PSEUDO PARALLEL CR-SUBMANIFOLDS IN A NON-FLAT COMPLEX SPACE FORM

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Abstract. We classify pseudo parallel proper CR-submanifolds in a non-flat complex space form with CR-dimension greater than one. With this result, the non-existence of recurrent as well as semi parallel proper CR-submanifolds in a non-flat complex space form with CR-dimension greater than one can also be obtained.

1. Introduction

Let M be an isometrically immersed submanifold in a Riemannian manifold $\hat{M}$. Denote by $\langle , \rangle$ the metric tensor of $\hat{M}$ as well as that induced on M. Then M is said to be *pseudo parallel* if its second fundamental form h satisfies the following condition

$$\bar{R}(X,Y)h = f((X \wedge Y)h)$$

for all vectors X, Y tangent to M, where f, called the *associated function*, is a smooth function on M, $\bar{R}$ is the curvature tensor corresponding to the van der Waerden-Bortolotti connection $\bar{\nabla}$ and

$$(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y.$$

In particular, when the associated function $f = 0$, M is called a *semi parallel* submanifold. It is called *recurrent* if and only if $(\bar{\nabla}_X h)(Y,Z) = \tau(X)h(Y,Z)$, where τ is a 1-form.

Pseudo parallel submanifolds is a generalization of semi parallel and parallel submanifolds. Parallel submanifolds in a real space form was completely classified in [12], [24]. Semi parallel and pseudo parallel submanifolds in a real space form were also studied extensively by many researchers (cf. [1], [2], [9], [10], [18], [20]).

By n-dimensional complex space forms $\hat{M}_n(c)$, we mean complete and simply connected n-dimensional Kaehler manifolds with constant holomorphic sectional curvature $4c$. For each real number c, up to holomorphic isometries, $\hat{M}_n(c)$ is a complex projective space $\mathbb{C}P_n$, a complex Euclidean

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space $\mathbb{C}_n$ or a complex hyperbolic space $\mathbb{C}H_n$ depending on whether c is positive, zero or negative, respectively.

It is known that a parallel submanifold of a non-flat complex space form $\hat{M}_n(c)$, $c \neq 0$, is either holomorphic or totally real (cf. [7]). As a result, there does not exist any parallel real hypersurface in $\hat{M}_n(c)$, $c \neq 0$. Further, the non-existence of semi parallel real hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$, $n \geq 2$, was proved by Ortega (cf. [23]). Nevertheless, there do exist pseudo parallel real hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$. Indeed, Lobos and Ortega gave a classification of pseudo parallel real hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$, $n \geq 2$, as below:

Theorem 1.1 ([17])**.** *Let M be a connected pseudo parallel real hypersurface in $\hat{M}_n(c)$, $n \geq 2$, $c \neq 0$, with associated function f. Then f is constant and positive, and M is an open part of one of the following real hypersurfaces:*

(a) *For $c = -1 < 0$:*
  (i) *A geodesic hypersphere of radius $r > 0$ with $f = \coth^2 r$.*
  (ii) *A tube of radius $r > 0$ over $\mathbb{C}H_{n-1}$ with $f = \tanh^2 r$.*
  (iii) *A horoshpere with $f = 1$.*

(b) *For $c = 1 > 0$:*
  (i) *A geodesic hypersphere of radius $r \in]0, \pi/2[$ with $f = \cot^2 r$.*

Note that a real hypersurface in a Kaehler manifold is a CR-submanifold of codimension one. A natural problem arisen is to generalize these known results on real hypersurfaces in $\hat{M}_n(c)$ into the content of CR-submanifolds. For technical reasons, certain additional restrictions such as the semi-flatness assumptions on the normal curvature tensor (cf. [25]), or restriction on the CR-codimension (cf. [11], [19]), have been imposed while dealing with CR-submanifolds of higher codimension. It would be interesting to see if any nice results on CR-submanifolds could be obtained without these restrictions.

In this paper, we study pseudo parallel proper CR-submanifolds in $\hat{M}_n(c)$, $c \neq 0$, with none of the above mentioned restrictions. More precisely, we prove the following:

Theorem 1.2. *Let M be a connected proper CR-submanifold in $\hat{M}_n(c)$, $c \neq 0$. Suppose that $\dim_{\mathbb{C}} \mathcal{D} = p \geq 2$. If M is pseudo parallel with associated function f, then f is a positive constant and M is an open part of one of the following spaces:*

(a) *For $c = -1 < 0$:*
  (i) *A geodesic hypersphere in $\mathbb{C}H_{p+1} \subset \mathbb{C}H_n$ of radius $r > 0$ with $f = \coth^2 r$.*
  (ii) *A tube over $\mathbb{C}H_p$ in $\mathbb{C}H_{p+1} \subset \mathbb{C}H_n$ of radius $r > 0$ with $f = \tanh^2 r$.*
  (iii) *A horoshpere in $\mathbb{C}H_{p+1} \subset \mathbb{C}H_n$ with $f = 1$.*

(b) *For $c = 1 > 0$:*
  (i) *A geodesic hypersphere in $\mathbb{C}P_{p+1} \subset \mathbb{C}P_n$ of radius $r \in]0, \pi/2[$ with $f = \cot^2 r$.*

(ii) *An invariant submanifold in a geodesic hypersphere in* $\mathbb{C}P_n$ *of radius* $r \in]0, \pi/2[$ *with* $f = \cot^2 r$.

