Riemann-Liouville and higher dimensional Harday operators for non-negative decreasing function in $L^{p(\cdot)}$ spaces

Ghulam Murtaza and Muhammad Sarwar

Abstract. In this paper one-weight inequalities with general weights for Riemann-Liouville transform and $n-$ dimensional fractional integral operator in variable exponent Lebesgue spaces defined on $\mathbb{R}^n$ are investigated. In particular, we derive necessary and sufficient conditions governing one-weight inequalities for these operators on the cone of non-negative decreasing functions in $L^{p(x)}$ spaces.

2000 Mathematics Subject Classification: 42B20, 42B25, 46E30.

Key Words and Phrases: Variable exponent Lebesgue spaces, Riemann-Liouville transform, $n-$ dimensional fractional integral operator, one–weight inequality.

1. Introduction

We derive necessary and sufficient conditions governing the one-weight inequality for the Riemann-Liouville operator

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \qquad 0 < \alpha < 1,$$

and $n-$dimensional fractional integral operator

$$I_\alpha g(x) = \frac{1}{|x|^\alpha} \int\limits_{|y|<|x|} \frac{g(t)}{|x-t|^{n-\alpha}} dt \quad 0 < \alpha < n,$$

on the cone of non-negative decreasing function in $L^{p(x)}$ spaces.

In the last two decades a considerable interest of researchers was attracted to the investigation of the mapping properties of integral operators in so called Nakano spaces $L^{p(\cdot)}$ (see e.g., the monographs [5], [7] and references therein). Mathematical problems related to these spaces arise in applications to mechanics of the continuum medium. For example, M. Ružička [19] studied the problems in the so called rheological and electrorheological fluids, which lead to spaces with variable exponent.

Weighted estimates for the Hardy transform

$$(Hf)(x) = \int\limits_0^x f(t)dt, \quad x > 0,$$

in $L^{p(\cdot)}$ spaces were derived in the papers [8] for power-type weights and in [11], [12], [15], [6], [17] for general weights. The Hardy inequality for non-negative decreasing functions was studied in [3], [4].

Weighted problems for the Riemann-Liouville transform in $L^{p(x)}$ spaces were explored in the papers [10], [11], [2], [14] (see also the monograph [18]).

Historically, one and two weight Hardy inequalities on the cone of non-negative decreasing functions defined on $\mathbb{R}_+$ in the classical Lebesgue spaces were characterized by M. A. Arino and B. Muckenhoupt [1] and E. Sawyer [22] respectively.

It should be emphasized that the operator $I_\alpha f(x)$ is the weighted truncated potential. The trace inequity for this operator in the classical Lebesgue spaces was established by E. Sawyer [21] (see also the monograph [13], Ch.6 for related topics).

In general, the modular inequality

$$\int\limits_0^1 \Big| \int\limits_0^x f(t)dt \Big|^{q(x)} v(x)dx \le c \int\limits_0^1 \big| f(t) \big|^{p(t)} w(t)dt \tag{$*$}$$

for the Hardy operator is not valid (see [23] , Corollary 2.3, for details). Namely the following fact holds: if there exists a positive constant c such that inequality (*) is true for all $f \ge 0$, where q; p; w and v are non-negative measurable functions, then there exists $b \in [0 \;\; 1]$ such that $w(t) > 0$ for almost every $t < b$; $v(x) = 0$ for almost every $x > b$, and $p(t)$ and $q(x)$ take the same constant values almost everywhere for $t \in (0; b)$ and $x \in (0; b) \cap \{v \ne 0\}$.

To get the main result we use the following pointwise inequities

$$c_1(Tf)(x) \le (R_\alpha f)(x) \le c_2(Tf)(x),$$
$$c_3(Hg)(x) \le (I_\alpha g)(x) \le c_4(Hg)(x),$$

for non-negative decreasing functions, where c_1, c_2, c_3 and c_4 are constants are independents of f, g and x, and

$$Tf(x) = \frac{1}{x} \int\limits_0^x f(t)dt, \qquad Hg(x) = \frac{1}{|x|^n} \int\limits_{|y|<|x|} g(y)dy.$$

In the sequel by the symbol $Tf \approx Tg$ we means that there are positive constants c_1 and c_2 such that $c_1 Tf(x) \le Tg(x) \le c_2 Tf(x)$. Constants in inequalities will be mainly denoted by c or C; the symbol $\mathbb{R}_+$ means the interval $(0, +\infty)$.

