Diversified homotopic behavior of closed orbits of some **R**-covered Anosov flows

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Abstract − We produce infinitely many examples of Anosov flows in closed 3-manifolds where the set of periodic orbits is partitioned into two infinite subsets. In one subset every closed orbit is freely homotopic to infinitely other closed orbits of the flow. In the other subset every closed orbit is freely homotopic to only one other closed orbit. The examples are obtained by Dehn surgery on geodesic flows. The manifolds are toroidal and have Seifert pieces and atoroidal pieces in their torus decompositions.

1   Introduction

This article deals with the question of free homotopies of closed orbits of Anosov flows [An] in 3-manifolds. In particular we deal with the following question: how many closed orbits are freely homotopic to a given closed orbit of the flow? Suspension Anosov flows have the property that an arbitrary closed orbit is not freely homotopic to any other closed orbit. Geodesic flows have the property that every closed orbit (which corresponds to a geodesic in the surface) is only freely homotopic to one other closed orbit. The other orbit corresponds to the same geodesic in the surface, but being traversed in the opposite direction. About twenty years ago the author proved that there is an infinite class of Anosov flows in closed hyperbolic 3-manifolds satisfying the property that every closed orbit is freely homotopic to infinitely many other closed orbits [Fe1]. Obviously this was diametrically opposite to the behavior of the previous two examples and it was also quite unexpected. This property in these examples is strongly connected with the large scale properties of Anosov flows when lifted to the universal cover. In particular in these examples the property of infinitely orbits which are freely homotopic to each other implies that the flows are not quasigeodesic, that is, orbits are not uniformly efficient in measuring length in the universal cover [Fe1]. This provided the first examples of Anosov flows in hyperbolic 3-manifolds which are not quasigeodesic. The analysis of freely homotopic closed orbits of Anosov flows is also extremely important in other situations, for example: 1) Analysing the interaction between incompressible tori in the manifold and the Anosov flow [Barb-Fe], 2) Studying the structure of the Anosov flow when "restricted" to a Seifert piece of the torus decomposition of the manifold [Barb-Fe].

We define the *free homotopy class* of a closed orbit of a flow to be the collection of closed orbits which are freely homotopic to the original closed orbit. For an Anosov flow each free homotopy class is at most infinite countable as there are only countably many closed orbits of the flow [An]. In this article we are concerned with the cardinality of free homotopy classes. Suspensions have all free homotopy classes with cardinality one and geodesic flows have all free homotopy classes with cardinality two. In the hyperbolic examples mentioned above every free homotopy class has infinite cardinality. In addition if we do finite covers of geodesic flows, where we "unroll the fiber direction", then we can get examples satisfying the property that every free homotopy class has cardinality $2n$ where n is a positive integer. The question we ask is whether we can have mixed behavior for an Anosov flow. In other words, can some free homotopy classes be infinite while others have finite cardinality? In this article we produce infinitely many examples where this indeed occurs.

Main theorem − There are infinitely many examples of Anosov flows Φ in closed 3-manifolds so that the set of closed orbits is partitioned in two infinite subsets A and B so that the following happens. Every closed orbit in A has infinite free homotopy class. Every closed orbit in B has free homotopy class of cardinality two. The examples are obtained by Dehn surgery on closed orbit of geodesic flows.

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2   Previous results and definitions

A three manifold M is *irreducible* if every sphere bounds a ball [He]. An *incompressible* torus is the image of an embedding $f : T^2 \to M$ which does not have compressing disks [He]. A 3-manifold is *homotopically atoroidal* if every π_1-injective map $f : T^2 \to M$ is homotopic into the boundary. The manifold is *geometrically atoroidal* if every incompressible torus is homotopic to the boundary. A 3-manifold M is *Seifert fibered* if it has a 1-dimensional foliation by circles [He, Ep]. The *torus decomposition* states that every compact, irreducible 3-manifold M can be decomposed by finitely incompressible tori $T_1, ..., T_k$ so that the closure of every component of $(M - \cup T_k)$ is either Seifert fibered or atoroidal [Ja, Ja-Sh]. A *graph manifold* is an irreducible 3-manifold whose pieces of the torus decomposition are all Seifert fibered. The base space of a Seifert fibered space is the quotient of M by the Seifert fibration. It is a 2-dimensional orbifold with finitely many cone points. The Seifert space is called *small* if the base is either the disk with less than 3 cone points or the sphere with less than 4 cone points.

If M closed admits an Anosov flow then M is irreducible [Ro]. But M may be toroidal, for example this happens in the case of geodesic flows or suspensions. In the case of geodesic flows the whole manifold is Seifert fibered.

2.1   Anosov flows and R-covered Anosov flows

Let Φ be an Anosov flow in M^3 and let Λ^s, Λ^u be its stable and unstable foliations respectively. The leaves of Λ^s, Λ^u can only be planes, annuli and Möbius bands [An]. A leaf L of Λ^s or Λ^u is an annulus or Möbius band if and only L contains a closed orbit of Φ [An]. Let $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ be the lifted foliations to the universal cover $\widetilde{M}$. Let F be a leaf of $\widetilde{\Lambda}^s$ or $\widetilde{\Lambda}^u$. By the above, the leaf F has non trivial stabilizer if and only if $\pi(F)$ has a closed orbit of Φ. Here the map $\pi : \widetilde{M} \to M$ is the universal covering map. The stabilizer of F is $\{g \in \pi_1(M) \mid g(F) = F\}$.

Theorem 2.1. *([Fe2, Fe3]) Suppose that Φ is an Anosov flow in M^3 and suppose that α and β are closed orbits of Φ which are freely homotopic to each other, as oriented curves. Then there is γ periodic orbit of Φ so that α is freely homotopic to γ^{-1} as oriented curves.*

For an arbitrary Anosov flow Φ the orbit space $\mathcal{O}$ of the lifted flow $\widetilde{\Phi}$ to the universal cover is homeomorphic to the plane $\mathbf{R}^2$ [Fe1]. The lifted foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ are invariant by the flow $\widetilde{\Phi}$ so they induce one dimensional foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O}$.