From the above theorem, we see that the associated function f is a non-zero constant for pseudo parallel proper CR-submanifolds in $\hat{M}_n(c)$, $c \neq 0$. Hence we have

Corollary 1.1. *There does not exist any semi parallel proper CR-submanifold* M *in* $\hat{M}_n(c)$*,* $c \neq 0$*, with* $\dim_{\mathbb{C}} \mathcal{D} \geq 2$.

This corollary generalizes the non-existence of semi parallel real hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$ (cf. [23]) and improves a result in [16]: There does not exist any semi parallel proper CR-submanifold in $\hat{M}_n(c)$, $c \neq 0$, with semi-flat normal connection.

By applying Corollary 1.1, we can then prove the non-existence of proper recurrent CR-submanifolds in $\hat{M}_n(c)$, $c \neq 0$, with $\dim_{\mathbb{C}} \mathcal{D} \geq 2$ (cf. Corollary 5.1).

The paper is organized as follows:
In Section 2, we fix some notations and recall some basic material of CR-submanifolds in a Kaehler manifold which we use later. A fundamental property of Hopf hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$, is that the principal curvature α corresponding to the Reeb vector field ξ is constant. Moreover, the other principal curvatures can be related to α by a nice formula (cf. [22]). We generalize these results to mixed-geodesic CR-submanifolds of maximal CR-dimension in $\tilde{M}_n(c)$ in Section 3. In Section 4 we present the proof of Theorem 1.2. In the last section, recurrence and semi-parallelism have been discussed in the context of Riemannian vector bundles. We show that for any homomorphism of Riemannian vector bundles, recurrence directly implies semi-paralellism and thus conclude that there does not exist any proper recurrent CR-submanifold M in $\tilde{M}_n(C)$, $c \neq 0$, with $\dim_{\mathbb{C}} \mathcal{D} \geq 2$ (cf. Corollary 5.1).

2. CR-submanifolds in a Kaehler manifold

Let $\hat{M}$ be a Riemannian manifold, and let M be a connected Riemannian manifold isometrically immersed in $\hat{M}$. For a vector bundle $\mathcal{V}$ over M, we denote by $\Gamma(\mathcal{V})$ the $\Omega^0(M)$-module of cross sections on $\mathcal{V}$, where $\Omega^k(M)$ denotes the space of k-forms on M.

Denote by $\langle , \rangle$ the Riemannian metric of $\hat{M}$ and M as well, h the second fundamental form and A_σ the shape operator of M with respect to a vector σ normal to M. Also, let ∇ denote the Levi-Civita connection on the tangent bundle TM of M and $\nabla^\perp$, the induced normal connection on the normal bundle $TM^\perp$ of M. The second fundamental form h and the shape operator A_σ of M with respect to $\sigma \in \Gamma(TM^\perp)$ is related by the following equation

$$\langle h(X, Y), \sigma \rangle = \langle A_\sigma X, Y \rangle$$

for any $X, Y \in \Gamma(TM)$.

Let R and $R^\perp$ be the curvature tensors associated with ∇ and $\nabla^\perp$ respectively. We denote by $\bar\nabla$ the van der Waerden-Bortolotti connection and $\bar R$ its corresponding curvature tensor. Then we have

$$\begin{aligned}(\bar R(X,Y)A)_\sigma Z &= R(X,Y)A_\sigma Z - A_\sigma R(X,Y)Z - A_{R^\perp(X,Y)\sigma}Z,\\ (\bar R(X,Y)h)(Z,W) &= R^\perp(X,Y)h(Z,W) - h(R(X,Y)Z,W)\\ &\quad -h(Z,R(X,Y)W),\end{aligned}$$

for any $X,Y,Z,W \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$. It can be verified that

$$\langle(\bar R(X,Y)h)(Z,W),\sigma\rangle = \langle(\bar R(X,Y)A)_\sigma Z, W\rangle.$$

A submanifold M is said to be *pseudo parallel* if

$$(\bar R(X,Y)h)(Z,W) = f[(X\wedge Y)h](Z,W)$$

for any $X,Y,Z,W \in \Gamma(TM)$, where $f \in \Omega^0(M)$, is called the *associated function*, and

$$\begin{aligned}(X\wedge Y)Z &= \langle Y,Z\rangle X - \langle X,Z\rangle Y,\\ [(X\wedge Y)h](Z,W) &= -h((X\wedge Y)Z,W) - h(Z,(X\wedge Y)W),\\ [(X\wedge Y)A]_\sigma Z &= (X\wedge Y)A_\sigma Z - A_\sigma(X\wedge Y)Z.\end{aligned}$$

If the associated function $f = 0$ then the submanifold M is said to be *semi parallel*.

