Counting tuples restricted by coprimality conditions

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Abstract

Given a set $A = \{(i_1, j_1), \ldots, (i_m, j_m)\}\$ we say that (a_1, \ldots, a_v) exhibits pairwise coprimality if $gcd(a_i, a_j) = 1$ for all $(i, j) \in A$. For a given positive x we give an asymptotic formula for the number of (a_1, \ldots, a_v) with $1 \leq a_1, \ldots, a_v \leq x$ that exhibit pairwise coprimality. Our error term is better than that of Hu.

1 Introduction

We study tuples whose elements are positive integers of maximum value x and impose certain coprimality conditions on pairs of elements. Toth $[10]$ used an inductive approach to give an asymptotic formula for the number of height constrained tuples that exhibit pairwise coprimality. For a generalisation from pairwise coprimality to v-wise coprimality see $[6]$.

Recently Fernández and Fernández, in [\[1\]](#page-14-0) and in subsequent discussions with the second author, have shown how to calculate the probability that v positive integers of any size exhibit coprimality across given pairs. Their approach is non-inductive. Hu [\[7\]](#page-15-2) has estimated the number of (a_1, \ldots, a_v) with $1 \leq a_1, \ldots, a_v \leq x$ that satisfy given coprimality conditions on pairs of elements of the v-tuple. His inductive approach gives an asymptotic formula with an upper bound on the error term of $O(x^{v-1} \log^{v-1} x)$.

Coprimality across given pairs of elements of a v -tuple is not only interesting in its own right. To date it has been necessary for quantifying v -tuples that are totally pairwise non-coprime, that is, $gcd(i, j) > 1$ for all $1 \leq i, j \leq v$ (see $[7], [5]$ $[7], [5]$ and $[8]$ and its comments regarding $[2]$).

Our main result gives a better error term than that of [\[7\]](#page-15-2). Unlike [\[7\]](#page-15-2) our approach is non-inductive.

We use a graph to represent the required primality conditions as follows. Let $G = (V, E)$ be a graph with v vertices and e edges. The set of vertices, V, will be given by $V = \{1, \ldots, v\}$ whilst the set of edges of G, denoted by E , is a subset of the set of pairs of elements of V . That is, $E \subset \{\{1,2\},\{1,3\},\ldots,\{r,s\},\ldots,\{v-1,v\}\}.$ We admit isolated vertices (that is, vertices that are not adjacent to any other vertex). An edge is always of the form $\{r, s\}$ with $r \neq s$ and $\{r, s\} = \{s, r\}$. For each real $x > 0$ we define the set of all tuples that satisfy the primality conditions by

$$
G(x) := \{ (a_1, \ldots, a_v) \in \mathbb{N}^v : a_r \le x, \ \gcd(a_r, a_s) = 1 \ \text{if} \ \{r, s\} \in E \}.
$$

We also let $q(x) = \text{card}(G(x))$, and denote with d the maximum degree of the vertices of G. Finally, let $Q_G(x) = 1 + a_2x^2 + \cdots + a_vx^v$ be the polynomial associated to the graph G defined in Section [2.](#page-2-0)

Our main result is as follows.

Theorem 1. For real $x > 0$ we have

$$
g(x) = x^v \rho_G + O(x^{v-1} \log^d x),
$$

where

$$
\rho_G = \prod_{p \ prime} Q_G \left(\frac{1}{p}\right).
$$

2 Preparations

As usual, for any integer $n \geq 1$, let $\omega(n)$ and $\sigma(n)$ be the number of distinct prime factors of n and the sum of divisors of n respectively (we also set $\omega(1) = 0$. We also use μ to denote the Möbius function, that is, $\mu(n) =$ $(-1)^{\omega(n)}$ if *n* is square free, and $\mu(n) = 0$ otherwise. $P^+(n)$ denotes the largest prime factor of the integer $n > 1$. By convention $P^+(1) = 1$. We recall that the notation $U = O(V)$ is equivalent to the assertion that the inequality $|U| \le c|V|$ holds for some constant $c > 0$. We will denote the least common multiple of integers x_1, \ldots, x_v by $[x_1, \ldots, x_v]$.

