COMPLETE λ-HYPERSURFACES OF WEIGHTED VOLUME-PRESERVING MEAN CURVATURE FLOW

QING-MING CHENG AND GUOXIN WEI

Abstract. In this paper, we introduce a definition of λ-hypersurfaces of weighted volume-preserving mean curvature flow in Euclidean space. We prove that λ-hypersurfaces are critical points of the weighted area functional for the weighted volume-preserving variations. Furthermore, we classify complete λ-hypersurfaces with polynomial area growth and $H-\lambda \geq 0$, which are generalizations of the results due to Huisken [19], Colding-Minicozzi [11]. We also define a $\mathcal{F}$-functional and study $\mathcal{F}$-stability of λ-hypersurfaces, which extend a result of Colding-Minicozzi [11]. Lower bound growth and upper bound growth of the area for complete and non-compact λ-hypersurfaces are also studied.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth n-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. A family $X(\cdot, t)$ of smooth immersions:

$$X(\cdot, t) : M \to \mathbb{R}^{n+1}$$

with $X(\cdot, 0) = X(\cdot)$ is called a mean curvature flow if they satisfy

$$\frac{\partial X(p,t)}{\partial t} = \mathbf{H}(p,t),$$

where $\mathbf{H}(t) = \mathbf{H}(p,t)$ denotes the mean curvature vector of hypersurface $M_t = X(M^n, t)$ at point $X(p,t)$. Huisken [17] proved that the mean curvature flow M_t remains smooth and convex until it becomes extinct at a point in the finite time. If we rescale the flow about the point, the rescaling converges to the round sphere. An immersed hypersurface $X : M \to \mathbb{R}^{n+1}$ is called *a self-shrinker* if

$$H + \langle X, N \rangle = 0,$$

where H and N denote the mean curvature and the unit normal vector of $X : M \to \mathbb{R}^{n+1}$, respectively. $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^{n+1}$. It is known that self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow ups at a given singularity of the mean curvature flow.

For $n = 1$, Abresch and Langer [1] classified all smooth closed self-shrinker curves in $\mathbb{R}^2$ and showed that the round circle is the only embedded self-shrinker. For

2001 *Mathematics Subject Classification*: 53C44, 53C42.

Key words and phrases: the weighted volume-preserving mean curvature flow, the weighted area functional, $\mathcal{F}$-stability, weak stability, λ-hypersurfaces.

The first author was partially supported by JSPS Grant-in-Aid for Scientific Research (B): No. 24340013 and Challenging Exploratory Research No. 25610016. The second author was partly supported by grant No. 11371150 of NSFC.

$n \geq 2$, Huisken [19] studied compact self-shrinkers. He proved that if M is an n-dimensional compact self-shrinker with non-negative mean curvature in $\mathbb{R}^{n+1}$, then $X(M) = S^n(\sqrt{n})$. In the remarkable paper [11], Colding and Minicozzi have classified complete self-shrinkers with non-negative mean curvature and polynomial area growth (which is called polynomial volume growth in [11] and [20]) in $\mathbb{R}^{n+1}$. We should remark that Huisken [20] proved the same results if the squared norm of the second fundamental form is bounded. Colding and Minicozzi [11] have introduced a notation of $\mathcal{F}$-functional and computed the first and the second variation formulas of the $\mathcal{F}$-functional. They have proved that an immersed hypersurface $X : M \to \mathbb{R}^{n+1}$ is a self-shrinker if and only if it is a critical point of the $\mathcal{F}$-functional. Furthermore, they have given a complete classification of the $\mathcal{F}$-stable complete self-shrinkers with polynomial area growth.

On the other hand, Huisken [18] studied the volume-preserving mean curvature flow

$$\frac{\partial X(t)}{\partial t} = -h(t)N(t) + \mathbf{H}(t),$$

where $X(t) = X(\cdot, t)$, $h(t) = \frac{\int_M H(t)d\mu_t}{\int_M d\mu_t}$ and $N(t)$ is the unit normal vector of $X(t) : M \to \mathbb{R}^{n+1}$. He proved that if the initial hypersurface is uniformly convex, then the above volume-preserving mean curvature flow has a smooth solution and it converges to a round sphere. Furthermore, by making use of the Minkowski formulas, Guan and Li [16] have studied the following type of mean curvature flow

$$\frac{\partial X(t)}{\partial t} = -nN(t) + \mathbf{H}(t),$$

which is also a volume-preserving mean curvature flow. They have gotten that the flow converges to a solution of the isoperimetric problem if the initial hypersurface is a smooth compact, star-shaped hypersurface.

