphism

$$\boldsymbol{\Upsilon}\ :\ \mathcal{H}(G, \lambda_{\max}) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\Lambda) \longrightarrow e_{\max} \star \mathcal{H}(G) \star e_{\max}\ .$$

Identifying $\mathrm{End}_{\mathbb{C}}(\Lambda)$ with $\Lambda \otimes_{\mathbb{C}} \check{\Lambda}$, $\boldsymbol{\Upsilon}$ is given by

$$\boldsymbol{\Upsilon}(\Phi \otimes w \otimes \check{w})(g) = \dim(\lambda_{\max})\mathrm{Tr}(w \otimes \Phi(g)\check{w})$$

for $g \in G,\ w \in \Lambda,\ \check{w} \in \check{\Lambda},\ \Phi \in \mathcal{H}(G, \lambda_{\max})$. In particular we have:

$$\mathrm{Support}\big(\boldsymbol{\Upsilon}(\Phi \otimes w \otimes \check{w})\big) \subset \mathrm{Support}(\Phi)\ ,\ w \in \Lambda,\ \check{w} \in \check{\Lambda},\ \Phi \in \mathcal{H}(G, \lambda_{\max})\ .$$

It follows that any non-zero element of $e_{\max} \star \mathcal{H}(G) \star e_{\max}$ has support in $I_G(\lambda_{\max})$ as required.

ii) Recall that, for $\varphi \in \mathcal{H}(G)$ and $g \in G$, we write ${}^g\varphi \in \mathcal{H}(G)$ for the function ${}^g\varphi(x) = \varphi(g^{-1}x)$. Then straightforward computations show that $\pi(e_{\max}) \circ \pi(x) \circ \pi(e_{\max}) = \pi(e_{\max} \star {}^x e_{\max})$ and that $e_{\max} \star {}^x e_{\max} \in e_{\max} \star \mathcal{H}(G) \star e_{\max}$. Moreover $e_{\max} \star {}^x e_{\max}$ clearly has support in $J_{\max} x J_{\max}$, whence is zero since $x \notin I_G(\lambda_{\max})$. So $p_x = \pi(e_{\max} \star {}^x e_{\max})$ is the zero map.

(X.1.2) Proposition. *Let x be a fixed element of $I_G(\lambda_{\max})$.*

i) There exist $m \geq 1$, $u_1, ..., u_m$, $v_1, ..., v_m \in J_{\max}$, $\gamma_1, ..., \gamma_m \in \mathbb{C}$, such that

$$\sum_{i=1}^{m} \gamma_i\, e_{\max} \star {}^{u_i x v_i} e_{\max}$$

is an invertible element of $e_{\max} \star \mathcal{H}(G) \star e_{\max}$.

ii) There exist $m \geq 1$, $u_1, ..., u_m$, $v_1, ..., v_m \in J_{\max}$, $\gamma_1, ..., \gamma_m \in \mathbb{C}$, *such that, for* $(\pi, \mathcal{V})$ *any smooth representation of* G

$$\sum_{i=1}^{m} \gamma_i \, \pi(e_{\max}) \circ \pi(u_i x v_i) \circ \pi(e_{\max})$$

induces a $\mathbb{C}$*-linear isomorphism* $\mathcal{V}_{\max}^{\lambda} \longrightarrow \mathcal{V}_{\max}^{\lambda}$.

Proof. To make the notation lighter, we shall set $K = J_{\max}$, $\rho = \lambda_{\max}$, $e = e_{\max}$.

Since

$$\sum_{i=1}^{m} \gamma_i \, \pi(e) \circ \pi(u_i x v_i) \circ \pi(e) = \pi\left(\sum_{i=1}^{m} \gamma_i e \star {}^{u_i x v_i} e\right)$$

assertion ii) is a consequence of i).

Via $\boldsymbol{\Upsilon}^{-1} : e \star \mathcal{H}(G) \star e \longrightarrow \mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\Lambda)$, an element $\varphi \in e \star \mathcal{H}(G) \star e$ corresponds to the element of $\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\Lambda)$ given as follows (see the proof of [BK] Proposition (4.2.4), pages 149–150). Fix a basis $\{w_1, ..., w_n\}$ of Λ and let $\{\check{w}_1, ..., \check{w}_n\}$ be the corresponding dual basis of $\check{\Lambda}$, so that $\langle w_i, \check{w}_j \rangle = \delta_{ij}$ (Kronecker's delta symbol). For each pair of indices (i, j) and for $g \in G$, define an operator $\Phi_{ij}(g) \in \mathrm{End}_{\mathbb{C}}(\check{\Lambda})$ by the formula:

$$(1) \qquad \langle w, \Phi_{ij}(g) \check{w} \rangle = \int_K \int_K \varphi(kgl) \langle \rho(l) w_i, \check{w} \rangle \langle w, \check{\rho}(k^{-1}) \check{w}_j \rangle dk dl \ ,$$

for all $w \in \Lambda$, $\check{w} \in \check{\Lambda}$. Then the function $g \mapsto \Phi_{ij}(g)$ lies in $\mathcal{H}(G, \rho)$, and we have

$$\boldsymbol{\Upsilon}^{-1}(\varphi) = \frac{\dim(\rho)}{\mu(K)^2} \sum_{i,j=1}^{n} \Phi_{ij} \otimes w_j \otimes \check{w}_i \ .$$

Assume now that $\varphi \in e \star \mathcal{H}(G) \star e$ has support in KxK. Then from formula (1), the Φ_{ij} have support in KxK. We need the following result.

(X.1.3) Lemma. *i) The* $\mathbb{C}$*-vector space*

$$\{\Phi \in \mathcal{H}(G, \lambda_{\max}) \ ; \ \mathrm{Support}(\Phi) \subset J_{\max} x J_{\max}\}$$

has dimension 1.

ii) Any non-zero Φ *in* $\mathcal{H}(G, \lambda_{\max})$ *with support* $J_{\max} x J_{\max}$ *is invertible.*

Proof. By [BK1](5.5), the G-intertwining of λ and $\lambda_{\max}$ are JG_LJ and $J_{\max}G_LJ_{\max}$ respectively, where L/E is the unramified extension introduced in §VIII and G_L the centralizer of L in G. Moreover by [BK1](5.5.13), there is a canonical algebra isomorphism $\mathcal{H}(G, \lambda) \longrightarrow \mathcal{H}(G, \lambda_{\max})$ which preserves supports in the following sense: if $y \in G_L$ and $\varphi \in \mathcal{H}(G,\lambda)$ has support JyJ, then its image $\varphi' \in \mathcal{H}(G,\lambda_{\max})$ has support $J_{\max}yJ_{\max}$. Moreover consider the Iwahori subgroup of G_L given by $I_L = U(\mathfrak{C}_{\min}) = U(\mathfrak{B}_{\min} \cap G_L$ and let $\mathcal{H}_0 = \mathcal{H}(G_L, I_L)$ be the corresponding affine Hecke algebra of type A formed of (locally constant) bi-I_L-invariant compactly supported functions on G_L. By Theorem (5.6.6) of [BK1], the algebras $\mathcal{H}(G,\lambda)$ and $\mathcal{H}_0$ are isomorphic in a support preserving way: there is (a non-canonical) isomorphism of $\mathbb{C}$-algebras Ψ: $\mathcal{H}_0 \longrightarrow \mathcal{H}(G,\lambda)$ such that for all $y \in G_L$ and for all $\varphi \in \mathcal{H}_0$ with support I_LyI_L, $\Psi(\varphi)$ has support JyJ. As a consequence, there exists an algebra isomorphism Ψ': $\mathcal{H}_0 \longrightarrow \mathcal{H}(G,\lambda_{\max})$ enjoying the same support preservation property.

Now assertions i) and ii) of our lemma hold for the corresponding assertions hold true for the standard affine Hecke algebra $\mathcal{H}_0$. Indeed if $y \in G_L$, we have:

i) $\{\varphi \in \mathcal{H}_0 \ ; \ \text{Support}(\varphi) \subset I_LyI_L\}$ is the line spanned by the characteristic function of I_LyI_L,

ii) it is a standard fact that any $\varphi \in \mathcal{H}_0$ with support I_LyI_L is invertible.

Let us fix a non-zero element Φ_0 in $\mathcal{H}(G,\rho)$ with support KxK. Then

$$\Upsilon^{-1}(e \star {}^x e) = \frac{\dim(\rho)}{\mu(K)^2}\Phi_0 \otimes (\sum_{i,j=1}^{n} \gamma_{ij} w_j \otimes \check{w}_i)$$

where γ_{ij} is defined by $\Phi_{ij} = \gamma_{ij}\Phi_0$, $i,j \in \{1,...,n\}$. For the same reason, for all $u,v \in K$, there exists a vector $\zeta(u,v) \in \Lambda \otimes \check{\Lambda}$ such that

$$\Upsilon^{-1}(e \star {}^{uxv} e) = \frac{\dim(\rho)}{\mu(K)^2}\Phi_0 \otimes \zeta(u,v) \ .$$

(X.1.4) Lemma. *For all $u,v \in K$, we have*

$$\zeta(u,v) = [\rho(u) \otimes \check{\rho}(v^{-1})]\zeta(1,1) \ .$$

Take this last lemma for granted. Since the representation $\rho \otimes \check{\rho}$ of $K \times K$ in $\Lambda \otimes \check{\Lambda}$ is irreducible, it is generated by the non-zero vector $\zeta(1,1)$. We may find $m \geq 1$, $u_i, v_i \in K$, $\gamma_i \in \mathbb{C}$, $i = 1,...,m$, such that $\sum_{i=1}^{m} \gamma_i \zeta(u_i,v_i)$ is an arbitrary element of $\Lambda \otimes \check{\Lambda} \simeq \text{End}_{\mathbb{C}}(\Lambda)$. In particular we may choose this element invertible in $\text{End}_{\mathbb{C}}(\Lambda)$. It follows that

$$\Upsilon^{-1}(\sum_{i=1} m\gamma_i e \star {}^{u_i x v_i} e)$$

is invertible. This finishes the proof of Proposition (X.1.2)(ii).

Proof of Lemma (X.1.4). The proof is somewhat technical but straightforward. It is inspired from the calculation of [BK], pages 232–233.

Write $\Phi_{ij}^{uv} \in \mathcal{H}(G,\rho)$ for the functions attached to $\varphi = e \star {}^{uxv}e$ via formula (1). For $g \in G$, we have

$$\varphi(g) = \int_K e_\rho(y) e_\rho((uxv)^{-1} y^{-1} g) dy$$

$$= \frac{\dim(\rho)^2}{\mu(K)^2} \sum_{b,c=1}^{n} \int_K \langle \rho(y^{-1}) w_b, \check{w}_b \rangle . \langle \rho(g^{-1} yuxv) w_c, \check{w}_c \rangle dy \ .$$

So for $w \in \Lambda$ and $\check{w} \in \check{\Lambda}$, we have

$$\frac{\mu(K)^2}{\dim(\rho)^2} \langle w, \Phi_{ij}^{uv} \check{w} \rangle =$$

$$\sum_{b,c=1}^{n} \int_{K^3} \langle \rho(y^{-1}) w_b, \check{w}_b \rangle \langle \rho(l^{-1} g^{-1} k^{-1} yuxv) w_c, \check{w}_c \rangle \langle \rho(l) w_i, \check{w} \rangle \langle w, \check{\rho}(k^{-1}) \check{w}_j \rangle dk dl dy$$

Integrating with respect to l and using the Schur orthogonality relation, we obtain:

$$\frac{\mu(K)}{\dim(\rho)} \langle w, \Phi_{ij}^{uv} \check{w} \rangle =$$

$$\sum_{b,c=1}^{n} \int_{K^2} \langle \rho(y^{-1}) w_b, \check{w}_b \rangle \langle \rho(g^{-1} k^{-1} yuxv) w_c, \check{w} \rangle \langle w_i, \check{w}_c \rangle \langle w, \check{\rho}(k^{-1}) \check{w}_j \rangle dk dy$$

$$= \sum_{b=1}^{n} \int_{K^2} \langle \rho(y^{-1}) w_b, \check{w}_b \rangle \langle \rho(g^{-1} k^{-1} yuxv) w_i, \check{w} \rangle \langle w, \check{\rho}(k^{-1}) \check{w}_j \rangle dk dy$$

We now make the change of variable $(k')^{-1} = k^{-1} yu$ and this last expression becomes:

$$\sum_{b=1}^{n} \int_{K^2} \langle \rho(y^{-1}) w_b, \check{w}_b \rangle \langle \rho(g^{-1} (k')^{-1} x) \rho(v) w_i, \check{w} \rangle \langle w, \check{\rho}((k')^{-1} u^{-1} y^{-1}) \check{w}_j \rangle dk' dy$$

$$= \sum_{b=1}^{n} \int_{K^2} \langle \rho(y^{-1}) w_b, \check{w}_b \rangle \langle \rho(g^{-1} (k')^{-1} x) \rho(v) w_i, \check{w} \rangle \langle \rho(y) \rho(uk') w, \check{w}_j \rangle dk' dy \ .$$

Using again the Schur orthogonality relation, we obtain:

$$\langle w, \Phi_{ij}^{uv}(g) \check{w} \rangle = \sum_{b=1}^{n} \int_K \langle w_b, \check{w}_j \rangle \langle \rho(uk') w, \check{w}_b \rangle \langle \rho(g^{-1} (k')^{-1} x) w_i, \check{w} \rangle dk'$$

$$= \int_K \langle \rho(k)w, \check{\rho}(u^{-1})\check{w}_j\rangle\langle\rho(g^{-1}k^{-1}x)\rho(v)w_i, \check{w}\rangle dk$$

Let (V_{ij}) (resp. (U_{ij})) be the matrix of $\rho(v)$ (resp. $\check{\rho}(u^{-1})$) in the basis $\{w_i\}$ (resp. in the basis $\{\check{w}_i\}$). We have

$$\langle w, \Phi_{ij}^{uv}(g)\check{w}\rangle = \sum_{\alpha,\beta=1}^{n} V_{\alpha i}U_{\beta j}\int_K \langle\rho(y)w, \check{w}_\beta\rangle\langle\rho(g^{-1}y^{-1}x)w_\alpha, \check{w}\rangle dk$$

$$= \sum_{\alpha,\beta=1}^{n} V_{\alpha i}U_{\beta j}\langle w, \Phi_{\alpha\beta}^{11}\check{w}\rangle \ .$$

In other words, we have proved that

$$\Phi_{ij}^{uv} = \sum_{\alpha,\beta=1}^{n} V_{\alpha i}U_{\beta j}\Phi_{\alpha,\beta}^{11} = \left(\sum_{\alpha,\beta=1}^{n} V_{\alpha i}U_{\beta j}\gamma_{\alpha\beta}\right)\Phi_0 \ .$$

Hence we obtain:

$$\boldsymbol{\Upsilon}^{-1}(e \star {}^{(uxv)}e) = \frac{\dim(\rho)}{\mu(K)^2}\Phi_0 \otimes \sum_{i,j=1}^{n}\sum_{\alpha,\beta=1}^{n} V_{\alpha i}U_{\beta j}\gamma_{\alpha\beta}w_j \otimes \check{w}_i$$

$$= \frac{\dim(\rho)}{\mu(K)^2}\Phi_0 \otimes \sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\left(\sum_{j=1}^{n} U_{\beta j}w_j\right) \otimes \left(\sum_{i=1}^{n} V_{\alpha i}\check{w}_i\right)$$

$$= \frac{\dim(\rho)}{\mu(K)^2}\Phi_0 \otimes \sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}\rho(u)(w_\beta) \otimes \check{\rho}(v^{-1})(\check{w}_\alpha)$$

$$= \frac{\dim(\rho)}{\mu(K)^2}\Phi_0 \otimes [\rho(u) \otimes \check{\rho}(v^{-1})]\left(\sum_{\alpha,\beta=1}^{n}\gamma_{\alpha\beta}w_\beta \otimes \check{w}_\alpha\right)$$

as required. (We have used that the matrix of $\rho(u)$ with respect to the basis $\{w_i\}$ is the transpose of the matrix of $\check{\rho}(u^{-1})$ with respect to the dual basis $\{\check{w}_i\}$.) This finishes the proof of Lemma (X.1.4).

