

AUSLANDER REGULARITY OF NORM BASED EXTENSIONS OF WEYL ALGEBRAS

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ABSTRACT. We first prove that for a Weyl algebra over a field of positive characteristic, its norm based extension is locally Auslander regular. We then prove that given an algebra which is Zariski locally isomorphic to the Weyl algebra, its norm based extension similarly defined is locally Auslander regular if and only if it is isomorphic to the original Weyl algebra.

1. INTRODUCTION

Let A be a Weyl algebra over a field of positive characteristic. It is finite over its center $Z(A)$ and may thus be studied by looking at the sheaf A_X of algebras over $X = \text{Spec}(Z(A)) \cong \mathbb{A}^{2n}$. It is known that the object is deeply related to Dixmier conjecture and Jacobian problem (See for example [17][18][4][2].) In [21], the author defined the “norm based extension” (NBE) $A_{\bar{X}}$ of A_X as an algebra sheaf over a projective space $\bar{X} = \mathbb{P}^{2n}$. In this paper we show that the norm based extension $A_{\bar{X}}$ is locally Auslander regular, in the sense that every stalk of it is Auslander regular.

Next we take an algebra B which is Zariski locally isomorphic to A , that means, B is an algebra over Z whose corresponding sheaf B_X on X is Zariski locally isomorphic to A_X . Such algebra is considered for example in [3]. Then we show that the norm based extension $B_{\bar{X}}$ defined in a similar manner as $A_{\bar{X}}$ is locally Auslander regular if and only if B is isomorphic to A as a Z -algebra.

2. LOCAL RINGS IN THE SENSE OF THIS PAPER

In this paper, by a ring we mean a unital associative ring. By a module we mean a left module unless otherwise specified.

Unfortunately, there are several definitions of “local rings”. In several literature, the term “local ring” is meant to stand for a ring with only one maximal left-ideal. See for example [10, Proposition 10.1.1] for details. In contrast, we are interested in the rings with only one

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maximal (two-sided) ideal. So we introduce here somewhat unusual terms ‘demi local rings’, ‘NAK local rings’, and, following Ramras, a term ‘quasi local rings’ to make a distinction.

In this paper, we mainly deal with rings A which are finite over their central subring R . In such cases, the two theory of ‘local rings’ are not very far. Indeed, as we will illustrate in Lemma 2.5, maximal left ideals of A are pretty well understood by looking at maximal ideals.

2.1. demi local rings and quasi local rings.

DEFINITION 2.1. Let us call a unital associative ring A a *demi local ring* if A has only one maximal ideal \mathfrak{m} . As is customary, we often say that (A, \mathfrak{m}) is a demi local ring to express this fact.

DEFINITION 2.2. Let I be an ideal of a ring A . We say that *Nakayama’s lemma holds* for (A, I) if for any finitely generated left A -module M , we have

$$IM = M \implies M = 0.$$

For any ring A , let us denote by $\text{rad}(A)$ its Jacobson radical. Namely, $\text{rad}(A)$ is the intersection of all maximal left ideal of A . It is well known that for any ring A , its radical is in fact an ideal of A and that the Nakayama’s lemma holds for $(A, \text{rad}(A))$. (See for example [10, Lemma 3.4.4, Lemma 3.4.11])

We note:

Lemma 2.3. *Let (A, \mathfrak{m}) be a demi local ring. The following conditions are equivalent.*

- (1) *Nakayama’s lemma holds for (A, \mathfrak{m}) .*
- (2) *Every maximal left ideal J of A contains \mathfrak{m} as a submodule.*
- (3) $\mathfrak{m} = \text{rad}(A)$.

Proof. That (3) implies (1) is a consequence of the usual Nakayama’s lemma cited above.

(1) \implies (2): Let J be a maximal left ideal of A . If $J \not\supseteq \mathfrak{m}$, then by the maximality of J we have $J + \mathfrak{m} = A$. In other words, we have $\mathfrak{m}A/J = 0$. By the assumption we have $A/J = 0$, which is a contradiction.

(2) \implies (3) is a consequence of the definition of $\text{rad}(A)$ and the fact that $\text{rad}(A)$ is an ideal of A . \square

DEFINITION 2.4. Let us call a demi local ring (A, \mathfrak{m}) a *NAK local ring* if it satisfies the equivalent conditions of Lemma 2.3. If in addition A/\mathfrak{m} is simple Artinian, we call it a *quasi local ring*.

The term quasi local ring appears in papers of Ramras. See for example [16]. The term also appears in papers such as [13],[14].

2.2. demi local rings which are finite over the center. We are mainly interested in cases where our ring A is finite over its center $Z(A)$. For technical reasons we consider a slightly general situation and employ a central subring R of A (that means, a subalgebra of $Z(A)$) instead of $Z(A)$ itself.

Lemma 2.5. *Let (A, \mathfrak{m}) be a demi local ring which is finite over its central subring R . Then:*

- (1) $\mathfrak{m}_R = \mathfrak{m} \cap R$ is the unique maximal ideal of R (so that (R, \mathfrak{m}_R) is a local ring in the usual sense in commutative algebra).
- (2) A/\mathfrak{m} is isomorphic to a full matrix algebra $M_{k_0}(D)$ over a skew field D . (In other words, A/\mathfrak{m} is simple Artinian.)
- (3) If furthermore A is a quasi local ring, then maximal left ideals of A in one to one correspondence with points of the projective space $\mathbb{P}^{k_0-1}(D)$ of dimension $k_0 - 1$ over D .

Proof. (1) For any maximal ideal \mathfrak{a} of R , we have

$$\mathfrak{a}A \neq A$$

by the usual Nakayama's lemma for the commutative ring (R, \mathfrak{m}_R) . By the maximality of \mathfrak{m} we have $\mathfrak{a}A \subset \mathfrak{m}$, so that we have $\mathfrak{a} \subset \mathfrak{m} \cap R = \mathfrak{m}_R$. By the maximality of \mathfrak{a} we have $\mathfrak{a} = \mathfrak{m}_R$.

(2) A/\mathfrak{m} is a finite dimensional simple algebra over a field R/\mathfrak{m}_R . Thus we may use Wedderburn's theorem.

(3) Since every maximal left ideal J contains \mathfrak{m} , it corresponds bijectively to a maximal left ideals of $A/\mathfrak{m} \cong M_{k_0}(D)$. \square

Lemma 2.6. *Let (A, \mathfrak{m}) be a demi local ring which is finite over its central subalgebra R . We put $\mathfrak{m}_R = \mathfrak{m} \cap R$. (Note that (R, \mathfrak{m}_R) is a local ring in view of Lemma 2.5.) Then (A, \mathfrak{m}) is quasi local if and only if the \mathfrak{m} -adic topology on A coincides with the $\mathfrak{m}_R A$ -adic topology. Namely, if $\mathfrak{m}^{k_0} \subset \mathfrak{m}_R A$ for some $k_0 \gg 0$.*

Proof. By Lemma 2.5 we see that A/\mathfrak{m} is simple Artinian.

Assume $\mathfrak{m}^{k_0} \subset \mathfrak{m}_R A$ for some $k_0 \gg 0$. Since A is finite over R , M is a finite R -module. On the other hand $M = \mathfrak{m}M$ implies $M = \mathfrak{m}^{k_0}M$, so that we have $M = \mathfrak{m}_R M$. Thus we see that $M = 0$ by using the usual Nakayama's lemma for (R, \mathfrak{m}_R) .

Conversely, let us assume (A, \mathfrak{m}) is quasi local. $A/\mathfrak{m}_R A$ is a module of finite rank over a field R/\mathfrak{m}_R . Thus a decreasing sequence $\{\mathfrak{m}^k(A/\mathfrak{m}_R A)\}_{k=1}^\infty$ stops. In other words, there exists some integer k_0 such that

$$N = \bigcap_k \mathfrak{m}^k(A/\mathfrak{m}_R A) = \mathfrak{m}^{k_0}(A/\mathfrak{m}_R A)$$

holds. The module N clearly is a finitely generated over A and satisfies $\mathfrak{m}N = N$. By the assumption we have $N = 0$. \square

Note: There exists a demi local ring finite over its center which is not a quasi local ring. Indeed, let us consider a semisimple Lie algebra \mathfrak{g} over a field

k of positive characteristic and consider its universal enveloping algebra $U = U(\mathfrak{g})$. Then U is finite over its center. The localization $A = U_{(0)}$ of U at the origin is a demi local ring but is not quasi local, since we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ so that the unique maximal ideal \mathfrak{m} of A satisfies $\mathfrak{m}^2 = \mathfrak{m}$.

2.3. Rings which are finite over their centers. In the world of commutative algebraic geometry, it is useful to consider commutative rings at each of their maximal ideals. We localize a commutative ring R at its maximal ideal \mathfrak{m} and obtain the localization $R_{\mathfrak{m}}$. $R_{\mathfrak{m}}$ carries the local information of R around \mathfrak{m} . It is also useful to consider the completion $\hat{R}_{\mathfrak{m}}$ to obtain the local information. We would like to extend this idea to deal with non commutative rings which have sufficiently many maximal ideals.

