# Factorization of 4d $\mathcal{N} = 1$ superconformal index

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#### Abstract

We study the factorization of four dimensional  $\mathcal{N} = 1$  superconformal index for U(N)(SU(N)) SQCD with  $N_F$  fundamental and anti-fundamental chiral multiplets. When both the anomaly free R-charge assignment and the traceless condition for total vorticities are satisfied, we find that the superconformal index factorizes to a pair of the elliptic uplift of the vortex partition functions. We also study the relation between open topological string and the the elliptic uplift of the vortex partition functions. In the three dimensional limit, we show index for U(N) theory reduces to the factorized form of the partition function on the three dimensional squashed sphere.

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### 1 Introduction

The localization computation of three dimensional  $\mathcal{N} \geq 3$  supersymmetric theories are performed in [1, 2]. The results are generalized to  $\mathcal{N} = 2$  with general charge assignments [3, 4, 5, 6]. In the localization calculation, the path integrals reduce to the matrix model like multi-contour integrals.

It is revealed in [7] that U(1) gauge theories on the three dimensional squashed sphere  $S_b^3$  possess a remarkable vortex and anti-vortex factorization property by performing contour integral. Such a vortex and anti-vortex factorization is generalized to G = U(N)gauge group in [8, 9]. See also Abelian quiver case [10]. In general, the partition functions of three dimensional  $\mathcal{N} = 2$  theories on  $S^1 \times S^2$  or  $S_b^3$  are believed to be factorized, at least theory has sufficient global symmetries to make the vacua gapped. The fundamental building block for the partition functions for  $\mathcal{N} = 2$  theories in the three dimension is called holomorphic block [11]. The difference between partition function on  $S^1 \times S^2$  and that on  $S_b^3$  is only sewing procedure and holomorphic block is the universal.

It is natural to think that the partition function of  $\mathcal{N} = 1$  theories in four dimensions also possess factorization properties and there also exists four dimensional analog of holomorphic block which becomes fundamental building block for the partition function on  $S^1 \times S^3$  or  $T^2 \times S^2$ . In this article, we present the first evidence for factorization for  $\mathcal{N} = 1$  superconformal index in four dimensions. We consider the U(N) SQCD with  $N_F$ -flavors fundamental chiral multiplets and anti-fundamental chiral multiplets without the superpotential. We perform the contour integrals for the superconformal index and study relation to the vortex and anti-vortex factorization.

This article is organized as follows. In section 2, we introduce  $\mathcal{N} = 1$  the superconformal index in four dimensions. In section 3, we first evaluate the contour integral for the formal superconformal index for the U(1) gauge theory and show that the vortex and anti-vortex factorization only occurs when the anomaly free R-charge charge assignment is satisfied. Next we generalize the calculation to non-Abelian gauge group U(N). In this case, we find that the factorization only occurs when both the anomaly free R-charge charge assignment and traceless condition for total vorticities are satisfied. The vortex partition function for the four dimensional theory becomes elliptic (theta function) uplift of two dimensional vortex partition function. In section 4, we study the open topological string which give the elliptic uplift of vortex partition function. In section 5, we take the three dimensional limit and study the relation between superconformal index and the factorized partition function on  $S_b^3$ . The section 6 is devoted to summary.

# 2 $\mathcal{N} = 1$ superconformal index in four dimensions

The partition function on  $S^1 \times S^3$  with twisted periodic boundary condition along  $S^1$  which respect supersymmetries define the certain BPS index. The index called superconformal index in four dimensions is introduced in [12, 13]. The  $\mathcal{N} = 1$  superconformal index in four dimensions is defined by

$$\mathcal{I} = \operatorname{tr}\left((-1)^{F} e^{D - \frac{3}{2}R - 2J_{L}} s^{2J_{L} + 2J_{R} - \frac{R}{2}} t^{2J_{R} - 2J_{L} - \frac{R}{2}} \prod_{I} z_{I}^{F_{I}}\right).$$
(2.1)

Here D, R,  $J_L(J_R)$  and  $F_I$  are the dilation, the R-charge, the Cartan generators of left(right) SU(2) isometry of  $S^3$  and the flavor charges. The superconformal index counts the BPS operators which saturate the bound  $D - \frac{3}{2}R - 2J_L \ge 0$ . The superconformal index can be expressed in multi-contour representation as

$$\mathcal{I} = \frac{((s;s)_{\infty}(t;t)_{\infty})^r}{|W|} \oint_{\mathbb{T}^r} \prod_{a=1}^r \frac{dx_a}{2\pi i x_a} Z_{\text{vec}}^{1-\text{loop}} Z_{\text{chi}}^{1-\text{loop}}.$$
(2.2)

Here r is the rank of gauge group G.  $Z_{\text{vec}}^{1-\text{loop}}$  is the one-loop determinant of the vector multiplet and  $Z_{\text{chi}}^{1-\text{loop}}$  is the one-loop determinant of the chiral multiplets. They are given

by

$$Z_{\rm vec}^{1-\rm loop} = \prod_{\alpha>0} \theta(e^{i\alpha(y)}; s)\theta(e^{-i\alpha(y)}; t), \qquad (2.3)$$

$$Z_{\rm chi}^{1-\rm loop} = \prod_{\rho \in R} \prod_{I} \Gamma((st)^{\frac{R}{2}} e^{-i\rho(y)} z_{I}^{F_{I}}; s, t), \qquad (2.4)$$

with  $x_a = e^{iy_a}$ . The definition of the theta function  $\theta(x, q)$  and the elliptic gamma function  $\Gamma(x; s, t)$  are summarized in appendix. The integration contours are taken as unit circles.

We can also introduce FI-term for over all U(1) factor of gauge group

$$\mathcal{L}_{\rm FI} = \zeta \operatorname{Tr}(\frac{2i}{r_3}A_4 - D), \qquad (2.5)$$

which only contributes to the saddle point value in the localization calculation. Here  $\zeta$  is the FI-parameter and  $r_3$  is the radius of  $S^3$ .  $A_4$  is the gauge field along the  $S^1$  circle. For simplicity, we omit the FI-term in the later calculation, but it is easy to recover its contribution. We consider the FI-term contribution in section 5; we study the relation between the superconformal index in four dimensions and the partition function in three dimensions.