For each $F \subset E$, a subset of the edges of G, let $v(F)$ be the number of non-isolated vertices of F. We define two polynomials $Q_G(x)$ and $Q_G^+(x)$ by

$$
Q_G(x) = \sum_{F \subset E} (-1)^{\text{card}(F)} x^{v(F)}, \qquad Q_G^+(x) = \sum_{F \subset E} x^{v(F)}.
$$

In this way we associate two polynomials to each graph. It is clear that the only $F \subset E$ for which $v(F) = 0$ is the empty set. Thus the constant term of $Q_G(x)$ and $Q_G^+(x)$ is always 1. If F is non-empty then there is some edge $a = \{r, s\} \in F$ so that $v(F) \geq 2$. Therefore the coefficient of x in $Q_G(x)$ and $Q_G^+(x)$ is zero. Since we do not allow repeated edges the only case in which $v(F) = 2$ is when F consists of one edge. Thus the coefficient of x^2 in $Q_G^+(x)$ is e, that is, the number of edges e in G . The corresponding x^2 coefficient in $Q_G(x)$ is $-e$.

As a matter of notation we shall sometimes use r and s to indicate vertices. The letter v will always denote the last vertex and the number of vertices in a given graph. Edges will sometimes be denoted by a or b . As previously mentioned, we use d to denote the maximum degree of any vertex and e to denote the number of edges. We use terms like e_i to indicate the j-th edge.

We associate several multiplicative functions to any graph. To define these functions we consider functions $E \to \mathbb{N}$, that is, to any edge a in the graph we associate a natural number n_a . We call any of these functions, $a \mapsto n_a$, an edge numbering of the graph. Given an edge numbering we assign a corresponding vertex numbering function $r \mapsto N_r$ by the rule $N_r =$ $[n_{b_1},\ldots,n_{b_u}]$, where $E_r = \{b_1,\ldots,b_u\} \subset E$ is the set of edges incident to r. We note that in the case where r is an isolated vertex we will have $E_r = \emptyset$ and $N_r = 1$. With these notations we define

$$
f_G(m) = \sum_{N_1N_2\cdots N_v=m} \mu(n_1)\cdots\mu(n_e), \quad f_G^+(m) = \sum_{N_1N_2\cdots N_v=m} |\mu(n_1)\cdots\mu(n_e)|,
$$

where the sums extend to all possible edge numberings of G.

The following is interesting in its own right but will also be used to prove Theorem [1.](#page-1-0)

Proposition 2. Let $f : \mathbb{N} \to \mathbb{C}$ be a multiplicative function. For any graph G the function

$$
g_{f,G}(m) = \sum_{N_1N_2\cdots N_v=m} f(n_1)\cdots f(n_e)
$$

is multiplicative.

Proof. Let $m = m_1 m_2$ where $gcd(m_1, m_2) = 1$. Let us assume that for a given edge numbering of G we have $N_1 \cdots N_v = m$. For any edge $a = \{r, s\}$ we have $n_a|N_r$ and $n_a|N_s$. Therefore $n_a^2|m$. It follows that we may express n_a as $n_a = n_{1,a} n_{2,a}$ with $n_{1,a}|m_1$ and $n_{2,a}|m_2$. In this case $gcd(n_{1,a}, n_{2,a}) = 1$, and we will have

$$
N_r = [n_{b_1}, \dots, n_{b_v}] = [n_{1,b_1}, \dots, n_{1,b_v}][n_{2,b_1}, \dots, n_{2,b_v}],
$$

$$
f(n_1) \cdots f(n_e) = f(n_{1,1}) \cdots f(n_{1,e}) \cdot f(n_{2,1}) \cdots f(n_{2,e}).
$$

Since each edge numbering n_a splits into two edge numberings $n_{1,a}$ and $n_{2,a}$, we have

$$
m_1 = N_{1,1} \cdots N_{1,v}, \quad m_2 = N_{2,1} \cdots N_{2,v}.
$$

Thus

$$
g_{f,G}(m_1m_2) = g_{f,G}(m)
$$

=
$$
\sum_{N_1N_2\cdots N_v=m} f(n_1)\cdots f(n_e)
$$

=
$$
\sum_{N_{1,1}\cdots N_{1,v} \cdot N_{2,1}\cdots N_{2,v}=m_1m_2} f(n_{1,1})\cdots f(n_{1,e}) \cdot f(n_{2,1})\cdots f(n_{2,e})
$$

=
$$
\sum_{N_{1,1}\cdots N_{1,v}=m_1} f(n_{1,1})\cdots f(n_{1,e}) \sum_{N_{2,1}\cdots N_{2,v}=m_2} f(n_{2,1})\cdots f(n_{2,e})
$$

=
$$
g_{f,G}(m_1)g_{f,G}(m_2),
$$

which completes the proof.

We now draw the link between $f_G^+(p^k)$ and $Q_G^+(x)$.