Now, let $\hat M$ be a Kaehler manifold with complex structure J. For any $X \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$, we denote the tangential (resp. normal) part of JX and $J\sigma$ by ϕX and $B\sigma$ (resp. ωX and $C\sigma$) respectively. From the parallelism of J, we have (cf. [25, pp. 77])

$$(\bar\nabla_X\phi)Y = A_{\omega Y}X + Bh(X,Y) \tag{2.1}$$

$$(\bar\nabla_X\omega)Y = -h(X,\phi Y) + Ch(X,Y) \tag{2.2}$$

for any X, $Y \in \Gamma(TM)$.

The maximal J-invariant subspace $\mathcal{D}_x$ of the tangent space T_xM, $x \in M$ is given by

$$\mathcal{D}_x = T_xM \cap JT_xM.$$

Definition 2.1 ([6])**.** *A submanifold M in a Kaehler manifold $\hat M$ is called a* generic submanifold *if the dimension of $\mathcal{D}_x$ is constant along M. The distribution $\mathcal{D} : x \to \mathcal{D}_x$, $x \in M$ is called the* holomorphic distribution (or Levi distribution) *on M and the complex dimension of $\mathcal{D}$ is called the CR-dimension of M.*

Definition 2.2 ([4])**.** *A generic submanifold M in a Kaehler manifold $\hat M$ is called a* CR-submanifold *if the orthogonal complementary distribution $\mathcal{D}^\perp$ of $\mathcal{D}$ in TM is totally real, i.e., $J\mathcal{D}^\perp \subset TM^\perp$. The real dimension of $\mathcal{D}^\perp$ is called the CR-codimension of M.*

If $\mathcal{D}^\perp = \{0\}$ (resp. $\mathcal{D} = \{0\}$), the CR-submanifold M is said to be holomorphic *(resp.* totally real*). A CR-submanifold M is said to be* proper *if it* *is neither holomorphic nor totally real. Let* ν *be the orthogonal complementary distribution of* $J\mathcal{D}^{\perp}$ *in* $TM^{\perp}$*. Then an* anti-holomorphic *submanifold* M *is a CR-submanifold with* $\nu = \{0\}$*, i.e.,* $J\mathcal{D}^{\perp} = TM^{\perp}$*. A* real hypersurface *is a proper CR-submanifold of codimension one.*

For a local frame of orthonormal vectors $E_1, E_2, \cdots, E_{2p}$ in $\Gamma(\mathcal{D})$, where $p = \dim_{\mathbb{C}} \mathcal{D}$, we define the *$\mathcal{D}$-mean curvature vector* $H_{\mathcal{D}}$ by

$$H_{\mathcal{D}} = \frac{1}{2p} \sum_{j=1}^{2p} h(E_j, E_j).$$

Lemma 2.1 ([19])**.** *Let* M *be a CR-submanifold in a Kaehler manifold* $\hat{M}$*. Then* $\langle (\phi A_\sigma + A_\sigma \phi)X, Y\rangle = 0$*, for any* $X, Y \in \Gamma(\mathcal{D})$ *and* $\sigma \in \Gamma(\nu)$*. Moreover, we have* $CH_{\mathcal{D}} = 0$*.*

If $h(\mathcal{D}^{\perp}, \mathcal{D}) = 0$, the CR-submanifold M is said to be *mixed totally geodesic*. M is said to be *mixed foliate* if it is mixed totally geodesic and $\mathcal{D}$ is integrable.

The following lemma characterizes mixed foliate CR-submanifolds in a Kaehler manifold.

Lemma 2.2 ([5])**.** *A CR-submanifold* M *in a Kaehler manifold is mixed foliate if and only if* $Bh(\phi X, Y) = Bh(X, \phi Y)$*, for any* $X, Y \in \Gamma(\mathcal{D})$ *and* $h(\mathcal{D}^{\perp}, \mathcal{D}) = 0$*.*

Now suppose the ambient space is an n-dimensional complex space form $\hat{M}_n(c)$ with constant holomorphic sectional curvature $4c$. The curvature tensor $\hat{R}$ of $\hat{M}_n(c)$ is given by

$$\hat{R}(X, Y)Z = c(X \wedge Y + JX \wedge JY - 2\langle JX, Y\rangle J)Z$$

for any $X, Y, Z \in \Gamma(T\hat{M}_n(c))$. The equations of Gauss, Codazzi and Ricci are then given respectively by

$$R(X, Y)Z = c(X \wedge Y + \phi X \wedge \phi Y - 2\langle \phi X, Y\rangle \phi)Z + A_{h(Y,Z)}X - A_{h(X,Z)}Y$$

$$(\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) = c\{\langle \phi Y, Z\rangle \omega X - \langle \phi X, Z\rangle \omega Y - 2\langle \phi X, Y\rangle \omega Z\}$$

$$R^{\perp}(X, Y)\sigma = c(\omega X \wedge \omega Y - 2\langle \phi X, Y\rangle C)\sigma + h(X, A_\sigma Y) - h(Y, A_\sigma X)$$

for any X, Y, $Z \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^{\perp})$.

We now recall the following known result.