When the gauge group is G = U(N) and the matter chiral multiplets are the  $N_F$ -flavors with fundamental representation and  $N_F$ -flavors with anti-fundamental representation, the superconformal index is written as

$$\mathcal{I}_{N_{F}}^{U(N)} = \frac{(s;s)^{N}(t;t)^{N}}{N!} \oint_{\mathbb{T}^{N}} \prod_{a=1}^{N} \frac{dx_{a}}{2\pi i} \prod_{a>b} \theta(x_{a}x_{b}^{-1};s)\theta(x_{a}^{-1}x_{b};t)$$
$$\prod_{a=1}^{N} \prod_{I=1}^{N_{F}} \Gamma((st)^{\frac{R}{2}}x_{a}^{-1}z_{I};s,t)\Gamma((st)^{\frac{\tilde{R}}{2}}x_{a}\tilde{z}_{I};s,t).$$
(2.6)

In the next section, we evaluate the above multi-contour integrals and study the relation to the vortex partition function and anti-vortex partition function factorization.

### **3** Factorization of superconformal index

#### 3.1 Abelian case

In this subsection, we consider the Abelian gauge group G = U(1). Since the Abelian gauge theories in the four dimensions is infrared free, the index (2.6) of this theory is the formal object, but it is useful to see the factorization property and to generalize to non-Abelian case. We assume that fugacities are analytically continued to the region  $|pqz_I| < 1$  and the residues are evaluated at the poles of the one-loop determinant of  $N_F$  fundamental chiral multiplets. As in the case of 3d  $\mathcal{N} = 2$  superconformal indices [8], By shifting  $z_I \to z_I(st)^{c_I}$ , we can set R = 0 and  $\tilde{R} = 0$ . In the same reason, we have omitted the Baryonic charges from the beginning. Then we evaluate residues at pole  $x = z_{I'}s^jt^k$ ,  $(j,k \in \mathbb{Z}_{\geq 0}, I = 1, \dots, N_F)$  in the one-loop determinant of the chiral multiplets.

The index is written as

$$\mathcal{I}_{N_F}^{U(1)} = -\sum_{I'=1}^{N_F} \sum_{j,k=0}^{\infty} (s;s)_{\infty}(t;t)_{\infty} \operatorname{Res}_{x=z_{I'}t^jt^k} \prod_{I=1}^{N_F} \Gamma(x^{-1}z_I;s,t)\Gamma(x\tilde{z}_I;s,t).$$
(3.1)

The contributions from one-loop determinant of the fundamental chiral multiplet are given by

$$\prod_{\substack{I=1\\I\neq I'}}^{N_F} \Gamma(x^{-1}z_I;s,t) \Big|_{x=z_{I'}s^{j}t^{k}} = \prod_{\substack{I=1\\I\neq I'}}^{N_F} \Gamma(s^{-j}t^{-k}z_{I'}^{-1}z_I;s,t) \\
= \prod_{\substack{I=1\\I\neq I'}}^{N_F} (-z_{I'}^{-1}z_I)^{-jk}s^{k\frac{j(j+1)}{2}}t^{j\frac{k(k+1)}{2}} \\
\prod_{l=1}^{j} \theta^{-1}(s^{-l}z_{I'}^{-1}z_I;t) \prod_{m=1}^{k} \theta^{-1}(t^{-m}z_{I'}^{-1}z_I;s)\Gamma(z_{I'}^{-1}z_I;s,t). \\$$
(3.2)

Here we have used an identity (A.12) for the elliptic gamma function. The residue is evaluated as

$$\begin{aligned} &\operatorname{Res}_{y=s^{-j}t^{-k}} \Gamma(y;s,t) \\ &= (-y)^{jk} s^{k\frac{j(j-1)}{2}} t^{j\frac{k(k-1)}{2}} \prod_{l=0}^{j-1} \theta^{-1}(s^{l}y;t) \prod_{m=0}^{k-1} \theta^{-1}(t^{m}y;s) \Big|_{y=s^{-j}t^{-k}} \operatorname{Res}_{y=s^{-j}t^{-k}} \Gamma(s^{j}t^{k}y;s,t) \\ &= (-1)^{jk+1} s^{k\frac{j(j+1)}{2}} t^{j\frac{k(k+1)}{2}}(s;s)_{\infty}^{-1}(t;t)_{\infty}^{-1} \prod_{l=1}^{j} \theta^{-1}(s^{-l};t) \prod_{m=1}^{k} \theta^{-1}(t^{-m};s). \end{aligned}$$
(3.3)

From the second line to third line in (3.3), we have used the relation (A.5) and

$$\operatorname{Res}_{y=1} \Gamma(y; s, t) = -(s; s)_{\infty}^{-1}(t; t)_{\infty}^{-1}.$$
(3.4)

In a similar manner, we can evaluate the contribution from the anti-fundamental chiral

multiplets as

$$\prod_{I=1}^{N_{F}} \Gamma(y\tilde{z}_{I};s,t) \Big|_{y=z_{I'}s^{j}t^{k}} = \prod_{I=1}^{N_{F}} \Gamma(s^{j}t^{k}z_{I'}\tilde{z}_{I};s,t) \\
= \prod_{I=1}^{N_{F}} (-z_{I'}\tilde{z}_{I})^{-jk}s^{-k\frac{j(j-1)}{2}}t^{-j\frac{k(k-1)}{2}} \\
\prod_{I=0}^{j-1} \theta(s^{l}z_{I'}\tilde{z}_{I};t) \prod_{m=0}^{k-1} \theta(t^{m}z_{I'}\tilde{z}_{I};s)\Gamma(z_{I'}\tilde{z}_{I};s,t).$$
(3.5)

From (3.2), (3.3) and (3.5), we obtain the index as

$$\mathcal{I}_{N_{F}}^{U(1)} = \sum_{I'=1}^{N_{F}} \left( \prod_{I=1}^{N_{F}} \Gamma(z_{I'}\tilde{z}_{I};s,t) \right) \left( \prod_{\substack{I=1\\I\neq I'}}^{N_{F}} \Gamma(z_{I'}^{-1}z_{I};s,t) \right) \sum_{j,k=0}^{\infty} \left( \prod_{I=1}^{N_{F}} (z_{I}\tilde{z}_{I})^{-1}st \right)^{jk} \\ \left( \prod_{l=1}^{j} \frac{\prod_{I=1}^{N_{F}} \theta(s^{l-1}z_{I'}\tilde{z}_{I};t)}{\theta(s^{-l};t) \prod_{\substack{I=1\\I\neq I'}}^{N_{F}} \theta(s^{-l}z_{I'}^{-1}z_{I};t)} \right) \left( \prod_{m=1}^{k} \frac{\prod_{I=1}^{N_{F}} \theta(t^{m-1}z_{I'}\tilde{z}_{I};s)}{\theta(t^{-m}z_{I'}^{-1}z_{I};s)} \right).$$
(3.6)