X.2. Orientation of $X[L]$.

In order to work with a simpler version of the chain complex of §IX, we are going to show that, as a simplicial complex, $X[L]$ has a G°-invariant labelling, where

$$G^\circ = \{g \in G \ ; \mathrm{Det}(g) \in \mathfrak{o}_F^\times\}$$

Recall [Brown] that a labelling of a d-dimensional simplicial complex Y is a simplicial map $\mathbf{l} : Y \longrightarrow \Delta_d$, from Y to the standard d-dimensional simplex, such that $\dim(\mathbf{l}(\sigma)) = \dim(\sigma)$ for any simplex σ of Y.

Fix a chamber C of X. It is classical that the action of G° on X has the following property: any simplex σ of X has a unique G°-conjugate that lies in (the closure of) C. In particular the stabilizer of C in G° fixes C pointwise. Even though $X[L]$ is not a building in general, its maximal simplices have the same dimension and we call them *chambers*.

We now fix the chamber C so that $C \cap X_L \neq \emptyset$. It is false in general that G° acts transitively on the chambers of $X[L]$. For instance, if L/F is a maximal unramified extension of F in A, then $X[L]$ is 0-dimensional and consists of the vertices of X. But the action of G° on the vertices of X is not transitive.

Let us notice that $C \cap X[L]$ is a sub-simplicial complex of $X[L]$. Indeed, passing to the first barycentric subdivisions, we first have that $\mathrm{sd}(C) \cap \mathrm{sd}(X[L]) = \mathrm{sd}(C) \cap X(L)$ is a sub-simplicial complex of $X(L)$. To get our assertion, it suffices to prove that if $\mathrm{sd}(C) \cap X(L)$ contains a vertex x_σ corresponding to the isobarycenter of a simplex σ of $X[L]$, then $\sigma \subset C \cap X[L]$. The interior σ° of σ is of the form $\Sigma^\circ \cap X[L]$, where Σ is some simplex of X. We have $x_\sigma \in \sigma^\circ \subset \Sigma^\circ$ and $x_\sigma \in C \cap X[L] \subset C$. In particular $\Sigma^\circ \cap C \neq \emptyset$ and this forces the containment $\Sigma \subset C$. Therefore $\sigma \subset C$ as required.

(X.2.1) Lemma. *The simplicial subcomplex $C \cap X[L]$ of $X[L]$ is a disjoint union of $f(L/F)$ chambers of $X[L]$.*

Proof. First we prove that any vertex of $C \cap X[L]$ is contained in a chamber of $X[L]$ which is itself contained in C. Let s be such a vertex. There exist a field extension $L'/F \subset A$ and an order $\mathfrak{A} \subset A$ such that:

– $e(L'/F) = e(L/F)$ and $f(L'/F) = f(L/F)$,

– the order $\mathfrak{A}$ lies in $\mathrm{Her}(A)^{L'^\times}$,

– s is the vertex of $X_{L'}$ attached to the maximal order $\mathfrak{A} \cap \mathrm{End}_{L'}(V)$.

Let $(N_k)_{k\in\mathbb{Z}}$ be a chain in V corresponding to $\mathfrak{A}$. It must have $\mathfrak{o}_{L'}$-period 1, whence it has $\mathfrak{o}_F$-period $e(L'/F)$. Assume that C corresponds to a lattice chain $(L_k)_{k\in\mathbb{Z}}$ in V of $\mathfrak{o}_F$-period N. There exists an integer k_o such that:

$$N_k = L_{k_o+kN/e(L/F)}\ ,\ k \in \mathbb{Z}\ .$$

Since $f(L/F)$ divides $N/e(L/F)$, we have the containments:

$$\{L_{k_o+k.N/e(E/F)}\ ;\ k \in \mathbb{Z}\} \subset \{L_{k_o+kf(L/F)}\ ;\ k \in \mathbb{Z}\} \subset \{L_k\ ;\ k \in \mathbb{Z}\}\ .$$

By the numerical criterion of (I.3.5), the set of lattices $\{L_{k_o+kf(L/F)}\ ;\ k\in\mathbb{Z}\}$ corresponds to a chamber C_L of $X[L]$ and the previous containments mean that $s\in C_L\subset C$.

Let C_L be a chamber of $X[L]$. There exist a field extension $L'/F\subset A$ and an order $\mathfrak{A}\subset A$ such that:

– $e(L'/F)=e(L/F)$ and $f(L'/F)=f(L/F)$,

– the order $\mathfrak{A}$ lies in $\mathrm{Her}(A)^{L'^\times}$,

– C_L is the chamber of $X_{L'}$ attached to the order $\mathfrak{A}\cap End_{L'}(V)$.

Let $(M_k)_{k\in\mathbb{Z}}$ be a chain in V corresponding to $\mathfrak{A}$ and $\mathfrak{B}$. It has $\mathfrak{o}_{L'}$-period $N/[L:F]$. So it has $\mathfrak{o}_F$-period $e(L/F).N/[L:F]=N/f(L/F)$. Moreover, for all $k\in\mathbb{Z}$, we have:

$$\dim_{\mathbb{F}_F}(M_k/M_{k+1})=f(L/F)\dim_{\mathbb{F}_{L'}}(M_k/M_{k+1})=f(L/F).1=f(L/F)\ .$$

Assume now that C_L lies in the chamber C of X. According to the previous discussion, there exists a coset Γ of $f(L/F)\mathbb{Z}/N\mathbb{Z}$ in $\mathbb{Z}/N\mathbb{Z}$, such that:

$$\{M_k\ ;\ k\in\mathbb{Z}\}=\{L_l\ ;\ l\in\mathbb{Z}\text{ and } l \bmod N\mathbb{Z}\in\Gamma\}\ .$$

Conversely, using proposition (I.3.5), we have that for all such coset Γ, the lattice chain whose lattice set is given by $\{L_l\ ;\ l\in\mathbb{Z}$ and l mod $N\mathbb{Z}\in\Gamma\}$ correspond to a chamber of $X[L]$ contained in C. Indeed if F_Γ is the simplex of X corresponding to the lattice set $\{L_l\ ;\ l\in\mathbb{Z}$ and l mod $N\mathbb{Z}\in\Gamma\}$, then the corresponding (closed) chamber of $X[L]$ is $F_\Gamma\cap X[L]$. When Γ runs over the $f(L/F)$ cosets of $f(L/F)\mathbb{Z}/N\mathbb{Z}$ in $\mathbb{Z}/N\mathbb{Z}$, the corresponding (closed) chambers are disjoint, as required.

Since the simplicial complex $X[L]\cap C$ is a disjoint union of (closed) chambers, it is trivially labelable. Let us fix a labelling $\mathbf{l}_C\ :\ X[L]\cap C\longrightarrow\Delta_C$, where Δ_C is the standard simplex of dimension $\dim X[L]=N/[L:F]-1$. For any simplex σ of $X[L]$, we define a simplex $\mathbf{l}(\sigma)$ of Δ_C by $\mathbf{l}(\sigma)=\mathbf{l}_C(\sigma_C)$, where σ_C is the unique simplex of $X[L]\cap C$ which is a conjugate of σ under the action of G°.

(X.2.2) Lemma. *The map $\mathbf{l}\ :\ X[L]\longrightarrow\Delta_C$ is a labelling. It is invariant under the action of G°.*

Proof. Obvious from the properties of the action of G° on X.

From now on, we fix the G°-invariant labelling $\mathbf{l}$ of $X[L]$ (by fixing $\mathbf{l}_C$). It gives rise to a G°-invariant orientation of the simplicial complex $X[L]$ as well as G°-invariant incidence numbers $[\sigma:\tau]$ for any pair of simplices $\tau\subset\sigma$ of $X[L]$ with τ of codimension 1 in σ.

X.3 Another chain complex.

We fix a smooth complex representation $(\pi, \mathcal{V})$ in $\mathcal{R}_{(J,\lambda)}G$ and consider the coefficient system $\mathcal{C} = (\mathcal{V}[\sigma])_\sigma = \mathcal{C}_{(J,\lambda)}(\mathcal{V})$ of §VIII.

For $q = 0, \dots, N/[L:F]-1$, let $X[L]_q$ denote the set of q-simplices of $X[L]$. The space $C_q(X[L], \mathcal{C})$ of (unoriented) q-chains of $X[L]$ with coefficient in $\mathcal{C}$ is the $\mathcal{C}$-vector space of all maps $\omega : X[L]_q \longrightarrow \mathcal{V}$ such that ω has finite support and $\omega(\sigma) \in \mathcal{V}[\sigma]$, for all $\sigma \in X[L]_q$. The group G acts smoothly on $C_q(X[L], \mathcal{C})$ via $(g\omega)(\sigma) := g(\omega(g^{-1}\sigma))$. The orientation of $X[L]$ gives rise to boundary maps:

$$
\begin{array}{rcl}
\partial \; : \; C_{q+1}(X[L], \mathcal{C}) & \to & C_q(X[L], \mathcal{C}) \\
\omega & \mapsto & [\;\sigma \mapsto \displaystyle\sum_{\tau \in X[L]_q\ ,\ \tau \subset \sigma} [\sigma : \tau]\omega(\tau)\;]
\end{array}
$$

We obtain an augmented chain complex of G°-modules:

(X.3.1) $$C_{N/[L:F]-1}(X[L], \mathcal{C}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(X[L], \mathcal{C}) \xrightarrow{\epsilon} \mathcal{V}$$

where $\epsilon(\omega) = \sum_{\sigma \in X[L]_0} \omega(\sigma) \in \mathcal{V}$.

(X.3.2) Lemma. *As augmented chain complexes of G°-modules, the complexes (IX.1) and (X.3.1) are canonically isomorphic.*

Proof. By standard arguments.

X.4 $J_{\max}$-orbits of simplices.

Fix $q \in \{0, \dots, N[L:F]-1\}$. For any subset Σ of $X[L]_q$, we denote by $C_q(\Sigma, \mathcal{C})$ the subspace of $C_q(X[L], \mathcal{C})$ formed of those q-chains with support in Σ.

Let Ω_q be the set of orbits of $J_{\max}$ in $X[L]_q$. As a $J_{\max}$-module, $C_q(X[L], \mathcal{C})$ decomposes as

$$C_q(X[L], \mathcal{C}) = \coprod_{\Sigma \in \Omega_q} C_q(\Sigma, \mathcal{C}) \ .$$

Fix $\Sigma \in \Omega_q$. There exist $\mathfrak{C} \in \mathrm{Her}(C)$ satisfying $\mathfrak{C}_{\min} \subset \mathfrak{C} \subset \mathfrak{C}_{\max}$ and $x \in G$ such that $\Sigma = J_{\max} x.\sigma_{\mathfrak{C}}$.We have the disjoint union:

$$\Sigma = \bigcup_{j \in J_{\max}/J_{\max} \cap U(\mathfrak{A})^x} \{jx\sigma_{\mathfrak{C}}\} \ ,$$

where $\mathfrak{A} = \mathfrak{A}(\mathfrak{B})$ and $\mathfrak{B} = \mathfrak{B}(\mathfrak{C})$, from which we deduce the following isomorphisms of $J_{\max}$-modules:

$$C_q(\Sigma, \mathcal{C}) = \coprod_{j \in \in J_{\max}/J_{\max} \cap U(\mathfrak{A})^x} C_q(jx\sigma_{\mathfrak{C}}, \mathcal{C}) = \coprod_{j \in \in J_{\max}/J_{\max} \cap U(\mathfrak{A})^x} jxC_q(\sigma_{\mathfrak{C}}, \mathcal{C}) \ .$$

We have a natural $J_{\max}$-homomorphism $S_\Sigma : C_q(\Sigma, \mathcal{C}) \longrightarrow \mathcal{V}$, given by

$$S_\Sigma(\omega) = \sum_{\sigma \in \Sigma} \omega(\sigma) \ .$$

In other words:

$$S_\Sigma\Big(\bigoplus_{j \in J_{\max}/J_{\max} \cap U(\mathfrak{A})^x} jx\omega_j\Big) = \sum_{j \in J_{\max}/J_{\max} \cap U(\mathfrak{A})^x} jx\omega_j(\sigma_{\mathfrak{C}}) \ , \ \omega_j \in \mathcal{V}[\sigma_{\mathfrak{C}}] \ .$$

We set $K_\Sigma = \mathrm{Ker} S_\sigma$. We have the following exact sequences of $J_{\max}$-modules and $\mathbb{C}$-vector spaces respectively:

$$0 \longrightarrow K_\Sigma \longrightarrow C_q(\Sigma, \mathcal{C}) \longrightarrow \sum_{j \in J_{\max}} jx\mathcal{V}[\sigma_{\mathfrak{C}}] \longrightarrow 0$$

$$0 \longrightarrow K_\Sigma^{\lambda_{\max}} \longrightarrow C_q(\Sigma, \mathcal{C})^{\lambda_{\max}} \longrightarrow \big(\sum_{j \in J_{\max}} jx\mathcal{V}[\sigma_{\mathfrak{C}}]\big)^{\lambda_{\max}} \longrightarrow 0$$