Definition 2.2. *A foliation $\mathcal{F}$ in M is $\mathbf{R}$-covered if the leaf space of the lifted foliation $\widetilde{\mathcal{F}}$ to the universal cover $\widetilde{M}$ is homeomorphic to the real numbers $\mathbf{R}$ [Fe1].*

Definition 2.3. *An Anosov flow is $\mathbf{R}$-covered if its stable foliation (or equivalently its unstable foliation [Ba1, Fe1]) is $\mathbf{R}$-covered.*

Examples of $\mathbf{R}$-covered Anosov flows are suspensions and geodesic flows [Fe1]. In [Fe1] the author showed that there is an infinite class of $\mathbf{R}$-covered Anosov flows on hyperbolic 3-manifolds. Barbot [Ba2] proved that the Handel-Thurston Anosov flows [Ha-Th] are $\mathbf{R}$-covered, as well as an infinite class of Anosov flows in graph manifolds. The Anosov flows in the Main theorem are $\mathbf{R}$-covered. On the other hand the class of non $\mathbf{R}$-covered Anosov flows is extremely large. For example Barbot [Ba1] proved that $\mathbf{R}$-covered Anosov flows are transitive. Hence the intransitive Anosov flows constructed by Franks and

Williams [Fr-Wi] are not **R**-covered. In addition the Bonatti and Langevin examples [Bo-La] are transitive Anosov flows which are not covered. Barbot [Ba2] constructed many other examples of transitive Anosov flows in graph manifolds and these examples were greatly generalized in [Barb-Fe]. In all of these examples the underlying manifold is toroidal and consequently not hyperbolic.

2.2   R-covered Anosov flows

Suppose that Φ is an **R**-covered Anosov flow. Then there are two possibilities for the topological structure of the lifted stable and unstable foliations $\widetilde{\Lambda}^s, \widetilde{\Lambda}^u$ to $\widetilde{M}$ [Ba1, Fe1] or equivalently for the topological structure of the foliations $\mathcal{O}^s, \mathcal{O}^u$ in $\mathcal{O} \cong \mathbf{R}^2$.

- Suppose that every leaf of $\widetilde{\Lambda}^s$ intersects every leaf of $\widetilde{\Lambda}^u$. In this case Φ is said to have the *product* type. Then Barbot [Ba1] showed that Φ is topologically equivalent to a suspension Anosov flow. This implies that M fibers over the circle with fiber a torus.

- The other possibility is that Φ is a *skewed* **R**-covered Anosov flow. This means that $\mathcal{O}$ has a model homeomorphic to an infinite strip $(0, 1) \times \mathbf{R}$. In addition the model satisfies the following properties. The stable foliation $\mathcal{O}^s$ in $\mathcal{O}$ is a foliation by horizontal segments in $(0, 1) \times \mathbf{R}$. The unstable foliation is a foliation by parallel segments in $(0, 1) \times \mathbf{R}$ which make and angle θ which is not $\pi/2$ with the horizontal. That is, they are not vertical and hence an unstable leaf does not intersect every stable leaf and vice versa. We refer to figure 1. Notice that every unstable leaf u of $\mathcal{O}^u$ intersects an interval J_u of stable leaves of $\mathcal{O}^s$. This is a strict subset of the leaf space of $\mathcal{O}^s$. The model implies that different leaves u of $\mathcal{O}^u$ generate different intervals J_u.

In [Fe3] the author proved that if Φ is a skewed **R**-covered Anosov flow then the underlying manifold is orientable. With this one can also produce infinitely many examples of transitive non **R**-covered Anosov flows where the underlying manifold is hyperbolic.

Proposition 2.4. *Proposition 3.1 of [Ba-Fe] Let γ be a closed orbit of an Anosov flow Φ. Then any lift $\widetilde{\gamma}$ of γ to $\widetilde{M}$ is unknotted. In particular $\pi_1(\widetilde{M} - \widetilde{\gamma})$ is $\mathbf{Z}$ and $\widetilde{M} - \widetilde{\gamma}$ is an open solid torus.*

2.3   Dehn surgery and Fried's flow Dehn surgery

Dehn filling - Let N be a 3-manifold which is the interior of a compact 3-manifold $\hat{N}$ so that the boundary of $\hat{N}$ is a union of tori $P_1, ..., P_k$. Choose simple closed curve generators a_i, b_i for each $\pi_1(P_i)$. Let $N_{(x_1,y_1),...,(x_k,y_k)}$ be the manifold obtained by *Dehn filling* N (or more specifically $\hat{N}$) with k solid tori V_i as follows: for each i glue the boundary of the solid torus V_i to P_i by a homeomorphism so that the meridian in V_i is glued to the curve $x_i a_i + y_i b_i$ in P_i. We require that the pair of integers x_i, y_i are relatively prime to ensure that $x_i a_i + y_i b_i$ is a simple closed curve in P_i. The topological type of the Dehn filled manifold is completely determined by the collection of pairs of integers (x_i, y_i). They are called the Dehn surgery coefficients.

Dehn surgery - Let M be a closed 3-manifold and γ an orientation preserving simple closed curve in M. Let $N(\gamma)$ be a solid torus neighborhood of γ and $M' = M - \mathring{N}(\gamma)$. Choose a pair of generators for $\partial N(\gamma) \subset \partial M'$. Then *Dehn surgery* on γ with coefficients x, y is the Dehn filled manifold $M'_{(x,y)}$.