Since we have shifted fugacity as  $z_{Iold} \rightarrow z_{Inew}(st)^{c_I}$  to set the R-charges zero, it satisfies  $(z_I \tilde{z}_I)_{new}^{-1} st = (z_I \tilde{z}_I)_{old}^{-1} (st)^{-(\frac{R}{2} + \frac{\tilde{R}}{2})+1}$ . In order to occur complete factorization, the R-charges have to satisfy  $R = \tilde{R} = 1$ . In this case, we find the complete factorization occurs, because the original flavor fugacities for  $SU(N_F) \times SU(N_F)$  satisfy  $\prod_{I=1}^{N_F} z_{I,old} = \prod_{I=1}^{N_F} \tilde{z}_{I,old} = 1$ .  $R = \tilde{R} = 1$  is precisely the R-charge assignments determined from the anomaly free condition. In this R-charge assignments, the superconformal index has completely factorized form:

$$\mathcal{I}_{N_{F}}^{U(1)} = \sum_{I'=1}^{N_{F}} \left(\prod_{I=1}^{N_{F}} \Gamma(z_{I'}\tilde{z}_{I};s,t)\right) \left(\prod_{\substack{I=1\\I\neq I'}}^{N_{F}} \Gamma(z_{I'}^{-1}z_{I};s,t)\right) \\
\left[\sum_{j=0}^{\infty} \prod_{l=1}^{j} \frac{\prod_{I=1}^{N_{F}} \theta(s^{l-1}z_{I'}\tilde{z}_{I};t)}{\theta(s^{-l};t) \prod_{\substack{I=1\\I\neq I'}}^{N_{F}} \theta(s^{-l}z_{I'}^{-1}z_{I};t)}\right] \left[\sum_{k=0}^{\infty} \prod_{m=1}^{k} \frac{\prod_{I=1}^{N_{F}} \theta(t^{m-1}z_{I'}\tilde{z}_{I};s)}{\theta(t^{-m}z_{I'}^{-1}z_{I};s)}\right].$$
(3.7)

Here we emphasize that U(1) formal superconformal index never factorize, if the number of fundamental chiral multiplets is different from that of anti-fundamental chiral multiplets. This is quite different structure from two or three dimensional theories. In two or three dimensions, the factorization occurs when the number of fundamental chiral multiplets is different from that of anti-fundamental chiral multiplets. Next, we study the relation between the superconformal index and vortex partition functions [14, 15, 16]. In the two dimensions, it is shown in [17, 18] that partition functions on  $S^2$  factorized to vortex and anti-vortex partition functions. In the three dimensional case, the factorized partition functions become trigonometric (hyperbolic) uplift of the vortex partition function [7, 8, 9]. Thus we expect the above factorized form is related to elliptic uplift of the vortex partition function. To see this, we introduce  $s = e^{i\varepsilon}, z_I = e^{im_I}, \tilde{z}_I = e^{-i\tilde{m}_I}$ . Then a factorized part is written as

$$\prod_{l=1}^{j} \frac{\prod_{I=1}^{N_{F}} \theta(s^{l-1} z_{I'} \tilde{z}_{I}; t)}{\theta(s^{-l}; t) \prod_{I=1 \atop I \neq I'}^{N_{F}} \theta(s^{-l} z_{I'}^{-1} z_{I}; t)} = \prod_{l=1}^{j} \frac{\prod_{I=1}^{N_{F}} \theta(e^{i(l-1)\varepsilon + m_{I'} - \tilde{m}_{I}}; t)}{\theta(e^{-il\varepsilon}; t) \prod_{I=1 \atop I \neq I'}^{N_{F}} \theta(e^{i(-l\varepsilon + m_{I} - m_{I'})}; t)}.$$
 (3.8)

By changing  $\theta(e^{ix}; t) \to x$ , we find that (3.8) agrees with the two dimensional U(1) vortex partition function of  $N_F$  fundamental flavors and anti-fundamental flavors with vorticity j at a vacuum labeled by twisted mass  $m_{I'}$ . Thus (3.8) is precisely the elliptic uplift of vortex partition function in two dimensions. In section 5, we will also show that the elliptic uplift of the vortex partition functions reduce to the three dimensional vortex partition function by the dimensional reduction.

#### 3.2 Non-Abelian case

In this subsection, we generalize the result in the previous subsection to non-Abelian gauge group U(N). As in the case of three dimensions [9], by use of the Cauchy determinant formula of the theta function, we find that the poles in the vector multiplet one-loop determinant do not contribute to the evaluation. Thus, It is enough to evaluate the residues at pole  $x_a = z_{I_a} s^{j_a} t^{k_a}$ ,  $(a = 1, \dots, N)$ ,  $j_a, k_a \geq \mathbb{Z}_{\geq 0}$  in the chiral multiplet oneloop determinant. The evaluation of the residues for chiral multiplet is quite parallel to the Abelian case in the previous section. The contribution of vector multiplet is as follows. Substituting  $x_a = z_{I_a} s^{j_a} t^{k_a}$  into the vector multiplet one-loop determinant, we obtain

$$\begin{split} &\prod_{a>b} \theta(z_{I_a} z_{I_b}^{-1} s^{j_a - j_b} t^{k_a - k_b}; s) \theta(z_{I_a}^{-1} z_{I_b} s^{j_b - j_a} t^{k_b - k_a}; t) \\ &= \prod_{a>b} (-z_{I_a} z_{I_b}^{-1})^{j_b - j_a - k_a + k_b} s^{\frac{1}{2}(j_a - j_b)(j_a - j_b - 1)} t^{\frac{1}{2}(k_b - k_a)(k_b - k_a - 1)} (st)^{(j_b - j_a)(k_a - k_b)} \\ &\left(\prod_{b=1}^{N} (-1)^{j_b} s^{-\frac{1}{2}j_b(j_b + 1)}\right) \left(\prod_{a>b} s^{-j_a(j_b + 1)}\right) \left(\prod_{a,b=1}^{N} s^{j_b(j_a + 1)}\right) \\ &\left(\prod_{a=1}^{N} (-1)^{k_a} t^{-\frac{1}{2}k_a(k_a + 1)}\right) \left(\prod_{a>b} t^{-k_b(k_a + 1)}\right) \left(\prod_{a,b=1}^{N} t^{k_a(k_b + 1)}\right) \\ &\left(\prod_{a,b=1}^{N} \prod_{m=0}^{k_a - 1} \frac{\theta(z_{I_a} z_{I_b}^{-1} t^{-m-1}; s)}{\theta(z_{I_a} z_{I_b}^{-1} t^{-m+k_b}; s)}\right) \left(\prod_{a>b} \theta(z_{I_a} z_{I_b}^{-1}; s)\right) \\ &\left(\prod_{a,b=1}^{N} \prod_{m=0}^{j_b - 1} \frac{\theta(z_{I_b} z_{I_a}^{-1} s^{-l-1}; t)}{\theta(z_{I_b} z_{I_a}^{-1} s^{-m+j_a}; t)}\right) \left(\prod_{a>b} \theta(z_{I_b} z_{I_a}^{-1}; t)\right). \end{split}$$
(3.9)