Let A be a ring with a maximal ideal \mathfrak{M} . We use its \mathfrak{M} -adic completion

$$\hat{A}_{\mathfrak{M}} = \varprojlim_n A/\mathfrak{M}^n$$

as a substitute for “localization of A at \mathfrak{M} .” However, when we deal with non commutative rings, whether we use localization or completion, these notions are sometimes insufficient. For example, when A is equal to $U(\mathfrak{g})$, the universal enveloping algebra of a simple Lie algebra \mathfrak{g} over a field \mathbb{k} of characteristic $p \neq 0$, then a maximal ideal $\mathfrak{M} = \mathfrak{g}U(\mathfrak{g})$ satisfies $\mathfrak{M}^2 = \mathfrak{M}$, which is in deep contrast with the NAK lemma for quasi local ring as developed in subsection 2.1. As a result, the completion $\varprojlim_n A/\mathfrak{M}^n$ is equal to the “small” residue ring A/\mathfrak{M} in this case.

Localizations and formal completions of such algebras behaves much like the ones in the commutative case. Let us distinguish between such behaviors. The idea is follows. Given a ring A as in the definition, we may surely localize it with respect to \mathfrak{m} and obtain $A \otimes_R R_{\mathfrak{m}}$. We may also consider the completion $A \otimes_R \hat{R}_{\mathfrak{m}}$. Such localizations and completions are common in the usual commutative algebraic geometry. Then we examine if the completion $A \otimes_R R_{\mathfrak{m}}$ is a good one. We first note:

Lemma 2.7. *Let A be a ring which is finite over its central subring R . Then:*

(1) For any maximal ideal \mathfrak{M} of A , $\mathfrak{a} = \mathfrak{M} \cap R$ is a maximal ideal of R .

(2) For any maximal ideal \mathfrak{m} of R , the set

$$S = \{\mathfrak{M} \in \text{Spm}(A); \mathfrak{M} \cap R = \mathfrak{m}\}$$

of maximal ideals which lie over \mathfrak{m} is a finite set.

(3) The intersection $\mathfrak{N} = \bigcap_{\mathfrak{M} \in S} \mathfrak{M}$ satisfies $\mathfrak{N}^{k_0} \subset \mathfrak{m}A$ for some positive integer k_0 .

Proof. (1) By using the usual Nakayama lemma for a finite module R/\mathfrak{M} over the commutative ring R , we see that there exist a maximal left ideal \mathfrak{b} of R such that $(A/\mathfrak{M})\mathfrak{b} \neq A/\mathfrak{M}$. It follows that $\mathfrak{b} \subset \mathfrak{M} \cap R = \mathfrak{a}$. By using the maximality of \mathfrak{b} we have $\mathfrak{a} = \mathfrak{b}$.

(2) The algebra $\bar{A} = A/\mathfrak{m}$ is an Artinian ring which is finite over a field $\bar{R} = R/\mathfrak{m}$. Thus we see that the set S is a finite set.

(3) According to a theorem of Hopkins [10, Theorem 3.5.1], the Jacobson radical $\text{rad}(\bar{A})$ of \bar{A} is nilpotent. It follows that any maximal ideal $\bar{\mathfrak{M}}$ of \bar{A} contains $\text{rad}(\bar{A})$ as a subset. On the other hand, according to a theorem [10, Theorem 3.5.5] on semisimple Artinian algebras, we see that $\bar{A}/\text{rad}(\bar{A})$ is a direct sum of full matrix algebras over skew fields. We therefore easily see that $\bar{\mathfrak{N}} = \text{rad}(\bar{A})$ and $\bar{\mathfrak{N}}$ is therefore nilpotent. \square

We then employ the following definition.

DEFINITION 2.8. Let A be a ring which is finite over its central subring R . Let \mathfrak{m} be a maximal ideal of R . We know that there exists only a finite maximal ideals $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_s$. Let us say that A has a *formally completely decomposable fiber* at \mathfrak{m} if the natural homomorphism

$$A \otimes_R \hat{R}_{\mathfrak{m}} \rightarrow \prod_j \hat{A}_{\mathfrak{M}_j}$$

is bijective. When there exists only one maximal ideal of A which lies over \mathfrak{m} , (that means, when $s = 1$), then we say that A has a *formally indecomposable fiber* at \mathfrak{m} .

Note: In the above definition, if A has formally completely decomposable fiber at \mathfrak{m} , then each \hat{A}_{M_j} is flat over A . Indeed, $A \otimes_R \hat{R}_{\mathfrak{m}}$ is flat over A as the usual commutative ring theory tells. \hat{A}_{M_j} is a direct summand of $A \otimes_R \hat{R}_{\mathfrak{m}}$ as an A -module and is hence an A -flat module.

The author admits that the definition above is a temporary one. There should be a better way to manage the situation which is independent of the choice of the center R .

As a result of a usual argument of topology, we obtain the following:

Lemma 2.9. *Let A be a ring which is finite over its central subring R . Let \mathfrak{m} be a maximal ideal of R . Let $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_s$ be the maximal ideals which lies over \mathfrak{m} . Then A has a formally completely decomposable fiber at \mathfrak{m} if and only if the following condition is satisfied.*

$$\forall k > 0 \exists N > 0; \quad (\mathfrak{M}_1^N \cap \dots \cap \mathfrak{M}_s^N) \subset (\mathfrak{M}_1 \cap \dots \cap \mathfrak{M}_s)^k$$

□

Corollary 2.10. *Let A be a ring which is finite over its central subring R . Let \mathfrak{m} be a maximal ideal of R . If there exists only one maximal ideal of A which lies over \mathfrak{m} , then A has a formally indecomposable fiber at \mathfrak{m} .*

□

Proposition 2.11. *If A is a commutative ring which is finite over a noetherian subring R , then A has a formally completely decomposable fiber at every closed point of R .*

Proof. This is a direct consequence of the NAK lemma for commutative rings. □

EXAMPLE 2.12. Let $R = \mathbb{k}$ be field. Let $A \subset M_n(\mathbb{k})$ be the ring of upper triangular matrices over R :

$$A = \{(a_{ij}) \in M_n(R) \mid a_{ij} = 0 \text{ if } i > j\}$$

Then the maximal ideals which are on \mathfrak{m} are $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n$ defined as follows.

$$\mathfrak{M}_k = \{(a_{ij}) \in A; a_{kk} = 0\}$$

We then note that $A \otimes_{\mathbb{k}} \hat{\mathbb{k}}_{\mathfrak{m}} = A$ does not decompose into a direct sum of rings. A in this case therefore does not have completely decomposable fiber at \mathfrak{m} . It might be also useful to note that $\mathfrak{M}_k^2 = \mathfrak{M}_k$ holds for any k so that each completion $\varprojlim A/\mathfrak{M}_k^n$ is isomorphic to a rather “small” module $A/\mathfrak{M}_k (\cong \mathbb{k})$.

3. AUSLANDER REGULARITY

In this section we summarize some results concerning Auslander regularity of non commutative rings. It may be helpful to read an article[7] for an introduction to the topic.

Let us begin by giving the definition of Auslander regularity.

DEFINITION 3.1. Let A be a ring. Then the grade $j(M)$ of a left A -module M is defined by

$$j(M) = \inf\{i \mid \text{Ext}_A^i(M, A) \neq 0\}.$$

(By convention we define $\inf \emptyset = +\infty$.)

DEFINITION 3.2. Let A be a ring. Then:

- (1) We say that A satisfies the Auslander condition if for every Noetherian left A -module M and for all $i \geq 0$, $j(N) \geq i$ for all right $-A$ submodules $N \subset \text{Ext}_A^i(M, A)$.
- (2) A is said to be *Auslander-Gorenstein* if A is left Noetherian, satisfies the Auslander condition, and has finite left injective dimension;
- (3) A is said to be *Auslander regular* if it is Auslander-Gorenstein and has finite global dimension.

A commutative ring is Auslander regular if and only if it is a regular ring in the usual sense of commutative ring theory. It is also easy to see that the following lemma holds. The author could not find a right reference. For the sake of self-containedness, let us give here a proof.