Theorem 2.1 ([5], [8])**.** *There does not exist any proper mixed foliate CR-submanifold in* $\hat{M}_n(c)$*,* $c \neq 0$*.*

3. Mixed-totally geodesic CR-submanifolds in a complex space form

A real hypersurface M in a Kaehler manifold is said to be *Hopf* if it is mixed-totally geodesic. A fundamental property of Hopf hypersurfaces in $\hat{M}_n(c)$, $c \neq 0$, is that the principal curvature α corresponds to the Reeb vector field ξ is constant. Moreover, the other principal curvatures could be related to α by a nice formula (cf. [22]). In this section, we show that these properties hold for mixed-totally geodesic proper CR-submanifolds of maximal CR-dimension.

Suppose M is a real $(2p+1)$-dimensional CR-submanifold in $\hat{M}_n(c)$ of maximal CR-dimension, that is, $\dim_{\mathbb{C}} \mathcal{D} = p$ and $\dim \mathcal{D}^\perp = 1$. Let $N \in \Gamma(J\mathcal{D}^\perp)$ be a unit vector field, $\xi = -JN$ and η the 1-form dual to ξ. Then we have

$$\phi^2 X = -X + \eta(X)\xi \tag{3.1}$$

$$\omega X = \eta(X)N; \quad B\sigma = -\langle \sigma, N\rangle \xi \tag{3.2}$$

for any $X \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$. It follows from (2.1) and (2.2) that

$$(\nabla_X \phi)Y = \eta(Y)A_N X - \langle A_N X, Y\rangle \xi \tag{3.3}$$

$$\nabla_X \xi = \phi A_N X; \quad \nabla^\perp_X N = Ch(X, \xi) \tag{3.4}$$

$$h(X, \phi Y) = -\langle \phi A_N X, Y\rangle N - \eta(Y)Ch(X, \xi) + Ch(X, Y) \tag{3.5}$$

for any X, $Y \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$.

The equations of Codazzi and Ricci can also be reduced to

$$\begin{aligned}(\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z) \; = \; & c\{\eta(X)\langle \phi Y, Z\rangle - \eta(Y)\langle \phi X, Z\rangle \\ & -2\eta(Z)\langle \phi X, Y\rangle\}N \end{aligned} \tag{3.6}$$

$$R^\perp(X, Y)\sigma = -2c\langle \phi X, Y\rangle C\sigma + h(X, A_\sigma Y) - h(Y, A_\sigma X) \tag{3.7}$$

for any X, Y, $Z \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$.

Lemma 3.1. *Let M be a mixed-totally geodesic proper CR-submanifold of maximal CR-dimension in $\hat{M}_n(c)$, $c \neq 0$, and let $\alpha = \langle h(\xi, \xi), N\rangle$. Then*
(a) $2A_N\phi A_N - \alpha(\phi A_N + A_N\phi) - 2c\phi = 0$;
(b) *if $A_N Y = \lambda Y$ and $A_N \phi Y = \lambda^* \phi Y$, where $Y \in \Gamma(\mathcal{D})$, then* $(2\lambda - \alpha)(2\lambda^* - \alpha) = \alpha^2 + 4c$;
(c) *α is a constant.*

Proof. By the hypothesis,

$$h(Y, \xi) = \eta(Y)h(\xi, \xi) \tag{3.8}$$

for any $Y \in \Gamma(TM)$. Differentiating covariantly both sides of (3.8) in the direction of $X \in \Gamma(TM)$, we get

$$(\bar{\nabla}_X h)(Y, \xi) + h(\phi A_N X, Y) = \langle \phi A_N X, Y\rangle h(\xi, \xi) + \eta(Y)\nabla^\perp_X h(\xi, \xi).$$

By applying the Codazzi equation and this equation, we have

$$\begin{aligned} h(\phi A_N X, Y) - h(X, \phi A_N Y) - \langle (\phi A_N + A_N\phi)X, Y\rangle h(\xi, \xi) \\ -2c\langle \phi X, Y\rangle N = \eta(Y)\nabla^\perp_X h(\xi, \xi) - \eta(X)\nabla^\perp_Y h(\xi, \xi). \end{aligned} \tag{3.9}$$

By putting $Y = \xi$ in this equation, we obtain

$$\nabla^\perp_X h(\xi, \xi) = \eta(X)\nabla^\perp_\xi h(\xi, \xi) \tag{3.10}$$

and

$$h(\phi A_N X, Y) - h(X, \phi A_N Y) - \langle(\phi A_N + A_N\phi)X, Y\rangle h(\xi,\xi) \\ = 2c\langle \phi X, Y\rangle N. \tag{3.11}$$

By taking inner product of (3.11) with N, we get

$$2A_N\phi A_N - \alpha(\phi A_N + A_N\phi) - 2c\phi = 0.$$

Statement (b) is directly from this equation. Next, it follows from (3.4), (3.8), and (3.10) that

$$Y\alpha = Y\langle h(\xi,\xi), N\rangle = g\eta(Y)$$

for any $Y \in \Gamma(TM)$, where $g = \xi\alpha$, i.e., $d\alpha = g\eta$. Hence

$$0 = d^2\alpha = dg \wedge \eta + gd\eta.$$

Since $2d\eta(X,\xi) = \langle(\phi A_N + A_N\phi)X, \xi\rangle = 0$ and $Xg - (\xi g)\eta(X) = dg\wedge\eta(X,\xi)$, for any $X \in \Gamma(TM)$, we have $dg = (\xi g)\eta$. Hence we have $gd\eta = 0$. This implies that $g = 0$ (for otherwise, if $d\eta = 0$ then $\mathcal{D}$ is integrable. It follows that M is mixed foliate but this contradicts Theorem 2.1). Hence we have $d\alpha = 0$ or α is a constant. □

4. Proof of Theorem 1.2

Throughout this section, suppose M is a $(2p+q)$-dimensional pseudo parallel proper CR-submanifold in $\hat{M}_n(c)$, $c \neq 0$, where $\dim_{\mathbb{C}} \mathcal{D} = p \geq 2$ and $\dim_{\mathbb{R}} \mathcal{D}^\perp = q$.