In this paper, we consider a new type of mean curvature flow:

$$\frac{\partial X(t)}{\partial t} = -\alpha(t)N(t) + \mathbf{H}(t), \tag{1.1}$$

with

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}}d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}}d\mu},$$

where N is the unit normal vector of $X : M \to \mathbb{R}^{n+1}$. We define *a weighted volume* of M_t (see, section 2) by

$$V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}}d\mu.$$

We can prove that the flow (1.1) preserves the weighted volume $V(t)$. Hence, we call the flow (1.1) *a weighted volume-preserving mean curvature flow*.

The properties of solutions of the weighted volume-preserving mean curvature flow (1.1) will be studied in Cheng and Wei [9].

This paper is organized as follows. In section 2, we give a definition of the weighted volume and the first variation formula of the weighted area functional for all weighted volume-preserving variations is given. As critical points of it, λ-hypersurface is defined. Self-similar solutions of the weighted volume-preserving mean curvature flow is considered. In section 3, the basic properties of λ-hypersurfaces are studied. In section 4, we give a classification for compact λ-hypersurfaces with $H-\lambda\geq 0$. In sections 5 and 6, we define $\mathcal{F}$-functional. The first and second variation formulas of $\mathcal{F}$-functional are proved. Notation of $\mathcal{F}$-stability and $\mathcal{F}$-unstability of λ-hypersurfaces are introduced. We prove that spheres $S^n(r)$ with $r\leq\sqrt{n}$ or $r>\sqrt{n+1}$ are $\mathcal{F}$-stable and spheres $S^n(r)$ with $\sqrt{n}<r\leq\sqrt{n+1}$ are $\mathcal{F}$-unstable. In section 7, we study the weak stability of the weighted area functional for the weighted volume-preserving variations. In section 8, a classification for complete and non-compact λ-hypersurfaces with polynomial area growth and $H-\lambda\geq 0$ is given. In sections 9 and 10, the area growth of complete and non-compact λ-hypersurfaces are studied.

Acknowledgement. A part of this work was finished when the first author visited to Beijing Normal University. We would like to express our gratitude to Professor Tang Zizhou and Dr. Yan Wenjiao for warm hospitality.

2. The first variation formula and λ-hypersurfaces

Let $X:M^n\to\mathbb{R}^{n+1}$ be an n-dimensional connected hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. We choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+1}$ in $\mathbb{R}^{n+1}$ with dual coframe field $\{\omega_A\}_{A=1}^{n+1}$, such that, restricted to M^n, $e_1,\cdots,e_n$ are tangent to M^n. Then we have

$$dX=\sum_i\omega_ie_i,\quad de_i=\sum_j\omega_{ij}e_j+\omega_{in+1}e_{n+1}$$

and

$$de_{n+1}=\sum_i\omega_{n+1i}e_i.$$

We restrict these forms to M^n, then

$$\omega_{n+1}=0,\quad \omega_{n+1i}=-\sum_{j=1}^n h_{ij}\omega_j,\quad h_{ij}=h_{ji},$$

where h_{ij} denotes the component of the second fundamental form of $X:M^n\to\mathbb{R}^{n+1}$. $\mathbf{H}=\sum_{j=1}^n h_{jj}e_{n+1}$ is the mean curvature vector field, $H=|\mathbf{H}|=\sum_{j=1}^n h_{jj}$ is the mean curvature and $II=\sum_{i,j}h_{ij}\omega_i\otimes\omega_je_{n+1}$ is the second fundamental form of $X:M^n\to\mathbb{R}^{n+1}$. Let

$$f_{,i}=\nabla_if,\ f_{,ij}=\nabla_j\nabla_if,\quad h_{ijk}=\nabla_kh_{ij}\ \text{ and }\ h_{ijkl}=\nabla_l\nabla_kh_{ij},$$

where ∇_j is the covariant differentiation operator.The Gauss equations and Codazzi equations are given by

$$R_{ijkl}=h_{ik}h_{jl}-h_{il}h_{jk}, \tag{2.1}$$

$$h_{ijk}=h_{ikj}, \tag{2.2}$$

where R_{ijkl} and h_{ijk} denote components of curvature tensor and components of the covariant derivative of h_{ij}, respectively. Furthermore, we have the Ricci formula:

$$h_{ijkl} - h_{ijlk} = \sum_{m=1}^{n} h_{im} R_{mjkl} + \sum_{m=1}^{n} h_{mj} R_{mikl}. \tag{2.3}$$