Remarks 1) More generally one can do Dehn surgery on γ if M has boundary, or is not compact. 2) Dehn surgery is very general: any closed orientable 3-manifold can be obtained from the 3-sphere by iterated Dehn surgery [Li].

Hyperbolic Dehn surgery [Th1, Th2, Be-Pe] - Let M be a complete hyperbolic manifold with finite volume and not compact. Then M is homeomorphic to the interior of a compact manifold $\hat{M}$ with boundary components $P_1, ..., P_k$ which we assume are tori. Each such torus corresponds to a cusp in M.

As above one can do Dehn filling to obtain the closed manifold $M_{(x_1,y_1),...,(x_k,y_k)}$. Thurston proved except for finitely many choices of the integers x_i, y_i the manifold $M_{(x_1,y_1),...,(x_k,y_k)}$ admits a hyperbolic structure. If one fills only one of the cusps, or a subset of the cusps, then for big enough surgery coefficients, the resulting manifold is hyperbolic but not closed, it still has some cusps. The reference [Th1] has a very extensive and detailed proof of this in the case of the figure eight knot complement in the sphere $\mathbf{S}^3$. The book [Be-Pe] has a proof in the general case.

Fried's flow Dehn surgery [Fr] - Suppose that Φ is an Anosov flow and that γ is a closed orbit which is orientation preserving in M, and so that the stable leaf $\Lambda^s(\gamma)$ of γ is an annulus (as opposed to a Möbius band). Let $N(\gamma)$ be a solid torus neighborhood of γ. Choose generators for $\pi_1(P)$ where $P = \partial(N(\gamma))$ as follows. Let a be a meridian in $N(\gamma)$ − unique up to inverse in $\pi_1(P)$. Let b be the intersection of the local stable leaf of γ with $\partial N(\gamma)$. Choose the orientation in b to be the one induced by the positive flow direction in γ. This is the "longitude" in $\partial N(\gamma)$ in this case. Do $(1,n)$ Dehn surgery on γ. Fried [Fr] showed how to do this along the flow: blow up the orbit γ (producing a boundary torus). Then blow down this boundary torus to a closed curve according to the surgery coefficients $(1,n)$. The resulting flow is still an Anosov flow and is denoted by $\Phi_{(1,n)}$. The orbit γ blows up to a torus and then blows down to an orbit of $\Phi_{(1,n)}$. With this construction notice that there is a bijection between the orbits of Φ and the orbits of the surgered flow $\Phi_{(1,n)}$.

Theorem 2.5. *(Fe1) Let Φ be an $\mathbf{R}$-covered Anosov flow in M^3 of skewed type. Let γ be a closed orbit so that $\Lambda^s(\gamma)$ is an annulus. Then under a positivity condition, every Anosov flow $\Phi_{(1,n)}$ is $\mathbf{R}$-covered and of skewed type. Under appropriate choices of the basis of $\pi_1(N(\gamma))$ the positivity condition is satisfied for all positive n. In particular this is true if Φ is the geodesic flow in T_1S where S is a closed hyperbolic surface.*

In general the positivity condition is satisfied either for all positive n or for all negative n [Fe1]. The issue is that the meridian is well undefined up to inverse. Hence there are two possibilities for the basis, and one of them satisfies the positivity condition for every positive n. As it is not needed in this article we do not specify exactly when the positivity condition holds.

Theorem 2.6. *([Fe1]) Let $M = T_1S$ where S is a closed, orientable hyperbolic surface and let Φ be the geodesic flow in M. Let γ be a closed geodesic in S which fills S. Let γ_1 be a periodic orbit of Φ which projects to γ in S. Do $(1,n)$ Dehn surgery on γ_1 satisfying the positivity condition to generate manifold M_s and Anosov flow Φ_s. By theorem 2.5 Φ_s is $\mathbf{R}$-covered. Suppose that n is big enough so that M_s is hyperbolic. Then every closed orbit of Φ is freely homotopic to infinitely many other closed orbits.*

3   Atoroidal submanifolds of unit tangent bundles of surfaces

Let S be a closed hyperbolic surface and let $M = T_1S$ be the unit tangent bundle of S. Notice that M is orientable, whether S is orientable or not. In the next section we will do Dehn surgery on a closed orbit of the geodesic flow to obtain the examples of flows for our Main theorem. We use the following notation to denote the projection map

$$\tau: \ M = T_1S \ \to \ S$$

which is the projection of a unit tangent vector to its basepoint in S.

Let Φ be the geodesic flow in M. It is well known that Φ is an Anosov flow [An]. Let α be a closed geodesic in S. This geodesic of S generates two orbits of Φ, let α_1 be one such orbit. This is equivalent to picking an orientation along α. Let S_1 be a subsurface of S that α fills. If α is simple then S_1 is an annulus. If α fills S then $S_1 = S$. Let S_2 be the closure in S of $S - S_1$ − which may be empty. Let $M_i = T_1S_i, \ i = 1, 2$. Notice that both M_1 and M_2 are Seifert fibered (with boundary).

The purpose of this section is to prove the following result.

Proposition 3.1. *(atoroidal) The submanifold $M_1 - \alpha_1$ is homotopically atoroidal.*

Proof. We will prove that $M_1 - \alpha_1$ is geometrically atoroidal. This statement means the following: notice that $M_1 - \alpha_1$ is not compact, but $M_1 - \overset{\circ}{N}(\alpha_1)$ is compact. The statement means that $M_1 - \overset{\circ}{N}(\alpha_1)$ is homotopically atoroidal. Notice that $M_1 - \alpha_1$ is irreducible [He]. Gabai [Ga] showed that since $M_1 - \alpha_1$ is not a small Seifert fibered space then $M_1 - \alpha_1$ is also homotopically atoroidal.