Moreover, by lemmas (IX.4) and (IX.5), we have

$$\mathcal{V}[\sigma_{\mathfrak{C}}] = \sum_{g \in U(\mathfrak{A})/J_{\max}} g\mathcal{V}^{\eta(\mathfrak{B}_{\min}, \mathfrak{B}_{\max})} = \sum_{g \in U(\mathfrak{A})/J_{\max}} g\mathcal{V}^{\lambda_{\max}} \ .$$

Therefore we have

$$\sum_{j \in J_{\max}} jx\mathcal{V}[\sigma_{\mathfrak{C}}] = \sum_{j \in J_{\max}} \sum_{g \in U(\mathfrak{A})} jxg\mathcal{V}^{\lambda_{\max}} \subset \mathcal{V} \ .$$

By proposition (X.1.1), for all $j \in J_{\max}$, $g \in U(\mathfrak{A})$, we have

$$e_{\max} \star \{jxg\mathcal{V}^{\lambda_{\max}}\} = \begin{cases} \mathcal{V}^{\lambda_{\max}} & \text{if } jxg \in I_G(\lambda_{\max}) \\ 0 & \text{otherwise} \end{cases}$$

We deduce that

$$\big(\sum_{j \in J_{\max}} jx\mathcal{V}[\sigma_{\mathfrak{C}}]\big)^{\lambda_{\max}} = \begin{cases} \mathcal{V}^{\lambda_{\max}} & \text{if } \exists\ g \in U(\mathfrak{A}),\ j \in J_{\max} \text{ s.t. } jxg \in I_G(\lambda_{\max}) \\ 0 & \text{otherwise} \end{cases}$$

Since the G-intertwining of $\lambda_{\max}$ is $J_{\max}G_LJ_{\max}$, this may be rewritten:

$$\Big(\sum_{j\in J_{\max}} jx\mathcal{V}[\sigma_{\mathfrak{C}}]\Big)^{\lambda_{\max}} = \begin{cases} \mathcal{V}^{\lambda_{\max}} & \text{if } x \in J_{\max}G_LU(\mathfrak{A}) \\ 0 & \text{otherwise} \end{cases}$$

(X.4.1) Conjecture. *For any $\Sigma \in \Omega_q$, we have $K_\Sigma^{\lambda_{\max}} = 0$.*

(X.4.2) Corollary. *Assume that conjecture (X.4.1) holds.*

i) If $\Sigma \cap X_L \neq \emptyset$, then S_Σ induces an isomorphism of $\mathbb{C}$-vector spaces: $C_q(\Sigma, \mathcal{C})^{\lambda_{\max}} \longrightarrow \mathcal{V}^{\lambda_{\max}}$.

ii) If $\Sigma \cap X_L = \emptyset$, then $C_q(\Sigma, \mathcal{C})^{\lambda_{\max}} = 0$.

Indeed we have $J_{\max}x\sigma_{\mathfrak{C}} \cap X_L \neq \emptyset$ if and only if there exist $\mathfrak{C}' \in \mathrm{Her}(C)$ and $j \in J_{\max}$ such that $jx\sigma_{\mathfrak{C}} = \sigma_{\mathfrak{C}'}$. By lemma (I.3.1) and (I.3.3), this is equivalent to the existence of $z \in G_L$ and $g \in U(\mathfrak{A})$ such that $jx = zg$, as required.

From now on we fix an apartment $\mathcal{A}_L$ of X_L containing the chamber $\sigma_{\mathfrak{C}_{\min}}$.

(X.4.3) Lemma. *Let $\Sigma \in \Omega_q$. Assume that $\Sigma \cap X_L \neq \emptyset$. Then $\Sigma \cap \mathcal{A}_L \neq \emptyset$ and the intersection $\Sigma \cap \mathcal{A}_L$ is reduced to a single simplex. Moreover $\Sigma \cap X_L$ is a single $U(\mathfrak{C}_{\min})$-orbit.*

If $\Sigma \cap X_L \neq \emptyset$, then $\Sigma = J_{\max}.\sigma_L$ for some $\sigma_L \in (X_L)_q$. Since J_m contains the Iwahori subgroup $U(\mathfrak{C}_{\min})$ (of $\mathrm{Aut}_L(V)$) and that $(\mathcal{A}_L)_q$ is a system of representatives of the $U(\mathfrak{C}_{\min})$-orbits in $(X_L)_q$, we have $\Sigma \cap \mathcal{A}_L \neq \emptyset$. At this stage we need the following technical result.

(X.4.4) Lemma. *Let σ, τ be simplices of X_L. Then if they are conjugate under the action of $U(\mathfrak{A}_{\min})$, they are conjugate under the action of $U(\mathfrak{C}_{\min})$.*

Lemma (X.4.3) follows from the previous lemma by observing that $J_{\max}$ is contained in $U(\mathfrak{C}_{\min})U^1(\mathfrak{A}_{\max}) \subset U(\mathfrak{A}_{\min})$.

Proof of Lemma (X.4.4). Let C_0 be the chamber of X_L fixed by $U(\mathfrak{C}_{\min})$. Fix an apartment $\mathcal{A}_L$ of X_L containing C_0 and σ. Let x_σ the barycenter of σ. Then there exists a point $x_0 \in C_0^\circ$ such that the geodesic segment $[x_0, x_\sigma) \subset \mathcal{A}_L$ does not intersect any simplex of X_L of codimension greater than or equal to 2. Indeed consider the subsets of $\mathcal{A}_L$ of the form $C^\bullet = \mathrm{Cvx}\{x_\sigma, F\}\backslash\{x_\sigma\}$, where F is a simplex of codimension greater than or equal to 2 in $\mathcal{A}_L$ and where Cvx denotes a convex hull. The set of such subsets is countable. Moreover these subsets have empty interiors and by Baire's theorem their union has empty interior. It follows that this union cannot contain C_0° as required.

Let Γ be the set of chambers D in $\mathcal{A}_L$ such that $D \cap [x_0, x_\sigma) = D^\circ \cap [x_0, x_\sigma) \neq \emptyset$. Then it is easy to see that there exists an indexation $\Gamma = \{D_i\ ;\ i = 0, ..., r\}$ of the elements of Γ such that $(D_i)_{i=0,..,r}$ is a gallery satisfying: $D_0 = C_0$ and D_r contains x_σ, whence contains σ. We can be more precise: for $i = 0, ..., r-1$, D_{i+1} is the unique chamber adjacent to D_i and intersecting $[y, x]$, where $[x_0, y] =$

$[x_0, x] \cap (\cup_{j=0,\dots,i} D_j)$. Moreover, for $i = 0, ..., r-1$, let H_i be the wall separating D_i and D_{i+1}. It defines two roots $H_i^{\pm}$ (half-spaces with boundary H_i), such that H_i^- contains x_0 and H_i^+ contains x. Then the gallery $(D_i)_{i=0,\dots,r}$ is constructed in such a way that $\bigcup_{j=0}^{i} D_j \subset H_i^-$ and $\bigcup_{j=i+1}^{r} D_j \subset H_i^+$.

Let $g \in U(\mathfrak{C}_{\min})$ be such that $g\sigma = \tau$. Then g fixes C_0 pointwise. Recall that by (I.2.3), there exist normalizations of metrics on X_L and X such that the embedding $X_L \subset X$ is isometric. It follows that the set $g[x_0, x_\sigma]$ is the geodesic segment in X_L linking $g.x_0 = x_0$ and $g.x_\sigma \in X_L$. Recall that X_L is a simplicial subcomplex of $\mathrm{sd}(X)$. For $i = 0, ..., r$, gD_i is a simplex of $\mathrm{sd}(X)$ whose interior intersects X_L. So this simplex belongs to X_L. It follows that $(gD_i)_{i=0,\dots,r}$ is a gallery in X_L satisfying $gD_0 = C_0$ and $gD_r \supset \tau$.

We are going to prove by induction on $t \in \{0, ..., r\}$ that there exists $g_t \in G_L$ such that $g_t D_i = gD_i$, $i = 0, ..., t$. We will then have $g_t \in U(\mathfrak{C}_{\min})$ and $g_t^{-1} g D_r = D_r$. Since $g_t^{-1} g \in U(\mathfrak{C}_{\min})$ is a compact element of G_L, it must fix D_r pointwise. It will follow that $g_t^{-1} g\sigma = \sigma$, that is $g_t \sigma = \tau$, as required.

The result is obvious when $t = 0$. Assume $t \in \{0, .., r-1\}$ and that the result is proved for t. Replacing τ by $g_t^{-1}\tau$, g by $g_t^{-1} g$, we may assume that $gD_i = D_i$, $i = 0, ..., t$. The chamber D_{t+1} does not belong to H_t^- and has a codimension 1 face contained in H_t. The chamber gD_{t+1} has a codimension 1 face contained in H_t and does not belong to H_t^-, otherwise this would contradict the fact that $g[x_0, x_\sigma]$ is a geodesic segment. Let ς be the codimension 1 simplex $H_t \cap D_{t+1} = H_t \cap gD_{t+1}$. Then the pointwise fixator of H_t^- in G_L acts transitively on the set of chambers containing ς and not contained in H_t^- (an easy exercise left to the reader). It follows that there exists g_{t+1} fixing H_t^- pointwise such that $g_{t+1} D_{t+1} = gD_{t+1}$, as required.

X.5 Comparison of chain complexes

As in the previous section, we fix an apartment $\mathcal{A}_L$ containing $\sigma_{\mathfrak{C}_{\min}}$. As a subcomplex of $\mathcal{A}_L$, the topological space $\mathcal{A}_L$ is equipped with its canonical triangulation. We denote by $\underline{\mathcal{V}}^{\lambda_{\max}}$ the constant coefficient system on $\mathcal{A}_L$ such that for any simplex σ, $\underline{\mathcal{V}}^{\lambda_{\max}}[\sigma] = \mathcal{V}^{\lambda_{\max}}$. It gives rise to the chain complex $C_\bullet(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}})$, with an augmentation map: $C_\bullet(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}}) \xrightarrow{\epsilon_L} \mathcal{V}^{\lambda_{\max}}$. This complex is exact since the topological space $\mathcal{A}_L$ is contractible (more precisely it is homeomorphic to a finite dimensional affine space). We shall denote by ∂_L the boundary maps of that complex.

Denote by $C_\bullet(X[L], \mathcal{C})^{\lambda_{\max}} \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$ the augmented chain complex obtain by applying the functor of $\lambda_{\max}$-isotypic components to the augmented complex

(X.3.1) (or equivalently to the augmented complex of (IX.1)). It lies in the category of left $e_{\max} \star \mathcal{H}(G) \star e_{\max}$-modules. Since the functor

$$\begin{array}{ccc} \mathcal{R}_{(J,\lambda)}(G) & \longrightarrow & e_{\max} \star \mathcal{H}(G) \star e_{\max} - \mathrm{Mod} \\ \mathcal{W} & \mapsto & \mathcal{W}^{\lambda_{\max}} \end{array}$$

is an equivalence of categories, we have, using proposition (IX.2), that the complex is exact if and only if $C_\bullet(X[L], \mathcal{C})^{\lambda_{\max}} \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$ is exact.

(X.5.1) Proposition. *Assume that the representation $(\pi, \mathcal{V})$ satisfies conjecture (X.4.1). The augmented chains complexes $C_\bullet(X[L], \mathcal{C})^{\lambda_{\max}} \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$ and $C_\bullet(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}}) \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$ are then naturally isomorphic as complexes of $\mathbb{C}$-vector spaces.*

Remark. There is maybe a more precise result to prove. Indeed there should be a natural action of the scalar Hecke algebra on $C_\bullet(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}}) \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$ such that the complexes are isomorphic as complexes of $e_{\max} \star \mathcal{H}(G) \star e_{\max}$-modules.

As a corollary, we have:

(X.5.2) Theorem . *Let (J, λ) be a simple type of G. Let $(\pi, \mathcal{V})$ be a smooth complex representation in $\mathcal{R}_{(J,\lambda)}(G)$ satisfying conjecture (X.4.1). Then the augmented chain complex*

$$C_\bullet(X[L], \mathcal{C}_{(J,\lambda)}(\mathcal{V})) \xrightarrow{\epsilon} \mathcal{V}$$

is a resolution of $\mathcal{V}$ in the category $\mathcal{R}_{(J,\lambda)}(G)$. In particular, as a G-module, the space $\mathcal{V}$ is given by the homology module $H_0(X[L], \mathcal{C}_{(J,\lambda)}(\mathcal{V}))$.

Proof of proposition (X.5.1). We are going to construct a natural isomorphism of complexes from $C_\bullet(X[L], \mathcal{C})^{\lambda_{\max}} \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$ to $C_\bullet(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}}) \xrightarrow{\epsilon} \mathcal{V}^{\lambda_{\max}}$. This is a collection of isomorphisms : $[(\varphi_q)_{q\geq 0}, \psi]$, where

$$\varphi_q \in \mathrm{Hom}_{\mathbb{C}}\big(C_q(X[L], \mathcal{C})^{\lambda_{\max}}, C_q(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}})\big) \ , \ \psi \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}^{\lambda_{\max}}, \mathcal{V}^{\lambda_{\max}}) \ ,$$

and where the obvious square diagrams are commutative. We first take ψ to be the identity map of $\mathcal{V}^{\lambda_{\max}}$. To define φ_q, we note that

$$C_q(X[L], \mathcal{C}) = \coprod_{\Sigma \in \Omega_q} C_q(\Sigma, \mathcal{C})$$

and that, by corollary (X.4.2)(ii), we have:

$$C_q(X[L], \mathcal{C})^{\lambda_{\max}} = \coprod_{\Sigma \in \Omega_q \ , \ \Sigma \cap X_L \neq \emptyset} C_q(\Sigma, \mathcal{C})^{\lambda_{\max}} \ .$$

For any simplex σ of $\mathcal{A}_L$, we let Σ_σ denote the $J_{\max}$-orbit of simplices through σ, so that:

$$C_q(X[L],\mathcal{C})^{\lambda_{\max}} = \coprod_{\sigma\in(\mathcal{A}_L)q} C_q(\Sigma_\sigma,\mathcal{C})^{\lambda_{\max}} \ .$$

We now define $\varphi_q : C_q(X[L],\mathcal{C})^{\lambda_{\max}} \longrightarrow C_q(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}})$ by

$$\varphi_q(\omega)(\sigma) = S_{\Sigma_\sigma}(\omega|\ \Sigma_\sigma)\ , \sigma\in(\mathcal{A}_L)_q\ .$$

By corollary (X.4.2)(i), the map φ_q is clearly an isomorphism of $\mathbb{C}$-vector spaces.