For a constant vector $a \in \mathbb{R}^{n+1}$, one has

$$\langle X, a\rangle_{,i} = \langle e_i, a\rangle, \quad \langle N, a\rangle_{,i} = -\sum_j h_{ij}\langle e_j, a\rangle,$$

$$\langle X, a\rangle_{,ij} = h_{ij}\langle N, a\rangle,$$

$$\langle N, a\rangle_{,ij} = -\sum_k h_{ijk}\langle e_k, a\rangle - \sum_k h_{ik}h_{jk}\langle N, a\rangle.$$

We call $X(t)$ is a variation of X if $X(t) : M \to \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$ is a family of immersions with $X(0) = X$. For $X_0 \in \mathbb{R}^{n+1}$ and a real number t_0, we define *a weighted area function* $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$A(t) = \int_M e^{-\frac{|X(t)-X_0|^2}{2t_0}} d\mu_t,$$

where $d\mu_t$ is the area element of M in the metric induced by $X(t)$. The *weighted volume function* $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is defined by

$$V(t) = \int_M \langle X(t) - X_0, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu.$$

In this paper, we only consider compactly supported variations. By a direct calculations, we have the following first variation formulas of $A(t)$ and $V(t)$:

Lemma 2.1.

$$\frac{dA(t)}{dt} = \int_M \Big(-\frac{\langle X(t) - X_0, \frac{\partial X(t)}{\partial t}\rangle}{t_0} - H(t)\langle \frac{\partial X(t)}{\partial t}, N(t)\rangle \Big) e^{-\frac{|X(t)-X_0|^2}{2t_0}} d\mu_t, \tag{2.4}$$

$$\frac{dV(t)}{dt} = \int_M \langle \frac{\partial X(t)}{\partial t}, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu. \tag{2.5}$$

Let $\frac{\partial X(t)}{\partial t} = W(t)$. Then the vector field $\frac{\partial X(t)}{\partial t}|_{t=0} = W(0) = W$ is called *a variation vector field*. Set $f(t) = \langle W(t), N(t)\rangle$, where $N(t)$ is the normal vector of M_t, $N(0) = N$. In this paper, we only consider the normal variation vector field, which can be expressed as $\frac{\partial X(t)}{\partial t}|_{t=0} = fN$. We say a variation of X is *a weighted volume-preserving variation* if $V(t) = V(0)$ for all t, that is

$$\begin{aligned} 0 = \frac{dV(t)}{dt} &= \int_M \langle \frac{\partial X(t)}{\partial t}, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\ &= \int_M f(t)\langle N(t), N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu. \end{aligned} \tag{2.6}$$

We can prove the following lemma using the same method as that of the lemma 2.4 of [3].

Lemma 2.2. *Given a smooth function $f : M \to \mathbb{R}$ with $\int_M f e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0$, there exists a weighted volume-preserving normal variation such that its variation vector field is fN.*

Let

$$\lambda = \frac{1}{A}\int_M (\langle \frac{X-X_0}{t_0}, N\rangle + H) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu,$$

with

$$A = \int_M e^{-\frac{|X-X_0|^2}{2t_0}} d\mu$$

and define $J : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$J(t) = A(t) + \lambda V(t),$$

for constant λ. Then, one has

Proposition 2.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an immersion. The following statements are equivalent with each other:*

(1) $\langle \frac{X-X_0}{t_0}, N\rangle + H = \lambda$.
(2) *For all weighted volume-preserving variations,* $A'(0) = 0$.
(3) *For all arbitrary variations,* $J'(0) = 0$.

Proof. From Lemma 2.1, we have (1)$\Rightarrow$ (3) and (3)$\Rightarrow$ (2). We next prove (2)$\Rightarrow$ (1). Assume that at a point $p \in M$, we have $(\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda)(p) \neq 0$. We can assume that $(\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda)(p) > 0$. Let

$$M^+ = \{q \in M : (\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda)(p) > 0\},$$

$$M^- = \{q \in M : (\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda)(p) < 0\}.$$

Let φ and ψ be non-negative real smooth functions on M such that

$$p \in \mathrm{supp}\varphi \subset M^+, \quad \mathrm{supp}\psi \subset M^-,$$

and

$$\int_M (\varphi + \psi)(\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0.$$