Therefore superconformal index is written as

$$\mathcal{I}_{N_{F}}^{U(N)} = \sum_{1 \leq I_{1} < \cdots I_{N} \leq N_{F}} \left( \prod_{a > b} \theta(z_{I_{a}} z_{I_{b}}^{-1}; s) \theta(z_{I_{b}} z_{I_{a}}^{-1}; t) \right) \left( \prod_{a=1}^{N} \prod_{I=1}^{N_{F}} \Gamma(z_{I_{a}} \tilde{z}_{I}; s, t) \right) \left( \prod_{a=1}^{N} \prod_{I \neq I_{a}}^{N_{F}} \Gamma(z_{I_{a}}^{-1} z_{I}; s, t) \right) \\ \sum_{\{j\},\{k\}=0}^{\infty} \left( \prod_{I=1}^{N_{F}} z_{I} \tilde{z}_{I} st \right)^{j_{a}k_{a}} (st)^{\sum_{a > b}(j_{b} - j_{a})(k_{a} - k_{b})} \\ \left( \prod_{a > b} (-z_{I_{a}} z_{I_{b}}^{-1})^{j_{b} - j_{a}} \right) (-1)^{\sum_{b=1}^{N} j_{b}} s^{\sum_{a > b} \left( \frac{1}{2} (j_{a} - j_{b})(j_{a} - j_{b} - 1) - j_{a}(j_{b} + 1) \right) - \sum_{a=1}^{N} \frac{1}{2} j_{a}(j_{a} + 1) + \sum_{a, b=1}^{N} j_{b}(j_{a} + 1)} \\ \left( \frac{\prod_{I=1}^{N_{F}} \prod_{a=1}^{N} \prod_{I=1}^{M} \theta(s^{I-1} z_{I_{a}} \tilde{z}_{I}; t)}{\prod_{a, b=1}^{N} \prod_{l=0}^{I-1} \theta(z_{I_{b}} z_{I_{a}}^{-1} s^{-l + j_{a}}; t) \prod_{a=1}^{N} \prod_{I=0}^{I-1} \prod_{I \notin \mathcal{A}} \theta(s^{-l} z_{I_{a}}^{-1} z_{I}; t)} \right) \\ \left( \prod_{a > b} (-z_{I_{a}} z_{I_{b}}^{-1})^{k_{b} - k_{a}} \right) (-1)^{\sum_{a=1}^{N} k_{a}} t^{\sum_{a > b} \left( \frac{1}{2} (k_{b} - k_{a})(k_{b} - k_{a} - 1) - k_{b}(k_{a} + 1) \right) - \sum_{b=1}^{N} \frac{1}{2} k_{a}(k_{a} + 1) + \sum_{a, b=1}^{N} k_{a}(k_{b} + 1)} \\ \left( \frac{\prod_{a > b}^{N_{F}} \prod_{m=0}^{N_{F}} \prod_{a \in I}^{N} \prod_{m=1}^{M} \theta(t^{m-1} z_{I_{a}} \tilde{z}_{I}; s)}{\prod_{a = 1}^{N_{F}} \prod_{m=1}^{N} \prod_{m=1}^{K_{a}} \theta(t^{m-1} z_{I_{a}} \tilde{z}_{I}; s)} \right).$$

$$(3.10)$$

Here we defined  $\mathcal{A} := \{I_1, \cdots, I_N\}$ . The elliptic uplift of the U(N) vortex partition function with *a*-th U(1) vorticity  $j_a$  and a vacuum labeled by  $\mathcal{A}$  is given by

$$Z_{\{j_a\},\{I_a\}}^V = \frac{\prod_{I=1}^{N_F} \prod_{a=1}^{N} \prod_{l=1}^{j_a} \theta(s^{l-1} z_{I_a} \tilde{z}_I; t)}{\prod_{a,b=1}^{N} \prod_{l=0}^{j_b-1} \theta(z_{I_b} z_{I_a}^{-1} s^{-l+j_a}; t) \prod_{a=1}^{N} \prod_{l=0}^{j_a-1} \prod_{I \notin \mathcal{A}} \theta(s^{-l} z_{I_a}^{-1} z_I; t)},$$
(3.11)

and that of the anti-vortex partition function with *a*-th U(1) vorticity  $k_a$  and a vacuum labeled by  $\mathcal{A}$  is given by

$$Z_{\{k_a\},\{I_a\}}^{\bar{V}} = \frac{\prod_{I=1}^{N_F} \prod_{a=1}^{N} \prod_{m=1}^{k_a} \theta(t^{m-1} z_{I_a} \tilde{z}_I; s)}{\prod_{a,b=1}^{N} \prod_{m=0}^{k_b-1} \theta(z_{I_b} z_{I_a}^{-1} t^{-m+k_a}; s) \prod_{a=1}^{N} \prod_{m=0}^{k_a-1} \prod_{I \notin \mathcal{A}} \theta(t^{-m} z_{I_a}^{-1} z_I; s)}.$$
(3.12)

The (3.10) almost factorize, but contains the non-factorized factor with respect to the vorticity  $j_a$  and the anti-vorticity  $k_a$  which is given by

$$\left(\prod_{I=1}^{N_F} (z_I \tilde{z}_I)^{-1} st\right)^{\sum_{a=1}^{N} j_a k_a} (st)^{\sum_{a>b} (j_b - j_a)(k_a - k_b)}.$$
(3.13)

The non-factorizable factor is rewritten as

$$\left(\prod_{I=1}^{N_F} (z_I \tilde{z}_I)_{\text{new}}^{-1} st\right)^{\sum_{a=1}^{N} j_a k_a} (st)^{\sum_{a>b} (j_b - j_a)(k_a - k_b)} 
= (st)^{[-N_F(\frac{R}{2} + \frac{\tilde{R}}{2}) + (N_f - N)](\sum_{a=1}^{N} j_a k_a) + (\sum_{a=1}^{N} j_a)(\sum_{a=1}^{N} k_a)}.$$
(3.14)

Therefore, the conditions for the complete factorization becomes

$$R = \tilde{R} = 1 - \frac{N}{N_F},\tag{3.15}$$

$$\sum_{a=1}^{N} j_a = 0 \quad \text{and} \quad \sum_{a=1}^{N} k_a = 0.$$
 (3.16)

The condition (3.15) is again the R-charges assignments determined uniquely by the anomaly free condition for SU(N) SQCD with  $N_F$  fundamental and anti-fundamental chiral multiplets without superpotential. If the traceless condition is imposed for the gauge group U(N) (namely gauge group becomes SU(N)), the condition (3.16) is satisfied. When the delta function constraint  $\delta(\sum_{a=1}^{N} x_a)$  is inserted in the integrations (2.6), these conditions are satisfied. Up to the Weyl permutations, we assume that  $j_a, k_a (a = 1, \dots, N-1)$  run non-negative integers and  $j_N = -\sum_{a=1}^{N-1} j_a, k_N = -\sum_{a=1}^{N-1} k_a$ 

is imposed. It is interesting to study the the relation between index computation and the fact that the overall U(1) factor is decoupled in the infrared limit.