(X.5.3) Lemma. *Under the assumptions of Theorem (X.5.2), for* $q = 1, \ldots, N/[L:F]-1$, *the following diagram is commutative:*

$$\begin{array}{ccccc} & C_q(X[L],\mathcal{C})^{\lambda_{\max}} & \stackrel{\partial}{\longrightarrow} & C_{q-1}(X[L],\mathcal{C})\lambda_{\max} & \\ \varphi_q & \downarrow & & \downarrow & \varphi_{q-1} \\ & C_q(\mathcal{A}_L,\underline{\mathcal{V}}^{\lambda_{\max}}) & \stackrel{\partial_L}{\longrightarrow} & C_{q-1}(\mathcal{A}_L,\underline{\mathcal{V}}^{\lambda_{\max}}) & \end{array}$$

Fix $\omega\in C_q(X[L],\mathcal{C})^{\lambda_{\max}}$. We have

$$\varphi_q(\omega)(\beta) = \sum_{\tau\in\Sigma_\beta}\omega(\tau),\ \beta\in(\mathcal{A}_L)_q\ ,$$

and

$$(E1)\qquad \partial_L(\varphi_q(\omega))(\alpha) = \sum_{\beta\in(\mathcal{A}_L)_q\ ,\ \beta\supset\alpha}\Big\{\sum_{\tau\in\Sigma_\beta}[\beta:\alpha]\omega(\tau)\ ,\ \alpha\in(\mathcal{A}_L)_{q-1}\Big\}.$$

On the other hand we have

$$\partial\omega(\sigma) = \sum_{\theta\in X[L]_q,\ ,\theta\supset\sigma}[\theta:\sigma]\omega(\theta)\ ,\sigma\in X[L]_q\ ,$$

and

$$(E2)\qquad \varphi_{q-1}(\partial\omega)(\alpha) = \sum_{\sigma\in\Sigma_\alpha}\Big\{\sum_{\theta\in X[L]_q\ ,\ \theta\supset\sigma}[\theta:\sigma]\omega(\theta)\ ,\ \alpha\in(\mathcal{A}_L)_{q-1}\Big\}.$$

Fix $\alpha\in(\mathcal{A}_L)_{q-1}$. The set Θ of $\theta\in X[L]_q$ containing some $\sigma\in\Sigma_\alpha$ in general strictly contains the set of τ in $X[L]_q$ such that there exists $\beta\in(\mathcal{A}_L)_q$, $\beta\supset\alpha$ and $\tau\in\Sigma_\beta$. However the first set Θ is stable under $J_{\max}$ and splits into two disjoint subsets:

– the subset Θ_1 of those θ whose $J_{\max}$-orbits intersect $\mathcal{A}_L$;

– the complementary subset Θ_2.

Let $\theta \in \Theta_1$ and $\sigma \in \Sigma_\alpha$ such that $\theta \supset \sigma$. We have $\theta \in \Sigma_\beta$ for some simplex β of $\mathcal{A}_L$. The simplex β contains a $J_{\max}$-conjugate of σ lying in $\mathcal{A}_L$. By unicity in lemma (X.4.3), that simplex must be α. In other words θ lies in Σ_β for some $\beta \in (\mathcal{A}_L)_q$ containing α and there is a unique $\sigma \in \Sigma_\alpha$ such that $\theta \supset \sigma$: if $\theta = j\beta$, $j \in J_{\max}$, then $\sigma = j\alpha$. Since the action of $J_{\max}$ preserves the incidence numbers, we must have $[\theta : \sigma] = [\beta : \alpha]$.

From the previous discussion, we deduce:

$$\varphi_{q-1}(\partial\omega)(\alpha) = \partial_L(\varphi_q(\omega))(\alpha) + \sum_{\sigma\in\Sigma_\alpha} \Big\{ \sum_{\theta\in\Theta_2\ ,\ \theta\supset\sigma} [\theta : \sigma]\omega(\theta) \Big\} \ .$$

Note that if θ is a simplex of $X[L]$, there is at most one $\sigma \in \Sigma_\alpha$ such that $\theta \supset \sigma$. Indeed two such simplices contained in θ must be equal since they have the same label. In other words in the sum σ depends in a $J_{\max}$-equivariant way from σ; we shall write $\sigma = \sigma(\theta)$. Let $\Omega_q(\Theta_2)$ be the set of $J_{\max}$-orbits in Θ_2. We may write:

$$\varphi_{q-1}(\partial\omega)(\alpha) - \partial_L(\varphi_q(\omega))(\alpha) = \sum_{\Sigma\in\Omega_q(\Theta_2)} \sum_{\theta\in\Sigma} [\theta : \sigma(\theta)]\omega(\theta)$$

$$= \sum_{\Sigma\in\Omega_q(\Theta_2)} \epsilon_\Sigma \sum_{\theta\in\Sigma} \omega(\theta) \ ,$$

where ϵ is a sign depending only on Σ. For $\Sigma \in \Omega_q(\Theta_2)$, the restriction map:

$$\begin{array}{ccc} C_q(X[L],\mathcal{C}) & \longrightarrow & C_q(\Sigma,\mathcal{C}) \\ \omega & \mapsto & \omega_{|\Sigma} \end{array}$$

is $J_{\max}$-equivariant and its restriction to $C_q(X[L],\mathcal{C})^{\lambda_{\max}}$ must have image in $C_q(\Sigma,\mathcal{C})^{\lambda_{\max}}$. Since $\Sigma \cap \mathcal{A}_L = \emptyset$, by applying corollary (X.4.2)(ii), we obtain that $C_q(\Sigma,\mathcal{C})^{\lambda_{\max}} = 0$, whence $\sum_{\theta\in\Sigma} \omega(\theta) = 0$, for all $\Sigma \in \Omega_q(\Theta_2)$. Finally we get $\varphi_{q-1}(\partial\omega)(\alpha) - \partial_L(\varphi_q(\omega))(\alpha) = 0$ and the commutativity of the diagram.

Using a quite similar proof we have the following result.

(X.5.4) Lemma. *Under the assumptions of Theorem (X.5.2), the following diagram is commutative:*

$$\begin{array}{cccc} & C_0(X[L],\mathcal{C})^{\lambda_{\max}} & \stackrel{\epsilon}{\longrightarrow} & \mathcal{V}^{\lambda_{\max}} & \\ \varphi_0 & \downarrow & & \downarrow & \mathrm{Id} \\ & C_0(\mathcal{A}_L, \underline{\mathcal{V}}^{\lambda_{\max}}) & \stackrel{\epsilon_L}{\longrightarrow} & \mathcal{V}^{\lambda_{\max}} & \end{array}$$

This finishes the proof of proposition (X.5.1) and theorem (X.5.2).

XI. Acyclicity in the case of a discrete series representation.

The aim of this section is to prove conjecture (X.4.1) when the representation is irreducible and lies in the discrete series of G. More precisely we shall assume that our representation $(\pi, \mathcal{V})$ is *an unramified twist of a (irreducible unitary) discrete series representation* of G containing our fixed simple type (J, λ). In that case the chain complex attached to $\mathcal{C}_{(J,\lambda)}(\mathcal{V})$ may be entirely computed.

XI.1. Determination of the chain complex.

We keep the notation as in section IX. Let $\mathfrak{C}$ be a hereditary order of the L-algebra C satisfying $\mathfrak{C}_{\min} \subset \mathfrak{C} \subset \mathfrak{C}_{\max}$, and let $\sigma_{\mathfrak{C}}$ be the corresponding simplex in $X_L \subset X[L]$. We want to understand the $U(\mathfrak{A})$-module structure of

$$\mathcal{V}[\sigma_{\mathfrak{C}}] = \sum_{g \in U(\mathfrak{A})/U(\mathfrak{B})J^1(\mathfrak{B}_{\max})} g.\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{\max})} \ .$$

Let $\mathcal{W}$ an irreductible constituent of the $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$-module $\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{\max})}$. By Frobenius reciprocity $\mathcal{W}$ embeds in a representation of the form $\kappa_{\max} \otimes \tau$, where τ is an irreducible representation of $U(\mathfrak{B})U^1(\mathfrak{B})$ seen as a representation of $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$ trivial on $U^1(\mathfrak{B})J^1(\mathfrak{B}_{\max})$. The following result implies that $\mathcal{W}$ actually has the form $\kappa_{\max} \otimes \tau$.

(XI.1.1) Lemma. *If τ is an irreducible representation of $U(\mathfrak{B})/U^1(\mathfrak{B})$, then the $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$-module $\kappa_{\max} \otimes \tau$ is irreducible.*

Proof. By Schur Lemma, it suffices to prove that $\mathrm{End}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}\, \kappa_{\max} \otimes \tau$ is one-dimensional. For this we closely follow the proof of [BK] Proposition (5.3.2)(ii), page 176. Write X for the representation space of $\kappa_{\max}$ and Y for the representation space of τ. Let $\varphi \in \mathrm{End}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}\, \kappa_{\max} \otimes \tau$ that we write $\varphi = \sum_j S_j \otimes T_j$, where $S_j \in \mathrm{End}_{\mathbb{C}}\, X$, $T_j \in \mathrm{End}_{\mathbb{C}}\, Y$, and where the T_j are linearly independent. For $h \in J^1(\mathfrak{B}_{\max})$ we have

$$(\kappa_{\max} \otimes \tau)(h) \circ \varphi = \varphi \circ (\kappa_{\max} \otimes \tau)(h) \ .$$

Since $J^1(\mathfrak{B}_{\max}) \subset \mathrm{Ker}\,(\xi)$, we obtain

$$\sum_j (\kappa_{\max}(h) \circ S_j - S_j \kappa_{\max}(h)) \otimes T_j = 0 \ .$$

Since the T_j are linearly independent, we obtain that $S_j \in \mathrm{End}_{J^1(\mathfrak{B}_{\max})}\, \eta(\mathfrak{B}_{\max})$ for all j. But since $\eta(\mathfrak{B}_{\max})$ is irreducible, we have that $\mathrm{End}_{J^1(\mathfrak{B}_{\max})}\, \eta(\mathfrak{B}_{\max})$ and $\mathrm{End}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}\, \kappa_{\max}$ are equal and one-dimensional. So we may as well take $j = 1$, so that $\varphi = S \otimes T$, where $S \in \mathrm{End}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}\, \kappa_{\max}$ and $T \in \mathrm{End}_{\mathbb{C}}\, Y$. Now any $h \in U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$ must satisfy

$$(S \circ \kappa_{\max}(h)) \otimes (T \circ \xi(h)) = (\kappa_{\max}(h) \circ S) \otimes (T \circ \xi(h)) = (\kappa_{\max}(h) \circ S) \otimes (\xi(S) \circ T) \ .$$

But this implies that $T \in \mathrm{End}_{U(\mathfrak{B})}\,\tau$ and our result follows from the irreducibility of τ and Schur Lemma.

First consider the case $\mathfrak{C} = \mathfrak{C}_{\max}$, so that we have $U(\mathfrak{B})J^1(\mathfrak{B}_{\max}) = J(\mathfrak{B}_{\max})$ and $\eta(\mathfrak{B}_{\max}, \mathfrak{B}_{\max}) = \eta(\mathfrak{B}_{\max}) = (\kappa_{\max})_{|J^1(\mathfrak{B}_{\max})}$. Since $\mathcal{V}$ is admissible, $\mathcal{V}^{\eta(\mathfrak{B}_{\max})}$ is finite dimensional and, as a $J(\mathfrak{B}_{\max})$-module, decomposes as a finite sum of irreducibles submodules. By lemma (XI.1.1), these irreducible representations have the form $\kappa_{\max} \otimes \tau$, where τ is an irreducible representation of $J(\mathfrak{B}_{\max})/J^1(\mathfrak{B}_{\max}) \simeq U(\mathfrak{B}_{\max})/U^1(\mathfrak{B}_{\max})$. Moreover by [SZ] (see the discussion preceeding Lemma 2, page 176), for such a τ, we have:

$$(1) \quad \mathrm{Hom}_{J(\mathfrak{B}_{\max})}\left(\kappa_{\max} \otimes \tau, \mathcal{V}\right) = \mathrm{Hom}_{U(\mathfrak{B}_{\max})/U^1(\mathfrak{B}_{\max})}\left(\tau, \mathcal{V}(\mathfrak{B}_{\max})\right) .$$

Recall that $\mathcal{V}(\mathfrak{B}_{\max})$ is the $U(\mathfrak{B}_{\max})/U^1(\mathfrak{B}_{\max})$-module $\mathrm{Hom}_{J^1(\mathfrak{B}_{\max})}(\kappa_{\max}, \mathcal{V})$.

By considering $L_{\mathfrak{B}_0} = U(\mathfrak{B}_0)/U^1(\mathfrak{B}_0)$ as a Levi subgroup of $\bar{G} = U(\mathfrak{B}_{\max})/U^1(\mathfrak{B}_{\max})$, we may form the *generalized Steinberg representation* $\mathrm{St}(\mathfrak{B}_{\max}, \rho)$ with cuspidal support $(L_{\mathfrak{B}_0}, \rho)$. It may be defined in several ways. In particular it is the unique generic sub-$\bar{G}$-module of the representation of $\bar{G}$ parabolically induced from $(L_{\mathfrak{B}_0}, \rho)$. We then have the following crucial result.

(XI.1.2) Lemma. ([SZ], Proposition 6, page 179.) *As a $\bar{G}$-module, $\mathcal{V}(\mathfrak{B}_{\max})$ is isomorphic to* $\mathrm{St}(\mathfrak{B}_{\max}, \rho)$.

It follows from (1) and the previous lemma that the space $\mathrm{Hom}_{J(\mathfrak{B}_{\max})}(\kappa_{\max} \otimes \tau, \mathcal{V})$ is zero except when $\tau \simeq \mathrm{St}(\mathfrak{B}_{\max}, \rho)$ where it is 1-dimensional. We have proved the following result.