2. PRELIMINARIES

We say that a radial function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is decreasing if there is a decreasing function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $g(|x|) = f(x)$, $x \in \mathbb{R}^n$. We will denote g

again by f. Let $p : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a measurable function, satisfying the conditions $p^- = \operatorname{ess}\inf_{x\in\mathbb{R}^n} p(x) > 0$, $p^+ = \operatorname{ess}\sup_{x\in\mathbb{R}^n} p(x) < \infty$.

Given $p : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and a non-negative measurable function (weight) u in $\mathbb{R}^n$, let us define the following local oscillation of p :

$$\varphi_{p(\cdot),u(\delta)} = \operatorname{ess}\sup_{x\in B(0,\delta)\cap \text{ supp u}} p(x) \;-\; \operatorname{ess}\inf_{x\in B(0,\delta)\cap \text{ supp u}} p(x),$$

where $B(0,\delta)$ is the ball with center 0 and radius δ.

We observe that $\varphi_{p(\cdot),u(\delta)}$ is non-decreasing and positive function such that

$$\lim_{\delta\to\infty} \varphi_{p(\cdot),u(\delta)} = p_u^+ - p_u^-, \tag{1}$$

where p_u^+ and p_u^- denote the essential infimum and supremum of p on the support of u. respectively.

By the similar manner it is defined (see [3]) the function $\psi_{p(\cdot),u(\eta)}$ for an exponent $p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and weight v on $\mathbb{R}_+$:

$$\varphi_{p(\cdot),v(\varepsilon)} = \operatorname{ess}\sup_{x\in B(0,\delta)\cap \text{ supp v}} p(x) \;-\; \operatorname{ess}\inf_{x\in(0,\eta)\cap \text{ supp v}} p(x),$$

Let $D(\mathbb{R}_+)$ be the class of non-negative decreasing functions on $\mathbb{R}_+$ and let $DR(\mathbb{R}^n)$ be the class of all non–negative radially decreasing functions on $\mathbb{R}^n$. Suppose that u is measurable a.e. positive function (weight) on $\mathbb{R}^n$. We denote by $L^{p(x)}(u,\mathbb{R}^n)$, the class of all non–negative functions on $\mathbb{R}^n$ for which

$$S_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} u(x) d\mu(x) < \infty.$$

For essential properties of $L^{p(x)}$ spaces we refer to the papers [16] [20] and the monographs [7], [5].

Under the symbol $L^{p(x)}_{dec}(u,\mathbb{R}_+)$ we mean the class of non-negative decreasing functions on $\mathbb{R}_+$ from $L^{p(x)}(u,\mathbb{R}^n) \cap DR(\mathbb{R}^n)$.

Now we list the well-known results regarding one-weight inequality for the operator T. For the following statement we refer to [1].

Theorem A. *Let r be constant such that $0 < r < \infty$. Then the inequity*

$$\int_0^\infty v(x)(Tf(x))^r dx \leq c \int_0^\infty v(x)(f(x))^r dx, \qquad f \in L^r(v,\mathbb{R}_+),\; f\downarrow \tag{2}$$

for a weight v holds, if and only if there exists a positive constant C such that for all $s > 0$

$$\int_s^\infty \left(\frac{s}{x}\right)^r v(x)dx \le C \int_0^s v(x)dx. \tag{3}$$

Condition (3) is called B_r condition and was introduced in [1].

Theorem B[3]. *Let v be a weight on $(0,\infty)$ and $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $0 < p^- \le p^+ < \infty$, and assume that $\psi_{p(\cdot),v(0^+)} = 0$. The following facts are equivalent:*
(a) There exists a positive constant c such that for any $f \in D(\mathbb{R}_+)$,

$$\int_0^\infty \big(Tf(x)\big)^{p(x)} v(x)dx \le C \int_0^\infty \big(f(x)\big)^{p(x)} v(x)dx. \tag{4}$$

(b) For any $r, s > 0$,

$$\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} v(x)dx \le C \int_0^r \frac{v(x)}{s^{p(x)}}dx. \tag{5}$$

(c) $p_{|supp\ v} \equiv p_0$ a.e and $v \in B_{p_0}$.