Lemma 3.3. *Auslander regularity is Morita invariant.*

Proof. Assume that A is Morita equivalent to B and that B is Auslander regular. There exists a (left A -, right $-B$) module P and a (left B -, right $-A$) module Q such that they give an equivalence of categories

$$(\mathbb{L}) \quad (\text{left } A \text{ modules}) \begin{array}{c} \xrightarrow{Q \otimes_A \bullet} \\ \xleftarrow{P \otimes_B \bullet} \end{array} (\text{left } B \text{ modules})$$

and another equivalence of categories

$$(\mathbb{R}) \quad (\text{right } A \text{ modules}) \begin{array}{c} \xrightarrow{\bullet \otimes_A P} \\ \xleftarrow{\bullet \otimes_B Q} \end{array} (\text{right } B \text{ modules}).$$

Let M be a finitely generated left A -module. Let N be an A -right submodule of $\text{Ext}_A^i(M, A)$ for some integer i . The equivalence \mathbb{L} of categories gives rise to an isomorphism

$$\text{Ext}_A^i(M, A) \cong \text{Ext}_B^i(Q \otimes_A M, Q)$$

of extension groups for each i . These extension groups are right $-A$ modules and the isomorphism is a isomorphism of right $-A$ -modules. The right hand side corresponds, via the equivalence \mathbb{R} to a right $-B$ module

$$\text{Ext}_A^i(Q \otimes_A M, Q) \otimes_A P \cong \text{Ext}_B^i(Q \otimes_A M, Q \otimes_A P).$$

Note that $Q \otimes_A P \cong B$. So the inclusion $N \subset \text{Ext}_A^i(M, A)$ corresponds via \mathbb{R} to an inclusion

$$N \otimes_A P \subset \text{Ext}_B^i(Q \otimes_A M, B).$$

The Auslander condition for B implies

$$\text{Ext}_B^j(N \otimes_A P, B) = 0 \quad (\forall j \leq i).$$

Now by using the same idea as above and we obtain

$$\begin{aligned} & Q \otimes_A \text{Ext}_A^j(N, A) \\ & \cong Q \otimes_A \text{Ext}_B^j(N \otimes_A P, P) \\ & \cong \text{Ext}_B^j(N \otimes_A P, Q \otimes_A P) \\ & = 0 \quad (\text{if } i \geq j.) \end{aligned}$$

An A -module $\text{Ext}_A^j(N, A)$ corresponds to $Q \otimes_A \text{Ext}_A^j(N, A)$ and is therefore equal to zero if $j \leq i$. Thus A satisfies the Auslander condition.

That A has a finite projective dimension is obvious. □

Corollary 3.4. *A full matrix algebra $M_s(\mathbb{k}[t_1, \dots, t_n])$ over a polynomial algebra over a field \mathbb{k} is Auslander regular.*

The Auslander regularity might be a little difficult to grasp at a first glance. Fortunately, there is a good criterion for the Auslander regularity due to Björk. Namely, if A has a “good” filtration such that the associated algebra $\text{gr } A$ is Auslander regular, then A itself is Auslander regular. We may thus reduce the problem of judging the Auslander regularity of A to the same problem for $\text{gr } A$. Let us now begin by defining what “good” filtrations are.

DEFINITION 3.5. By a *filtration* of a ring A we mean an increasing sequence $\{A_i\}$ of additive subgroup of A such that

$$A = \bigcup_{i=0}^{\infty} A_i, \quad \bigcap_{i=0}^{\infty} A_i = 0, \quad A_n A_m \subset A_{n+m} \quad (\forall n, m), \quad 1 \in A_0$$

holds. We say that a *strong closure condition* is satisfied if for any finite elements a_1, a_2, \dots, a_s of A and for any integers i_1, i_2, \dots, i_s , submodules

$$\begin{aligned} & A_{i_1} a_{i_1} + \dots + A_{i_s} a_{i_s} \\ & a_{i_1} A_{i_1} + \dots + a_{i_s} A_{i_s} \end{aligned}$$

are closed in the filtrated topology.

It might be helpful to note that the associated graded ring $\text{gr } A$ is denoted by GA in several papers like [6], [7]. There are several other “closure conditions”. These conditions are verified easily for the rings of our concern:

Lemma 3.6. *Let (A, \mathfrak{m}) be a quasi local ring finite over its central Noetherian subalgebra R . If A is filtrated such that its topology coincides with a \mathfrak{n} -adic topology for some ideal $\mathfrak{n} \subset \mathfrak{m}$, then A with the \mathfrak{m} -adic filtration satisfies the strong closure condition.*

Proof. By imitating the proof of Lemma 2.6, we see that there exists a positive integer k_0 such that $\mathfrak{n}^{k_0} \subset \mathfrak{n} \cap R$ and that \mathfrak{n} -adic topology coincides with $\mathfrak{n} \cap R$ -adic topology. The lemma then follows from the Krull intersection theorem for commutative rings. \square

We now cite the following strong result of Björk.

Theorem 3.7 (Björk). [6] *Let A be a filtered ring such that “closure conditions” hold. If the associated graded algebra $\text{gr}(A)$ is Auslander regular, then the original ring A is also Auslander regular.*

By using Lemma 3.6, we obtain immediately the following corollary.

Corollary 3.8. *Let (A, \mathfrak{m}) be a Noetherian quasi local ring finite over its center. Then A is Auslander regular if the associated graded ring $\text{gr}_{\mathfrak{m}} A$ is Auslander regular.*

The Auslander regularity of a ring guarantees us several good properties of the ring. The following theorem of Ramras is an example. It plays an important role in our argument.

Theorem 3.9. [16, Theorem 2.16] *Let A be a quasi local ring finite over a central regular local ring R of dimension n . If the left injective dimension of A over A is finite, then A is R -free.*

The following fact seems to be well-known to the specialists.

Proposition 3.10. *Let A be a non commutative Noetherian ring. Then A is an Auslander regular ring if and only if the following conditions are fulfilled.*

- (1) *The global dimension of A is finite.*
- (2) *Let*

$$0 \rightarrow A \rightarrow E_0 \rightarrow E_1 \rightarrow \dots,$$

be the minimal injective resolution of A as an A -left module. Then the flat dimension of each E_i is less than or equal to i .

For the proof see for example [15]. \square

On the other hand, we know that taking injective hulls commutes with localizing modules with respect to a multiplicative subset of R , namely, the following lemma holds.

Lemma 3.11. *Let A be a ring which is finite over its noetherian central subalgebra R . Let S be a multiplicative subset of R . Then for any A -left module M , the localization $E(M)[S^{-1}]$ of the injective hull $E(M)$ of M is isomorphic to the injective hull of $M[S^{-1}]$ as an $A[S^{-1}]$ -module.*

Lemma 3.12. *Let A be a ring which is finite over its noetherian central subalgebra R . Let S be a multiplicative subset of R . If A is Auslander regular, then $A[S^{-1}]$ is also Auslander regular.*

With the above proposition and the lemma, we may easily prove the following proposition.

Proposition 3.13. *Let A be an algebra which is finite over its Noetherian central subalgebra R . Let us denote by A_X the sheaf of \mathcal{O}_X -algebras over $X = \text{Spec}(R)$ associated to A . Then the following are equivalent:*

- (1) A is an Auslander regular ring.
- (2) There exists an affine open covering $\{U_i\}$ of $\text{Spec}(R)$ such that each ring $\Gamma(U_i, A_X)$ is Auslander regular.
- (3) There exist elements $f_1, f_2, \dots, f_t \in R$ such that their zero-locus $V(f_1, \dots, f_t)$ is empty and each localization $A[f_j^{-1}]$ is Auslander regular.
- (4) For any prime ideal \mathfrak{p} of R , $A \otimes_R R_{\mathfrak{p}}$ is Auslander regular.

If in addition, R is a finitely generated commutative algebra over a field \mathbb{k} , the above conditions are also equivalent to:

- (5) For any maximal ideal \mathfrak{m} of R , $A \otimes_R R_{\mathfrak{m}}$ is Auslander regular.

Proof. We have trivial implications (1) \implies (2) \implies (3) \implies (4) \implies (5). To prove (4) \implies (1), we take an arbitrary finite A -module M and consider $\text{Ext}^{\bullet}(N, A)$ for $N \subset \text{Ext}^{\bullet}(M, A)$. Since these modules are finite, we may easily deduce the result. Similarly, when R is finitely generated over a field, then we have (5) \implies (1). \square

Looking at the proposition above, we may define Auslander regularity of sheaf of algebras as follows:

DEFINITION 3.14. Let X be a scheme. Let \mathcal{A} be a sheaf of \mathcal{O}_X -algebras such that \mathcal{O}_X is contained in the center of \mathcal{A} and \mathcal{A} is coherent as an \mathcal{O}_X -module. Then \mathcal{A} is said to be *Auslander regular* if for any point P of X , the stalk \mathcal{A}_P of \mathcal{A} is Auslander regular.

We see easily from Proposition 3.13 the following.

Proposition 3.15. *Let \mathcal{A} be a sheaf of algebras on a scheme X such that the assumptions of Definition 3.14 are satisfied. Then the following conditions are equivalent:*

- (1) \mathcal{A} is Auslander regular.
- (2) For each affine open subset U of X , the algebra $\mathcal{A}(U)$ of sections on U is Auslander regular.
- (3) There exists an affine open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that each $\mathcal{A}(U_\lambda)$ is Auslander regular.