Since $\int_M (\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0$, we know that such a choice is possible. Let $f = (\varphi + \psi)(\langle \frac{X-X_0}{t_0}, N\rangle + H - \lambda)$, then $\int_M f e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0$. By Lemma 2.2, we get a weighted volume-preserving variation such that its variation vector field is fN. From our assumption,

$$A'(0) = \int_M (-\frac{\langle X - X_0, N\rangle}{t_0} - H) f e^{-\frac{|X-X_0|^2}{2t_0}} d\mu = 0.$$

Hence, we have

$$
\begin{aligned}
0 &= \int_M f(\langle \frac{X - X_0}{t_0}, N\rangle + H - \lambda) e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&= \int_M (\varphi + \psi)(\langle \frac{X - X_0}{t_0}, N\rangle + H - \lambda)^2 e^{-\frac{|X-X_0|^2}{2t_0}} d\mu \\
&> 0.
\end{aligned} \tag{2.7}
$$

It is a contradiction. It follows that $\langle \frac{X-X_0}{t_0}, N\rangle + H = \lambda$. □

Definition 2.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional immersed hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If $\langle \frac{X-X_0}{t_0}, N\rangle + H = \lambda$ holds, we call $X : M \to \mathbb{R}^{n+1}$ a λ-hypersurface of the weighted volume-preserving mean curvature flow.*

Remark 2.1. *If $\lambda = 0$, then the λ-hypersurface is a self-shrinker of the mean curvature flow. Hence, we know that the notation of the λ-hypersurface is a generalization of the self-shrinker.*

Theorem 2.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an immersed hypersurface. The following statements are equivalent with each other:*

1. *$X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface.*
2. *$X : M \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume-preserving variations.*
3. *$X : M \to \mathbb{R}^{n+1}$ is a hypersurface with constant weighted mean curvature $H_w = \lambda$ in $\mathbb{R}^{n+1}$ with respect to the metric $g_{AB} = e^{-\frac{|X-X_0|^2}{nt_0}}\delta_{AB}$, where the weighted mean curvature and the mean curvature H are related by $H_w = e^{-\frac{|X-X_0|^2}{2nt_0}} H$.*

Example 2.1. *The n-dimensional sphere $S^n(r)$ with radius $r > 0$ is a compact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{n}{r} - r$. It should be remarked that the sphere $S^n(\sqrt{n})$ is the only self-shrinker sphere in $\mathbb{R}^{n+1}$.*

Example 2.2. *For $1 \le k \le n-1$, the n-dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with radius $r > 0$ is a complete and non-compact λ-hypersurface in $\mathbb{R}^{n+1}$ with $\lambda = \frac{k}{r} - r$. We should notice that the cylinder $S^k(\sqrt{k}) \times \mathbb{R}^{n-k}$ is the only self-shrinker cylinder in $\mathbb{R}^{n+1}$.*

From [7], Chang has proved there exist a lot of complete embedded λ-curves Γ in $\mathbb{R}^2$. Hence we have

Example 2.3. *The n-dimensional hypersurfaces $\Gamma \times \mathbb{R}^{n-1}$ are complete embedded λ-hypersurfaces, which are not self-shrinkers, in $\mathbb{R}^{n+1}$.*

Remark 2.2. *For 1-dimensional self-shrinker in $\mathbb{R}^2$, Abresch and Langer* [1] *proved the circle is the only compact embedded self-shrinker. But for λ-curve in $\mathbb{R}^2$, Chang* [7] *has proved, for $\lambda < 0$, there are many compact embedded λ-curves other than the circle. From the above examples, we know that there are a lot of examples of complete embedded λ-hypersurfaces, which are not self-shrinkers, in $\mathbb{R}^{n+1}$.*

Proposition 2.2. *Let $X : M \to \mathbb{R}^{n+1}$ be a λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If the mean curvature H is constant, then $X : M \to \mathbb{R}^{n+1}$ is isometric to $S^k(r) \times \mathbb{R}^{n-k}$, $0 \le k \le n$, locally.*