Proposition 2.1. *For the operators T, H, R_α and I_α, the following relations hold:*
(a)

$$R_\alpha f \approx Tf, \qquad 0 < \alpha < 1, \qquad f \in D(\mathbb{R}_+);$$

(b)

$$I_\alpha g \approx Hg, \qquad 0 < \alpha < n, \qquad g \in DR(\mathbb{R}^n).$$

Proof. (*a*) Upper estimate. Represent $R_\alpha f$ as follows:

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}}dt + \frac{1}{x^\alpha} \int_{x/2}^x \frac{f(t)}{(x-t)^{1-\alpha}}dt = S_1(x) + S_2(x).$$

Observe that if $t < x/2$, then $x/2 < x - t$. Hence

$$S_1(x) \le c\frac{1}{x} \int_0^{x/2} f(t)dt \le cTf(x),$$

where the positive constant c does not depend on f and x. Using the fact that f is decreasing we find that

$$S_2(x) \le cf(x/2) \le cTf(x).$$

Lower estimate follows immediately by using the fact that f is non-negative and the obvious estimate $x-t \le x$ and $0<t<x$.

(b) Upper estimate. Let us represent the operator I_α as follows:

$$I_\alpha g(x) = \frac{1}{|x|^\alpha} \int\limits_{|y|<|x|/2} \frac{g(y)}{|x-y|^{n-\alpha}} dy + \frac{1}{|x|^\alpha} \int\limits_{|x|/2<|y|<|x|} \frac{g(y)}{|x-y|^{n-\alpha}} dy$$
$$=: S_1'(x) + S_2'(x).$$

Since $|x|/2 \le |x-y|$ for $|y|<|x|/2$ we have that

$$S_1'(x) \le \frac{c}{|x|^n} \int\limits_{|y|<|x|/2} g(y)dy \le cHg(x).$$

Taking into account the fact that f is radially decreasing on $\mathbb{R}^n$ we find that there is a decreasing function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$S_2'(x) \le f(|x|/2) \cdot \frac{1}{|x|^\alpha} \int\limits_{|x|/2<|y|<|x|} |x-y|^{\alpha-n} dy$$

Let $F_x = \{y : |x|/2 < |y| < |x|\}$. Then we have

$$\int\limits_{F_x} |x-y|^{\alpha-n} dy = \int\limits_0^\infty \big|\{y \in F_x : |x-y|^{\alpha-n} > t\}\big| dt$$
$$\le \int\limits_0^{|x|^{\alpha-n}} \big|\{y \in F_x : |x-y|^{\alpha-n} > t\}\big| dt + \int\limits_{|x|^{\alpha-n}}^\infty \big|\{y \in F_x : |x-y|^{\alpha-n} > t\}\big| dt$$
$$=: I_1 + I_2.$$

It is easy to see that

$$I_1 \le \int\limits_0^{|x|^{\alpha-n}} |B(0,|x|)| dt = c|x|^\alpha,$$

while using the fact that $\frac{n}{n-\alpha} > 1$ we find that

$$I_2 \le \int\limits_{|x|^{\alpha-n}}^\infty \big|\{y \in F_x : |x-y| \le t^{\frac{1}{\alpha-n}}\}\big| dt \le c \int\limits_{|x|^{\alpha-n}}^\infty t^{\frac{n}{\alpha-n}} dt = c_{\alpha,n}|x|^\alpha.$$

Finally we conclude that

$$S_2'(x) \le cf(|x|/2) \le cHf(x).$$

Lower estimate follows immediately by using the fact that f is non-negative and the obvious estimate $|x-y| \le |x|$, where $0<|y|<|x|$. $\square$

We will also need the following statement:

Lemma 2.2. *Let r be a constant such that $0<r<\infty$. Then the inequality*

$$\int_{\mathbb{R}^n} \big(Hf(x)\big)^r u(x)dx \le C \int_{\mathbb{R}^n} \big(f(x)\big)^r u(x)dx, \qquad f \in L^r_{dec}(u,\mathbb{R}^n) \tag{6}$$

holds, if and only if there exists a positive constant C such that for all $s>0$,

$$\int_{|x|>s} \Big(\frac{s}{|x|}\Big)^r |x|^{r(1-n)} u(x)dx \le C \int_{|x|<s} |x|^{r(1-n)} u(x)dx. \tag{7}$$

Proof. We shall see that inequality (6) is equivalent to the inequality

$$\int_0^\infty \tilde{u}(t)\big(T\bar{f}(t)\big)^r dt \le C \int_0^\infty \tilde{u}(t)\big(\bar{f}(t)\big)^r dt,$$

where $\tilde{u}(t) = t^{(n-1)(1-r)}\bar{u}(t)$, $\bar{f}(t) = t^{n-1}f(t)$ and $\bar{u}(t) = \int_{S_0} u(t\bar{x})d\sigma(\bar{x})$.