 \Box

Lemma 3. For any graph G and prime p the value $f_G^+(p^k)$ is equal to the coefficient of x^k in $Q_G^+(x)$. In the same way the value of $f_G(p^k)$ is equal to the coefficient of x^k in $Q_G(x)$.

Proof. First we consider the case of $f_G(p^k)$. Recall that

$$
Q_G(x) = \sum_{F \subset E} (-1)^{\text{card}(F)} x^{v(F)}, \qquad f_G(p^k) = \sum_{N_1 \cdots N_v = p^k} \mu(n_1) \cdots \mu(n_e),
$$

where the last sum is on the set of edge numberings of G . In the second sum we shall only consider edge numberings of G giving a non null term. This means that we only consider edge numberings with n_a squarefree numbers. Notice also that if $N_1 \cdots N_v = p^k$, then each $n_a \mid p^k$. So the second sum extends to all edge numbering with $n_a \in \{1, p\}$ for each edge $a \in E$ and satisfying $N_1 \cdots N_v = p^k$.

We need to prove the equality

$$
\sum_{F \subset E, \ v(F)=k} (-1)^{\text{card}(F)} = \sum_{N_1 \cdots N_v = p^k} \mu(n_1) \cdots \mu(n_e). \tag{1}
$$

To this end we shall define for each $F \subset E$ with $v(F) = k$ a squarefree edge numbering $\sigma(F) = (n_a)$ with $N_1 \cdots N_v = p^k$, $n_a \in \{1, p\}$ and such that $(-1)^{\text{card}(F)} = \mu(n_1) \cdots \mu(n_e)$. We will show that σ is a bijective mapping between the set of $F \subset E$ with $v(F) = k$ and the set of edge numberings (n_a) with $N_1 \cdots N_v = p^k$. Thus equality [\(1\)](#page-4-0) will be established and the proof finished.

Assume that $F \subset E$ with $v(F) = k$. We define $\sigma(F)$ as the edge numbering (n_a) defined by

$$
n_a = p
$$
 for any $a \in F$, $n_a = 1$ for $a \in E \setminus F$.

In this way it is clear that $\mu(n_1)\cdots\mu(n_e) = (-1)^{\text{card}(F)}$. Also $N_r = p$ or $N_r = 1$. We will have $N_r = p$ if and only if there is some $a = \{r, s\} \in F$. So that $N_1 \cdots N_v = p^{v(F)}$ because by definition $v(F)$ is the cardinality of the union $\bigcup_{\{r,s\}\in F}\{r,s\}.$

The map σ is invertible. For let (n_a) be an edge numbering of squarefree numbers with $N_1 \cdots N_v = p^k$ and $n_a \in \{1, p\}$. If $\sigma(F) = (n_a)$ necessarily we will have $F = \{a \in E : n_a = p\}$. It is clear that defining F in this way we will have $v(F) = k$ and $\sigma(F) = (n_a)$.

Therefore the coefficient of x^k in $Q_G(x)$ coincide with the value of $f_G(p^k)$. The proof for f_G^+ is the same observing that for $\sigma(F) = (n_a)$ we will have $1 = |(-1)^{\text{card}(F)}| = |\mu(n_1) \cdots \mu(n_e)|.$ \Box

3 Proof of Theorem [1](#page-1-0)

We prove the theorem in the following steps:

1. We show that

$$
g(x) = \sum_{n_1,\dots,n_e} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^v \left\lfloor \frac{x}{N_r} \right\rfloor.
$$

2. We show that

$$
g(x) = x^v \sum_{n_1=1}^{\infty} \cdots \sum_{n_e=1}^{\infty} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^{v} \frac{1}{N_r} + R + O\left(x^{v-1} \log^d x\right),
$$

where

$$
|R| \leq x^{v-1} \sum_{j=1}^e \sum_{n_1=1}^\infty \cdots \sum_{n_{j-1}=1}^\infty \sum_{n_j > x} \sum_{n_{j+1}=1}^\infty \cdots \sum_{n_e=1}^\infty \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^v \frac{1}{N_r}.
$$

3. We show that $|R| = O(x^{v-1} \log^d x)$.

We start with the following sieve result which generalises the sieve of Eratosthenes.