Let T be an incompressible torus in $M_1 - \alpha_1$. We think of T as contained in M. There are 2 possibilities:

<u>Case 1</u> $- T$ is π_1-injective in M.

Here T is contained in $M - M_2$. We use that M is Seifert fibered. In addition T is incompressible, so it is an essential lamination in M [Ga-Oe]. By Brittenham's theorem [Br] T is isotopic to either a vertical torus or a horizontal torus in M. Vertical torus means it is a union of S^1 fibers of the Seifert fibration. Horizontal torus means that it is transverse to these fibers. Since S is a hyperbolic surface, there is no horizontal torus in $M = T_1 S$. It follows that T is isotopic to a vertical torus T'. In addition since T itself is disjoint from M_2 and M_2 is saturated by the Seifert fibration, we can push the isotopy away from M_2 and suppose it is contained in M_1. Finally the isotopy forces an isotopy of the orbit α_1 into a curve α' disjoint from T'. This isotopy projects by τ to an homotopy in S from α to a curve α^* and the image of this homotopy is contained in S_1, since the isotopy in M has image in M_1. The curve α^* is disjoint from the projection $\tau(T')$. Since T' is vertical this projection is a simple closed curve β in S_1. Since α fills S_1, α^* is homotopic to α in S, and β is disjoint from α^*, it now follows that β is a peripheral curve in S_1. By another isotopy we can assume that β does not intersect α or that T' does not intersect α_1.

In addition the isotopy from T to T' can be extended to an isotopy from M_1 to itself. The geometric intersection number of T, T' with α_1 is zero. So we can adjust the isotopy so that the images of T under the isotopy never intersect α_1, and consequently we can further adjust it so that it leaves α_1 fixed pointwise. In other words this induces an isotopy in $M_1 - \alpha_1$ from T to T'. This shows that T is peripheral in $M_1 - \alpha_1$. This finishes the proof in this case.

<u>Case 2</u> $- T$ is not π_1-injective in M.

In particular since T is two sided (as M is orientable), then T is compressible [He]. This means that there is a closed disk D which compresses T [He], chapter 6. Since T is incompressible in $M_1 - \alpha_1$, then D intersects α_1. Let D_1, D_2 be parallel isotopic copies of D very near D which also are compressing disks for T. Then D_1, D_2 intersect T in two curves which partition T into two annuli. One annulus is very near both D_1 and D_2, we call it A_1, let A be the other annulus which is almost all of T. Then $A \cup D_1 \cup D_2$ is an embedded two dimensional sphere W. Since M is irreducible then W bounds a 3-ball B. There are two possibilities for the sphere W and ball B. In addition $A_2 \cup D_1 \cup D_2$ also obviously bounds a ball B_1 which is very near the disk D.

Suppose first that the ball B contains the torus T. This means that A_2 and consequently also B_1, are both contained in B. In addition B_1 is a regular tubular neighborhood of a properly embedded arc γ in B. The intersection of α_1 with B_1 is a collection of arcs which are isotopic to the core γ of B_1. Let δ_1 be one such arc. By Proposition 2.4 flow lines of Anosov flows lift to unknotted curves in $\widetilde{M}$. This implies that γ is unknotted in B and also implies that $\pi_1(B - \gamma)$ is **Z**. In particular the torus T is compressible in $B - B_1$, that is, the closure of $B - B_1$ is a solid torus. It follows that the T is compressible in $M_1 - \alpha_1$. This contradicts the assumption that T is incompressible in $M_1 - \alpha_1$.

The second possibility is that the ball B does not contain T. In particular B and B_1 have disjoint interiors and the the union $B \cup B_1$ is a solid torus V with boundary T. The union of B and B_1 cannot be a solid Klein bottle because M is orientable. This solid torus lifts to an infinite solid tube $\widetilde{V}$ in $\widetilde{M}$ with boundary $\widetilde{T}$ which is an infinite cylinder. Notice that there is a lift $\widetilde{\alpha}_1$ of α_1 contained in $\widetilde{V}$ so $\widetilde{T}$ cannot be compact. Again by the result of Proposition 2.4, the infinite curve $\widetilde{\alpha}_1$ is unknotted in $\widetilde{M}$ and hence it is isotopic to the core of $\widetilde{V}$.

Let β be a simple closed curve in V which is isotopic to the core of V. If α_1 is not isotopic to β then it is homotopic to a power β^n where $n > 1$. Projecting β to $\tau(\beta)$ in S we obtain a closed curve in S so that $(\tau(\beta))^n$ is freely homotopic to α. But α is an <u>indivisible</u> closed geodesic and represents an indivisible element of $\pi_1(S)$. It follows that this cannot happen. We conclude that α_1 is isotopic to the core of V. It follows that T is isotopic to the boundary of a regular neighborhood of α_1 in $M_1 - \alpha$ and hence again T is peripheral in $M_1 - \alpha_1$.

This finishes the proof of proposition 3.1. $\square$

Remark In the case that α fills S Proposition 3.1 is well known and there is a written proof by Foulon and Hasselblatt in [Fo-Ha].

Since $M_1 - \alpha_1$ is atoroidal the geometrization theorem in the Haken case [Th1, Th2] shows that $M_1 - \alpha_1$ admits a hyperbolic structure. The hyperbolic Dehn surgery theorem of Thurston implies that for almost all Dehn fillings along α_1, the resulting manifold M_s is hyperbolic. Notice that since M_1 has boundary, the statement M_s is hyperbolic means that the interior of M_s has a complete hyperbolic structure of finite volume, and each (torus) component of M_1 generates a cusp in the hyperbolic structure in the interior of M_s.