(XI.1.3) Lemma. *We have an isomorphism of $J(\mathfrak{B}_{\max})$-modules:*

$$\mathcal{V}^{\eta(\mathfrak{B}_{\max})} \simeq \kappa_{\max} \otimes \mathrm{St}(\mathfrak{B}_{\max}, \rho) .$$

Similarly, as a $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$-module, $\mathcal{V}^{\eta(\mathfrak{B}, \mathfrak{B}_{\max})}$ is a finite sum of irreducible submodules of the form $\kappa_{\max} \otimes \tau$, where τ is an irreducible representation of $U(\mathfrak{B})/U^1(\mathfrak{B})$. For such a τ we have:

$$\begin{aligned}
\mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\kappa_{\max} \otimes \tau, \mathcal{V}) &= \mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\tau, \mathrm{Hom}_{U^1(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\kappa_{\max}, \mathcal{V})) \\
&= \mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\tau, \mathcal{V}(\mathfrak{B}_{\max})^{U^1(\mathfrak{B})J^1(\mathfrak{B}_{\max})}) \\
&= \mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\tau, \mathcal{V}(\mathfrak{B}_{\max})^{\mathbf{U}_{\mathfrak{B}}})
\end{aligned}$$

where $\mathcal{V}(\mathfrak{B}_{\max})^{\mathbf{U}_{\mathfrak{B}}}$ is the Jacquet module of $\mathcal{V}(\mathfrak{B}_{\max})$ with respect to the unipotent radical $\mathbb{U}_{\mathfrak{B}}$ of the parabolic subgroup $\mathbb{P}_{\mathfrak{B}}$ of $\bar{G}$ given by $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})/J^1(\mathfrak{B}_{\max})$. Hence we have:

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\kappa_{\max}\otimes\tau,\mathcal{V}) &= \mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\tau,\mathcal{V}(\mathfrak{B}_{\max})^{\mathbf{U}_{\mathfrak{B}}})\\ &= \mathrm{Hom}_{\mathbb{P}_{\mathfrak{B}}}(\tau,\mathcal{V}(\mathfrak{B}_{\max})^{\mathbf{U}_{\mathfrak{B}}})\\ &= \mathrm{Hom}_{\mathbb{L}_{\mathfrak{B}}}(\tau,\mathrm{St}(\mathfrak{B}_{\max},\rho)^{\mathbf{U}_{\mathfrak{B}}})\end{aligned}$$

Denote by $\mathrm{St}(\mathfrak{B},\rho)$ the generalized Steinberg representation of $\mathbb{L}_{\mathfrak{B}}$ with cuspidal support $(L_{\mathfrak{B}_0},\rho)$. It is classical that

$$\mathrm{St}(\mathfrak{B}_{\max},\rho)^{\mathbf{U}_{\mathfrak{B}}}\simeq\mathrm{St}(\mathfrak{B},\rho)$$

as $\mathbb{L}_{\mathfrak{B}}$-modules. It follows that

$$\mathrm{Dim}\,\mathrm{Hom}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}(\kappa_{\max}\otimes\tau,\mathcal{V})=\begin{cases}0 & \text{if } \tau\not\simeq\mathrm{St}(\mathfrak{B},\rho)\\ 1 & \text{if } \tau\simeq\mathrm{St}(\mathfrak{B},\rho)\end{cases}$$

As a consequence we have an isomorphism of $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$-modules:

$$\mathcal{V}^{\eta(\mathfrak{B},\mathfrak{B}_{\max})}\simeq\kappa_{\max}\otimes\mathrm{St}(\mathfrak{B},\rho)\ .$$

(XI.1.4) Proposition. (i) *The $U(\mathfrak{A})$-intertwing of $\kappa_{\max}\otimes\mathrm{St}(\mathfrak{B},\rho)$ is equal to $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$.*

(ii) *The representation of $U(\mathfrak{A})$ given by*

$$\lambda(\mathfrak{A}):=\mathrm{Ind}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}^{U(\mathfrak{A})}\kappa_{\max}\otimes\mathrm{St}(\mathfrak{B},\rho)$$

is irreducible.

(iii) *We have*

$$\mathcal{V}[\sigma_{\mathfrak{B}}]=\mathcal{V}^{\lambda(\mathfrak{A})}\simeq\lambda(\mathfrak{A}),$$

where the isomorphism is an isomorphism of $U(\mathfrak{A})$-modules.

Proof. The restriction of $\kappa_{\max}\otimes\mathrm{St}(\mathfrak{B},\rho)$ to $U^1(\mathfrak{B})J^1(\mathfrak{B}_{\max})$ is a multiple of $\eta(\mathfrak{B},\mathfrak{B}_{\max})$, so by Proposition (III.1.1)(v), we have

$$I_G(\kappa_{\max}\otimes\mathrm{St}(\mathfrak{B},\rho))\subset\mathrm{J}^1(\mathfrak{B}_{\max})\mathrm{B}^\times\mathrm{J}^1(\mathfrak{B}_{\max})\ .$$

In particular we have

$$I_{U(\mathfrak{A})}(\kappa_{\max}\otimes\mathrm{St}(\mathfrak{B},\rho))=\mathrm{U}(\mathfrak{B})\mathrm{J}^1(\mathfrak{B}_{\max})\ ,$$

and point (i) follows. Point (ii) is a consequence of Mackey irreducibility criterion and point (iii) of Lemma (V.2).

XI.2 Proof of conjecture (X.4.1) for irreducible discrete series representations.

Let $\mathfrak{C}$ be as before and x be an element of G. Write $q = \mathrm{Dim}\,\sigma_{\mathfrak{C}}$. Let Σ be the $J_{\max}$-orbit $J_{\max} x \sigma_{\mathfrak{C}}$. We must prove that $K_{\Sigma}^{\lambda_{\max}} = 0$.

Recall that we have the exact sequence of $J_{\max}$-modules

$$0 \longrightarrow K_{\Sigma}^{\lambda_{\max}} \longrightarrow C_q(\Sigma, \mathcal{C})^{\lambda_{\max}} \longrightarrow \mathcal{V}^{\lambda_{\max}} \longrightarrow 0\ ,$$

if $x \in J_{\max} G_L U(\mathfrak{A})$, and

$$0 \longrightarrow K_{\Sigma}^{\lambda_{\max}} \longrightarrow C_q(\Sigma, \mathcal{C})^{\lambda_{\max}} \longrightarrow 0\ ,$$

if $x \notin J_{\max} G_L U(\mathfrak{A})$.

Since $(\pi, \mathcal{V})$ is a discrete series representation, $\lambda_{\max}$ occurs in $\mathcal{V}$ with multiplicity 1, so that $\mathcal{V}^{\lambda_{\max}} \simeq \lambda_{\max}$ (see e.g. the discussion in [SZ] following the proof of Lemma 4, page 178). So we are reduced to proving the following result.

(XI.2.1) Proposition. *We have*

$$\mathrm{Dim}\,\mathrm{Hom}_{J_{\max}}\left(\lambda_{\max}, C_q(\Sigma, \mathcal{C})\right) \leq \begin{cases} 1 & \text{if } x \in J_{\max} G_L U(\mathfrak{A}) \\ 0 & \text{otherwise} \end{cases}$$

The rest of this section will be devoted to the proof of this proposition. Recall that

$$C_q(\Sigma, \mathcal{C}) = \coprod_{j \in J_{\max}/J_{\max} \cap U(\mathfrak{A})^x} j x C_q(\sigma_{\mathfrak{C}}, \mathcal{C}) = \mathrm{Ind}_{J_{\max} \cap U(\mathfrak{A})^x}^{J_{\max}} x C_q(\sigma_{\mathfrak{C}}, \mathcal{C})\ .$$

Using Proposition (XI.1.4), we obtain:

$$\begin{aligned} x C_q(\sigma_{\mathfrak{C}}, \mathcal{C}) &= \mathrm{Ind}_{J_{\max} \cap U(\mathfrak{A})^x}^{J_{\max}} x \mathrm{Ind}_{U(\mathfrak{B}) J^1(\mathfrak{B}_{\max})}^{U(\mathfrak{A})} \kappa_{\max} \otimes \mathrm{St}(\mathfrak{B}, \rho) \\ &= \mathrm{Ind}_{J_{\max} \cap U(\mathfrak{A})^x}^{J_{\max}} \mathrm{Ind}_{U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x}^{U(\mathfrak{A})^x} \kappa_{\max}^x \otimes \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}}\ . \end{aligned}$$

Mackey's restriction formula gives

$$\left(\mathrm{Ind}_{U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x}^{U(\mathfrak{A})^x} \kappa_{\max}^x \otimes \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}}\right)_{|\mathrm{J}_{\max} \cap \mathrm{U}(\mathfrak{A})^{\mathrm{x}}}$$

$$= \bigoplus_{u\in U} \mathrm{Ind}_{J_{\max}\cap U(\mathfrak{A})^x\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}}^{J_{\max}\cap U(\mathfrak{A})^x} \kappa_{\max}^{ux} \otimes \mathrm{St}(\mathfrak{B},\rho)^{\mathrm{ux}}$$

$$= \bigoplus_{u\in U} \mathrm{Ind}_{J_{\max}\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}}^{J_{\max}\cap U(\mathfrak{A})^x} \kappa_{\max}^{ux} \otimes \mathrm{St}(\mathfrak{B},\rho)^{\mathrm{ux}} \ .$$

where U is the double coset set

$$U = J_{\max} \cap U(\mathfrak{A})^x \backslash U(\mathfrak{A})^x / U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x \ .$$

By Frobenius reciprocity we have:

$$\mathrm{Hom}_{J_{\max}}(\lambda_{\max}, C_q(\Sigma, \mathcal{C}))$$

$$= \bigoplus_{u\in U} \mathrm{Hom}_{J_{\max}\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}}(\lambda_{\max}, \kappa_{\max}^{ux} \otimes \mathrm{St}(\mathfrak{B},\rho)^{\mathrm{ux}})$$

By definition of the cuspidal support of a representation of $U(\mathfrak{B})/U^1(\mathfrak{B})$, we have that $\kappa_{\max} \otimes \mathrm{St}(\mathfrak{B},\rho)$ embeds in

$$\mathrm{Ind}_{U(\mathfrak{B}_0)J^1(\mathfrak{B}_{\max})}^{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})} \kappa_{\max} \otimes \rho = \mathrm{Ind}_{J_{\max}}^{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})} \lambda_{\max}$$

as a $U(\mathfrak{B})J^1(\mathfrak{B}_{\max})$-module. It follows that $\mathrm{Hom}_{J_{\max}}(\lambda_{\max}, C_q(\Sigma, \mathcal{C}))$ embeds in the $\mathbb{C}$-vector space

$$\bigoplus_{u\in U} \mathrm{Hom}_{J_{\max}\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}}(\lambda_{\max}, \mathrm{Ind}_{J_{\max}^{ux}}^{U(\mathfrak{B})^{ux}J^1(\mathfrak{B})^{ux}} \lambda_{\max}^{ux})$$

Using Mackey's restriction formula again, we obtain:

$$\left(\mathrm{Ind}_{J_{\max}^{ux}}^{U(\mathfrak{B})^{ux}J^1(\mathfrak{B})^{ux}} \lambda_{\max}^{ux})\right)_{|J_{\max}\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}}$$

$$= \bigoplus_{v\in V_u} \mathrm{Ind}_{J_{\max}\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}\cap J_{\max}^{vux}}^{J_{\max}\cap U(\mathfrak{b})^{ux}J^1(\mathfrak{B}_{\max})^{ux}} \lambda_{\max}^{vux}$$

$$= \bigoplus_{v\in V_u} \mathrm{Ind}_{J_{\max}\cap J_{\max}^{vux}}^{J_{\max}\cap U(\mathfrak{B})^{ux}J^1(\mathfrak{B}_{\max})^{ux}} \lambda_{\max}^{vux} \ ,$$

where

$$V_u = J_{\max} \cap U(\mathfrak{B})^{ux} J^1(\mathfrak{B}_{\max})^{ux} \backslash U(\mathfrak{B})^{ux} J^1(\mathfrak{B}_{\max})^{ux} / J_{\max}^{ux} \ .$$

Hence it follows by Frobenius reciprocity that $\mathrm{Hom}_{J_{\max}}(\lambda_{\max}, C_q(\Sigma, \mathcal{C}))$ embeds in

$$\bigoplus_{u\in U} \bigoplus_{v\in V_v} \mathrm{Hom}_{J_{\max}\cap J_{\max}^{vux}}(\lambda_{\max}, \lambda_{\max}^{vux}) \ .$$

asz a $\mathbb{C}$-vector space. As a consequence, if $\mathrm{Hom}_{J_{\max}}(\lambda_{\max}, C_q(\Sigma, \mathcal{C}))$ is non-zero, there exist $u \in U$, $v \in V_u$ such that vux intertwines $J_{\max}$, that is

$$vux \in J_{\max} G_L J_{\max} \ .$$

For such u and v, we have $u \in xU(\mathfrak{A})x^{-1}$ and

$$v \in uxU(\mathfrak{B})J^1(\mathfrak{B}_{\max})x^{-1}u^{-1}$$

so that

$$vux \in uxU(\mathfrak{B})J^1(\mathfrak{B}_{\max}) \subset xU(\mathfrak{A})U(\mathfrak{B})J^1(\mathfrak{B}_{\max}) = xU(\mathfrak{A}) \ .$$

Hence we have $xU(\mathfrak{A}) \cap J_{\max}G_L J_{\max} \neq \emptyset$, that is $x \in J_{\max}G_L U(\mathfrak{A})$. As a consequence Proposition (XI.2.1) holds when $x \notin J_{\max}G_L U(\mathfrak{A})$.

Now let us assume that $x \in J_{\max}G_L U(\mathfrak{A})$. Writing $x = jx_L u$, $j \in J_{\max}$, $x_L \in G_L$ and $u \in U(\mathfrak{A})$, we have that

$$\Sigma = J_{\max} x \sigma_{\mathfrak{C}} = J_{\max} x_L \sigma_{\mathfrak{C}}$$

so that we may as well assume that $x \in G_L$.