If in addition, X is a scheme of finite type over a field \mathbb{k} , then the above conditions are also equivalent to:

- (4) For any closed point \mathfrak{m} of R , the stalk $A_{\mathfrak{m}}$ is Auslander regular.

□

Proposition 3.16. *Let A be a ring which is finite over its central noetherian subring R . We assume A has formally completely indecomposable fiber at each closed point $\mathfrak{m} \in \text{Spm}(R)$. Then the algebra A is Auslander regular if and only if its completion $\hat{A}_{\mathfrak{M}}$ is Auslander regular for any maximal ideal \mathfrak{M} of A .*

Proof. In view of the previous Proposition 3.15, we see that A is Auslander regular if and only if each fiber $A_{\mathfrak{m}}$ is Auslander regular at each closed point $\mathfrak{m} \in \text{Spm}(R)$. Since $\hat{R}_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$, the Auslander regularity of $A_{\mathfrak{m}}$ is the same as that of $A \otimes_R \hat{R}_{\mathfrak{m}}$. By the complete decomposability, we see that the property is also equivalent to the regularity of $\hat{A}_{\mathfrak{M}}$ for each maximal ideal \mathfrak{M} of A which lies over \mathfrak{m} . □

4. REGULARITY OF NORM BASED EXTENSIONS OF WEYL ALGEBRAS

In this section we describe the main result of this paper. We first show that the norm based extension of the Weyl algebra is Auslander regular. We then show that the Auslander regularity distinguish the Weyl algebra A with other algebras B which are Zariski locally isomorphic to A .

4.1. Norm based extensions of Weyl algebras.

4.1.1. *definition of Norm based extensions of Weyl algebras.* In [21], the author defined the “norm based extension” of a Weyl algebra A . Let us make a short review.

For the rest of the paper we denote by A the Weyl algebra $A = A_n(\mathbb{k})$ over a field \mathbb{k} of positive characteristic. It is an algebra generated by elements $\{\gamma_i\}_{i=1}^{2n}$ which satisfy the following commutation relations,

$$[\gamma_i, \gamma_j] = h_{ij} \quad (i, j = 1, 2, \dots, 2n)$$

where $(h_{ij})_{i,j=1}^{2n}$ is a skew symmetric non degenerate constant matrix. Such generators are referred to as CCR generators. We denote by $X = \text{Spec}(R)$ the affine spectrum of the center $R = Z(A)$ of A . X carries a sheaf of \mathcal{O}_X -algebras corresponding to A , which we denote by A_X .

Now, using the set $\{\gamma_i\}_{i=1}^{2n}$ of CCR generators, we notice that the center $Z(A)$ of A has “linear coordinates” $\{\gamma_i^p\}_{i=1}^{2n}$ which gives rise to an identification

$$X = \text{Spec } Z(A) \cong \mathbb{A}^{2n}.$$

We may then consider a compactification of X by putting

$$\bar{X} = \mathbb{P}^{2n}.$$

It should be noted that the compactification $\bar{X} \supset X$ depends on the choice of the CCR generators of A .

Let us extend the sheaf A_X on X to \bar{X} . To do that, we notice that the algebra A admits a norm $N_A : A \rightarrow Z(A)$ (or, to call it more appropriately in accordance with the general theory of Azumaya algebra, a “reduced norm”) which is essentially given by considering determinants on fibers. It gives rise to a morphism

$$N_A : A_X \rightarrow \mathcal{O}_X.$$

of sheaves on X . Noting that A is “asymptotically commutative” near the boundary of X and that N_A gives the asymptotic behavior of elements of A , we are able to define an extension of the sheaf A_X on X to the whole of \bar{X} :

DEFINITION 4.1. [21, Definition2.3] The *norm based extension* $A_{\bar{X}}$ of the Weyl algebra A is an $\mathcal{O}_{\bar{X}}$ -algebra on the projective space \bar{X} whose sections on an open set U of \bar{X} are given as follows:

$$A_{\bar{X}}(U) = \{s \in A_X(U \cap X); N_A(s) \text{ extends to an element of } \mathcal{O}_X(U)\}.$$

It turns out that $A_{\bar{X}}$ is a sheaf of $\mathcal{O}_{\bar{X}}$ -algebras.

4.1.2. Coordinates of the norm based extension at infinity. To describe $A_{\bar{X}}$ at a point \mathfrak{m} on the boundary $H = \bar{X} \setminus X$ more closely, let us describe it in terms of coordinates. It is convenient to introduce (non commutative) “projective coordinates”. Namely, we introduce variables $\Gamma_1, \dots, \Gamma_{2n}, Z$ such that:

- (1) Z is central.
- (2) $\gamma_i = \Gamma_i/Z$ ($i = 1, 2, \dots, 2n$).
- (3) $[\Gamma_i, \Gamma_j] = h_{ij}Z^2$ ($i, j = 1, 2, \dots, 2n$).

By making a linear change of coordinates if necessary, we may assume without loss of generality that $\Gamma_1 \neq 0$ at \mathfrak{m} . We then employ a local coordinate system $(v, u, \{\bar{\gamma}_i\}_{i=3}^{2n})$ around \mathfrak{m} defined as follows.

$$v = -\Gamma_2\Gamma_1^{-1}, \quad u = Z\Gamma_1^{-1}, \quad \bar{\gamma}_i = \Gamma_i\Gamma_1^{-1} \quad (i = 3, 4, \dots, 2n).$$

We may also assume that h_{ij} is in a “normal form” such that the coordinates are subject to the following commutation relations.

$$\begin{aligned} [v, u] &= u^3, & [u, \bar{\gamma}_i] &= 0, & [v, \bar{\gamma}_i] &= u^2\bar{\gamma}_i, & [\bar{\gamma}_i, \bar{\gamma}_j] &= h_{ij}u^2 \\ & & & & & & & (i, j = 3, 4, \dots, 2n). \end{aligned}$$

To sum up, we have proved the following:

Proposition 4.2. *Let U be an affine piece of the projective space \bar{X} . Then the algebra of sections $A_{\bar{X}}(U)$ of $A_{\bar{X}}$ on U is generated by $2n$ generators. Namely, we have $A_{\bar{X}}(U) = \mathbb{k}\langle u, v, \bar{\gamma}_3, \bar{\gamma}_4, \dots, \bar{\gamma}_{2n} \rangle$, where the generators are subject to the following relations.*

$$\begin{aligned} [v, u] &= u^3, & [u, \bar{\gamma}_i] &= 0, & [v, \bar{\gamma}_i] &= u^2\bar{\gamma}_i, & [\bar{\gamma}_i, \bar{\gamma}_j] &= h_{ij}u^2 \\ & & & & & & & (i, j = 3, 4, \dots, 2n). \end{aligned}$$

Using these generators, $\mathcal{O}_{\bar{X}}(U)$ is expressed as a polynomial algebra $\mathcal{O}_{\bar{X}}(U) = \mathbb{k}[v^p, u^p, \bar{\gamma}_3^p, \bar{\gamma}_4^p, \dots, \bar{\gamma}_{2n}^p]$. The defining equation of the hyperplane H at infinity in U is given by u^p .

The last two sentences are also checked easily. □

4.1.3. *Regularity of norm based extensions of Weyl algebras.* We now state a main result of this paper:

Proposition 4.3. *The norm based extension $A_{\bar{X}}$ of the Weyl algebra A is Auslander regular.*

Proof. As we have seen in Proposition 3.13, it is enough to prove that the stalk $A_{\bar{X}, \mathfrak{m}}$ of the sheaf $A_{\bar{X}}$ at each closed point \mathfrak{m} of \bar{X} is an Auslander regular ring. Let us first deal with the case where $\mathfrak{m} \in X$. \mathfrak{m} is then identified with the maximal ideal of $R = Z(A)$. Looking at Proposition 3.13 once again, we see that our claim in this case is a consequence of the fact that A is a regular ring which is already well-known. Indeed, since $A = A_n(\mathbb{k})$ has a filtration whose associated graded algebra is a polynomial ring, (putting aside of verifying the closure condition of such filtration,) we see by using the theorem of Björk (Theorem 3.7) that $A_n(\mathbb{k})$ is Auslander regular. Here, however, let us prove directly

that $A_{\bar{X}, \mathfrak{m}} = A \otimes_R R_{\mathfrak{m}}$ is Auslander regular. By using a structure theory of the Weyl algebras (such as [17, Lemma 3]), we see that $\text{gr}_{\mathfrak{m}} A_{\mathfrak{m}}$ is isomorphic to the full matrix algebra $M_{p^n}(R/\mathfrak{m})[t_1, \dots, t_{2n}]$ over a polynomial ring. Thus by using Corollary 3.4 we see that $\text{gr}_{\mathfrak{m}} A_{\mathfrak{m}}$ is Auslander regular and hence by using the theorem of Björk we conclude that $A_{\mathfrak{m}}$ is Auslander regular.