Under the condition (3.15) and (3.16), the superconformal index completely factorize as

$$\mathcal{I}_{N_F}^{SU(N)} = \sum_{1 \le I_1 < \cdots I_N \le N_F} Z_{\text{v.H}}^{1-\text{loop}} Z_{\text{chi.H}}^{1-\text{loop}} Z_{\text{a.chi.H}}^{1-\text{loop}} Z_V Z_{\bar{V}}, \qquad (3.17)$$

with

$$Z_{\text{v.H}}^{1-\text{loop}} = \prod_{a>b} \theta(z_{I_a} z_{I_b}^{-1}; s) \theta(z_{I_b} z_{I_a}^{-1}; t), \qquad (3.18)$$

$$Z_{\text{v.H}}^{1-\text{loop}} = \prod_{a=1}^{N} \prod_{\substack{I=1\\I \neq I_a}}^{N_F} \Gamma(z_{I_a}^{-1} z_I; s, t), \qquad (3.19)$$

$$Z_{\text{a.chi.H}}^{1-\text{loop}} = \prod_{a=1}^{N} \prod_{I=1}^{N_F} \Gamma(z_{I_a} \tilde{z}_I; s, t), \qquad (3.20)$$

$$Z_{V} = \sum_{\{j\}'} \left( \prod_{a=1}^{N} z_{I_{a}}^{Nj_{a}} \right) s^{\sum_{a>b} \left( \frac{1}{2} (j_{a} - j_{b}) (j_{a} - j_{b} - 1) - j_{a} (j_{b} + 1) \right) - \sum_{a=1}^{N} \frac{1}{2} j_{a}^{2}} \\ \left( \frac{\prod_{I=1}^{N} \prod_{a=1}^{N} \prod_{l=0}^{N} \prod_{l=1}^{N} \prod_{l=1}^{l} \theta(s^{l-1} z_{I_{a}} \tilde{z}_{I}; t)}{\prod_{a,b=1}^{N} \prod_{l=0}^{l-1} \theta(z_{I_{b}} z_{I_{a}}^{-1} s^{-l+j_{a}}; t) \prod_{a=1}^{N} \prod_{l=0}^{j_{a}-1} \prod_{I \notin \mathcal{A}} \theta(s^{-l} z_{I_{a}}^{-1} z_{I}; t)} \right), \quad (3.21)$$

$$Z_{\bar{V}} = \sum_{\{k\}'} \left( \prod_{a=1}^{N} z_{I_{a}}^{Nk_{a}} \right) t^{\sum_{a>b} \left( \frac{1}{2} (k_{b} - k_{a}) (k_{b} - k_{a} - 1) - k_{b} (k_{a} + 1) \right) - \sum_{b=1}^{N} \frac{1}{2} k_{a}^{2}} \\ \left( \frac{\prod_{a=1}^{N} \prod_{a=0}^{N} \prod_{a=1}^{N} \prod_{m=1}^{k_{a}} \theta(t^{m-1} z_{I_{a}} \tilde{z}_{I}; s)}{\prod_{a,b=1}^{N} \prod_{m=0}^{k_{b}-1} \theta(z_{I_{b}} z_{I_{a}}^{-1} t^{-m+k_{a}}; s) \prod_{a=1}^{N} \prod_{m=0}^{k_{a}-1} \prod_{I \notin \mathcal{A}} \theta(t^{-m} z_{I_{a}}^{-1} z_{I}; s)} \right). (3.22)$$

Here we defined  $\sum_{\{j\}'} := \sum_{a=1}^{N-1} \sum_{j_a=0}^{\infty}$ .

We refer to the relation with *Higgs branch localization* first introduced in [17]. In the ordinary localization, the saddle point values admit constant field (holonomy or the scalar in vector multiplet) configurations which lead to multi-contour integrals. On the other hand, in the Higgs branch localization, saddle point admit the discrete vacua of root of Higgs branch and the point like BPS (anti-BPS) equations at the north (south) pole of  $S^2$ . Again, the partition function can be evaluated in the WKB approximation around the discrete vacua. In the four dimensions, after the torus compactification, the combinations of the flavor holonomies along torus on  $T^2 \times \mathbb{R}^2$  play the role of twisted masses (real masses). Then, (3.18), (3.19) and (3.20) can be interpreted as the one-loop determinant

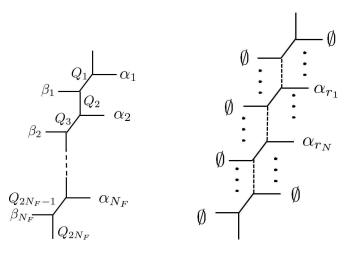


Figure 1: Left: The (p,q)-web for the Calabi-Yau 3-folds. Two vertical external legs are identified and becomes an internal line.  $Q_a, (a = 1, \dots, 2N_F)$  represents the exponentiated Kahler parameter of *a*-th internal line. Right: a A-brane is inserted at  $r_a$ -th right horizontal external leg  $(a = 1, \dots, N)$  and the representation is  $1^{k_a}$ . The other external legs are the trivial representation

of the vector multiplet, the fundamental chiral multiplets and anti-fundamental chiral multiplets in the Higgs branch localization, respectively. In the Higgs branch localization in three dimensions [19, 20], the point like vortices exist at the north pole of base space  $S^2$  of  $S^3$  and the point like anti-vortices exist at the south pole. Then vortex(anti-vortex) world volume becomes one-dimensional circle  $S^1$  which is the circle fiber at the north (south) pole, respectively. In the case of  $S^1 \times S^3$ , there is an additional trivial  $S^1$ -fiber and vortex world volume becomes two dimensional torus  $T^2$ . From the point like vortices, base  $S^2$  can be regarded as flat space  $\mathbb{R}^2$ , namely it is equivalently to consider the vortex partition function of the  $\mathcal{N} = 1$  theory on  $T^2 \times \mathbb{R}^2$ . Then Kaluza-Klein momenta along the  $T^2$  will provide the elliptic deformation of vortex partition function. This is quite analogous to the Nekrasov's instanton partition function is elliptically deformed, when  $\mathcal{N} = 2$  supersymmetric gauge theories is uplifted to torus compactified the six dimensional  $\mathcal{N} = (1, 0)$  theories on  $T^2 \times \mathbb{R}^4$  [21].