Assume that for some $u \in xU(\mathfrak{A})x^{-1}$, we have

$$\mathrm{Hom}_{J_{\max} \cap U(\mathfrak{B})^{ux} J^1(\mathfrak{B}_{\max})^{ux}}(\lambda_{\max}, \kappa_{\max}^{ux} \otimes \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{ux}}) \neq 0 \ .$$

Then by the preceeding discussion, there exists $u \in uxU(\mathfrak{B})J(\mathfrak{B}_{\max})(ux)^{-1}$ such that

$$vux \in uxU(\mathfrak{B})J^1(\mathfrak{B}_{\max}) \cap J_{\max}G_L J_{\max} \ .$$

This implies that

$$uxU(\mathfrak{B})J^1(\mathfrak{B}_{\max})x^{-1} \cap J_{\max}G_L J_{\max}^x \neq \emptyset$$

that is $u \in J_{\max}G_L U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x$. So without changing the double class $\bar{u}$ of u in U, we may as well assume that $u \in J_{\max}G_L$. Let us write $u = jg_L$, $j \in J_{\max}$, $g_L \in G_L$. Since $u \in xU(\mathfrak{A})x^{-1}$, we have

$$u(x\sigma_{\mathfrak{C}}) = x\sigma_{\mathfrak{C}} = j(g_L x \sigma_{\mathfrak{C}}) \ .$$

So $x\sigma_{\mathfrak{C}}$ and $g_L x\sigma_{\mathfrak{C}}$ are simplices of X_L conjugated under the action of $J_{\max} \subset U(\mathfrak{A}_{\min})$. By Lemma (X.4.4), there exists $i \in U(\mathfrak{C}_{\min})$ such that $x\sigma_{\mathfrak{C}} = ig_L x\sigma_{\mathfrak{C}}$. Hence $ig_L \in U(\mathfrak{A})^x \cap G_L = U(\mathfrak{C})^x$ and as a consequence $g_L \in U(\mathfrak{C}_{\min})U(\mathfrak{C})^x$. It follows that $u \in J_{\max}U(\mathfrak{C}_{\min})U(\mathfrak{C})^x = J_{\max}U(\mathfrak{C})^x$, and $u \in (J_{\max} \cap U(\mathfrak{A})^x)$. $U(\mathfrak{C})$. But this implies that the image $\bar{u}$ of u in

$$U = J_{\max} \cap U(\mathfrak{A})^x \backslash U(\mathfrak{A})^x / U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x$$

is $\bar{1}$. We have proved the following:

(XI.2.2) Lemma. *For all $x \in G_L$, we have*

$$\mathrm{Hom}_{J_{\max}}(\lambda_{\max}, C_q(\Sigma, \mathcal{C})) = \mathrm{Hom}_{J_{\max} \cap U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x}(\lambda_{\max}, \kappa_{\max}^x \otimes \mathrm{St}(\mathfrak{B}, \rho^{\mathrm{x}})$$

$$= \mathrm{Hom}_{U(\mathfrak{B}_0)J^1(\mathfrak{B}_{\max}) \cap U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x}(\kappa_{\max} \otimes \rho, \kappa_{\max}^x \otimes \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}}) \ .$$

We next prove:

(XI.2.3) Lemma. *For all $x \in G_L$, we have*

$$\mathrm{Dim}\,\mathrm{Hom}_{U(\mathfrak{B}_0)J^1(\mathfrak{B}_{\max}) \cap U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x}(\kappa_{\max} \otimes \rho, \kappa_{\max}^x \otimes \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}})$$
$$= \mathrm{Dim}\,\mathrm{Hom}_{U(\mathfrak{B}_0) \cap U(\mathfrak{B})^x}(\rho, \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}}) \ .$$

Proof. It is inspired from that of [BK](5.3.2), page 176. Abreviate $\rho_{\mathfrak{B}} = \mathrm{St}(\rho, \mathfrak{B})$. Write Y (resp. X_0, X) for the space of $\kappa_{\max}$ (resp. ρ, $\rho_{\mathfrak{B}}$). Let $\varphi \in \mathrm{Hom}_{\mathbb{C}}(Y \otimes X_0, Y \otimes X) = \mathrm{End}_{\mathbb{C}}(Y) \otimes \mathrm{Hom}_{\mathbb{C}}(X_0, X)$ and write

$$\varphi = \sum_{i \in I} S_i \otimes T_i$$

where $S_i \in \mathrm{End}_{\mathbb{C}}(Y)$, $T_i \in \mathrm{Hom}_{\mathbb{C}}(X_0, X)$, and where the T_i are linearly independent. Then φ intertwine $\kappa_{\max} \otimes \rho$ and $\kappa_{\max}^x \otimes \rho_{\mathfrak{B}}^x$ if and only if

$$\sum_{i \in} (S_i \circ \kappa_{\max}(u)) \otimes (T_i \circ \rho(u)) = \sum_{i \in I} (\kappa_{\max}^x(u) \circ S_i) \otimes (\rho_{\mathfrak{B}}^x(u) \circ T_i)$$

for all $u \in U(\mathfrak{B}_0)J^1(\mathfrak{B}_{\max}) \cap U(\mathfrak{B})^x J^1(\mathfrak{B}_{\max})^x$. In particular if φ intertwines these representations, for $u \in J^1(\mathfrak{B}_{\max}) \cap J^1(\mathfrak{B}_{\max})^x$, we must have

$$\sum_{i \in I} (S_i \circ \kappa_{\max}(u) - \kappa_{\max}^x(u) \circ S_i) \otimes T_i = 0$$

Since the T_i are linearly independent, we obtain

$$S_i \in \mathrm{Hom}_{J^1(\mathfrak{B}_{\max}) \cap J^1(\mathfrak{B}_{\max})^x}(\kappa_{\max}, \kappa_{\max}^x)$$

$$= \mathrm{Hom}_{J^1(\mathfrak{B}_{\max})\cap J^1(\mathfrak{B}_{\max})^x}\left(\eta_{\max}, \eta_{\max}^x\right) .$$

By [BK](5.1.8) and (5.2.7), the spaces

$$\mathrm{Hom}_{J(\mathfrak{B}_{\max})\cap J(\mathfrak{B}_{\max})^x}\left(\kappa_{\max}, \kappa_{\max}^x\right)$$

and

$$\mathrm{Hom}_{J^1(\mathfrak{B}_{\max})\cap J^1(\mathfrak{B}_{\max})^x}\left(\eta_{\max}, \eta_{\max}^x\right)$$

are equal and 1-dimensional. It follows that any φ in

$$\mathrm{Hom}_{J_{\max}\cap U(\mathfrak{B})^x J^1(\mathfrak{B})^x}\left(\lambda_{\max}, \kappa_{\max}^x \otimes \rho_{\mathfrak{B}}^x\right)$$

writes $\varphi = S \otimes T$, where $S \in \mathrm{Hom}_{J(\mathfrak{B}_{\max})\cap J(\mathfrak{B}_{\max})^x}, (\kappa_{\max}, \kappa_{\max}^x)$ and $T \in \mathrm{Hom}_{\mathbb{C}}(X_0, X)$. Writing that such a $S \otimes T$ does intertwine the representations, we easily obtain that

$$T \in \mathrm{Hom}_{U(\mathfrak{B}_0)\cap U(\mathfrak{B})^x}\left(\rho, \rho_{\mathfrak{B}}\right) .$$

It follows that we have a canonical isomorphism of $\mathbb{C}$-vector spaces:

$$\mathrm{Hom}_{J_{\max}\cap U(\mathfrak{B})^x J^1(\mathfrak{B})^x}\left(\lambda_{\max}, \kappa_{\max}^x \otimes \rho_{\mathfrak{B}}^x\right)$$

$$= \mathrm{Hom}_{J(\mathfrak{B}_{\max})\cap J(\mathfrak{B}_{\max})^x}\left(\kappa_{\max}, \kappa_{\max}^x\right) \otimes \mathrm{Hom}_{U(\mathfrak{B}_0)\cap U(\mathfrak{B})^x}\left(\rho, \rho_{\mathfrak{B}}^x\right) ,$$

with

$$\mathrm{Dim}\,\mathrm{Hom}_{J(\mathfrak{B}_{\max})\cap J(\mathfrak{B}_{\max})^x}\left(\kappa_{\max}, \kappa_{\max}^x\right) = 1$$

and the lemma follows.

To obtain Proposition (XI.2.1), we are now reduced to proving the following result.

(XI.2.3) Lemma. *For all $x \in G_L$, we have*

$$\mathrm{Dim}_{\mathbb{C}}\,\mathrm{Hom}_{U(\mathfrak{B}_0)\cap U(\mathfrak{B})^x}\left(\rho, \mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}}\right) \leq 1 .$$

Fix a level 0 discrete series representation $(\pi_0, \mathcal{V}_0)$ of G_E belonging to the Bernstein component of G_E defined by the type $(U(\mathfrak{B}_0), \rho)$. Applying the results of section (XI.1) to $(\pi_0, \mathcal{V}_0)$, we have that $\rho \simeq \mathcal{V}_0^{U^1(\mathfrak{B}_0)}$ as $U(\mathfrak{B}_0)$-modules and $\mathrm{St}(\mathfrak{B}, \rho)^{\mathrm{x}} \simeq \mathcal{V}_0^{\mathrm{U}^1(\mathfrak{B})^{\mathrm{x}}}$ as $U(\mathfrak{B})^x$-modules. Hence the statment of the lemma rewrites:

Let $\mathfrak{B}_1$ be a hereditary order lying in the image of the canonical map $\mathrm{Her}(C) \longrightarrow \mathrm{Her}(B)$*. Then*

$$\mathrm{Dim}_{\mathbb{C}}\, \mathrm{Hom}_{U(\mathfrak{B}_0)\cap U(\mathfrak{B}_1)}(\mathcal{V}_0^{U^1(\mathfrak{B}_0)}, \mathcal{V}_0^{U^1(\mathfrak{B}_1)}) \leq 1 \ .$$

We may write this in the language of simplicial complexes. For σ a simplex of X_E, write U_σ for the parahoric subgroup of G_E fixing σ and U_σ^1 for its pro-unipotent radical. Write $\sigma_0 = \sigma_{\mathfrak{B}_0}$. Then our lemma is equivalent to:

(XI.2.4) Lemma. *For all simplex τ lying in the image of the canonical simplicial map $X_L \longrightarrow X_E$, we have*

$$\mathrm{Dim}_{\mathbb{C}}\, \mathrm{Hom}_{U_{\sigma_0}\cap U_\tau}(\mathcal{V}_0^{U^1_{\sigma_0}}, \mathcal{V}_0^{U^1_\tau}) \leq 1 \ .$$

Fix an apartment $\mathcal{A}_L$ of X_L containing σ_0 and τ (we see $X_L \longrightarrow X_E$ as an inclusion). According to [Br] Lemma 4, there exists a unique chamber σ of $\mathcal{A}_L$ such that we have the containments

$$E[\sigma_0, \tau] \supset \sigma \supset \tau$$

where $E[\sigma,\tau]$ is the *enclos* of $\sigma \cup \tau$ in the sense of [BTI] Definition (2.4.1). Moreover by [Br] Lemma 5, we have that the simplex σ lies between σ_0 and τ in the sense of [SS] §2. This means that there exists points x_{σ_0} in $|\sigma_0|^{\mathrm{o}}$, x_σ in $|\sigma|^{\mathrm{o}}$, and x_τ in $|\tau|^{\mathrm{o}}$ such that x_σ belongs to the geometric segment $[x_{\sigma_0}, x_\tau]$. Since the embedding $X_L \longrightarrow X_E$ is simplicial and affine, we have that, as a simplex of X_E, σ lies between σ_0 and τ. We may then apply Proposition (2.5) of [SS]:

(XI.2.5) Lemma. *The image of $U_\sigma^1 = U_\sigma^1 \cap U_\tau$ in U_τ/U_τ^1 is contained in the image of $U_{\sigma_0}^1 \cap U_\tau$ in U_τ/U_τ^1.*

Next fix an L-basis $(v_1, ..., v_e)$ of V corresponding to the apartment $\mathcal{A}_L$. Moreover fix a basis $(\zeta_1, ..., \zeta_r)$ of the $\mathfrak{o}_E$-module $\mathfrak{o}_L$. Set

$$V_i = Lv_i = \mathrm{Vect}_E\langle \zeta_j v_i \ ; \ j = 1, ..., r\rangle \ , \ i = 1, ...e \ ,$$

and write M for the Levi subgroup of G_E corresponding to the decomposition $V = V_1 \oplus \cdots \oplus V_e$.

(XI.2.6) Lemma. *Let θ be a simplex of $\mathcal{A}_L$ (seen as a simplex of X_E).*

(i) *The intersection* $U_\theta \cap M$ *does not depends on* θ. *We denote it by* M^0. *It is given by*

$$\prod_{i=1,..,e} \mathrm{GL}(r, \mathfrak{o}_E)$$

where the ith copy of $\mathrm{GL}(r,\mathfrak{o}_E)$ *is the maximal compact subgroup of* $\mathrm{Aut}_E\,(V_i)$ *which is standard in the basis* $(\zeta_j v_i)_j$ *of* V_i.

(ii) *Assume moreover that* θ *is a chamber of* $\mathcal{A}_L$. *Then we have the Iwahori decomposition*

$$U_\theta = (U_\theta \cap M) U_\theta^1 \ .$$

Proof. This is an easy exercice in lattice chain theory and we only sketch the proofs.

The simplex θ corresponds to a certain $\mathfrak{o}_L$-lattice chain $\mathcal{N} = (N_k)_{k\in\mathbb{Z}}$ in V. The fact that θ lies in $\mathcal{A}_L$ exactly means that the chain $\mathcal{L}$ is split by the decomposition $V = \bigoplus V_i$, i.e. for k in $\mathbb{Z}$ we have:

$$N_k = \bigoplus N_k^i, \ N_k^i = N_k \cap V_i, \ i = 1, ...e.$$

Let $g \in U_\theta \cap M$ that we write $g = \bigoplus g_i$, $g_i \in \mathrm{End}_E\, V_i$, $i = 1, ..., e$. Then we get $g_i N_k^i = N_k^i$, $k \in \mathbb{Z}$, $i = 1, ..., e$, that is $g_i \mathfrak{o}_L v_i = \mathfrak{o}_L v_i$, $i = 1, ..., e$, and point (i) follows easily.

Assume moreover that θ is a chamber in $\mathcal{A}_L$. Since the identity of (ii) is invariant under the action of the affine Weyl group of this apartment (since it stabilizes M), we may as well assume that θ is the standard chamber attached to the lattice chain $\mathcal{N}$ defined by $N_k = \bigoplus N_k^i$ as above and, for $k = 0, ..., e-1$, $N_k^i = \mathfrak{o}_L v_i$, if $i \in \{0, ..., e-1-k\}$, $N_k^i = \mathfrak{p}_L v_i$, $i \in \{e-1-k+1, ..., e-1\}$. Then by a straightforward computation, we obtain that an element $g \in \mathrm{End}_E\, V$, with a block matrix $g = (g_{uv})_{u,v=1,...,e}$ in the decomposition $V = \bigoplus V_i$, lies in U_θ if and only if we have $g_{uu} \in \mathrm{GL}(r, \mathfrak{o}_E)$, $u = 1, ..., e$, $g_{uv} \in \mathrm{M}(r, \mathfrak{o}_E)$, if $v > u$, and $g_{uv} \in \mathfrak{p}_E \mathrm{M}(r, \mathfrak{o}_E)$, if $u > v$. It is then classical that such a matrix has an Iwahori decomposition as described by the identity of (ii).