Let us next deal with the case where \mathfrak{m} is on the hyper plane $H = \bar{X} \setminus X$ at infinity. In terms of the coordinate described in Proposition 4.2, the stalk $A_{\bar{X}, \mathfrak{m}}$ is a quasi local ring with the maximal ideal \mathfrak{M} generated $\{u, v\} \cup \{\tilde{\gamma}_i\}_{i=3}^{2n}$. The associated graded algebra $\text{gr}_{\mathfrak{M}}(A_{\bar{X}, \mathfrak{m}})$ is isomorphic to an algebra \tilde{A} generated by elements $\{\tilde{u}, \tilde{v}\} \cup \{\tilde{\gamma}_i\}_{i=3}^{2n}$ which satisfy the following relations.

$$[\tilde{v}, \tilde{u}] = 0, \quad [\tilde{\gamma}_i, \tilde{\gamma}_j] = h_{ij} \tilde{u}^2, \quad [\tilde{u}, \tilde{\gamma}_i] = 0, \quad [\tilde{v}, \tilde{\gamma}_i] = 0.$$

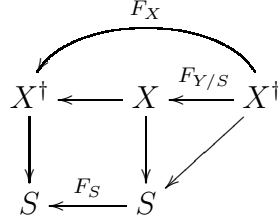
The algebra \tilde{A} again has an filtration by $u\tilde{A}$ whose associated graded ring $\text{gr}_{u\tilde{A}} \tilde{A}$ is a usual commutative polynomial ring. By the theorem of Björk (Theorem 3.7), we see that the algebra \tilde{A} is Auslander regular and then by the same theorem we see that A is also Auslander regular. \square

4.2. Inverse Frobenius pull-backs. Before proceeding further, we need to review the definition of “inverse Frobenius pull-backs” in [21]. In short, X^\dagger is obtained by adjoining p -th power roots of the coordinate functions of X .

In order to explain the notation more clearly, it might be helpful to compare it with a notation used in another paper. Let us compare here our notation with the one which appears in a paper of Illusie[12]. There one sees at (2.1.1) a notation such as $X^{(p)}$. In short, “ X^\dagger to X in our paper is what X is to $X^{(p)}$ in the paper of Illusie”. To avoid confusion, let us write Y in place of the X in the Illusie’s paper. The diagram (2.1.1) then looks like this.

$$\begin{array}{ccccc}
 & & & & F_Y \\
 & & & & \curvearrowright \\
 Y & \longleftarrow & Y^{(p)} & \xleftarrow{F_{Y/S}} & Y \\
 \downarrow & & \downarrow & \swarrow & \\
 S & \xleftarrow{F_S} & S & &
 \end{array}$$

In the present paper, we concentrate on a case where $S = \text{Spec}(\mathbb{k})$. Then the diagram above may be rewritten in the present paper’s notation provided we put $X^\dagger = Y$:



The same sort of idea applies and our spaces X, \bar{X} , and H has their own inverse Frobenius pull-backs who are denoted by $X^\dagger, \bar{X}^\dagger$, and H^\dagger .

4.3. $\mathcal{O}_{\bar{X}^\dagger}$ as a classical counterpart of $A_{\bar{X}}$. Since the inverse Frobenius pullback \bar{X}^\dagger and \bar{X} have the same underlying space $|\bar{X}^\dagger| = |\bar{X}|$, we identify $\mathcal{O}_{\bar{X}^\dagger}$ with a sheaf of $\mathcal{O}_{\bar{X}}$ -algebra on X . $\mathcal{O}_{\bar{X}^\dagger}$ is thought of as a “classical counterpart” of our sheaf $A_{\bar{X}}$. Indeed, on the affine piece X of \bar{X} , $\mathcal{O}_{\bar{X}^\dagger}$ is generated by $t_1 = (\gamma_1^p)^{1/p}, \dots, t_{2n} = (\gamma_{2n}^p)^{1/p}$. The affine coordinate ring $\mathbb{k}[t_1, t_2, \dots, t_{2n}]$ resembles the Weyl algebra $A_n(\mathbb{k})$ in many ways. For example, Both algebras are free module of rank p^{2n} over $Z = \text{Spec}(\mathbb{k}[t_1^p, \dots, t_{2n}^p])$. Moreover, there exists a filtration on $A_n(\mathbb{k})$ such that the associated graded algebra $\text{gr}(A_n(\mathbb{k}))$ is isomorphic to the polynomial ring $\mathbb{k}[t_1, t_2, \dots, t_{2n}]$. We may say that $A_n(\mathbb{k})$ is ‘asymptotically commutative’ and gradually looks like the polynomial ring $\mathbb{k}[t_1, t_2, \dots, t_{2n}]$.

Let us look at this by studying the behavior of the sheaf $A_{\bar{X}}$ around the hyperplane H at infinity. We have shown in [21] that there exists a specific ideal \mathfrak{J}_{H^\dagger} of $A_{\bar{X}}$. We may regard it as a non commutative analogue of the ideal sheaf of H^\dagger . It is related to the ideal sheaf \mathcal{I}_H of H by the following equation:

$$\mathfrak{J}_{H^\dagger}^p \cong \mathcal{I}_H A_{\bar{X}}.$$

The situation may probably be well-understood in terms of coordinates introduced in Proposition 4.2. \mathcal{I}_H is an ideal of $\mathcal{O}_{\bar{X}}$ generated by u^p , whereas \mathfrak{J}_{H^\dagger} is an ideal of $A_{\bar{X}}$ generated by u . The quotient $A_{\bar{X}}/\mathfrak{J}_{H^\dagger}$ is isomorphic to \mathcal{O}_{H^\dagger} . As a result, we obtain a “symbol map”

$$\rho : A_{\bar{X}} \rightarrow \mathcal{O}_{H^\dagger}$$

which essentially sends each element of $A_{\bar{X}}$ to its principal symbol restricted to H^\dagger . For the ease of future reference, let us write the fact down in a following:

Proposition 4.4. *There exists a “symbol map”*

$$\rho : A_{\bar{X}} \rightarrow \mathcal{O}_{H^\dagger}$$

which gives rise to an isomorphism

$$\bar{\rho} : A_{\bar{X}}/\mathfrak{J}_{H^\dagger} \rightarrow \mathcal{O}_{H^\dagger}.$$

The following diagram is commutative.

$$\begin{array}{ccc} A_{\bar{X}} & \xrightarrow{\rho} & \mathcal{O}_{H^\dagger} \\ N_A \downarrow & & \downarrow \bullet p^n \\ \mathcal{O}_{\bar{X}^\dagger} & \xrightarrow{\text{restr.}} & \mathcal{O}_{H^\dagger} \end{array}$$

(Note however that N_A is not additive.

□

5. A CHARACTERIZATION OF THE WEYL ALGEBRA AMONG ITS COUSINS

5.1. Algebras Zariski isomorphic to a Weyl algebra. We continue to consider a Weyl algebra A over a field of positive characteristic, and employ $X, \bar{X}, H = \bar{X} \setminus X$ as in the previous subsection. Let us study a “cousin” of A . Namely, we consider an Z -algebra B such that its associated sheaf B_X is a sheaf of algebra which is Zariski locally isomorphic to A_X as an \mathcal{O}_X -algebra. By abuse of language we call such B an algebra Zariski locally isomorphic to A .

As described for example in [3],[19], the algebra B is in one to one correspondence with an A -module W of rank one whose associated sheaf W_X on X is a locally free \mathcal{O}_X -module. The correspondence is given by the following relation.

$$B = \text{End}_A(W)$$

It is known that we have a bunch of non trivial locally free A -modules of rank 1. Indeed, when \mathbb{k} is a field of characteristic 0, projective $A_1(\mathbb{k})$ -modules of rank 1 are parametrized by “Calogero-Moser spaces” ([5]). We may then restrict the coefficient ring and consider them “mod p ” and obtain such modules for our case.) So A has a lot of such “cousins”.

We also note that a reflexive A -module W of rank one admits a norm $N_W : W \rightarrow Z$ and its associated sheaf W_X has a norm based extension $W_{\bar{X}}$, a sheaf on \bar{X} which extends W_X to \bar{X} (See [21].) Likewise, we see easily that an algebra B which is locally isomorphic to A admits a norm $N_B : B \rightarrow Z$ and that its associated sheaf B_X has a norm based extension $B_{\bar{X}}$ as a sheaf of $\mathcal{O}_{\bar{X}}$ -algebras in a similar manner.

The sheaf $B_{\bar{X}}$ is of course locally isomorphic to $A_{\bar{X}}$ on $X = \mathbb{A}^{2n}$. However, It is not always true that $B_{\bar{X}}$ is locally isomorphic to $A_{\bar{X}}$ on the whole of $\bar{X} = \mathbb{P}^{2n}$. The truth is far from that. In fact, $B_{\bar{X}}$ is Auslander regular only in the trivial case where B is isomorphic to A as we will prove later.