Lemma 4. Let X be a finite set, and let $A_1, A_2, \ldots, A_k \subset X$. Then

$$
\operatorname{card}\left(X\setminus\bigcup_{j=1}^k A_j\right) = \sum_{J\subset\{1,2,\ldots,k\}} (-1)^{\operatorname{card}(J)} \operatorname{card}(A_J),
$$

where $A_{\emptyset} = X$, and for $J \subset \{1, 2, ..., k\}$ nonempty

$$
A_J = \bigcap_{j \in J} A_j.
$$

To prove Theorem [1](#page-1-0) let X be the set

$$
X = \{(a_1, \ldots, a_v) \in \mathbb{N}^v : a_r \le x, 1 \le r \le v\}.
$$

Our set $G(x)$, associated to the graph G, is a subset of X. Now for each prime $p \leq x$ and each edge $a = \{r, s\} \in G$ define the following subset of X.

$$
A_{p,a} = \{(a_1, \ldots, a_v) \in X : p|a_r, p|a_s\}.
$$

Therefore the tuples in $A_{p,a}$ are not in $G(x)$. In fact it is clear that

$$
G(x) = X \setminus \bigcup_{\substack{a \in E \\ p \le x}} A_{p,a},
$$

where E denotes the set of edges in our graph G . We note that we have an $A_{p,a}$ for each prime number less than or equal to x and each edge $a \in E$. Denoting P_x as the set of prime numbers less than or equal to x we can represent each $A_{p,a}$ as A_j with $j \in P_x \times E$. We now apply Lemma [4](#page-5-0) and obtain

$$
g(x) = \sum_{J \subset P_x \times E} (-1)^{\operatorname{card}(J)} \operatorname{card}(A_J). \tag{2}
$$

We compute card (A_J) and then card (J) . For card (A_J) we have

$$
J = \{(p_1, e_1), \ldots, (p_m, e_m)\}, \quad A_J = \bigcap_{j=1}^m A_{p_j, e_j}.
$$

Therefore $(a_1, \ldots, a_v) \in A_J$ is equivalent to saying that $p_j | a_{r_j}, p_j | a_{s_j}$ for all $1 \leq j \leq m$, where $e_j = \{r_j, s_j\}$. We note that if $p_{i_1}, \ldots, p_{i_\ell}$ are the primes associated in J with a given edge $a = \{r, s\}$, then the product of $p_{i_1} \cdots p_{i_\ell}$ must also divide the values a_r and a_s associated to the vertices of a. Let $T_a \subset P_x$ consist of the primes p such that $(p, a) \in J$. In addition we define

$$
n_a = \prod_{p \in T_a} p,
$$

observing that when $T_a = \emptyset$ we have $n_a = 1$. Then $(a_1, \ldots, a_v) \in A_J$ is equivalent to saying that for each $a = \{r, s\}$ appearing in J we have $n_a | a_r$ and $n_a \mid a_s$. In this way we can define J by giving a number n_a for each edge a. We note that n_a will always be squarefree, and all its prime factors will be less than or equal to x. We also note that $(a_1, \ldots, a_v) \in A_J$ is equivalent to saying that $n_a|a_r$ for each edge a that joins vertex r with another vertex.

Then for each vertex r , consider all the edges a joining r to other vertices, and denote the least common multiple of the corresponding n_a 's by N_r . So $(a_1, \ldots, a_v) \in A_J$ is equivalent to saying that $N_r | a_r$. The number of multiples of N_r that are less than or equal to x is $\lfloor x/N_r \rfloor$, so we can express the number of elements of A_J as

$$
card(A_J) = \prod_{r=1}^{v} \left\lfloor \frac{x}{N_r} \right\rfloor.
$$
 (3)

We now compute $card(J)$. This is the total number of prime factors across all the n_j . As mentioned before n_j is squarefree, so

$$
(-1)^{\operatorname{card}(J)} = (-1)^{\sum_{j=1}^{e} \omega(n_j)} = \mu(n_1) \cdots \mu(n_e), \tag{4}
$$

where the summations are over all squarefree n_j with $P^+(n_j) \leq x$. Substituting (3) and (4) into (2) yields

$$
g(x) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_e=1}^{\infty} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^{v} \left\lfloor \frac{x}{N_r} \right\rfloor.
$$

At first the sum extends to the (n_1, \ldots, n_e) that are squarefree and have all prime factors less than or equal to x . But we may extend the sum to all (n_1, \ldots, n_e) , because if these conditions are not satisfied then the corresponding term is automatically 0. In fact we may restrict the summation to the $n_a \leq x$, because otherwise for $a = \{r, s\}$ we have $n_a \mid N_r$ and $\lfloor x/N_r \rfloor = 0$. Therefore

$$
g(x) = \sum_{1 \leq n_1 \leq x} \cdots \sum_{1 \leq n_e \leq x} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^v \left\lfloor \frac{x}{N_r} \right\rfloor.
$$

We now seek to express $g(x)$ as a multiple of x^v plus a suitable error term. Observe that for all real $z_1, z_2, z_3 > 0$,

$$
[z1][z2][z3] = z1z2z3 - z1z2{z3} - z1{z2][z3] - {z1][z2][z3],
$$

where $\{y\}$ denotes the fractional part of a number y.