Note that $\mathfrak{S}_{X,Y,Z}((X\wedge Y)h)(Z,W) = 0$ and

$$\mathfrak{S}_{X,Y,Z}(\bar{R}(X,Y)h)(Z,W) = \mathfrak{S}_{X,Y,Z}\{R^\perp(X,Y)h(Z,W) - h(Z, R(X,Y)W)\}$$

for any $X, Y, Z, W \in \Gamma(TM)$, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum over X, Y and Z. By the Gauss and Ricci equations, we obtain the following equation.

$$\begin{aligned}
&\langle \omega Y, h(Z,W)\rangle\langle \omega X,\sigma\rangle - \langle \omega X, h(Z,W)\rangle\langle \omega Y,\sigma\rangle - 2\langle \phi X, Y\rangle\langle Ch(Z,W),\sigma\rangle \\
&+ \langle \omega Z, h(X,W)\rangle\langle \omega Y,\sigma\rangle - \langle \omega Y, h(X,W)\rangle\langle \omega Z,\sigma\rangle - 2\langle \phi Y, Z\rangle\langle Ch(X,W),\sigma\rangle \\
&+ \langle \omega X, h(Y,W)\rangle\langle \omega Z,\sigma\rangle - \langle \omega Z, h(Y,W)\rangle\langle \omega X,\sigma\rangle - 2\langle \phi Z, X\rangle\langle Ch(Y,W),\sigma\rangle \\
&- \langle \phi Y, W\rangle\langle h(Z,\phi X),\sigma\rangle + \langle \phi X, W\rangle\langle h(Z,\phi Y),\sigma\rangle + 2\langle \phi X, Y\rangle\langle h(Z,\phi W),\sigma\rangle \\
&- \langle \phi Z, W\rangle\langle h(X,\phi Y),\sigma\rangle + \langle \phi Y, W\rangle\langle h(X,\phi Z),\sigma\rangle + 2\langle \phi Y, Z\rangle\langle h(X,\phi W),\sigma\rangle \\
&- \langle \phi X, W\rangle\langle h(Y,\phi Z),\sigma\rangle + \langle \phi Z, W\rangle\langle h(Y,\phi X),\sigma\rangle + 2\langle \phi Z, X\rangle\langle h(Y,\phi W),\sigma\rangle \\
&= 0.
\end{aligned} \tag{4.1}$$

for any $X, Y, Z, W \in \Gamma(TM)$ and $\sigma \in \Gamma(TM^\perp)$. By putting $Z \in \Gamma(TM)$, $W \in \Gamma(D^\perp)$, $Y = \phi X$, $X \in \Gamma(\mathcal{D})$ with $||X|| = 1$ and $X \perp Z, \phi Z$ in (4.1), we obtain

$$Ch(\mathcal{D}^\perp, TM) = 0. \tag{4.2}$$

Let $\{E_1, E_2, \cdots, E_{2p}\}$ be a local orthonormal frame on $\mathcal{D}$. By putting $X = E_j$, $Z = \phi E_j$ for $j \in \{1, 2, \cdots, 2p\}$ in (4.1), and then summing up these equations, with the help of (4.2), we obtain

$$(2p-2)Ch(Y,W) - 2p\langle \phi Y, W\rangle H_{\mathcal{D}} - h(\phi^2 W, \phi Y) \\ -2h(\phi^2 Y, \phi W) - (2p+1)h(Y, \phi W) = 0 \qquad (4.3)$$

for any $Y, W \in \Gamma(TM)$. By virtue of (4.2), after putting $Y \in \Gamma(\mathcal{D}^\perp)$ in the above equation, we have

$$h(\mathcal{D}^\perp, \mathcal{D}) = 0. \qquad (4.4)$$

This means that M is mixed-totally geodesic and so (4.3) reduces to

$$(2p-2)Ch(Y,W) - 2p\langle \phi Y, W\rangle H_{\mathcal{D}} + h(W, \phi Y) - (2p-1)h(Y, \phi W) = 0 \quad (4.5)$$

for any $Y, W \in \Gamma(TM)$. Next, we put $Y = W$ in the above equation to get $Ch(Y, Y) - h(Y, \phi Y) = 0$, then, combining with the linearity of C, h and ϕ, we obtain

$$2Ch(Y,W) - h(W, \phi Y) - h(Y, \phi W) = 0 \qquad (4.6)$$

for any $Y, W \in \Gamma(TM)$. It follows from this equation and (4.5) that

$$h(Y, \phi W) = \langle Y, \phi W\rangle H_{\mathcal{D}} + Ch(Y, W) \qquad (4.7)$$

for any $Y, W \in \Gamma(TM)$. From (4.1) and (4.7), we have

$$\langle \omega Y, h(Z,W)\rangle \omega X - \langle \omega X, h(Z,W)\rangle \omega Y + \langle \omega Z, h(X,W)\rangle \omega Y \\ - \langle \omega Y, h(X,W)\rangle \omega Z + \langle \omega X, h(Y,W)\rangle \omega Z - \langle \omega Z, h(Y,W)\rangle \omega X = 0$$

for any $X, Y, Z, W \in \Gamma(TM)$.