The following lemma is a consequence of the Auslander regularity of $A_{\bar{X}}$ and is a generalization of the equivalence (1) \iff (4) of [20, Theorem 2.4]

Lemma 5.1. *Let W be a reflexive A -module of rank one. Assume $W_{\bar{X}}$ is a locally free $\mathcal{O}_{\bar{X}}$ -module on a Zariski open subset U of \bar{X} . Then $W_{\bar{X}}$ is $A_{\bar{X}}$ -locally free on U .*

Proof. Let x be a closed point of U . Since the dimension of X is greater or equal to 2, we have $\dim \mathcal{O}_{\bar{X},x} \geq 2$. Since $W_{\bar{X}}$ is locally free on U , we have $\text{depth}_{\mathcal{O}_{\bar{X},x}} W_{\bar{X},x} \geq 2$. Since $A_{\bar{X}}$ is regular, we see by definition that every stalk of $A_{\bar{X}}$ has a finite global dimension. In particular, the projective dimension $\text{pd}_{A_{\bar{X},x}} W_{\bar{X},x}$ is finite and according to Ramras[16, Lemma 2.9], it is equal to the following quantity which is, under the notation of Ramras, denoted as “ $\text{Ext}_{A_{\bar{X},x}} - \dim W_{\bar{X},x}$ ”.

$$\sup\{j \geq 0; \text{Ext}_{A_{\bar{X},x}}^j(\dim W_{\bar{X},x}, A_{\bar{X},x}) \neq 0\}.$$

According to another Proposition of Ramras[16, proposition 2.14], the quantity is equal to 0. Thus the projective dimension of W_x is equal to zero and hence W_x is locally free at x . □

Corollary 5.2. *There exists a closed subset F of codimension at least 3 of \bar{X} such that $W_{\bar{X}}$ is $A_{\bar{X}}$ -locally free on $V_0 = \bar{X} \setminus F$. Accordingly, we have:*

- (1) $A_{\bar{X}}$ is locally isomorphic to $B_{\bar{X}}$ on $\bar{X} \setminus F$.
 - (2) $B_{\bar{X}}$ is a reflexive extension of $B_{\bar{X}}|_{V_0}$.
-

5.2. Summary of results on reflexive sheaves. In the sequel, we need to handle reflexive sheaves. Let us briefly summarize here some of the results on reflexive sheaves which appear in the usual commutative algebraic geometry..

Recall that a coherent sheaf \mathcal{F} on a scheme X is reflexive if it is isomorphic to its double dual.

Proposition 5.3. [9, Proposition 1.1] *A coherent sheaf \mathcal{F} on a noetherian integral scheme X is reflexive if and only if (at least locally) it*

can be included in an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{E} \rightarrow \mathcal{G} \rightarrow 0,$$

where \mathcal{E} is locally free and \mathcal{G} is torsion-free.

Proposition 5.4. [9, Proposition 1.9] *Assume that the base scheme X is integral and locally factorial. Then any reflexive sheaf of rank one on X is invertible.*

Recall that a coherent sheaf \mathcal{F} on X is normal if for every open set $U \subseteq X$ and every closed subset $Y \subseteq U$ of codimension ≥ 2 , the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(U \setminus Y)$ is bijective.

Proposition 5.5. [9, Proposition 1.6] *Let \mathcal{F} be a coherent sheaf on a normal integral scheme X . The following conditions are equivalent:*

- (i) \mathcal{F} is reflexive.
- (ii) \mathcal{F} is torsion-free and normal.
- (iii) \mathcal{F} is torsion free, and for each open $U \subseteq X$ and closed subset $Y \subseteq U$ of codimension ≥ 2 , $\mathcal{F}_U \cong j_*\mathcal{F}_{U \setminus Y}$, where $j : U \setminus Y \rightarrow U$ is the inclusion map.

Let us now cite here the following theorem which is a generalization of the Horrocks[11]. Although we only use it in an original assumption of Horrocks, it gave a good guideline for our study. It is also used in the study reflexive sheaves in the author's another paper[21].

Theorem 5.6. [1, Theorem 0.2] *Let \mathbb{k} be an algebraically closed field, n be an integer greater than or equal to 3, and let E be a reflexive sheaf on $\mathbb{P}_{\mathbb{k}}^n$ of rank $r(\geq 1)$. Then E splits into a direct sum of line bundles if and only if there exists a hyperplane $H \subset \mathbb{P}_{\mathbb{k}}^n$ such that $E|_H$ splits into a direct sum of line bundles.*

In the proof of the above-cited theorem, Abe and Yoshinaga shows the following theorem, which is also useful for us.

Theorem 5.7. ([1, Theorem 2.2]) *Let E and F be reflexive sheaves on $\mathbb{P}_{\mathbb{k}}^n$ ($n \geq 2$) and H be a hyperplane in $\mathbb{P}_{\mathbb{k}}^n$. Suppose $E|_H \cong F|_H$ and $\text{Ext}_{\mathbb{P}_{\mathbb{k}}^n}^1(F, E(-1)) = 0$. Then $E \cong F$. As is shown in the proof of [1, Theorem 2.2], for each isomorphism $\psi : E|_H \rightarrow F|_H$, there exists its extension $\Psi : E \rightarrow F$ which extends ψ .*

5.3. Serre twist of $B_{\bar{X}}$. As we have seen, there exists a specific ideal \mathfrak{J}_{H^\dagger} of $A_{\bar{X}}$. By taking suitable tensor power of \mathfrak{J}_{H^\dagger} , we may consider ‘‘Serre twist’’ $\mathfrak{J}_{H^\dagger}^c$ of $A_{\bar{X}}$ described at [21, lemma 2.6]. Likewise, there is also a ‘‘Serre twist’’ of the sheaf $B_{\bar{X}}$. To emphasize the similarity with Serre twist, we use notation $B_{\bar{X}}\langle k \rangle$ as the ‘‘B-version’’ of $\mathfrak{J}_{H^\dagger}^{-k}$.

(Note the sign convention which is used in order to fit the notation of the usual Serre twist.) To be more precise, we define as follows.

DEFINITION 5.8. For any $c \in \mathbb{Z}$ we define a presheaf $B_{\bar{X}}\langle k \rangle$ on \bar{X} by

$$B_{\bar{X}}\langle k \rangle(U) = \{s \in B_X(U \cap X); \text{ord}_{H^\dagger}(N_B(s)) \geq -kp^n\},$$

where ord_{H^\dagger} denotes the valuation with respect to the prime divisor H^\dagger . For $k = -1$, we also use the notation $\mathfrak{J}_{B,H^\dagger}$ for $B_{\bar{X}}\langle -1 \rangle$.

When $B = A$, $\mathfrak{J}_{B,H^\dagger}$ is identical with the \mathfrak{J}_{H^\dagger} defined as above.

Lemma 5.9. *The following facts are true.*

- (1) *For any integer k , the presheaf $B_{\bar{X}}\langle k \rangle$ defined in Definition 5.8 is actually a sheaf.*
- (2) *For any integer k , the sheaf $B_{\bar{X}}\langle k \rangle$ is an $B_{\bar{X}}$ -bimodule.*
- (3) *For any integer k , the sheaf $B_{\bar{X}}\langle k \rangle$ is $\mathcal{O}_{\bar{X}}$ -reflexive.*
- (4) *$\mathfrak{J}_{B,H^\dagger}$ is an ideal sheaf of $B_{\bar{X}}$. Moreover, for any positive integer k , we have a natural inclusion*

$$B_{\bar{X}}\langle -k \rangle \supset (\mathfrak{J}_{B,H^\dagger})^k$$

which is an isomorphism outside of a closed set of codimension at least 2 on \bar{X} .

- (5) *We have*

$$B_{\bar{X}}\langle -p \rangle = \mathcal{I}_H B_{\bar{X}}$$

Proof. (1),(2),(4) are easily verified by the definition.

- (3): It is also easily to check that $B_{\bar{X}}\langle k \rangle$ is normal and torsion free,
- (5): The right hand side is locally isomorphic to $B_{\bar{X}}$ as an \mathcal{O}_X -module and is in particular a reflexive $\mathcal{O}_{\bar{X}}$ -module. Since both hand sides are reflexive and are equal on outside of a closed subset of codimension at least 2, they are equal. \square

As a result of the lemma above, we have a descending chain $\{B_{\bar{X}}\langle k \rangle\}_{k \in \mathbb{Z}}$ of $B_{\bar{X}}$ -bimodules. The associated graded module plays an important role in this paper.

5.4. Formally indecomposability of $B_{\bar{X}}$. There exists a symbol map

$$\bar{\rho}_B : B_{\bar{X}} \rightarrow \mathcal{O}_{H^\dagger}.$$

which is defined as follows: there exists an open subset V_0 of \bar{X} where $B_{\bar{X}}$ is locally isomorphic to $A_{\bar{X}}$. Then on the open set V_0 , we may define ρ_B using the map ρ in Proposition 4.4 via such local isomorphisms. Note that the consistency of ρ and the norm map assures us that ρ_B so defined is independent of the choice of such local isomorphism. We may then extend ρ_B to the whole of \bar{X} by using the normality. The

kernel of ρ_B is equal to $\mathfrak{J}_{B,H^\dagger}$. This fact is also verified at first on V_0 and then verified on the whole of \bar{X} by using the Hartogs property.