Indeed, using polar the coordinates in $\mathbb{R}^n$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \big(Hf(x)\big)^r u(x)dx &= \int_{\mathbb{R}^n} u(x)\bigg(\frac{1}{|x|^n}\int_{|y|<|x|} f(y)dy\bigg)^r dx \\
&= \int_0^\infty t^{n-1}\bigg(\frac{1}{|t|^n}\int_{|y|<|x|} f(y)dy\bigg)^r \bigg(\int_{S_0} u(t\bar{x})d\sigma\bar{x}\bigg)dt \\
&= C\int_0^\infty t^{n-1}t^{-nr}t^r\bigg(\frac{1}{t}\int_0^t \tau^{n-1}f(\tau)d\tau\bigg)^r \bar{u}(t)dt \\
&= C\int_0^\infty t^{n-1}t^{r(1-n)}\bar{u}(t)\bigg(\frac{1}{t}\int_0^t \bar{f}(\tau)d\tau\bigg)^r dt \\
&\le C\int_0^t \tilde{u}(t)\big(\bar{f}(t)\big)^r dt \\
&= Ct^{(n-1)(1-r)}t^{(n-1)r}\big(f(t)\big)^r dt \\
&= C\int_{\mathbb{R}^n} \big(f(x)\big)^r u(x)dx.
\end{aligned}$$

□

3. THE MAIN RESULTS

To formulate the main results we need to prove

Proposition 3.1. *Let u be a weight on $\mathbb{R}^n$ and $p : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ such that $0 < p^- \le p^+ < \infty$, and assume that $\varphi_{p(\cdot),u(0+)} = 0$. The following statements are equivalent:*
(a) There exists a positive constant C such that for any $f \in DR(\mathbb{R}^n)$,

$$\int\limits_{\mathbb{R}^n} \big(Hf(x)\big)^{p(x)} u(x)dx \le C \int\limits_{\mathbb{R}^n} \big(f(x)\big)^{p(x)} u(x)dx. \tag{8}$$

(b) For any $r, s > 0$,

$$\int\limits_{|x|>r} \left(\frac{r}{s|x|^n}\right)^{p_0} u(x)dx \le C \int\limits_{B(0,r)} \frac{|x|^{(1-n)p_0} u(x)}{s^{p_0}} dx. \tag{9}$$

(c) $p_{|_{supp\ u}} \equiv p_0$ a.e and $u \in B_{p_0}$.

Proof. We use the arguments of [3]. To show that (a) implies (b) it is enough to test the modular inequality (8) for the function $f_{r,s}(x) = \frac{1}{s}\chi_{B(0,r)}(x)|x|^{1-n}$, $s, r > 0$. Indeed, it can be checked that

$$Hf_{r,s}(x) = \begin{cases} \frac{1}{|x|^n s} \int\limits_{|y|\le|x|} |y|^{1-n} dy, & \text{if } |x| \le r; \\ \\ \frac{1}{|x|^n s} \int\limits_{|y|\le r} |y|^{1-n} dy, & \text{if } |x| > r \end{cases}.$$

Further, we find that

$$\int\limits_{|x|>r} u(x)\big(Hf_{r,s}\big)^{p(x)} dx \le \int\limits_{\mathbb{R}^n} u(x)\big(Hf_{r,s}\big)^{p(x)} dx \le C \int\limits_{\mathbb{R}^n} u(x)\left(\frac{1}{s}\chi_{B(0,r)}(x)|x|^{1-n}\right)^{p(x)} dx.$$

Therefore

$$\int\limits_{|x|>r} u(x)\left(\frac{r}{s|x|^n}\right)^{p(x)} dx \le C \int\limits_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx.$$