The matter contents we considered in this section is the electric theory in the Seiberg duality. When we consider the the magnetic theory, the factorization again occurs, if and only if both the traceless condition and the correct R-charges assignments are satisfied.

### 4 Relation to open topological string

It is shown in [7, 9] that the vortex partition function in the factorized partition functions coincides the open topological string partition function of strip geometry. In this section, we study open topological string amplitude which give the elliptic uplift of vortex partition function (3.11). We consider the Calabi-Yau 3-fold described by (p,q)-web in the figure 1. The topological vertex [22] gives topological string amplitudes on this geometry. The (refined) topological string on the geometry we concern are studied in detail [23] [24] ( see also [21]). The open topological string amplitude for the left of figure 1 is given by

$$\frac{W_{\emptyset\cdots\emptyset}^{\alpha_{1}\cdots\alpha_{N_{F}}}}{W_{\emptyset\cdots\emptyset}^{\emptyset\cdots\emptyset}} = \left(\prod_{I=1}^{N_{F}} q^{\frac{||\alpha_{I}||^{2}}{2}} \tilde{Z}_{\alpha_{I}}(q)\right) \prod_{r,l=1}^{N_{F}} \frac{\mathcal{J}_{\emptyset\alpha_{r+l}}(Q_{2r,2r+2l-2};q)\mathcal{J}_{\alpha_{r}\emptyset}(Q_{2r-1,2r+2l-3};q)}{\mathcal{J}_{\alpha_{r}\alpha_{r+l}}(Q_{2r-1,2r+2l-2};q)}.$$

$$(4.1)$$

Here we defined

$$\mathcal{J}_{\mu\nu}(x,q) := \prod_{k=1}^{\infty} \prod_{(i,j)\in\mu} (1 - Q_U^{k-1} x q^{\mu_i + \nu_i^t - i - j + 1}) \prod_{(i,j)\in\nu} (1 - Q_U^{k-1} x q^{-\mu_j^t - \nu_i + i + j - 1}), \quad (4.2)$$

with  $Q_U = \prod_{r=1}^{2N} Q_r$  and  $Q_{a,b} = \prod_{r=a}^{b} Q_r$ . The  $\tilde{Z}_{\mu}(q)$  is defined by

$$\tilde{Z}_{\mu}(q) = \prod_{s \in \mu} (1 - q^{l_{\mu}(s) + a_{\mu}(s) + 1})^{-1}, \qquad (4.3)$$

with  $l_{\mu}(s) = \mu_i - j$ ,  $a_{\mu}(s) = \mu_j^t - i$  for s = (i, j) in the partition  $\mu$ .

We set the representation of  $r_a$ -th right external horizontal leg as  $\alpha_{r_a} = 1^{k_a}$  for  $a = 1, \dots N$  and the others as trivial representation  $\alpha_r = \beta_r = \emptyset$ . Then the non-trivial  $\mathcal{J}_{\mu\nu}(x,q)$  are following three types:

$$\mathcal{J}_{1^{k_a}\emptyset}(x,q) = \prod_{k=1}^{\infty} \prod_{i=1}^{k_a} (1 - xQ_U^{k-1}q^{1-i}),$$
  
$$\mathcal{J}_{\emptyset 1^{k_a}}(x,q) = \prod_{k=1}^{\infty} \prod_{i=1}^{k_a} (1 - xQ_U^{k-1}q^{-1+i}),$$
  
$$\mathcal{J}_{1^{k_a}1^{k_b}}(x,q) = \prod_{k=1}^{\infty} \prod_{i=1}^{k_a} (1 - xQ_U^{k-1}q^{1+k_b-i}) \prod_{j=1}^{k_b} (1 - xQ_U^{k-1}q^{-1-k_a+j}).$$
 (4.4)

It follows from (4.4) that

$$\prod_{r,l=1}^{N_F} \mathcal{J}_{\emptyset \alpha_{r+l}}(Q_{2r,2r+2l-2};q) \mathcal{J}_{\alpha_r \emptyset}(Q_{2r-1,2r+2l-3};q) = \prod_{r=1}^{N_F} \prod_{a=1}^{N} \prod_{i=0}^{k_a-1} \theta(q^i Q_{2r_a-1,2r-1};Q_U),$$

$$\prod_{r,l=1}^{N_F} \mathcal{J}_{\alpha_r \alpha_{r+l}}(Q_{2r-1,2r+2l-2};q)$$

$$= \left(\prod_{a,b=1}^{N} \prod_{i=1}^{k_a} \theta(Q_{2r_a-1,2r_b-2}q^{1+k_b-i};Q_U)\right) \left(\prod_{r \notin \{r_1,\cdots r_N\}} \prod_{a=1}^{N} \prod_{i=1}^{k_a} \theta(Q_{2r_a-1,2r-2}q^{1-i};Q_U)\right).$$
(4.5)

Here we used the relation  $Q_U^{-1}Q_{a,b} = Q_{b+1,a-1}$ . Therefore open topological string amplitude becomes

$$\frac{W_{\emptyset \dots \dots \dots \emptyset}^{\emptyset \dots \emptyset 1^{k_1} \dots 1^{k_N} \emptyset \dots \emptyset}}{W_{\emptyset \dots \emptyset}^{\emptyset \dots \emptyset}} = \left( \prod_{a=1}^{N} q^{\frac{k_a}{2}} \prod_{i=1}^{k_a} (1-q^i)^{-1} \right) \prod_{r=1}^{N_F} \prod_{a=1}^{N} \prod_{i=0}^{k_a-1} \theta(q^i Q_{2r_a-1,2r-1}; Q_U) \left( \prod_{a,b=1}^{N} \prod_{i=1}^{k_a} \theta(Q_{2r_a-1,2r_b-2}q^{1+k_b-i}; Q_U) \right) \left( \prod_{r \notin \{r_1, \dots r_N\}} \prod_{a=1}^{N} \prod_{i=1}^{k_a} \theta(Q_{2r_a-1,2r-2}q^{1-i}; Q_U) \right) \right) (4.6)$$

If we identify the parameters as q = s,  $Q_U = t$ ,  $Q_{2I_{a-1},2I-1} = z_{I_a}\tilde{z}_I$ ,  $k_a = j_a$  and  $Q_{2I_{a-1},2I-2} = z_{I_a}^{-1} z_I$ , it finds that the open topological string amplitude agrees with (3.11) up to the over all factor  $\prod_{a=1}^{N} q^{\frac{k_a}{2}} \prod_{i=1}^{k_a} (1-q^i)^{-1}$ .