DEFINITION 5.10. We put $\bar{B}_H = B_{\bar{X}}/\mathfrak{J}_{B,H^\dagger}$.

There exists a symbol map

$$\bar{\rho}_B : B_{\bar{X}} \rightarrow \mathcal{O}_{H^\dagger}.$$

Proposition 5.11. *Following facts are true.*

- (1) $\bar{B}_H = B_{\bar{X}}/\mathfrak{J}_{B,H^\dagger} \subset \mathcal{O}_{H^\dagger}$
- (2) *The reflexive hull of \bar{B}_H as an \mathcal{O}_H -module is isomorphic to \mathcal{O}_{H^\dagger} .*

Proof. (1): We consider the symbol map

$$\bar{\rho}_B : B_{\bar{X}} \rightarrow \mathcal{O}_{H^\dagger}$$

and we claim

$$\text{Ker}(\bar{\rho}_B) = \mathfrak{J}_{B,H^\dagger}.$$

Indeed, by using the local generators we may easily verify that the both hand sides are equal on V_0 . Since they are reflexive \mathcal{O}_X -modules, they are equal on the whole of \bar{X} .

(2): It goes without saying that \mathcal{O}_{H^\dagger} is locally free \mathcal{O}_H module. In particular, it is reflexive. On the other hand, the two sheaves \mathcal{O}_{H^\dagger} and \bar{B}_H coincide except at a locus of codimension 3 in \bar{X} . □

Proposition 5.12. *For any closed point \mathfrak{m} of \bar{X} , There exists only one maximal ideal of $B_{\bar{X}}$ which lies over \mathfrak{m} .*

Proof. When \mathfrak{m} is in the affine piece X , the result is a consequence of the structure theory of the sheaf of algebras A_X on X which is essentially well known (see for example [17]). Let us treat the case where \mathfrak{m} is on the hyperplane H at infinity. Let \mathfrak{M} be a maximal ideal of $B_{\bar{X}}$ which lies over \mathfrak{m} . Since \mathfrak{m} is on H , we have $\mathfrak{m} \supset \mathcal{S}_H A$ so that we have $\mathfrak{M} \supset \mathcal{S}_H B_{\bar{X}}$. By using Lemma 5.9, we have

$$\mathfrak{J}_{B,H^\dagger}^p \subset B_{\bar{X}}\langle -p \rangle = \mathcal{S}_H B_{\bar{X}} \subset \mathfrak{M}.$$

By using the maximality of \mathfrak{M} , we easily see that $\mathfrak{J}_{B,H^\dagger} \subset \mathfrak{M}$. (otherwise we would have $\mathfrak{J}_{B,H^\dagger} + \mathfrak{M} = B_{\bar{X}}$ which implies in particular that 1 is nilpotent in $B_{\bar{X}}$, which is absurd.) Thus \mathfrak{M} corresponds to a maximal ideal $\bar{\mathfrak{M}} = \mathfrak{M}/\mathfrak{J}_{B,H^\dagger}$ of $\bar{B} = B_{\bar{X}}/\mathfrak{J}_{B,H^\dagger}$. As stated in Proposition 5.11, \bar{B} is a subsheaf of \mathcal{O}_{H^\dagger} . Let us put

$$\mathfrak{m}^\dagger \stackrel{\text{def}}{=} \{f \in \mathcal{O}_{H^\dagger} \mid f^p \in \mathfrak{m}\} = \text{Ker}(\mathcal{O}_{H^\dagger} \xrightarrow{\text{Frob}} \mathcal{O}_H \rightarrow \mathcal{O}_H/\mathfrak{m}).$$

We may easily see from the definition that we have $\mathfrak{m}^\dagger \cap \bar{B} \subset \bar{\mathfrak{M}}$ and so we have the following inclusion of commutative domains.

$$\mathcal{O}_H/\mathfrak{m} \subset \bar{B}/(\mathfrak{m}^\dagger \cap \bar{B}) \subset \mathcal{O}_{H^\dagger}/\bar{\mathfrak{M}}.$$

Note that $\mathcal{O}_{H^\dagger}/\bar{\mathfrak{M}}$ is a finite extension field of the field $\mathcal{O}_H/\mathfrak{m}$. We conclude that the middle side of the above inclusion is also a field, and that $\mathfrak{m}^\dagger \cap \bar{B}$ is a maximal ideal of \bar{B} . We thus have $\mathfrak{m}^\dagger \cap \bar{B} = \bar{\mathfrak{M}}$. \square

5.5. Behavior of $B_{\bar{X}}$ on loci where $B_{\bar{X}}$ is Auslander regular. The following lemma, which is essentially a consequence of the theorem of Ramras (Theorem 3.9), is an central heart of our next result.

Lemma 5.13. *Let B be a ring Zariski locally isomorphic to a Weyl algebra $A = A_n(\mathbb{k})$ over a field of positive characteristic p . We equip $B_{\bar{X}}$ with the filtration $\{B_{\bar{X}}\langle -k \rangle\}_{k=0}^\infty$. Let U be a Zariski open subset of \bar{X} . We assume $B_{\bar{X}}$ is Auslander regular on U . Then the following facts are true on U :*

- (1) $B_{\bar{X}}$ is locally free as an $\mathcal{O}_{\bar{X}}$ -module.
- (2) $\mathrm{gr}_0(B/\mathcal{I}_H B) = \bar{B}_H = B/\mathfrak{J}_{H^\dagger}$ is isomorphic to \mathcal{O}_{H^\dagger} as an \mathcal{O}_H -algebra.
- (3) For each k , $\mathrm{gr}_k(B/\mathcal{I}_H B)$ is a locally free \mathcal{O}_{H^\dagger} -module of rank one.

Proof. (1): Let \mathfrak{m} be a closed point of U . As we proved in Proposition 5.12, there exists only one maximal ideal \mathfrak{M} of $B_{\bar{X}}$ which lies over \mathfrak{m} . The \mathfrak{M} -adic completion $\hat{B}_{\bar{X}, \mathfrak{M}} = \varprojlim_j B_{\bar{X}}/\mathfrak{M}^j$ of $B_{\bar{X}}$ is isomorphic to $B_{\bar{X}, \mathfrak{m}} \otimes_{\mathcal{O}_{\bar{X}, \mathfrak{m}}} \hat{\mathcal{O}}_{\bar{X}, \mathfrak{m}}$. It is a ring which is finite over a central local ring $\hat{\mathcal{O}}_{\bar{X}, \mathfrak{m}}$. By the theorem of Ramras (Theorem 3.9), we see that $\hat{B}_{\bar{X}, \mathfrak{M}}$ is free over $\hat{R}_{\mathfrak{m}}$. Thus $B_{\bar{X}}$ is locally free on U .

Let us prove (2). This is done in several steps.

Let us first take a look at a sheaf $\mathcal{M} = B_{\bar{X}}\langle -(p-1) \rangle / \mathcal{I}_H B_{\bar{X}}$. We note that the norm map $N_B : B_{\bar{X}} \rightarrow \mathcal{O}_{\bar{X}}$ factors through a map

$$\bar{N}_B : B_{\bar{X}}/\mathcal{I}_H B_{\bar{X}} \rightarrow \mathcal{O}_{\bar{X}}/\mathcal{I}_H^{p^n}$$

so that the sheaf \mathcal{M} may be rewritten as follows.

$$\mathcal{M} = \{s \in B_{\bar{X}}/\mathcal{I}_H B_{\bar{X}} \mid \mathrm{ord}_{H^\dagger}(\bar{N}_B(s)) \geq (p-1)p^n\}$$

Since the sheaf $B_{\bar{X}}/\mathcal{I}_H B_{\bar{X}}$ is locally free, by checking the normality and torsion freeness, we see that \mathcal{M} is a reflexive $\mathcal{O}_{\bar{X}}$ module.

Let us then focus on the action α of $\bar{B}_H = B_{\bar{X}}/\mathfrak{J}_{H^\dagger}$ on the sheaf \mathcal{M} :

$$\alpha : \bar{B}_H \otimes \mathcal{M} \rightarrow \mathcal{M}$$

Note that the reflexive extension of \bar{B}_H as a \mathcal{O}_H -module is equal to \mathcal{O}_{H^\dagger} (Proposition 5.11). For any $f \in \mathcal{O}_{H^\dagger}(U \cap H)$, the action α gives a local section $\alpha(f)$ outside of a closed subset of $U \cap H$ of codimension at least 2. Then by virtue of the \mathcal{O}_H -reflexivity of \mathcal{M} , $\alpha(f)$ extends to the whole of $U \cap H$. In this way, We may extend the action α to an action of \mathcal{O}_{H^\dagger} on \mathcal{M} . So \mathcal{M} is now an \mathcal{O}_{H^\dagger} module. Knowing that \mathcal{M} is normal and torsion free as an \mathcal{O}_H -module, (and that these conditions are irrelevant of whether we regard it as \mathcal{O}_H -module or \mathcal{O}_{H^\dagger} -module,) we see that \mathcal{M} is a reflexive \mathcal{O}_{H^\dagger} -module of rank one. As we reviewed in Proposition 5.4, this implies that the sheaf \mathcal{M} is a locally free \mathcal{O}_{H^\dagger} -module.