Applying a similar procedure, with v factors instead of 3, we get

$$
g(x) = \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^{v} \frac{x}{N_r}
$$

\n
$$
- \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} \mu(n_1) \cdots \mu(n_e) \left\{ \frac{x}{N_1} \right\} \prod_{r=2}^{v} \left\lfloor \frac{x}{N_r} \right\rfloor
$$

\n
$$
- \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} \mu(n_1) \cdots \mu(n_e) \frac{x}{N_1} \left\{ \frac{x}{N_2} \right\} \prod_{r=3}^{v} \left\lfloor \frac{x}{N_r} \right\rfloor
$$

\n
$$
\cdots
$$

\n
$$
- \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} \mu(n_1) \cdots \mu(n_e) \frac{x}{N_1} \cdots \frac{x}{N_{v-1}} \left\{ \frac{x}{N_v} \right\}
$$

\n
$$
= x^v \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^{v} \frac{1}{N_r} + \sum_{k=1}^{v} R_k,
$$
 (5)

where for $1 \leq k \leq v$,

$$
R_k = -\sum_{1 \leq n_1 \leq x} \cdots \sum_{1 \leq n_e \leq x} \mu(n_1) \cdots \mu(n_e) \frac{x}{N_1} \cdots \frac{x}{N_{k-1}} \left\{ \frac{x}{N_k} \right\} \left\lfloor \frac{x}{N_{k+1}} \right\rfloor \cdots \left\lfloor \frac{x}{N_v} \right\rfloor,
$$

with the obvious modifications for $j = 1$ and $j = v$. We then have

$$
|R_k| \leq \sum_{1 \leq n_1 \leq x} \cdots \sum_{1 \leq n_e \leq x} |\mu(n_1) \cdots \mu(n_e)| \frac{x}{N_1} \cdots \frac{x}{N_{k-1}} \frac{x}{N_{k+1}} \cdots \frac{x}{N_v}
$$

$$
\leq x^{v-1} \sum_{P^+(m) \leq x} \frac{C_{G,k}(m)}{m},
$$

where

$$
C_{G,k}(m) = \sum_{m=\prod_{1 \leq r \leq v, r \neq k} N_r} |\mu(n_1) \cdots \mu(n_e)|.
$$

By similar reasoning to that of Proposition [2](#page-3-0) the function $C_{G,k}(m)$ can be shown to be multiplicative. The numbers $C_{G,k}(p^{\alpha})$ do not depend on p, and $C_{G,k}(p^{\alpha})=0$ for $\alpha > v$. So we have

$$
\sum_{p+(m)\leq x} \frac{C_{G,k}(m)}{m} \leq \prod_{p\leq x} \left(1 + \frac{C_{G,k}(p)}{p} + \frac{C_{G,k}(p^2)}{p^2} + \cdots + \frac{C_{G,k}(p^v)}{p^v}\right)
$$

$$
= O(\log^{C_{G,k}(p)} x),
$$

where $C_{G,k}(m)$ is the number of solutions (n_1, \ldots, n_e) , with n_j squarefree, to

$$
\prod_{1 \le r \le v, r \ne k} N_r = m. \tag{6}
$$

Let h_k denote the degree of vertex k. It is easy to see that for a prime p we have $C_{G,k}(p) = h_k$. The solutions are precisely those with all $n_j = 1$, except one $n_\ell = p$, where ℓ should be one of the edges meeting at vertex k. Therefore the maximum number of solutions occurs when k is one of the vertices of maximum degree. So if we let d be this maximum degree, then the maximum value of $C_{G,k}(p)$ is d. Therefore

$$
|R_k| = O(x^{v-1} \log^d x). \tag{7}
$$

Substituting [\(7\)](#page-9-0) into [\(5\)](#page-8-0) we obtain

$$
g(x) = x^v \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^v \frac{1}{N_r} + O(x^{v-1} \log^d x). \tag{8}
$$

We require the following lemma.