### 5 Three dimensional limit

When the radius of  $S^1$  goes to zero, it shown in [25], [26], [27] (See also [29]) that the superconformal indices in four dimension reduces to the partition functions on the three dimensional (squashed) sphere. In this section, we consider the three dimensional limit of U(N) index (2.6) and study the relation to the partition function on three dimensional squashed sphere [27], [28]. We set parameter as

$$s = e^{\beta\omega_1}, \quad t = e^{\beta\omega_2}, \quad \omega_1\omega_2 = 1, \quad b := \sqrt{\frac{\omega_1}{\omega_2}}, \quad Q := b + \frac{1}{b}.$$
$$z_I = e^{i\beta m_I}, \quad \tilde{z}_I = e^{-i\beta\tilde{m}_I}.$$
(5.1)

Here  $\beta = \frac{r_1}{r_3}$  is the ratio of the  $S^1$  radius  $r_1$  and  $S^3$  radius  $r_3$ . In the limit  $r_1 \to 0$ , the theta function and elliptic gamma function reduce to a trigonometric and the double-sin function, respectively:

$$\lim_{\beta \to 0} \theta(e^{\beta\sigma}; e^{\beta\omega_1}) = 2\sin\pi b^{-1}\sigma, \quad \lim_{\beta \to 0} \theta(e^{\beta\sigma}; e^{\beta\omega_2}) = 2\sin\pi b\sigma, \tag{5.2}$$

$$\lim_{\beta \to 0} \Gamma(e^{\beta\sigma}; e^{\beta\omega_1}, e^{\beta\omega_2}) = s_b \left( i\sigma + i\frac{Q}{2} \right) = s_b^{-1} \left( -i\sigma - i\frac{Q}{2} \right).$$
(5.3)

Then it follows that

$$\lim_{\beta \to 0} \theta(z_{I_a} z_{I_b}^{-1}; s) \theta(z_{I_b} z_{I_a}^{-1}; t) = 4 \sinh \pi b^{-1} \left( m_{I_b} - m_{I_a} \right) \sinh \pi b \left( m_{I_a} - m_{I_b} \right), \quad (5.4)$$

$$\lim_{\beta \to 0} \Gamma(z_{I_a}^{-1} z_I; s, t) = s_b \left( m_{I_a} - m_I + i \frac{Q}{2} \right),$$
(5.5)

$$\lim_{\beta \to 0} \Gamma(z_{I_a} \tilde{z}_I; s, t) = s_b^{-1} \left( m_I - \tilde{m}_I - i \frac{Q}{2} \right),$$
(5.6)

and

$$\lim_{\beta \to 0} Z_{\{j_a\},\{I_a\}}^V = \frac{\prod_{I=1}^N \prod_{a=1}^N \prod_{l=1}^{j_a} 2\sinh \pi b \left(m_{I_a} - \tilde{m}_I - i(l-1)b\right)}{\prod_{a,b=1}^N \prod_{l=0}^{j_b-1} 2\sinh \pi b \left(m_{I_b} - m_{I_a} + i(l-j_a)b\right) \prod_{a=1}^N \prod_{l=0}^{j_a-1} \prod_{I \notin \mathcal{A}} 2\sinh \pi b \left(m_I - m_{I_a} + ilb\right)},$$
(5.7)

$$\lim_{\beta \to 0} Z_{\{k_a\},\{I_a\}}^{\bar{V}} = \frac{\prod_{I=1}^{N_F} \prod_{a=1}^{N} \prod_{l=1}^{k_a} 2 \sinh \pi b^{-1} \left( m_{I_a} - \tilde{m}_I - i(l-1)b^{-1} \right)}{\prod_{a,b=1}^{N} \prod_{l=0}^{k_b-1} 2 \sinh \pi b^{-1} \left( m_{I_b} - m_{I_a} + i(l-k_a)b^{-1} \right) \prod_{a=1}^{N} \prod_{l=0}^{k_a-1} \prod_{I \notin \mathcal{A}} 2 \sinh \pi b^{-1} \left( m_I - m_{I_a} + ilb^{-1} \right)}.$$
(5.8)

We find that the right hand sides of (5.4), (5.5) and (5.6) correctly reproduce the oneloop determinant of the three dimensional vector multiplet, the  $N_F$ -flavors fundamental chiral multiplets and  $N_F$ -flavors anti-fundamental chiral multiplets in the Higgs branch localization, respectively. Moreover, (5.7) and (5.8) also agree the vortex and anti-vortex partition functions on the squashed sphere, respectively. Next we consider the contribution of FI-parameter (2.5). We substitute the saddle point value at the  $x_a = z_{I_a} s^{j_a} t^{k_a}$  to the  $e^{iA_a}$ 

$$\lim_{\beta \to 0} \exp(-S_{FI}) = e^{-4\pi^2 r_3^2 \zeta \sum_a (m_{I_a} + bj_a + b^{-1}k_a)}$$
(5.9)

This also correctly reproduces the FI-term contribution in the three dimensions. Therefore, in the three dimensional limit, we find that the index of U(N) theory in four dimensions directly reduces to the factorized partition function of U(N) theory with  $N_F$ -flavors fundamental chiral multiples and  $N_F$ -flavors anti-fundamental chiral multiples on the three dimensional squashed sphere.

## 6 Summary

In this article, we have studied factorization properties of  $\mathcal{N} = 1$  superconformal index in the four dimensions. In the U(1) case, the index factorize, when the following two conditions are satisfied: the number of fundamental and anti-fundamental multiplets is same, the correct R-charge assignments are satisfied. In the factorized form, the elliptic uplift of the vortex partition function and anti-vortex function appear. In the non-Abelian case, it found that the superconformal index completely factorize to the elliptic uplift of the vortex partition function and anti-vortex function, only when the traceless condition in addition to the above two condition is satisfied. We comment on several future directions.

• Higgs branch localization.

In the direct contour integral evaluation, It is obscure the reason why the vortex and anti-vortex partition functions appear. Higgs branch localization directly explains them. It is interesting to perform the Higgs branch localization in four dimensions and explain the origin of the parameters in the factorized form.

• The relation between the elliptic uplift of the vortex partition and  $\mathcal{N} = (0, 2)$  elliptic genus.