Let us now take a point P on H and argue locally around P . There exists an affine open subset V of U and a section $s_1 \in B_{\bar{X}}(U)$ such that its residue \bar{s}_1 generates \mathcal{M} as an \mathcal{O}_{H^\dagger} -module. By shrinking V if necessary, we may assume that the section s_1 generates $B_{\bar{X}}\langle -(p-1) \rangle$ on V . The norm $N(\mathbf{m}_1)$ restricted to V has zeros only on H . We may also assume that \mathcal{I}_H is generated by a single element z (“a local defining function of H) on V . Let us consider a map φ defined as follows.

$$\varphi : \bar{B}_H \ni f \mapsto f \cdot s_1 \in \mathcal{M}.$$

Then the kernel of φ is equal to $\mathfrak{J}_{H^\dagger}/\mathcal{I}_H B_{\bar{X}}$. (This again is firstly trivially verified outside a locus of codimension greater than 2 and then we may extend the fact to the whole of H by using the reflexivity of \mathfrak{J}_{H^\dagger} .) Let us now prove that φ is surjective. $N(s_1)/z^{p^{n-1}(p-1)}$ is invertible on V . Thus $N(s_1)/z^{p^{n-1}(p-1)}$ is invertible on a neighborhood V_0 of P . Let $s \in \mathcal{M}_P$ be an arbitrary element of the stalk at P . There exists $m_s \in B_{\bar{X},P}$ such that $s = m_s \bmod \mathcal{I}_H$. By definition, m_s is a germ of a section (also denoted by m_s) of $B_{\bar{X}}$ on a Zariski open neighborhood V_1 of P . Let us then consider a section

$$b_s = m_s s_1^{-1}.$$

It is regular on $\mathbb{C}H \cap V_0 \cap V_1$. By looking at norms, and using the definition of norm based extensions, we see that b_s defines a regular section of $B_{\bar{X}}$ on $V_0 \cap V_1$. We see immediately that $\varphi(b_s) = s$. We thus conclude $B_{\bar{X}}/\mathfrak{J}_{H^\dagger}$ is locally free \mathcal{O}_{H^\dagger} -module.

Let us prove (3). By choosing a suitable products of s_1 and z , we may create several sections of $B_{\bar{X}}$. Indeed, for each $i \in \mathbb{Z}$, let us define a section $m_{(i)}$ of $B_{\bar{X}}\langle i \rangle$ as follows.

$$m_{(i)} = s_1^i / z^i$$

Then by using an similar argument as above we see that $m_{(i)}$ is a section of $B_{\bar{X}}\langle i \rangle$ and that it generates $B_{\bar{X}}\langle i \rangle$ as an \mathcal{O}_{H^\dagger} -module on $V_0 \cap V_1$. \square

5.6. A characterization of Weyl algebras using Auslander regularity. We now state the second main result of this paper.

Theorem 5.14. *Let $A = A_n(\mathbb{k})$ be the Weyl algebra over a field \mathbb{k} of positive characteristic p . We assume that the dimension $2n$ of the base space $X = \text{Spec}(Z(A))$ is greater than 2 (i.e. $n \geq 2$). Let B be an algebra which is Zariski locally isomorphic to A in the sense that B is an $Z(A)$ -algebra whose associated sheaf B_X on X is Zariski locally isomorphic to A_X . Then the norm based extension $B_{\bar{X}}$ of B_X is locally Auslander regular if and only if B is isomorphic to A as a $Z(A)$ -algebra.*

Proof. The “if” part is a consequence of Proposition 4.3. Let us prove the “only if” part. We assume that $B_{\bar{X}}$ is locally Auslander regular. We first consider the sheaf of algebras $B_{\bar{X}}/\mathcal{I}_H B_{\bar{X}}$ which has a filtration given by $\{B_{\bar{X}}\langle k \rangle/\mathcal{I}_H B_{\bar{X}}\}_{k=0}^{p-1}$. As we have seen in Lemma 5.13, for each $k \in \{0, 1, 2, \dots, p-1\}$, the sheaf $\text{gr}_k(B_{\bar{X}}/\mathcal{I}_H B_{\bar{X}})$ is a locally free \mathcal{O}_{H^\dagger} module of rank one. According to a theorem of Hartshorne ([8, corollary 6.4]), it is isomorphic to a direct sum of invertible \mathcal{O}_H -modules. Since the extension groups $\text{Ext}_{\mathcal{O}_H}^1(\mathcal{L}_1, \mathcal{L}_2)$ for any invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$ on $H \cong \mathbb{P}^{2n-1}$ which are involved are all equal to zero, we see that the whole of $B_{\bar{X}}/\mathcal{I}_H B_{\bar{X}}$ is isomorphic to a direct sum of invertible \mathcal{O}_H -module. Then by using the theorem of Horrocks (Theorem 5.6, we see that $B_{\bar{X}}$ is isomorphic to a direct sum of invertible $\mathcal{O}_{\bar{X}}$ -modules. Namely, we have an isomorphism

$$B_{\bar{X}} \cong \bigoplus_j \mathcal{O}_{\bar{X}}(c_j) \quad (\exists c_1, c_2, \dots \in \mathbb{Z})$$

as $\mathcal{O}_{\bar{X}}$ -modules. The numbers c_j are identical with the ones which appear in the splitting of $\bigoplus_{k=0}^{p-1} \mathcal{O}_{H^\dagger}(-kH^\dagger)$ over \mathcal{O}_H and are in particular independent of B . We thus have

$$\dim H^0(\bar{X}, B_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(kH)) = \dim H^0(\bar{X}, A_{\bar{X}} \otimes \mathcal{O}_{\bar{X}}(kH))$$

for any integer k . Furthermore, we know by using Theorem 5.7 that there exists an $\mathcal{O}_{\bar{X}}$ -module isomorphism Ψ such that the following diagram commutes.

$$\begin{array}{ccccc} B_{\bar{X}}\langle 1 \rangle & \longrightarrow & B_{\bar{X}}\langle 1 \rangle / B_{\bar{X}} & \xrightarrow{\cong} & \mathcal{O}_{H^\dagger}(1) \\ \downarrow \Psi & & & \nearrow \cong & \\ A_{\bar{X}}\langle 1 \rangle & \longrightarrow & A_{\bar{X}}\langle 1 \rangle / A_{\bar{X}} & & \end{array}$$

By a direct calculation (or by using the direct sum decomposition of $A_{\bar{X}}\langle 1 \rangle$), we see that $\Gamma(\bar{X}, A_{\bar{X}}\langle 1 \rangle)$ is a vector space with a basis

$1, \gamma_1, \gamma_2, \gamma_3, \dots, \gamma_{2n}$. Accordingly, $\Gamma(\bar{X}, B_{\bar{X}}\langle 1 \rangle)$ is a vector space with a basis $1, \Psi(\gamma_1), \Psi(\gamma_2), \Psi(\gamma_3), \dots, \Psi(\gamma_{2n})$.

For each i, j , commutator $[\beta_i, \beta_j]$ is apparently a section of $H^0(\bar{X}, B_{\bar{X}}\langle 2 \rangle)$.

By using the ‘‘asymptotic commutativity’’, we see that it is actually a section of $H^0(\bar{X}, B_{\bar{X}})$. Since $H^0(\bar{X}, A_{\bar{X}}) = \mathbb{k}$, we have $H^0(\bar{X}, B_{\bar{X}}) = \mathbb{k}$ so we conclude that there exist some constants $c_{ij} \in \mathbb{k}$ such that

$$[\beta_i, \beta_j] = c_{ij}$$

holds.

It is also easy to see that $\{\beta_i\}$ generates B . Then it follows that $\{\beta_i^p\}$ are in the center of B . $Z(B) = \mathbb{k}[\beta_1^p, \dots, \beta_{2n}^p]$. It also follows that $\{c_{ij}\}$ is non-degenerate. β_i^p are sections of $\mathcal{O}_{\bar{X}}(H)$. By looking at the behavior at H , we see

$$\beta_i^p = \gamma_i^p.$$

By a suitable linear change of coordinates we obtain a section $\tilde{\beta}_i$ of B such that

$$[\tilde{\beta}_i, \tilde{\beta}_j] = h_{ij}$$

holds. The shadow map should be symplectic. So we have $c_{ij} = h_{ij}$. In other words, B is isomorphic to A .

□

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