4 Diversified homotopic behavior of closed orbits

First we prove the statements about free homotopy classes of suspension Anosov flows and geodesic flows mentioned in the introduction. Suppose first that Φ is the geodesic flow in $M = T_1 S$, where S is a closed, orientable hyperbolic surface. Suppose that α, β are closed orbits of Φ which are freely homotopic to each other in M. Then the projections $\tau(\alpha), \tau(\beta)$ of these orbits to the surface S are freely homotopic in S. But $\tau(\alpha), \tau(\beta)$ are closed geodesics in a hyperbolic surface, so they are freely homotopic if and only if they are the same geodesic. If α and β are distinct, this can only happen if they represent the same geodesic $\tau(\alpha)$ of S which is being traversed in opposite directions. Conversely if $\tau(\alpha) = \tau(\beta)$ and they are traversed in opposite directions, there is a free homotopy from α to β. This is achieved by considering all unit tangent vectors to $\tau(\alpha)$ in the direction of α and then at time t, $0 \leq t \leq 1$, rotating all these vectors by an angle of $t\pi$. At $t = \pi$ we obtain the tangent vectors to $\tau(\alpha)$ pointing in the opposite direction, that is, the direction of β. This shows that every free homotopic class of the geodesic flow has exactly two elements. The orientability of S is used because if S is not orientable and $\tau(\alpha)$ is an orientation reversing closed geodesic, one cannot continuously turn the angle along $\tau(\alpha)$.

Now consider a suspension Anosov flow Φ. By Theorem 2.1, given an arbitrary Anosov flow which admits freely homotopic closed orbits, then the following happens. There are closed orbits α and β so that α is freely homotopic to β^{-1} as oriented periodic orbits. For suspension Anosov flows this is a problem as follows. This is because there is a cross section W which intersects all orbits of Φ. Suppose that the algebraic intersection number of α and W is positive. Then since α is freely homotopic to β^{-1} it follows that the algebraic intersection number of β and W is negative. But this is impossible as W is a cross section and transverse to Φ. This shows that every free homotopy class of a suspension is a singleton. Another proof of this fact is the following. There is a path metric in M which comes from a Riemannian metric in the universal cover $\widetilde{M} \cong \mathbf{R}^3$ with coordinates (x, y, t) given by the formula $ds^2 = \lambda_1^{2t} dx^2 + \lambda_2^{-2t} dy^2 + dt^2$ (1), where λ_1, λ_2 are real numbers > 1. The lifted flow $\widetilde{\Phi}$ has formula $\widetilde{\Phi}_t(x, y, t_0) = (x, y, t_0 + t)$ (2). If α, β are freely homotopic closed orbits of $\widetilde{\Phi}$, then they lift to two distinct orbits of $\widetilde{\Phi}$ which are a bounded distance from each other. But formulas (1) and (2) show that no two distinct entire orbits of $\widetilde{\Phi}$ are a bounded distance from each other. This also shows that free homotopy classes are singletons.

For the property of infinite free homotopy classes for the examples in hyperbolic 3-manifolds see Theorem 2.6.

We now proceed with the construction of the examples with diversified homotopic behavior and we prove the Main theorem.

Let S be a hyperbolic surface and α a closed geodesic that does not fill S. As in the previous section let S_1 be a subsurface that α fills and let S_2 be the closure of $S - S_1$. Let $M = T_1 S$ and Φ the geodesic flow of S in M. Let α_1 be an orbit of Φ so that $\tau(\alpha_1) = \alpha$. Let $M_i = T_1 S_i$, $i = 1, 2$. In the previous section we proved that $M_1 - \alpha_1$ is atoroidal.

Now we will do Fried's Dehn surgery on α_1. For simplicity we will assume that the unstable foliation of Φ (or equivalently the stable foliation of Φ) is transversely orientable. This is equivalent to the surface S being orientable. In particular this implies that the stable leaf of α_1 is a annulus. Let Z be the boundary of a small tubular solid torus neighborhood Z_0 of α_1 contained in M_1. Then Z is a two dimensional torus and we will choose a base for $\pi_1(Z) = H_1(Z)$. We assume that Z is transverse to the local sheet of the stable leaf of α_1. Then this local sheet intersects Z in a pair of simple closed curves. Each of these defines a longitude $(0, 1)$ in $\pi_1(Z)$, choose the direction which is isotopic to the flow forward direction along α_1. The boundary of a meridian disk in Z_0 defines the meridian curve $(0, 1)$ in $\pi_1(Z)$. The meridian is well defined up to sign. If the stable foliation of Φ were not transversely orientable and α were an orientation reversing curve, then the stable leaf of α would be a Möbius band and the intersection of the local sheet with Z would be a single closed curve. This closed curve would intersect the meridian twice and could not form a basis of $H_1(Z)$ jointly with the meridian. We do not want that, hence one of the reasons to restrict to S orientable.

Now we perform Fried's Dehn surgery on α_1 [Fr] as described in section 2. We do $(1, n)$ surgery on α_1, so that the following happens. The resulting flow is Anosov in the Dehn surgery manifold M_α. The meridian is chosen so that for any $n > 0$ the Dehn surgery flow Φ_α with new meridian the $(1, n)$ curve is an **R**-covered Anosov flow. Recall that there is a bijection between the orbits of the surgered flow Φ_α and the orbits of the original flow Φ. Given an orbit γ of Φ_α we let γ' be the corresponding orbit of Φ under this bijection.

We are now ready to prove the prove the main result of this article, which is restated with more detail below.