Lemma 5.

$$
\lim_{x \to \infty} \sum_{1 \leq n_1 \leq x} \cdots \sum_{1 \leq n_e \leq x} |\mu(n_1) \cdots \mu(n_e)| \prod_{r=1}^v \frac{1}{N_r} < +\infty.
$$

Proof. We have

$$
\lim_{x \to \infty} \sum_{1 \le n_1 \le x} \cdots \sum_{1 \le n_e \le x} |\mu(n_1) \cdots \mu(n_e)| \prod_{r=1}^v \frac{1}{N_r} = \sum_{m=1}^\infty \frac{f_G^+(m)}{m},\tag{9}
$$

where

$$
f_G^+(m) = \sum_{m=\prod_{r=1}^v N_r} |\mu(n_1) \cdots \mu(n_e)|.
$$

We note that $f_G^+(m)$ is multiplicative by Proposition [2.](#page-3-0) It is clear that $f_G^+(1) = 1$. Also, each edge joins two vertices r and s and thus $n_j | E_r$ and $n_j|E_s$. This means that

$$
n_j^2 \big| \prod_{r=1}^v N_r.
$$

It follows that

$$
\prod_{r=1}^{v} N_r \neq p,
$$

for any prime p and so $f_G^+(p) = 0$. We also note that a multiple (n_1, \ldots, n_e) only counts in $f_G^+(m)$ if $|\mu(n_1)\cdots\mu(n_e)|=1$. Therefore each n_j is squarefree. So each factor in

$$
\prod_{r=1}^{v} N_r \tag{10}
$$

brings at most a p. So the greatest power of p that can divide [\(10\)](#page-10-0) is p^v . So $f_G^+(p^{\alpha}) = 0$ for $\alpha > v$. Recall that $f_G^+(p^{\alpha})$ is equal to the coefficient of x^{α} in $Q_G^+(x)$. So, by Lemma [3,](#page-4-1) we note that $f_G^+(p^{\alpha})$ depends on α but not on p . Putting all this together we have

$$
\sum_{m=1}^{\infty} \frac{f_G^+(m)}{m} = \prod_{p \text{ prime}} \left(1 + \frac{f_G^+(p^2)}{p^2} + \ldots + \frac{f_G^+(p^v)}{p^v} \right) < +\infty.
$$
 (11)

Substituting [\(11\)](#page-10-1) into [\(9\)](#page-9-1) completes the proof.

Returning to [\(8\)](#page-9-2) it is now clear from Lemma [5](#page-9-3) that

$$
\rho_G = \lim_{x \to \infty} \sum_{1 \leq n_1 \leq x} \cdots \sum_{1 \leq n_e \leq x} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^v \frac{1}{N_r}
$$

is absolutely convergent. In fact,

$$
g(x) = x^v \rho_G + R + O(x^{v-1} \log^d x), \tag{12}
$$

where

$$
\rho_G = \sum_{n_1=1}^{\infty} \cdots \sum_{n_e=1}^{\infty} \mu(n_1) \cdots \mu(n_e) \prod_{r=1}^{v} \frac{1}{N_r},
$$

and

$$
|R| \leq x^{v-1} \sum_{j=1}^e \sum_{n_1=1}^\infty \cdots \sum_{n_{j-1}=1}^\infty \sum_{n_j > x} \sum_{n_{j+1}=1}^\infty \cdots \sum_{n_e=1}^\infty |\mu(n_1) \cdots \mu(n_e)| \prod_{r=1}^v \frac{1}{N_r}.
$$

$$
\Box
$$

Now

$$
\rho_G = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{N_1 \cdots N_v = m} \mu(n_1) \cdots \mu(n_e) = \sum_{m=1}^{\infty} \frac{f_G(m)}{m}.
$$

We note that $f_G(m)$ is multiplicative by Proposition [2.](#page-3-0) In a similar way to Lemma [5](#page-9-3) we have $f_G(1) = 1, f_G(p) = 0$ and $f_G(p^{\alpha}) = 0$, for all $\alpha > v$. Thus, by the multiplicativity,

$$
\rho_G = \sum_{m=1}^{\infty} \frac{f_G(m)}{m} = \prod_{p \text{ prime}} \left(1 + \frac{f_G(p^2)}{p^2} + \ldots + \frac{f_G(p^v)}{p^v} \right),
$$

Therefore, by Lemma [3,](#page-4-1) we have

$$
\rho_G = \prod_{p \text{ prime}} Q_G \left(\frac{1}{p}\right). \tag{13}
$$

Substituting [\(13\)](#page-11-0) into [\(12\)](#page-10-2), it only remains to show that $|R| = O(x^{v-1} \log^d x)$. We have

$$
|R| \leq x^{v-1} \sum_{j=1}^e \sum_{n_1=1}^\infty \cdots \sum_{n_{j-1}=1}^\infty \sum_{n_j > x} \sum_{n_{j+1}=1}^\infty \cdots \sum_{n_e=1}^\infty |\mu(n_1) \cdots \mu(n_e)| \prod_{r=1}^v \frac{1}{N_r}.
$$