The instanton partition functions on  $T^2 \times \mathbb{R}^4$  is equivariant elliptic genera of instanton moduli space. In the case of  $\mathcal{N} = 1$  supersymmetric theories in four dimensions, it is known that vortex world volume preserves  $\mathcal{N} = (0, 2)$  supersymmetry in two dimensions [30]. Thus, the elliptic uplift of the vortex partition should be  $\mathcal{N} = (0, 2)$ elliptic genera of vortex moduli space. The  $\mathcal{N} = (0, 2)$  flavored elliptic genus is introduced in [31], [32] and localization is studied in detail. It is interesting to derived the elliptic uplift of vortex partition function directly from elliptic genus of vortex moduli space via localization.

• Factorization of partition functions on  $T^2 \times S^2$  or  $S^1 \times S^3/\mathbb{Z}_n$ .

Recently, a partition function on  $T^2 \times S^2$  is studied in [33, 34]. From the Higgs branch localization perspective, partition function on  $T^2 \times S^2$  should also factorize. Because, point like vortices (anti-vortices) exist on  $S^2$  and the trivial  $T^2$ -fiber exists.

In [35], the factorization of partition functions on  $S^3/\mathbb{Z}_n$  is studied. This suggest that the partition function on  $S^1 \times S^3/\mathbb{Z}_n$  [36] can be regarded  $S^1$  uplift of  $S^3/\mathbb{Z}_n$ also factorized into two parts. Because we have found that U(N) theory correctly reduces to the factorized partition function in three dimensions.

In the case of  $S^1 \times S^3/\mathbb{Z}_n$ ,  $S^1$ -fiber is  $\mathbb{Z}_n$  orbifolded. Then the vortex world volume becomes  $S^1 \times S^1/\mathbb{Z}_n$ . This means that  $\mathcal{N} = (0, 2)$  elliptic genus of vortex moduli space is orbifolded elliptic genus and twisted sectors appear. The twisted sectors are combined to reproduce a weak Jacobi form or modular covariance in the Landau-Ginzburg orbifolds [37, 38]. It is interesting to study the modular properties of the orbifolded elliptic genus of vortex moduli space.

#### • Holomorphic blocks in four dimensions

To construct general theory of holomorphic blocks in four dimensions is one of the most interesting and challenging future directions. In three dimensions, the partition function on  $S^1 \times S^2$  is constructed from identity fusion of holomorphic blocks and the partition function on  $S_b^3$  is constructed from S-fusion. The holomorphic block is universal in these two spaces. We conjecture that the partition functions on  $T^2 \times S^2$  is identity fusion of holomorphic block in four dimensions and the partition functions on  $S^1 \times S^3$  is S-fusion of holomorphic block in four dimensions:

$$Z_{T^2 \times S^2} = \sum_{\mathcal{A}} ||\mathcal{B}^{\mathcal{A}}(s,t)||_{\mathrm{id}} = \sum_{\mathcal{A}} \mathcal{B}^{\mathcal{A}}(\tilde{s},\tilde{t})\mathcal{B}^{\mathcal{A}}(\tilde{s},\tilde{t})$$
(6.1)

$$Z_{S^1 \times S^3} = \sum_{\mathcal{A}} ||\mathcal{B}^{\mathcal{A}}(s,t)||_S = \sum_{\mathcal{A}} \mathcal{B}^{\mathcal{A}}(s,t)\mathcal{B}^{\mathcal{A}}(t,s)$$
(6.2)

Here  $\sum_{\mathcal{A}}$  runs all the possible choice of vacua up to the Weyl permutations.  $\tilde{s}, \tilde{t}$  in the identity fusion mean that  $\tilde{s} = e^{2\pi i \tau}, \tilde{t} = e^{2\pi i \sigma}$ .  $(\tau, \sigma)$  parametrizes complex structure of  $T^2 \times S^2$ .

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# A Conventions and useful formula

A q-pochhammer symbols is defined by

$$(a;q)_n = \prod_{i=1}^n (1 - aq^{i-1}) \tag{A.1}$$

The theta function is defined by

$$\theta(x;q) = \prod_{n=0}^{\infty} (1 - xq^n)(1 - x^{-1}q^{n+1}), \quad x \in \mathbb{C}^*, \ |q| < 1.$$
(A.2)

It satisfies the following relations

$$\theta(x;q) = \theta\left(\frac{q}{x};q\right) = -x\theta\left(\frac{1}{x};q\right)$$
(A.3)

From this, it follows for n > 0

$$\theta(q^{n}x;q) = (-x)^{-n} q^{-\frac{n(n-1)}{2}} \theta(x;q)$$
(A.4)

$$\theta(q^{-n}x;q) = (-x)^n q^{-\frac{n(n+1)}{2}} \theta(x;q)$$
(A.5)

The elliptic gamma function is defined by

$$\Gamma(x;s,t) = \prod_{j,k=0}^{\infty} \frac{1 - x^{-1} s^{j+1} t^{k+1}}{1 - x s^{j} t^{k}}, \quad |s|, |t| < 1.$$
(A.6)

The elliptic gamma function satisfies the following relations.

$$\Gamma(sx; s, t) = \theta(x; t)\Gamma(x; s, t)$$
(A.7)

$$\Gamma(tx; s, t) = \theta(x; s)\Gamma(x; s, t)$$
(A.8)

$$\Gamma(s^{-1}x; s, t) = \theta^{-1}(xs^{-1}; t)\Gamma(x; s, t)$$
(A.9)

$$\Gamma(t^{-1}x;s,t) = \theta^{-1}(xt^{-1};s)\Gamma(x;s,t)$$
(A.10)

For  $j, k \in \mathbb{Z}_{\geq 0}$ , the above equations lead to the following identities

$$\frac{\Gamma(s^{j}t^{k}x;s,t)}{\Gamma(x;s,t)} = (-x)^{-jk}s^{-k\frac{j(j-1)}{2}}t^{-j\frac{k(k-1)}{2}}\prod_{l=0}^{j-1}\theta(s^{l}x;t)\prod_{m=0}^{k-1}\theta(t^{m}x;s),$$
(A.11)

$$\frac{\Gamma(s^{-j}t^{-k}x;s,t)}{\Gamma(x;s,t)} = (-x)^{-jk} s^{k\frac{j(j+1)}{2}} t^{j\frac{k(k+1)}{2}} \prod_{l=1}^{j} \theta^{-1}(s^{-l}x;t) \prod_{m=1}^{k} \theta^{-1}(t^{-m}x;s).$$
(A.12)

The double-sin function is defined by

$$s_b(x) = \prod_{j,k=0}^{\infty} \frac{jb + kb^{-1} + Q/2 - ix}{jb + kb^{-1} + Q/2 + ix}$$
(A.13)

which satisfies the relation  $s_b(x)s_b^{-1}(x) = 1$ .

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