Theorem 4.1. *(diversified homotopic behavior) Let S be an orientable, closed hyperbolic surface with a closed geodesic α which does not fill S. Let S_1 be a subsurface of S which is filled by α and let S_2 be the closure of $S - S_1$. We assume also that S_2 is not a union of annuli. Let $M = T_1 S$ with geodesic flow Φ and let $M_i = T_1 S_i$, $i = 1, 2$. Let α_1 be a closed orbit of Φ which projects to α in S. Do $(1, n)$ Fried's Dehn surgery along α_1 to yield a manifold M_α and an Anosov flow Φ_α so that Φ_α is **R**-covered. Since M_2 is disjoint from α_1 it is unaffected by the Dehn surgery and we consider it also as a submanifold of M_α. Let M_3 be the closure of $M_\alpha - M_2$. We still denote by α_1 the orbit of Φ_α corresponding to α_1 orbit of Φ. Proposition 3.1 implies that $M_3 - \alpha_1$ is atoroidal and for n big the hyperbolic Dehn surgery theorem [Th1, Th2] implies that M_3 is hyperbolic. Choose one such n. Consider the bijection $\beta \to \beta'$ between closed orbits of Φ_α and those of Φ. Then the following happens:*

- *i) Let γ be a closed orbit of Φ_α so that the corresponding orbit γ' of Φ is homotopic into the submanifold M_2. Equivalently γ' projects to a geodesic in S which is disjoint from α in S. Then γ is freely homotopic in M_α to just one other closed orbit of Φ_α.*

- *ii) Let γ be a closed orbit of Φ_α which corresponds to a closed orbit γ' of Φ which is not homotopic into M_2. Equivalently γ' projects to a geodesic in S which transversely intersects α. Then γ is freely homotopic in M_α to infinitely many other closed orbits of Φ_α.*

- *In addition both classes i) and ii) have infinitely many elements.*

Proof. First we prove that both classes i) and ii) are infinite. Since orbits of Φ_α are in one to one correspondence with orbits of Φ, one can think of these as statements about closed orbits of Φ. Any closed geodesic of S which intersects α is in class ii). Clearly there are infinitely many such geodesics so class ii) is infinite. On the other hand since S_2 is not a union of annuli, there is a component S' which is
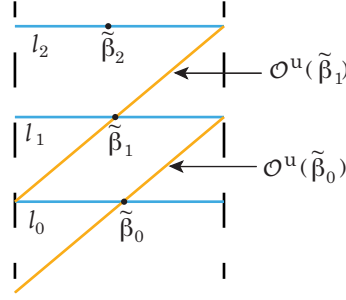
Figure 1: *The picture of the orbit space $\mathcal{O} \cong (0,1) \times \mathbf{R}$ of a skewed $\mathbf{R}$-covered Anosov flow. The stable foliation $\mathcal{O}^s$ is the foliation by horizontal segments in $(0,1) \times \mathbf{R}$. The unstable foliation $\mathcal{O}^u$ is the foliation by the parallel slanted segments. This picture also shows how to construct the orbits $\widetilde{\beta}_i$ starting with the orbit $\widetilde{\beta}_0$.*

not an annulus. Any geodesic β of S which is homotopic into S' creates an orbit in class i). Since S' is not an annulus, there are infinitely many such geodesics β. This proves that i) and ii) are infinite subsets.

An orbit δ of Φ which projects in S to a geodesic intersecting α cannot be homotopic into M_2. Otherwise the homotopy projects in S to an homotopy from a geodesic intersecting α to a curve in S_2 and hence to a geodesic not intersecting α. This is impossible as closed geodesics in hyperbolic surfaces intersect minimally. Conversely if an orbit δ projects to a geodesic not intersecting α, then this geodesic is homotopic to a geodesic contained in S_2. This homotopy lifts to a homotopy in M from δ to a curve in M_2. This proves the equivalence of the first 2 statements in i) and in ii).

Now we prove that conditions i), ii) imply the respective conclusions about the size of the free homotopy classes. Let $\widetilde{\Phi}_\alpha$ be the lifted flow to the universal cover $\widetilde{M}_\alpha$.

The flow Φ_α is $\mathbf{R}$-covered. As explained in section 2 there are two possibilites for Φ_α, either product or skewed. If Φ_α is product then M_α fibers over the circle with fiber a torus. But in our case, M_α has a torus decomposition with one hyperbolic piece M_3 and one Seifert piece M_2. Therefore it cannot fiber over the circle with fiber a torus. We conclude that this case cannot happen.

Therefore Φ_α is skewed.

Let then β_0 be a closed orbit of Φ_α. since Φ_α is an skewed $\mathbf{R}$-covered Anosov flow we will produce orbits β_i, $i \in \mathbf{Z}$ which are all freely homotopic to β_0. However it is not a priori true that all the orbits β_i are distinct from each other, this will be analysed later.

Here we identify the fundamental group of the manifold with the set of covering translations of the universal cover.

Construction of the orbits $\widetilde{\beta}_j$.

Lift β_0 to an orbit $\widetilde{\beta}_0$ contained in a stable leaf l_0 of $\mathcal{O}^s$. Let g be the deck transformation of $\widetilde{M}_\alpha$ which corresponds to β_0 in the sense that it generates the stabilizer of $\widetilde{\beta}_0$. Then $u = \mathcal{O}^u(\widetilde{\beta}_0)$ intersects an open interval J_u of stable leaves. This is a strict subset of the leaf space of $\mathcal{O}^s$ (equal to leaf space of $\widetilde{\Lambda}^s$) by the skewed property. Let l_1 be one of the two stable leaves in the boundary of this interval. The fact that there are exactly two boundary leaves in this interval is a direct consequence of the fact that $\mathcal{O}^s$ (or $\widetilde{\Lambda}^s$) has leaf space $\mathbf{R}$ and this fact is not true in general. Since $g(\mathcal{O}^u(\widetilde{\beta}_0)) = \mathcal{O}^u(\widetilde{\beta}_0)$ and g preserves the orientation of $\mathcal{O}^s$ (because Λ^s is transversely orientable), then $g(l_1) = l_1$. But this implies that there is an orbit $\widetilde{\beta}_1$ of $\widetilde{\Phi}_\alpha$ in l_1 so that $g(\widetilde{\beta}_1) = \widetilde{\beta}_1$. We refer to fig. 1 which shows how to obtain leaf l_1 and hence the orbit $\widetilde{\beta}_1$. This orbit projects to a closed orbit β_1 of Φ_α in M_α. Since both are associated to g, it follows that β_0, β_1 are freely homotopic. More specifically if we care about orientations then the positively oriented orbit β_0 is freely homotopic to the inverse of the positively oriented orbit β_1.