We claim that $q = 1$. Suppose the contrary that $q \geq 2$. By putting $Z = W \in \Gamma(\mathcal{D})$, $Y = BH_{\mathcal{D}}$ and $X \perp BH_{\mathcal{D}}$ a unit vector field in $\mathcal{D}^\perp$ in this equation, with the help of (4.6), we obtain $BH_{\mathcal{D}} = 0$. This, together with (4.6) imply that $h(\mathcal{D}, \mathcal{D}) = 0$ and hence, by Lemma 2.2 and (4.4), M is mixed foliate. This contradicts Theorem 2.1. Accordingly, $q = 1$.

Let $N \in \Gamma(J\mathcal{D}^\perp)$ be a unit vector field normal to M, and (ϕ, η, ξ) the almost contact structure on M as defined in Section 3. It follows from Lemma 2.1 and equations (3.1), (3.2), (4.2) and (4.4) that

$$H_{\mathcal{D}} = \lambda N, \qquad (4.8)$$
$$h(X, \xi) = \eta(X)h(\xi, \xi) = \alpha\eta(X)N$$

for any $X \in \Gamma(TM)$, where $\lambda = \langle H_{\mathcal{D}}, N\rangle$ and $\alpha = \langle h(\xi, \xi), N\rangle$. By using (4.6) and the above two equations, we obtain

$$h(X,Y) = h(X, -\phi^2 Y + \eta(Y)\xi) \\ = \{\lambda\langle X, Y\rangle + b\eta(X)\eta(Y)\}N - Ch(X, \phi Y) \qquad (4.9)$$

for any $X, Y \in \Gamma(TM)$, where $b = \alpha - \lambda$. From Lemma 3.1 and (4.9), we obtain

$$\lambda^2 - \alpha\lambda - c = 0 \qquad (4.10)$$

and so λ is a non-zero constant. Further, for any unit vector $Y \in \mathcal{D}$, we have

$$0 = \langle(\bar{R}(\xi, Y)h)(Y, \xi), N\rangle\rangle - f\langle((\xi \wedge Y)h)(Y, \xi), N\rangle = (\alpha - \lambda)(f - \alpha\lambda - c)$$

Hence, $f = \lambda^2$ is a positive constant.

We consider two cases: $Ch = 0$ and $Ch \neq 0$.

Case 1. $Ch = 0$.

By the hypothesis, (3.4) and the fact that $\lambda \neq 0$, the first normal space $\mathcal{N}^1_x = \mathbb{R}N_x$, $x \in M$, and $\mathcal{N}^1$ is a parallel normal subbundle of $TM^\perp$. Since ν is J-invariant, by Codimension Reduction Theorems (cf. [11], [15]), M is contained in a totally geodesic holomorphic submanifold $\hat{M}_{p+1}(c)$ as a real hypersurface.

Now, let ∇', A', *etc* denote the Levi-Civita connection on M induced by the Levi-Civita connection of $\hat{M}_{p+1}(c)$, the shape operator, *etc*, respectively. Since $\hat{M}_{p+1}(c)$ is totally geodesic in $\hat{M}_n(c)$, we can see that $\nabla'_X Y = \nabla_X Y$, $A' = A_N$ and $N' = N$. Further, as $\nabla^\perp N = 0$, we have $R^\perp(X, Y)N = 0$ and so $R'(X, Y)A = (\bar{R}(X, Y)A)_N$, for any X, Y tangent to M. Then M is a pseudo parallel real hypersurface in $\hat{M}_{p+1}(c)$ and by Theorem 1.1, we obtain List (a) and (b-i) in Theorem 1.2.

Case 2. $Ch \neq 0$.

Suppose $Ch \neq 0$ at a point $x \in M$. There is a number $a \neq 0$, $\sigma \in \nu_x$ and a unit vector $Y \in \mathcal{D}_x$ such that $A_\sigma Y = aY$. From Lemma 2.1, we have $A_\sigma \phi Y = -a\phi Y$. Then from $\langle(\bar{R}(\phi Y, Y)h)(Y, \phi Y), \sigma\rangle = f\langle((\phi Y \wedge Y)h)(Y, \phi Y), \sigma\rangle$, we obtain

$$a\{3c - 2\langle h(Y, \phi Y), h(Y, \phi Y)\rangle + \langle h(Y, Y), h(\phi Y, \phi Y)\rangle\} = af.$$

On the other hand, from (4.9), we have

$$\langle h(Y, \phi Y), h(Y, \phi Y)\rangle = \langle Ch(Y, Y), Ch(Y, Y)\rangle$$
$$\langle h(Y, Y), h(\phi Y, \phi Y)\rangle = \lambda^2 - \langle Ch(Y, Y), Ch(Y, Y)\rangle.$$

Since $a \neq 0$ and $f = \lambda^2$, these equations give $c = \langle Ch(Y, Y), Ch(Y, Y)\rangle$. Hence, we conclude that $c > 0$ (without loss of generality, we assume $c = 1$) and $||Ch|| > 0$ on the whole of M.