All terms in the sum on j are analogous; so assuming that the first is the largest, we have

$$
|R| \leq C_1 x^{v-1} \sum_{n_1 > x} \sum_{n_2=1}^{\infty} \sum_{n_{j+1}=1}^{\infty} \cdots \sum_{n_e=1}^{\infty} |\mu(n_1) \cdots \mu(n_e)| \prod_{r=1}^{v} \frac{1}{N_r},
$$

where C_1 is a function of e and not x. So it will suffice to show that

$$
R_1 := \sum_{n_1 > x} \sum_{n_2=1}^{\infty} \cdots \sum_{n_e=1}^{\infty} |\mu(n_1) \cdots \mu(n_e)| \prod_{r=1}^{v} \frac{1}{N_r} = O(\log^d x). \tag{14}
$$

We will treat an edge $e_1 = \{r, s\}$ differently to the other edges. For a given (n_1, \ldots, n_e) of squarefree numbers we have two special N_r ,

$$
N_r = [n_1, n_{\alpha_1}, \ldots, n_{\alpha_k}], \quad N_s = [n_1, n_{\beta_1}, \ldots, n_{\beta_k}].
$$

We also remark that we may have $N_r = [n_1]$ or $N_s = [n_1]$.

For any edge e_j with $2 \leq j \leq e$ we define $d_j = \gcd(n_1, n_j)$. Since the n_j are squarefree, we have

$$
n_j = d_j n'_j
$$
, $d_j | n_1$, $gcd(n_1, n'_j) = 1$.

Then it is clear that

$$
N_r = [n_1, d_{\alpha_1} n'_{\alpha_1}, \dots, d_{\alpha_k} n'_{\alpha_k}] = n_1 [n'_{\alpha_1}, \dots, n'_{\alpha_k}], \quad N_s = n_1 [n'_{\beta_1}, \dots, n'_{\beta_l}].
$$

For any other vertex with $t\neq r$ and $t\neq s,$ we have

$$
N_t = [n_{t_1}, \ldots, n_{t_m}] = [d_{t_1} n'_{t_1}, \ldots, d_{t_m} n'_{t_m}] = [d_{t_1}, \ldots, d_{t_m}] [n'_{t_1}, \ldots, n'_{t_m}],
$$

where m will vary with t. Substituting the equations for N_r , N_s and N_t into the definition of R_1 in [\(14\)](#page-11-1) we obtain

$$
R_{1} = \sum_{n_{1}>x} \sum_{n_{2}=1}^{\infty} \cdots \sum_{n_{e}=1}^{\infty} |\mu(n_{1}) \cdots \mu(n_{e})| \frac{1}{N_{r}} \frac{1}{N_{s}} \prod_{\substack{1 \leq t \leq v \\ t \neq r, t \neq s}} \frac{1}{N_{t}}
$$

\n
$$
= \sum_{n_{1}>x} \frac{|\mu(n_{1})|}{n_{1}^{2}} \sum_{d_{2}|n_{1}} \cdots \sum_{d_{e}|n_{1}} \sum_{n'_{2}=1}^{\infty} \cdots \sum_{n'_{e}=1}^{\infty} \frac{|\mu(n_{2}) \cdots \mu(n_{e})|}{[n'_{\alpha_{1}}, \cdots, n'_{\alpha_{k}}][n'_{\beta_{1}}, \cdots, n'_{\beta_{l}}]}
$$

\n
$$
\times \prod_{\substack{1 \leq t \leq v \\ t \neq r, t \neq s}} \frac{1}{[d_{t_{1}}, \cdots, d_{t_{m}}][n'_{t_{1}}, \cdots, n'_{t_{m}}]}
$$

\n
$$
= \sum_{n_{1}>x} \frac{|\mu(n_{1})|}{n_{1}^{2}} \sum_{d_{2}|n_{1}} \cdots \sum_{d_{e}|n_{1}} \prod_{\substack{1 \leq t \leq v \\ t \neq r, t \neq s}} \frac{1}{[d_{t_{1}}, \cdots, d_{t_{m}}]}
$$

\n
$$
\times \sum_{n'_{2}=1}^{\infty} \cdots \sum_{n'_{e}=1}^{\infty} \frac{|\mu(d_{2}n'_{2}) \cdots \mu(d_{e}n'_{e})|}{[n'_{\alpha_{1}}, \cdots, n'_{\alpha_{k}}][n'_{\beta_{1}}, \cdots, n'_{\beta_{l}}]} \prod_{\substack{1 \leq t \leq v \\ t \neq r, t \neq s}} \frac{1}{[n'_{t_{1}}, \cdots, n'_{t_{m}}]}
$$