To obtain (c) from (b) we are going to prove that condition (b) implies that $\varphi_{p(\cdot),u(\delta)}$ is a constant function, namely $\varphi_{p(\cdot),u(\delta)} = p_u^+ - p_u^-$ for all $\delta > 0$. This fact and the hypothesis on $\varphi_{p(\cdot),u(\delta)}$ implies that $\varphi_{p(\cdot),u(\delta)} \equiv 0$, and hence due to (1),

$$p_{|\text{supp u}} \equiv p_u^+ - p_u^- \equiv p_0 \quad \text{a.e.}.$$

Finally (9) means that $u \in B_{p_0}$. Let us suppose that $\varphi_{p(\cdot),u}$ is not constant. Then one of the following conditions hold:
(i) there exists $\delta > 0$ such that

$$\alpha = \operatorname*{ess\,sup}_{x \in B(0,\delta) \cap \text{supp u}} p(x) < p_u^+ < \infty, \tag{10}$$

and hence, there exists $\epsilon > 0$ such that

$$\left| \{|x| > \delta : p(x) \geq \alpha + \epsilon\} \cap \text{supp u} \right| > 0,$$

or
(ii) there exists $\delta > 0$ such that

$$\beta = \operatorname*{ess\,inf}_{x \in B(0,\delta) \cap \text{supp } u} p(x) > p_u^- > 0, \tag{11}$$

and then, for some $\epsilon > 0$,

$$\left| \{|x| > \delta : p(x) \leq \beta - \epsilon\} \cap \text{supp} u \right| > 0.$$

In the case (i) we observe that condition (b) for $r = \delta$, implies that

$$\int\limits_{|x|>\delta} \left(\frac{\delta}{s} \right)^{p(x)} \frac{u(x)}{|x|^{np(x)}} dx \leq C \int\limits_{B(0,\delta)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx.$$

Then using (10) we obtain, for $s < \min(1, \delta)$,

$$\left(\frac{\delta}{s} \right)^{\alpha+\epsilon} \int\limits_{\{|x| \geq \delta : p(x) \geq \alpha+\epsilon\}} \frac{u(x)}{|x|^{np(x)}} dx \leq \frac{C}{s^\alpha} \int\limits_{B(0,\delta)} u(x) |x|^{(1-n)p(x)} dx,$$

which is clearly a contradiction if we let $s \downarrow 0$. Similarly in the case (ii) let us consider the same condition (b) for $r = \delta$, and fix now $s > 1$. Taking into account (11) we find that:

$$\frac{1}{s^{\beta-\epsilon}} \int\limits_{\{|x| \geq \delta : p(x) \leq \beta-\epsilon\}} \left(\frac{\delta}{|x|^n} \right)^{p(x)} u(x) dx \leq \frac{C}{s^\beta} \int\limits_{B(0,\delta)} |x|^{(1-n)p(x)} u(x) dx,$$

which is a contradiction if we let $s \uparrow \infty$.
Finally, the fact that condition (c) implies (a) follows from[1,Theorem 1.7] □

Theorem 3.2. *Let u be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$. Assume that $\psi_{p(\cdot),v(0^+)} = 0$. The following facts are equivalent:*
(i) There exists a positive constant C such that for any $f \in D(\mathbb{R}_+)$,

$$\int\limits_{\mathbb{R}_+} \left(R_\alpha f(x) \right)^{p(x)} v(x) dx \leq C \int\limits_{\mathbb{R}_+} \left(f(x) \right)^{p(x)} v(x) dx.$$

(ii) condition (5) holds;

(*iii*) *condition* (*c*) *of Theorem B is be satiesfied.*

Proof. Proof follows by using Theorems B and Proposition 2.1(a). □

Theorem 3.3. *Let u be a weight on $\mathbb{R}^n$ and $p : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\varphi_{p(\cdot),u(0^+)} = 0$. The following facts are equivalent:*
(*i*) *There exists a positive constant C such that for any $f \in DR(\mathbb{R}^n)$,*

$$\int\limits_{\mathbb{R}^n} \big(I_\alpha f(x)\big)^{p(x)} u(x)dx \leq C \int\limits_{\mathbb{R}^n} \big(f(x)\big)^{p(x)} u(x)dx.$$

(*ii*) *condition* (9) *holds;*
(*iii*) *condition* (*c*) *of Proposition 3.1 holds.*

Proof. Proof follows by using Propositions 3.1 and Proposition 2.1 (b). □

Acknowledgement. The authors are grafeful to Prof. A. Meskhi for drawing out attention to the problem studied in this paper and helpful remarks.