Fixed $r > 0$ and let BM be the unit normal bundle over M. The focal map Φ_r is given by

$$BM \ni \sigma \xrightarrow{\Phi_r} \exp(r\sigma) \in \mathbb{C}P_n$$

where exp is the exponential map on $\mathbb{C}P_n$. For each $x \in M$ and unit vector $\sigma \in T_x M^\perp$, denote by $\gamma_\sigma(s)$ the normalized geodesic in $\mathbb{C}P_n$ passes through $x \in M$ at $s = 0$ with velocity σ. Let $\mathcal{Y}_X$ be the M-Jacobi field along γ_σ with initial values $\mathcal{Y}_X(0) = X \in T_x M$ and $\dot{\mathcal{Y}}_X(0) = -A_\sigma X$. Then (cf. [3, pp.225])

$$d\Phi_r(\sigma)X = \mathcal{Y}_X(r).$$

In view of (4.9), A_N has two distinct constant eigenvalues α and λ with eigenspaces $\mathbb{R}\xi$ and $\mathcal{D}_x$ respectively at each $x \in M$. We put $\alpha = 2\cot 2r$, $0 < r < \pi/2$. Then $\lambda = \cot r$ or $\lambda = -\cot(\frac{\pi}{2} - r)$ by (4.10).

Subcase 2-a. $\lambda = \cot r$.

Since λ is a nonzero constant, by (4.8), $N = \lambda^{-1} H_{\mathcal{D}}$ is globally defined on M. We may immerse M in BM as a submanifold in a natural way: $x \mapsto N_x$, $x \in M$.

We claim that $\Phi_r(M)$ is a singleton for a suitable choice of r. This can be done by showing that $d\Phi_r(N_x)T_xM = \{0\}$, for each $x \in M$. We first note that at each $z \in \mathbb{C}P_n$, the Jacobi operator $\hat{R}_\sigma := \hat{R}(\cdot, \sigma)\sigma$, $\sigma \in T_z\mathbb{C}P_n$, has eigenvalues 0, 4 and 1 with eigenspaces $\mathbb{R}\sigma$, $\mathbb{R}J\sigma$ and $(\mathbb{R}\sigma \oplus \mathbb{R}J\sigma)^\perp$ respectively, To compute $d\Phi_r(N_x)X$, $X \in T_xM$, we select the Jacobi field

$$\mathcal{Y}_X(t) = \begin{cases} \left(\cos 2t - \frac{\alpha}{2}\sin 2t\right)\mathcal{E}_X(t), & X = \xi \\ (\cos t - \lambda \sin t)\mathcal{E}_X(t), & X \in \mathcal{D}_x \end{cases}$$

where $\mathcal{E}_X$ is the parallel vector field along γ_{N_x} with $\mathcal{E}_X(0) = X$. Then we have $d\Phi_r(N_x)X = \mathcal{Y}_X(r) = 0$ and conclude that $\Phi_r(M) = \{z_0\}$.

Subcase 2-b. $\lambda = -\cot(\frac{\pi}{2} - r)$.

Note that $\cot 2r = -\cot 2(\frac{\pi}{2} - r)$. By selecting the Jacobi field

$$\mathcal{Y}_X(t) = \begin{cases} \left(\cos 2t + \frac{\alpha}{2}\sin 2t\right)\mathcal{E}_X(t), & X = \xi \\ (\cos t + \lambda \sin t)\mathcal{E}_X(t), & X \in \mathcal{D}_x \end{cases}$$

we can see that $d\Phi_{\pi/2-r}(-N_x)X = 0$, for $X \in T_xM$ and hence $\Phi_{\pi/2-r}(M) = \{z_0\}$.

We have shown that $\Phi_r(M) = \{z_0\}$ for some $r \in]0, \pi/2[$ in both cases. By checking the Jacobi fields of $\mathbb{C}P_n$ (cf. [13, pp.149]), there is no conjugate point for z_0 along any geodesic in $\mathbb{C}P_n$ of length $r \in]0, \pi/2[$ starting at z_0, we conclude that M lies in a geodesic hypersphere M' around z_0 in $\mathbb{C}P_n$ with almost contact structure (ϕ', η', ξ'), where $\xi' = -JN'$, η' the 1-form dual to ξ', $\phi' = J_{|TM'} - \eta' \otimes N'$ and N' a unit vector field normal to M'. By the construction of M', we have $N = N'$, $\xi = \xi'$ and $\phi = \phi'$ on M. It follows that $\phi' TM \subset TM$ and so M is an invariant submanifold of M' (cf. [25]). Hence we obtain List (b-ii) in Theorem 1.2.