\n
$$
\times \sum_{n_{1}>x} \frac{|\mu(n_{1})|}{n_{1}^{2}} \sum_{d_{2}|n_{1}} \cdots \sum_{d_{e}|n_{1}} |\mu(d_{2}) \cdots \mu(d_{e})| \prod_{\substack{1 \leq t \leq v \\ t \neq r, t \neq s}} \frac
$$

The product

$$
\sum_{n'_2=1}^{\infty} \cdots \sum_{n'_e=1}^{\infty} \frac{|\mu(n'_2) \cdots \mu(n'_e)|}{[n'_{\alpha_1}, \dots, n'_{\alpha_k}][n'_{\beta_1}, \dots, n'_{\beta_l}]} \prod_{\substack{1 \leq t \leq v \\ t \neq r, t \neq s}} \frac{1}{[n'_{t_1}, \dots, n'_{t_m}]}
$$

is finite by Lemma [5](#page-9-3) (but this time considering the graph G without the edge ${r, s}$). Thus, for some constant C_1 , we have

$$
R_1 \leq C_2 \sum_{n_1 > x} \frac{|\mu(n_1)|}{n_1^2} \sum_{d_2 | n_1} \cdots \sum_{d_e | n_1} |\mu(d_2) \cdots \mu(d_e)| \prod_{\substack{1 \leq t \leq v \\ t \neq r, \ t \neq s}} \frac{1}{[d_{t_1}, \dots, d_{t_m}]}
$$
\n
$$
= C_2 \sum_{n_1 > x} \frac{|\mu(n_1)|}{n_1^2} f_{G,e}(n_1), \tag{15}
$$

where the arithmetic function $f_{G,e}$ is defined as follows.

$$
f_{G,e}(n) = \sum_{d_2|n} \cdots \sum_{d_e|n} |\mu(d_2)\cdots\mu(d_e)| \prod_{\substack{1 \leq t \leq v \\ t \neq r, \ t \neq s}} \frac{1}{[d_{t_1},\ldots,d_{t_m}]}.
$$

We note that there is a factor $[d_{t_1}, \ldots, d_{t_m}]$ for each vertex other than r or s. The function $f_{G,e}$ is a multiplicative function. We have $f_{G,e}(p^k) = f_{G,e}(p)$ for any power of a prime p with $k \geq 2$, because in the definition of $f_{G,e}(p^k)$ only the divisors 1 and p of p^k give non null terms. When $n = p$ we have

$$
f_{G,e}(p) = 1 + \frac{A_1}{p} + \dots + \frac{A_{v-2}}{p^{v-2}},
$$

where A_i is the number of ways that

$$
\prod_{\substack{1 \leq t \leq v \\ t \neq r, \ t \neq s}} |\mu(d_2) \cdots \mu(d_e)| [d_{t_1}, \ldots, d_{t_m}] = p^i,
$$

where every divisor in the product $d_h | n = p$ can only be 1 or p. Clearly $A_i \leq 2^{e-1}$ do not depend on p, and so there must be a number w, independent of p , such that \sqrt{w}

$$
f_{G,e}(p^k) = f_{G,e}(p) \le \left(1 + \frac{1}{p}\right)^w.
$$

Since $f_{G,e}$ is multiplicative we have, for any squarefree n,

$$
f_{G,e}(n) \le \prod_{p|n} \left(1 + \frac{1}{p}\right)^w = \left(\frac{\sigma(n)}{n}\right)^w, \quad |\mu(n)| = 1. \tag{16}
$$

Substituting [\(16\)](#page-14-3) into [\(15\)](#page-13-0) yields

$$
R_1 \leq C_2 \sum_{n>x}^{\infty} \frac{|\mu(n)|}{n^2} \left(\frac{\sigma(n)}{n}\right)^w \leq C_2 \sum_{n>x}^{\infty} \frac{1}{n^2} \left(\frac{\sigma(n)}{n}\right)^w.
$$

It is well known that $\sigma(n) = O(n \log \log n)$ (see, for example, [\[3\]](#page-14-4)), and thus

$$
R_1 = O\left(\frac{(\log \log x)^w}{x}\right). \tag{17}
$$

Comparing [\(17\)](#page-14-5) with [\(14\)](#page-11-1) completes the proof of Theorem [1.](#page-1-0)

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