We have $M^0 \subset U_{\sigma_0} \cap U_\tau$ and, as a M^0-module, $\mathcal{V}_0^{U^1_{\sigma_0}}$ is isomorphic to the irreducible representation $\rho^{\otimes e}$. So in order to prove Lemma (XI.2.3), it suffices, by Schur Lemma, to show that $\mathrm{Im}\,\varphi = \varphi(\mathcal{V}_0^{U^1_{\sigma_0}})$ is independent of the choice of a non-zero intertwining operator φ in $\mathrm{Hom}_{U_{\sigma_0}\cap U_\tau}(\mathcal{V}_0^{U^1_{\sigma_0}}, \mathcal{V}_0^{U^1_\tau})$.

Let φ such a non-zero intertwining operator. Then $\mathcal{W} := \varphi(\mathcal{V}_0^{U^1_{\sigma_0}})$ is a sub-M^0-module of $\mathcal{V}_0^{U^1_\tau}$ equivalent to $\rho^{\otimes e}$. The groups $U^1_{\sigma_0} \cap U_\tau$ and U^1_τ act trivially on $\mathcal{W}$. Moreover by Lemma (XI.2.5), $U^1_\sigma \subset (U^1_{\sigma_0} \cap U_\tau)U^1_\tau$. It follows that U^1_σ

acts trivially on $\mathcal{W}$ and that the action of U_σ/U_τ^1 on that space is the action inflated from the representation $\rho^{\otimes e}$ of U_τ/U_τ^1; write ρ_σ for the corresponding representation.

As a U_σ-module, $\mathcal{V}_0^{U_\tau^1}$ is equivalent to the generalized Steinberg representation with cuspidal support $(\bar{M}_0, \rho^{\otimes e})$, where $\bar{M}_0$ is the image of M_0 in the quotient U_σ/U_σ^1. It is well known that this Steinberg representation occurs with multiplicity 1 in the parabolically induced representation $\mathrm{Ind}_{U_\sigma}^{U_\tau}\rho_\sigma$. It follows by Frobenius reciprocity that ρ_σ occurs in $\mathcal{V}_0^{U_\tau^1}$ with multiplicity 1. It follows that $\mathcal{W}$ is the unique sub-U_σ-module of $\mathcal{V}_0^{U_\tau^1}$ isomorphic to ρ_σ and Lemma (XI.2.4) is proved.

We have unconditionnaly proved the following result.

(XI.2.7) Theorem. *Assume that $(\pi, \mathcal{V})$ is an unramified twist of an irreducible unitary discrete series representation lying in the Bernstein block $\mathcal{R}_{(J,\lambda)}$. Then the augmented chain complex (IX.1) is exact.*

XII. Explicit pseudo-coefficients for discrete series representations.

Let $(\pi, \mathcal{V})$ be an irreducible (unitary) discrete series representation of G. In this section, following [SS2]§II.4, we show that Theorem (XI.2.7) leads to an explicit pseudo-coefficient φ_π for π. We then show how to derive an explicit formula for the value of the Harish-Chandra character of π at an elliptic regular element.

XII.1 The coefficient system $\mathcal{C}(\pi)$.

Recall that, with the notation of §XI, the coefficient system $\mathcal{C} = \mathcal{C}(\pi)$ canonically attached to π is given on a part of X_L by $V[\sigma_{\mathfrak{C}}] = \mathcal{V}^{\lambda(\mathfrak{A})} \simeq \lambda(\mathfrak{A})$, where

– $\sigma = \sigma_{\mathfrak{C}}$ is any simplex of X_L satisfying $\mathfrak{C}_{\min} \subset \mathfrak{C} \subset \mathfrak{C}_{\max}$,
– $\mathfrak{A} = \mathfrak{A}(\mathfrak{C})$,
– $\lambda(\mathfrak{A}) = \mathrm{Ind}_{U(\mathfrak{B})J^1(\mathfrak{B}_{\max})}^{U(\mathfrak{A})}\kappa_{\max} \otimes \mathrm{St}(\mathfrak{B}, \rho)$.

Since the coefficient system $\mathcal{C}$ is G-equivariant, for any order $\mathfrak{A}$ as above, the representation $\lambda(\mathfrak{A})$ extends to a representation of $\mathcal{K}(\mathfrak{A}) = \mathcal{N}_G(\sigma)$ that we still denote by $\lambda(\mathfrak{A})$. In the sequel we shall also write

$$\mathcal{N}_G(\sigma) = \mathcal{K}_\sigma \text{ and } \lambda(\mathfrak{A}) = \lambda_\sigma \ .$$

By equivariance, we may define an irreducible smooth representation λ_σ of $\mathcal{K}_\sigma$ for *any* simplex σ of $X[L]$, and by equivariance of $\mathcal{C}$ we have $\mathcal{V}[\sigma] = \mathcal{V}^{\lambda_\sigma} \simeq \lambda_\sigma$.

XII.2 Euler-Poincaré functions.

Let χ be the central character of π. All representations that we consider will lie in the category $\mathcal{S}_\chi(G)$ of those smooth representations admitting a central character equal to χ. If $\mathcal{V}', \mathcal{V}'' \in \mathcal{S}_\chi(G)$ with $\mathcal{V}'$ of finite length and $\mathcal{V}''$ admissible, we define the Euler-Poincaré characteristic :

$$\mathrm{EP}\,(\mathcal{V}', \mathcal{V}'') = \sum_{q\geq 0} (-1)^q \dim \mathrm{Ext}^q_{\mathcal{S}_\chi(G)}\,(\mathcal{V}', \mathcal{V}'')\ .$$

We denote by Z the center of G and fix a Haar measure $\mu_{G/Z}$ on G/Z. We denote by $\mathcal{H}_\chi(G)$ the convolution Hecke algebra of locally constant functions $f : G \longrightarrow \mathbb{C}$ satisfying

– $f(zf) = \chi^{-1}(z)f(g)$, $z\in Z$, $g\in G$,

– f has compact support modulo Z.

Representations in $\mathcal{S}_\chi(G)$ are naturally left $\mathcal{H}_\chi(G)$-modules. The *character* of an admissible representation $(\pi', \mathcal{V}')$ in $\mathcal{S}_\chi(G)$ is the functionnal

$$\mathrm{Tr}_{\mathcal{V}'}\ :\ \mathcal{H}_\chi(G) \longrightarrow \mathbb{C}\ ,\ \psi \longrightarrow \mathrm{tr}\,(\pi'(\psi))\ ,$$

where $\pi'(\psi)$ is the endomorphism of $\mathcal{V}'$ attached to π'; it is formally given by the integral

$$\pi'(\psi) = \int_{G/Z} \psi(g)\pi'(g)\, d\mu_{G/Z}\,(\dot g)\ .$$

We set $d = \dim X[L]$. For $q = 0, ..., d$, we fix a set $\mathcal{F}_q$ of representatives of G-orbits in the set $X[L]_q$ of q-simplices in $X[L]$. If $\sigma = \sigma_{\mathfrak{C}}$ is a simplex of $X[L]$, we denote by $\epsilon_\sigma : \mathcal{K}_\sigma \longrightarrow \{\pm 1\}$ the abelian character defined as follows. If $g \in \mathcal{K}_\sigma$, $\epsilon_\sigma(g)$ if the sign of the permutation of the vertex set of σ induced by the action of g. Moreover for such a simplex σ, we denote by $\tau_\sigma^{\mathcal{V}}$ the character of the representation $(\mathcal{K}_\sigma, \lambda_\sigma)$. For all simplices of $X[L]$ we extend the class functions ϵ_σ and $\tau_\sigma^{\mathcal{V}}$ by zero to functions on G. Following Kottwitz [Kot] and Schneider and Stuhler [SS2], we define the *Euler-Poincaré function* attached to $(\pi, \mathcal{V})$ by the formula:

$$f_{\mathrm{EP}}^{\mathcal{V}} := \sum_{q=0}^{d} \sum_{\sigma\in\mathcal{F}_q} (-1)^q .\, \mu_{G/Z}(\mathcal{K}_\sigma/Z)^{-1}.\bar\tau_\sigma^{\mathcal{V}}.\epsilon_\sigma\ .$$

Remark. The Euler-Poincaré function $f_{\mathrm{EP}}^{\mathcal{V}}$ does depend on the choices of representative sets $\mathcal{F}_q$, $q = 0, ..., d$.

(XII.2.1) Proposition. *For all admissible representations* $(\pi', \mathcal{V}')$ *in* $\mathcal{S}_\chi(G)$*, we have*

$$\mathrm{Tr}_{\mathcal{V}'}\left(f_{\mathrm{EP}}^{\mathcal{V}}\right) = \mathrm{EP}_{\mathcal{S}_\chi(G)}\left(\mathcal{V}, \mathcal{V}'\right) .$$

Proof. We have the decomposition.

$$C_c^{\mathrm{or}}(X_{(q)}, \mathcal{C}(\mathcal{V})) = \bigoplus_{\sigma \in \mathcal{F}_q} C_c^{\mathrm{or}}(G.\sigma, \mathcal{C}(\mathcal{V})) ,$$

where $C_c^{\mathrm{or}}(G.\sigma, \mathcal{C}(\mathcal{V}))$ denotes the G-space of oriented chains with support in $G.\{(\sigma, o_1), (\sigma, o_2)\}$, o_1 and o_2 denoting the two possible orientations of σ. Since

$$C_c^{\mathrm{or}}(X_{(\bullet)}, \mathcal{C}(\mathcal{V})) \xrightarrow{\epsilon} \mathcal{V}$$

is a projective resolution of $\mathcal{V}$ in $\mathcal{S}(G)_\chi$, Lefschetz formula gives :

$$\mathrm{EP}_{\mathcal{S}(G)_\chi}\left(\mathcal{V}, \mathcal{V}'\right) = \sum_{q=0}^{d} (-1)^q \sum_{\sigma \in \mathcal{F}_q} \dim \mathrm{Hom}_G\left(C_c^{\mathrm{or}}(G.\sigma, \mathcal{C}(\mathcal{V})), \mathcal{V}'\right) .$$

By definition of compact induction we have:

$$C_c^{\mathrm{or}}(G.\sigma, \mathcal{C}(\mathcal{V})) = \mathrm{c} - \mathrm{Ind}_{\mathcal{K}_\sigma}^{G} C_c^{\mathrm{or}}(\sigma, \mathcal{C}(\mathcal{V})) ,$$

where $\mathcal{C}_c^{\mathrm{or}}(\sigma, \mathcal{C}(\mathcal{V}))$ denotes the $\mathcal{K}_\sigma$-space of chains withs support in $\{(\sigma, o_1), (\sigma, o_2)\}$. Moreoer, again by definition, we have the isomorphism of $\mathcal{K}_\sigma$-modules:

$$C_c^{\mathrm{or}}(\sigma, \mathcal{C}(\mathcal{V})) = \lambda_\sigma \otimes \epsilon_\sigma .$$

Using Frobenius reciprocity for compact induction, we obtain

$$\mathrm{Hom}_G\left(C_c^{\mathrm{or}}(\sigma, \mathcal{C}(\mathcal{V})), \mathcal{V}'\right) = \mathrm{Hom}_{\mathcal{K}_\sigma}(\lambda_\sigma \otimes \epsilon_\sigma, \mathcal{V}') .$$

Moreover $\dim \mathrm{Hom}_{\mathcal{K}_\sigma}\left(\lambda_\sigma \otimes \epsilon_\sigma, \mathcal{V}'\right)$ is nothing other than the multiplicity of $\lambda_\sigma \otimes \epsilon_\sigma$ in the isotypic component $(\mathcal{V}')^{\lambda_\sigma \otimes \epsilon_\sigma}$:

$$\dim \mathrm{Hom}_{\mathcal{K}_\sigma}\left(\lambda_\sigma \otimes \epsilon_\sigma, \mathcal{V}'\right) = \frac{1}{\dim \lambda_\sigma} \dim\left(\mathcal{V}'\right)^{\lambda_\sigma \otimes \epsilon_\sigma} .$$

Hence we have obtained

$$\mathrm{EP}_{\mathcal{S}(G)_\chi}(\mathcal{V}, \mathcal{V}') = \sum_{q=0}^{d} \sum_{\sigma \in \mathcal{F}_q} \frac{(-1)^q}{\dim \lambda_\sigma} \dim\left(\mathcal{V}'\right)^{\lambda_\sigma \otimes \epsilon_\sigma} .$$

We need to compare this with $\mathrm{Tr}_{\mathcal{V}'}(f_{\mathrm{EP}}^{\mathcal{V}})$. For this we have to compute $\mathrm{tr}_{\mathcal{V}'}(\bar{\tau}_{\sigma}^{\mathcal{V}}.\epsilon_{\sigma})$, for all q and $\sigma \in \mathcal{F}_q$. Recall that for such q and σ,

$$E_{\sigma} := \frac{1}{\mu_{G/Z}(\mathcal{K}_{\sigma})}\, \bar{\tau}_{\sigma}^{\mathcal{V}} \epsilon_{\sigma}.\dim(\lambda_{\sigma}\epsilon_{\sigma})$$

is an idempotent of $\mathcal{H}(G)_{\chi}$, and that E_{σ} seen as an endomorphism of $\mathcal{V}'$ is the projection of the $\lambda_{\sigma} \otimes \epsilon_{\sigma}$-isotypic component $(\mathcal{V}')^{\lambda_{\sigma}\otimes\epsilon_{\sigma}}$. Hence we have that $\mathrm{Tr}_{\mathcal{V}'}(E_{\sigma}) = \dim(\mathcal{V}')^{\lambda_{\sigma}\otimes\epsilon_{\sigma}}$ and the proposition follows.

We shall need the following result.

(XII.2.2) Theorem. *Let $\mathcal{V}'$ be an irreducible tempered representation in $\mathcal{S}_{\chi}(G)$. Then :*

$$\mathrm{EP}_{\mathcal{S}_{\chi}(G)}(\mathcal{V}, \mathcal{V}') = \begin{cases} 1 & \text{if } \mathcal{V}' \simeq \mathcal{V} , \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is shown in [SZ2] Prop. 9.3 and subsequent remark (based upon a result of R. Meyer in [Me]) that

$$\mathrm{Ext}^*_{\mathcal{S}_{\chi}(G)}(\mathcal{V}, \mathcal{V}') = \mathrm{Ext}^*_{\mathcal{S}_{\chi}^{temp}(G)}(\mathcal{V}, \mathcal{V}') ,$$

where $\mathcal{S}_{\chi}^{temp}(G)$ denotes the category of all tempered smooth representations with central character χ. But by a variant of [SZ2] Prop. 2.3 the representation $\mathcal{V}$ is a projective object in $\mathcal{S}_{\chi}^{temp}(G)$.