5. Recurrent CR-submanifolds in a non-flat complex space form

In this section, wel show that there are no proper recurrent CR-submanifolds in $\hat{M}_n(c)$, $n \neq 0$. We first discuss the ideas of recurrence and semi-parallelism in a general setting.

Let M be a Riemannian manifold and $\mathcal{E}_j$ a Riemannian vector bundle over M with linear connection ∇^j, $j \in \{1, 2\}$. It is known that $\mathcal{E}_1^* \otimes \mathcal{E}_2$ is isomorphic to the vector bundle $Hom(\mathcal{E}_1, \mathcal{E}_2)$, consisting of homomorphisms from $\mathcal{E}_1$ into $\mathcal{E}_2$. We denote by the same $\langle, \rangle$ the fiber metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$

as well as that induced on $Hom(\mathcal{E}_1, \mathcal{E}_2)$. The connections ∇^1 and ∇^2 induce on $Hom(\mathcal{E}_1, \mathcal{E}_2)$ a connection $\bar{\nabla}$, given by

$$(\bar{\nabla}_X F)V = (\bar{\nabla} F)(V; X) = \nabla^2_X FV - F\nabla^1_X V$$

for any $X \in \Gamma(TM)$, $V \in \Gamma(\mathcal{E}_1)$ and $F \in \Gamma(Hom(\mathcal{E}_1, \mathcal{E}_2))$.

A section F in $Hom(\mathcal{E}_1, \mathcal{E}_2)$ is said to be *recurrent* if there exists $\tau \in \Omega^1(M)$ such that $\bar{\nabla} F = F \otimes \tau$. We may regard parallelism as a special case of recurrence, that is, the case $\tau = 0$. Let $\bar{R}$, R^1 and R^2 be the curvature tensor corresponding to $\bar{\nabla}$, ∇^1 and ∇^2 respectively. Then we have

$$(\bar{R} \cdot F)(V; X, Y) = (\bar{R}(X, Y)F)V = R^2(X, Y)FV - FR^1(X, Y)V$$

for any $X, Y \in \Gamma(TM)$, $V \in \Gamma(\mathcal{E}_1)$ and $F \in \Gamma(Hom(\mathcal{E}_1, \mathcal{E}_2))$.

We begin with the following result.

Lemma 5.1. *Let M be a connected Riemannian manifold, $\mathcal{E}_j$ a Riemannian vector bundle over M, $j \in \{1, 2\}$ and $F \in \Gamma(Hom(\mathcal{E}_1, \mathcal{E}_2))$. If F is recurrent then F is semi-parallel.*

Proof. Suppose F is recurrent, that is, $\bar{\nabla} F = F \otimes \tau$, for some $\tau \in \Omega^1(M)$. It is trivial if $F = 0$. Suppose that $\mu := ||F|| \neq 0$ on an open set $U \subset M$. Then the line bundle $\mathbb{R} \otimes F \to U$, spanned by F, is a parallel subbundle of $Hom(\mathcal{E}_1, \mathcal{E}_2)_{|U}$. Consider the unit section $E := \mu^{-1} F$ of $\mathbb{R} \otimes F$. Then

$$\bar{\nabla} E = \mu^{-1} \bar{\nabla} F + F \otimes d(\mu^{-1}) = F \otimes (\mu^{-1}\tau + d(\mu^{-1})) = E \otimes (\tau - \mu^{-1} d\mu).$$

Hence, E is also recurrent and $\bar{\nabla} E = E \otimes \lambda$, where $\lambda = \tau - \mu^{-1} d\mu \in \Omega^1(U)$. It follows that

$$0 = d\langle E, E\rangle = 2\langle \bar{\nabla} E, E\rangle = 2\langle E, E\rangle\lambda = 2\lambda.$$

Hence E is a flat section. This implies that $\mathbb{R} \otimes F$ is a flat bundle. Hence, $\bar{R} \cdot F = 0$ on U. By a standard topological argument, we conclude that $\bar{R} \cdot F = 0$ on M. □

Geometrically, Lemma 5.1 tells us that the line subbundle of $(Hom(\mathcal{E}_1, \mathcal{E}_2), \bar{\nabla})$, spanned by a nonvanishing recurrent section is a flat bundle.

A submanifold M of a Riemannian manifold $\hat{M}$ is said to be *recurrent* if its second fundamental form h is recurrent. Since every $T_x M^\perp$-valued bilinear map on $T_x M$ naturally induces a homomorphism from $T_x M \otimes T_x M$ to $T_x M^\perp$, $x \in M$, we may identify h as a section of $Hom(TM \otimes TM, TM^\perp)$. Accordingly, the following result can be obtained immediately from Corollary 1.1 and Lemma 5.1.

Corollary 5.1. *There does not exist any proper recurrent CR-submanifold M in $\hat{M}_n(c)$, $c \neq 0$, with $\dim_{\mathbb{C}} \mathcal{D} \geq 2$.*

Remark 5.1. *The above corollary generalizes the non-existence of recurrent real hypersurfaces in a non-flat complex space form (cf.* [14], [21]*).*

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