Remark − Transverse orientability of Λ^s is necessary for this. If for example Λ^s were not transversely orientable and the unstable leaf of β_0 were a Möbius band then the transformation g as constructed

above does not preserve the leaf l_1 as constructed above. Therefore β_0 is not freely homotopic to β_1 as unoriented curves. But g^2 preserves l_1 and from this it follows that the square β_0^2 (as a non simple closed curve) is freely homotopic to β_1^2.

We proceed with the construction of freely homotopic orbits of Φ_α. From now on we iterate the procedure above: use $\mathcal{O}^u(\widetilde{\beta}_1)$ to produce a leaf l_2 of $\mathcal{O}^s$ invariant by g, and a closed orbit β_2 freely homotopic to β_1 − if we consider them just as simple closed curves. We again refer to fig. 1. Now iterate and produce $\widetilde{\beta}_i, i \in \mathbf{Z}$ orbits of $\widetilde{\Phi}_\alpha$ so that they are all invariant under g and project to closed orbits β_i of Φ_α which are all freely homotopic to β_0 as unoriented curves.

Orbits freely homotopic to β_0.

The covering translation g preserves the leaf $\mathcal{O}^u(\widetilde{\beta}_0)$. Since β_0 is the only periodic orbit in $\Lambda^u(\beta_0)$, it follows that g only preserves the orbit $\widetilde{\beta}_0$ in $\mathcal{O}^u(\widetilde{\beta}_0)$. Therefore g does not leave invariant any stable leaf between $\mathcal{O}^s(\widetilde{\beta}_0)$ and $\mathcal{O}^s(\widetilde{\beta}_1)$ and similarly g does not leave invariant any stable leaf between $\mathcal{O}^s(\widetilde{\beta}_i)$ and $\mathcal{O}^s(\widetilde{\beta}_{i+1})$ for any $i \in \mathbf{Z}$. It follows that the collection $\{\mathcal{O}^s(\widetilde{\beta}_i), i \in \mathbf{Z}\}$ is exactly the collection of stable leaves left invariant by g.

Suppose now that δ is an orbit of Φ_α which is freely homotopic to β_0. We can lift the free homotopy so that β_0 lifts to $\widetilde{\beta}_0$ and δ lifts to $\widetilde{\delta}$. In particular g leaves invariant $\widetilde{\delta}$ and hence leaves invariant $\mathcal{O}^s(\widetilde{\delta})$. It follows that $\mathcal{O}^s(\widetilde{\delta}) = \mathcal{O}^s(\widetilde{\beta}_i)$ for some $i \in \mathbf{Z}$. As a consequence $\widetilde{\delta} = \widetilde{\beta}_i$ for $\widetilde{\beta}_i$ is the only orbit of $\widetilde{\Phi}_\alpha$ left invariant by g in $\mathcal{O}^s(\widetilde{\beta}_i)$. It follows that δ is one of $\{\beta_j, \ j \in \mathbf{Z}\}$.

<u>Conclusion</u> − The free homotopy class of β_0 is finite if and only if the collection $\{\beta_i, i \in \mathbf{Z}\}$ is finite.

Suppose now that $\beta_i = \beta_j$ for some i, j distinct. Hence there is $f \in \pi_1(M_\alpha)$ with $f(\widetilde{\beta}_i) = \widetilde{\beta}_j$. Then f sends $\mathcal{O}^u(\widetilde{\beta}_i)$ to $\mathcal{O}^u(\widetilde{\beta}_j)$. By the definition of $\widetilde{\beta}_{i+1}$ it follows that f sends $\widetilde{\beta}_{i+1}$ to $\widetilde{\beta}_{j+1}$. Iterating this procedure shows that f preserves the collection $\{\widetilde{\beta}_k, \ k \in \mathbf{Z}\}$. In addition it follows easily that f sends $\widetilde{\beta}_0$ to $\widetilde{\beta}_k$ for $k = j - i$. The free homotopy from β_0 to $\beta_k = \beta_0$ produces a π_1-injective map of either the torus or the Klein bottle into M. We have to consider the Klein bottle because the free homotopy may be from β_0 to the inverse of β_0 when we account for orientations along orbits. Taking the square of this free homotopy if necessary we produce a π_1-injective map of the torus into M. The torus theorem [Ja, Ja-Sh] shows that the free homotopy is homotopic into a Seifert piece of the torus decomposition of M_α. Therefore in our situation the homotopy is freely homotopic into M_2. It follows that the orbit β_0' of Φ associated to β_0 is freely homotopic into M_2. Therefore the geodesic $\tau(\beta_0')$ of S does not intersect α.

This proves part ii) of the theorem: If the geodesic $\tau(\beta_0')$ intersects α then the orbit β_0 of Φ_α is freely homotopic to infinitely many other closed orbits of Φ_α.