Recall that a function $f \in \mathcal{H}(G)_{\chi}$ is a pseudo-coefficient of (π, V) if for any irreducible tempered representation in $\mathcal{S}_{\chi}(G)$, we have

$$\mathrm{Tr}_{\mathcal{V}'}(f) = \begin{cases} 1 & \text{if } \mathcal{V}' \simeq \mathcal{V} , \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence of (XII.2.1) and (XII.2.2) we have :

(XII.2.3) Theorem. *The Euler-Poincaré function $f_{\mathrm{EP}}^{\mathcal{V}}$ is a pseudo-coefficient of $(\pi, \mathcal{V})$.*

In [Br2], the first author obtained pseudo-coefficients for discrete series representations of G using a quite different approach (but also based on Bushnell and Kutzko type theory). Our pseudo-coefficients are likely to be very close to those of [Br2], but the comparison has yet to be done.

XII.3. An explicit character formula.

If $\psi \in \mathcal{H}(G)_\chi$ and $h \in G$ is a regular elliptic element, the orbital integral

$$\overset{\vee}{\psi}(h) := \int_{G/Z} \psi(g^{-1}hg) d\mu_{G/Z}(\dot{g})$$

is known to converge (see e.g. [SS2], page 140 in the case of a reductive group with compact center, the non-compact case being similar).

Let Θ_π denote the Harish-Chandra character of $(\pi, \mathcal{V})$. This is a locally constant function on the set $G^{\rm reg}$ of regular semisimple elements of G. The following result relates values of Θ_π with the orbital integral of a pseudo-coefficient of π.

(XII.3.1) Theorem. (Kazhdan-Badulescu) *Let f_0 be a pseudo-coefficient of $(\pi, \mathcal{V})$. Then for all regular elliptic element h of G, we have*

$$\Theta_\pi(h) = \overset{\vee}{f_0}(h^{-1}) \ .$$

Remark. This theorem is due to Kazhdan ([Ka], Prop. 3, page 28) for a reductive group with compact center when F has characteristic 0. It is due to Badulescu ([Ba] Théorème (4.3)(ii), page 64) for our group G without restriction on F.

Let $h \in G$ be a regular elliptic element. To obtain a formula for $\Theta_\pi(h)$ it suffices to compute $(f_{\rm EP}^{\mathcal{V}})^\vee(h^{-1})$ explicitely. For this we closely follow the proof of Lemma (III.4.10) of [SS2] where a similar computation is done.

If $|X|$ denotes the geometric realization of the building of G, it is known that $|X|^h$ is compact (see e.g. [SS2], page 141). Hence so is $|X[L]|^h$ the set of h-fixed points in the geometric realization of $X[L]$ since the subset $X[L] \subset X$ is closed. Let us sketch the proof of this latter fact. Let $x_n = g_n.c_n$ be a converging sequence of points in $X[L]$ with limit x where, for all n, g_n is in G and c_n lies in some fixed (closed) chambre C_L of X_L. Then (c_n) has a convergent subsequence and replacing (x_n) by a subsequence we may assume that c_n converges to some $c \in C_L$. Let d be a G-invariant metric on G. we have $d(x, g_n.c_n) = d(g_n^{-1}.x, c_n) \longrightarrow 0$. Hence $d(c_n, G.x) \longrightarrow 0$ and $d(c, G.x) = 0$. But it is an easy exercice in Bruhat-Tits theory (left to the reader) that the G-orbit of any point of X is closed in X. Hence $c \in G.x$, that is $x \in G.c \subset X[L]$ as required.

It follows that their exists a finite number of simplices σ in $X[L]$ such that $h.\sigma = \sigma$. For such a σ, the intersection $\sigma \cap |X[L]|^h$ is non-empty. The collection of $\sigma(h)$ where σ runs over the simplices of $X[L]$ globally fixed by h endows the compact topological set $|X[L]|^h$ with a simplicial structure. As noticed by Kottwitz ([Kot], page 635), it is an easy exercise to check that for all σ in $X[L]$ fixed by h we have

$$\epsilon_\sigma(h) = (-1)^{\dim\sigma - \dim\sigma(h)} \ .$$

(XII.3.2) Theorem. *For all regular elliptic element h of G, we have*

$$\Theta_\pi(h) = \sum_{q=0}^{\dim|X[L]|^h} \sum_{\sigma(h)\in|X[L]|_q^h} (-1)^q \,\mathrm{Tr}\,(h,\lambda_\sigma) \ ,$$

where $|X[L]|_q^h$ denotes the set of q-simplices in $|X[L]|^h$.

Proof. We have to prove that

$$(f_{\mathrm{EP}}^{\mathcal{V}})^\vee(h) = \sum_{q=0}^{d} \sum_{\sigma(h)\in|X[L]|_q^h} (-1)^q \bar\tau_\sigma(h) \ .$$

Let $\psi \in \mathcal{H}(G)_\chi$ be any function with support in $\mathcal{K}_\sigma$, for some q-dimensional simplex σ of $X[L]$, such that $\psi_{|\mathcal{K}_\sigma}$ is a class function. Let $(G.\sigma)^h$ be the set of simplices in the G-orbit of σ that are fixed by h. Finally let G_h denote the centralizer of h in G. Following [SS2], page 141, we write

$$\begin{aligned}\int_{G/Z} \psi(g^{-1}hg)d\mu_{G/Z}(\dot g) &= \sum_{g\in G_h\backslash G/\mathcal{K}_\sigma,\ g^{-1}hg\in\mathcal{K}_\sigma} \psi(g^{-1}hg)\mu_{G/Z}(G_hg\mathcal{K}_\sigma/Z) \\ &= \sum_{g\sigma\in G_h\backslash(G.\sigma)^h} \psi(g^{-1}hg)\mu_{G/Z}(\mathcal{K}_\sigma/Z).[G_h : G_h\cap\mathcal{K}_{g\sigma}] \\ &= \mu_{G/Z}(\mathcal{K}_\sigma/Z). \sum_{g\sigma\in(G.\sigma)^h} \psi(g^{-1}hg) \ .\end{aligned}$$

We then apply this to each component of our Euler-Poincaré function $f_{\mathrm{EP}}^{\mathcal{V}}$:

$$\begin{aligned}(f_{\mathrm{EP}}^{\mathcal{V}})^\vee(h) &= \sum_{q=0}^{d} \sum_{\sigma\in\mathcal{F}_q} (-1)^q. \sum_{g\sigma\in(G.h)^h} (\bar\tau_\sigma^{\mathcal{V}}.\epsilon_\sigma)(g^{-1}hg) \\ &= \sum_{q=0}^{d} \sum_{\sigma\in\mathcal{F}_q} \sum_{g\sigma\in(G.\sigma)^h} (-1)^q.\epsilon_{g\sigma}(h)\bar\tau_{g\sigma}^{\mathcal{V}}(h) \\ &= \sum_{q=0}^{d} \sum_{\sigma\in(X[L]_q)^h} (-1)^q.\epsilon_\sigma(h).\bar\tau_\sigma^{\mathcal{V}}(h) \\ &= \sum_{q=0}^{\dim|X[L]|^h} \sum_{\sigma(h)\in|X[L]|_q^h} (-1)^{\dim\sigma(h)}.\bar\tau_\sigma^{\mathcal{V}}(h)\end{aligned}$$

and we are done.

XII.4 The character of discrete series representations at minimal elements.

In this section we prove that the character formula of theorem (XII.3.2) takes a striking simple form under a simple assumption on the regular elliptic element h.

Let $\gamma \in G$ satisfying: the algebra $K := F[\gamma] \subset A$ is a field (we shall assume later that the extension K/F is separable, but we do not need this hypothesis for the moment). Let v_K denote the normalized valuation of K. Following [BK1](1.4.14), one says that γ is *minimal over* F if it satifies:

(i) $\gcd(v_K(\gamma), e(K/F)) = 1$,

(ii) $\varpi_F^{-v_K(\gamma)}\gamma^{e(K/F)} + \mathfrak{p}_K$ generates the extension of residue fields $\mathbb{F}_K/\mathbb{F}$.

Here ϖ_F is some uniformizer of F that we fix once for all.

¿From [BK1], Exercice (1.5.6), page 44, we have the following result.

(XII.4.1) Lemma. *Assume that $\gamma \in G$ is minimal over F and let $\mathfrak{A}$ be a hereditary order of A. Then γ normalizes $\mathfrak{A}$ if, and only if, $K^\times$ normalizes $\mathfrak{A}$.*

Our next result is a more precise version of this lemma.

(XII.4.2) Proposition. *Assume that $\gamma \in G$ is minimal over F.*

(i) *We have $X^\gamma = X^{K^\times}$ (fixed points set in the geometric realizations). In particular X^γ coincides with the canonical image of X_K in X (cf. Theorem (I.2.1)).*

(ii) *In particular, if K/F is a maximal subfield extension of A, then X^γ reduces to a single point x_γ, isobarycenter of simplex corresponding to the unique hereditary order $\mathfrak{A}_\gamma$ normalized by $K^\times$ (it has $\mathfrak{o}_F$-period $e(K/F)$.*

Proof. We use the lattice model of the geometric realization of X given in [BL] §I. Let us describe this model. Let $L(V)$ denote the set of $\mathfrak{o}_F$-lattices in V. Let $\mathrm{Latt}^1_{\mathfrak{o}_F}(V)$ denote the set of functions $\Lambda \ : \ \mathbb{R} \longrightarrow L(V)$ satisfying:

– Λ is non-increasing, that is $\Lambda(r) \subset \Lambda(s)$, if $r \geq s$,

– Λ is periodical, that is $\Lambda(r+1) = \mathfrak{p}_F\Lambda(r)$, $r \in \mathbb{R}$,

– Λ is left-continuous for the discrete topology on $L(V)$: for all $r \in \mathbb{R}$, there exists $\epsilon > 0$, such that Λ is constant on the segment $[r-\epsilon, r]$.

We let G acts on $\mathrm{Latt}^1_{\mathfrak{o}_F}(V)$ by

$$(g.\Lambda)(r) = g.\Lambda(r),\ g \in G,\ r \in \mathbb{R}\ .$$

We define the set $\mathrm{Latt}_{\mathfrak{o}_F}(V)$ of *lattice functions* in V as the quotient $\mathrm{Latt}^1_{\mathfrak{o}_F}(V)/\sim$ for the equivalence relation defined by $\Lambda_1 \sim \Lambda_2$, if there exists $s \in \mathbb{R}$ such that $\Lambda_1(r) = \Lambda_2(r+s)$, for all $r \in \mathbb{R}$. Then $\mathrm{Latt}_{\mathfrak{o}_F}(V)$ is a G-set in an obvious way.

The point of [BL] §I is that, as a G-set, the geometric realization of X is naturally isomorphic to $\mathrm{Latt}_{\mathfrak{o}_F}(V)$.

Let $\bar\Lambda$ be a lattice function, with representative $\Lambda \in \mathrm{Latt}^1_{\mathfrak{o}_F}(V)$. Assume that $\gamma.\bar\Lambda = \bar\Lambda$. We must prove that $\bar\Lambda$ is fixed by $K^\times$. Consider the lattice chain $\mathcal{L} = \{\Lambda(r)\ ;\ r \in \mathbb{R}\}$, and let $\mathfrak{A}(\mathcal{L})$ and $\sigma_{\mathcal{L}}$ be the associate hereditary order and simplex respectively. Then by [BL] Proposition (3.1), $\bar\Lambda$ lies in the interior of the simplex $\sigma_{\mathcal{L}}$. It follows that $\sigma_{\mathcal{L}}$ is stabilized by γ and therefore that $\mathfrak{A}(\mathcal{L})$ is normalized by γ. Applying Lemma (XII.4.1), we obtain that $\mathfrak{A}(\mathcal{L})$ is normalized by $K^\times$. In particular it follows that $\mathcal{L}$ is a chain of $\mathfrak{o}_K$-lattices in V, and that for all $r \in \mathbb{R}$, $\Lambda(r)$ is fixed by $\mathfrak{o}_K^\times$. Hence $\bar\Lambda$ is fixed by $\mathfrak{o}_K^\times$. By condition (i) in the definition of a minimal element, there exist integers r, s such that $\varpi_K := \varpi_F^r\gamma^s$ is a uniformizer of K, and it follows that $\bar\Lambda$ is fixed by ϖ_K. Hence it is fixed by $K^\times = \langle\varpi_K\rangle\mathfrak{o}_K^\times$, as required.

With the notation as above, we fix an unramified twist of an irreducible discrete series representation $(\pi, \mathcal{V})$ of G with type (J, λ). Its coefficient system $\mathcal{C}(\pi)$ has support $X[L]$. We also fix an elliptic regular element $\gamma \in G$ assumed to be minimal over F. In other words, γ is minimal over F and the field extension K/F is separable and maximal. In particuler the fixed point set X^γ is reduced to a single point x_γ, isobarycenter of simplex σ_γ attached to a principal hereditary order $\mathfrak{A}_\gamma$ with $\mathfrak{o}_F$-period $e(K/F)$.

(XII.4.3) Lemma. *With the notation as above, we have that $x_\gamma \in X[L]$ if, and only if, $f(L/F)|f(K/F)$ and $e(L/F)|e(K/F)$.*

Proof. Using the numerical criterion (I.3.5), we have that $x_\gamma \in X[L]$ (that is $\mathfrak{A}_\gamma$ has a G-conjugate normalized by $L^\times$) if, and only if, with the notation of §I, we have:

i) $f(L/F)|d(\mathfrak{A}_\gamma)_k$, for all $k \in \mathbb{Z}$,

ii) $e(L/F)|e(\mathfrak{A}_\gamma/\mathfrak{o}_F)/p(\mathfrak{A}_\gamma)$.

But $\mathfrak{A}_\gamma$ being principal with period $e(\mathfrak{A}_\gamma/\mathfrak{o}_F) = e(K/F)$, we easily see that $(d(\mathfrak{A}_\gamma)_k$ is constant with value $f(K/F)$ and that $p(\mathfrak{A}_\gamma) = 1$. The lemma follows.

As a straightforward consequence of the previous lemma and theorem (XII.3.2), we obtain the following simple formula for the value of the Harish-Chandra character at a minimal element.

(XII.4.4) Proposition. *The Harish-Chandra character of the discrete series representation* $(\pi, \mathcal{V})$ *satisfies :*

$$\Theta_\pi(\gamma) = \begin{cases} \mathrm{Tr}\,(\gamma, \lambda_{\sigma_\gamma}) & \textit{if } f(L/F)|f(K/F) \textit{ and } e(L/F)|f(K/F), \\ 0 & \textit{otherwise.} \end{cases}$$

In some particular cases, the same formula was obtained by the first author in [Br2] using a different approach.

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