References

[1] M. A. Arino and B. Muckenhoupt, Maximal functions on clasical Lorentz spaces and Hardy's inequality with weight for nonincreasing functios *Trans. Amer. Math. soc.* **320** (1990), 727-735.

[2] U. Ashraf, V. Kokilashvili and A. Meskhi, Weight characterization of the trace inequality for the generalized Riemann-Liouville transform in $L^{p(x)}$ spaces, *Math. Ineq. Appl.* **13**(2010), No.1, 63–81.

[3] S. Boza and J. Soria Weighted Hardy modular inequalities in variable L^p spaces for decreasing functions *J. Math. Appl.* **348** (2008), 383-388.

[4] S. Boza and J. Soria, Weighted weak modular and norm inequalities for the Hardy operator in variable L^p spaces of monotone functions, *Revista Math. Compl.*, **25** (2012), 459-474

[5] D. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue spaces*, Birkhäuser, Springer, Basel, 2013.

[6] D. Cruz-Uribe and F. Mamedov, On a general weighted Hardy type inequality in variable exponent Lebesgue spaces, *Rev. Mat. Complut.***25**(2012), 335–367.

[7] L. Diening, P. Harjulehto, P. Hästö and M.Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, Vol. 2017, Springer, Heidelberg, 2011.

[8] L. Diening and S. Samko, Hardy inequality in variable exponent Lebesgue spaces,*Fract. Calc. Appl.* Anal. **10**(2007), 1-18.

[9] L. Diening and S. Samko, Hardy inequality in variable exponent Lebesgue spaces, Frac. Calc. Appl. Anal. **10** (2007), 1–18.

[10] D.E. Edmunds and A. Meskhi, Potential-type operators in $L^{p(x)}$ spaces., *Z. Anal. Anwend* **21** (2002), 681–690.

[11] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, On the boundedness and compactness of the weighted Hardy operators in $L^{p(x)}$ spaces. *Georgian Math. J.* **12** (2005), 27–44.

[12] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, Two-weight estimates in $L^{p(\cdot)}$ spaces with applications to Fourier series. *Houston J. Math.* **35**(2009) , No. 2, 665–689.
[13] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, *Bounded and compact integral operators, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, Boston, London*, 2002.
[14] V. Kokilashvili, A. Meskhi and M. Sarwar,One and Two-weight norm estimates for one-sided operator in $L^{p(x)}$ spaces, *Eurasian Math. J.,*1(2010), N0. 1, 73-110.
[15] T. S. Kopaliani, On some structural properties of Banach function spaces and boundedness of certain integral operators, *Czechoslovak Math. J.* **54**(2004), No. 3, 791-805.
[16] O. Kovácik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.* **41**(1991), No.4, 592–618.
[17] F. I. Mamedov and A. Harman, On a Hardy type general weighted inequality in spaces, *Integral Equ. Oper. Th.* **66**(2010), No.4, 565–592
[18] A. Meskhi, *Measure of Non-compactness for Integral Operators in Weighted Lebesgue Spaces.* Nova Science Publishers, New York , 2009.
[19] M. Ružička, *Electrorheological fluids: modeling and mathematical theory.* Lecture Notes in Mathematics, 1748, Berlin, Springer, 2000.
[20] S. Samko, Convolution type operators in $L^{p(x)}$, *Integral Transforms Spec. Funct.* **7**(1998), Nos 1-2, 123-144.
[21] E. T. Sawyer, Multipliers of Besov and power weighted L^2 spaces, *Indiana Univ. Math. J.*, **33**(1984),No. 3, 353–366.
[22] E. Sawyer, Boundedness of classical operators on classical Lorentz spaces, *Studia Math.* **96**(1990), No. 2, 145–158.
[23] G. Sinnamon, Four questions related to Hardy's inequality. *Function spaces and applications (Delhi, 1997)*, 255-266, *Narosa, New Delhi,* 2000.

Authors' Addresses:

G. Murtaza:
Department of Mathematics,
GC University, Faisalabad, Pakistan
Email: gmnizami@@googlemail.com
M. Sarwar:
Department of Mathematics,
University of Malakand, Chakdara,
Dir Lower, Khyber Pakhtunkhwa, Pakistan
E-mail: sarwar@uom.edu.pk; sarwarswati@gmail.com