Consider now a closed orbit β_0 of Φ_α so that it corresponds to a geodesic in S which does not intersect α. This geodesic is $\tau(\beta_0')$ which we denoted by γ. There is a non trivial free homotopy in $M = T_1 S$ from β_0' to itself with the same orientation, obtained by turning the angle along γ by a full turn, from 0 to 2π. Notice that this free homotopy at some point is exactly β_0' and at another point it is exactly the orbit corresponding to the geodesic γ being traversed in the opposite direction. This free homotopy is entirely contained in M_2 and therefore this free homotopy survives in the Dehn surgered manifold M_α. By construction the image of the free homotopy in M_α contains two distinct closed orbits of Φ_α, one of which is β_0. In particular the free homotopy class of β_0 has at least two elements. In addition the free homotopy produces a π_1-injective map from T^2 into M. Choose a basis g, f for $\pi_1(T^2)$ (seen as covering translations in $\widetilde{M}$) so that g leaves invariant a lift $\widetilde{\beta}_0$ of β_0. Then

$$gf(\widetilde{\beta}_0) \ = \ fg(\widetilde{\beta}_0) \ = \ f(\widetilde{\beta}_0).$$

So g also leaves invariant $f(\widetilde{\beta}_0)$. As seen in the paragraphs "Orbits freely homotopic to β_0", it follows that $f(\widetilde{\beta}_0) = \widetilde{\beta}_j$ for some j in $\mathbf{Z}$. This implies that the free homotopy class of β_0 is finite.

Let now δ be a closed orbit of Φ_α which is freely homotopic to β_0. In particular the free homotopy class of δ is the same as the free homotopy class of β_0 and in the part entitled "Orbits freely homotopic

to β_0" we showed that this free homotopy class is finite in this case. In addition from what we already proved in the theorem, it follows that δ is isotopic into M_2 and choosing M_2 appropriately we can assume that β_0, δ are contained in M_2. Let the free homotopy from β_0 to δ be realized by a π_1-injective annulus A which is in general position. The annulus A is a priori only immersed. Let $T = \partial M_3 = \partial M_2$ an embedded torus in M_α which is π_1-injective. Put A in general position with respect to T and analyse the self intersections. Any component which is null homotopic in T can be homotoped away because M_α is irreducible [He, Ja]. After this is eliminated each component of $A - T$ is an a priori only immersed annulus. But since M_3 is a hyperbolic manifold with a single boundary torus T it follows that M_3 is acylindrical [Th1, Th2]. This means that any π_1-injective properly immersed annulus is homotopic rel boundary into the boundary. This is because parabolic subgroups of the fundamental group of M_3 − as a Kleinian group, have an associated maximal $\mathbf{Z}^2$ parabolic subgroup [Th1, Th2]. In particular this implies that the annulus A can be homotoped away from M_3 to be entirely contained in M_2. Therefore the free homotopy represented by the annulus A survives if we undo the Dehn surgery on α. This produces a free homotopy between β_0' and δ' in $M = T_1S$. But the free homotopy classes of geodesic flows all have exactly two elements. Therefore there is only one possibility for δ if δ is distinct from β_0. This shows that the free homotopy class of β_0 has exactly two elements.

This finishes the proof of theorem 4.1   □

5   Generalizations

There are a few ways to generalize the main result of this article. Here we mention two of them.

1) Finite covers and Dehn surgery

Let $M = T_1S$ where S is a closed orientable surface. Let Φ be the geodesic flow in M. First take a finite cover of order n of M unrolling the circle fibers. Let this be the manifold M_1 with lifted Anosov flow Φ_1. Then every closed orbit of Φ_1 is freely homotopic to $2n - 1$ other closed orbits of the flow. The flow Φ_1 is a skewed $\mathbf{R}$-covered Anosov flow. We use the covering map $\eta : M_1 \to M$ and the projection $\tau : M \to S$. We do Dehn surgery on closed orbits of Φ_1. Essentially the same proof as the Main theorem yields the following result.

Theorem 5.1. *Let S be a closed orientable surface and $M = T_1S$ with Anosov flow Φ. Let M_1 be a finite cover of M where we unroll the Seifert fibers and let Φ_1 be the lifted flow to M_1. Do $(1,n)$ Fried's flow Dehn surgery on a closed orbit γ of the Φ_1 so $\tau \circ \eta(\gamma)$ is a closed geodesic in S which does not fill S and some complementary component of $\tau \circ \eta(\gamma)$ in S is not an annulus and so that the surgery satisfies the positivity condition. Let Φ_s be the resulting Anosov flow in the surgery manifold. Given β a closed orbit of Φ_s, it has an associated unique orbit of Φ_1 which in turn projects under $\tau \circ \eta$ to a closed geodesic β' of S. Then i) If β' does not intersect $\tau \circ \eta(\gamma)$ it follows β is freely homotopic to exactly $2n - 1$ other orbits of Φ_s. In addition ii) If β' intersects $\tau \circ \eta(\gamma)$ then β is freely homotopic to infinitely many other closed orbits of Φ_s.*

2) Dehn surgery on more than one closed orbit

In essentially the same way as in the proof of the Main theorem We obtain a result similar to the Main theorem under Dehn surgery on finitely many closed orbits as follows.

Theorem 5.2. *Let S be a closed orientable surface and $M = T_1S$ with geodesic flow Φ. Let $\{\alpha_i, \ 1 \leq i \leq i_0\}$ be a finite collection of disjoint closed geodesics in S which are pairwise disjoint and some component of the complement of their union is not an annulus. Let γ_i be a closed orbit of Φ which projects to α_i in S. For each i do $(1,n_i)$ Fried's flow Dehn surgery on γ_i to yield an Anosov flow Φ_s in the Dehn surgery manifold M_s and so that the surgery satisfies the positivity condition. Then Φ_1 is an $\mathbf{R}$-covered Anosov flow. There is a bijection between orbits of Φ_1 and orbits of Φ. Given an orbit β of Φ_s consider the orbit of Φ associated to it and project it to a closed geodesic β' in S. Then the following happens. i) If β' is*

disjoint from the union of the $\{\alpha_i\}$ then β is freely homotopic to a single other closed orbit of Φ_s. ii) If β' intersects the union of $\{\alpha_i\}$ then β is freely homotopic to infinitely many other closed orbits of Φ_s.

One can also combine the two constructions above.

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