Proof. Since $X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface, we have $\langle X, N\rangle + H = \lambda$. If H is constant, we get, for any $1 \le i \le n$,

$$\nabla_i \langle X, N\rangle = -\lambda_i \langle X, e_i\rangle = 0,$$

where λ_i is the principal curvature of the λ-hypersurface. If $\lambda_{i_0} \neq 0$ at a point p for some i_0, there exists a neighborhood U of p such that $\lambda_{i_0} \neq 0$ in U. Hence, we know $\langle X, e_{i_0}\rangle = 0$ in U. Thus,

$$X = \sum_{j \neq i_0} \langle X, e_j\rangle e_j + \langle X, N\rangle N.$$

We obtain

$$e_{i_0} = \nabla_{i_0} X = -\langle X, N\rangle \lambda_{i_0} e_{i_0},$$

that is, $\lambda_{i_0}(H - \lambda) = 1$ is constant. Thus, on U, λ_{i_0} is constant. Therefore, the λ-hypersurface is isoparametric. We obtain that $X : M \to \mathbb{R}^{n+1}$ is isometric to $S^k(r) \times \mathbb{R}^{n-k}$, $0 \le k \le n$, locally. $\square$

Definition 2.2. *A family of n-dimensional immersed hypersurfaces $X(t) : M \to \mathbb{R}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$ is called a self-similar solution of the weighted volume-preserving mean curvature flow if $X(t) = \beta(t)X$ holds, where $\beta(t) > 0$.*

Proposition 2.3. *A family of n-dimensional immersed hypersurfaces $X(t) : M \to \mathbb{R}^{n+1}$ in the Euclidean space $\mathbb{R}^{n+1}$ is a self-similar solution of the weighted volume-preserving mean curvature flow if and only if $X(t) = \sqrt{1 + \beta_0 t}X$, where β_0 is a constant.*

Proof. If $X(t) : M \to \mathbb{R}^{n+1}$ is a self-similar solution of the weighted volume-preserving mean curvature flow, we have $X(t) = \beta(t)X$. Hence, the mean curvature $H(t)$ of $X(t)$ satisfies

$$H(t) = \frac{H}{\beta(t)}.$$

Thus,

$$\alpha(t) = \frac{\int_M H(t)\langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu}{\int_M \langle N(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu} = \frac{\int_M H e^{-\frac{|X|^2}{2}} d\mu}{\beta(t) \int_M e^{-\frac{|X|^2}{2}} d\mu}.$$

From the equation of the weighted volume-preserving mean curvature flow, we have

$$\frac{\partial \beta(t)}{\partial t} X^{\perp} = \frac{1}{\beta(t)} \big(-\alpha(0) N + \mathbf{H}\big). \tag{2.8}$$

We obtain $\dfrac{\partial \beta(t)^2}{\partial t} = \beta_0 =$ constant. Since $\beta(0) = 1$, we have $\beta(t) = \sqrt{1 + \beta_0 t}$. The inverse is obvious. $\square$

Proposition 2.4. *Let $X : M \to \mathbb{R}^{n+1}$ be a λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. If $X(t) = \sqrt{1 + \beta_0 t}X$ is a self-similar solution of the weighted volume-preserving mean curvature flow, then $X : M \to \mathbb{R}^{n+1}$ is isometric to $S^k(r) \times \mathbb{R}^{n-k}$, $0 \le k \le n$, locally or $V(0) = 0$ and $\beta_0 = -2$.*

Proof. Since $X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface, we have $\langle X, N\rangle + H = \lambda$ and

$$V(t) = \int_M \langle X(t), N\rangle e^{-\frac{|X|^2}{2}} d\mu = \sqrt{1+\beta_0 t}V(0).$$

Since $X(t) = \sqrt{1+\beta_0 t}X$ is a self-similar solution of the weighted volume-preserving mean curvature flow, then $\beta_0 = 0$ or $V(0) = 0$. If $\beta_0 = 0$, then H is constant from (2.8). According to the proposition 2.2, we know that $X : M \to \mathbb{R}^{n+1}$ is isometric to $S^k(r) \times \mathbb{R}^{n-k}$, $0 \le k \le n$, locally. If $\beta_0 \neq 0$, we have $V(0) = 0$ since $V(t)$ is constant. The (2.8) gives $\beta_0 = -2$.

□

Definition 2.3. *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say that M has polynomial area growth if there exist constant C and d such that for all $r \geq 1$,*

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \le Cr^d, \tag{2.9}$$

where $B_r(0)$ is a standard ball in $\mathbb{R}^{n+1}$ with radius r and centered at the origin.

3. Properties of λ-hypersurfaces

In this section, we give several properties of λ-hypersurfaces. We define an elliptic operator $\mathcal{L}$ by

$$\mathcal{L}f = \Delta f - \langle X, \nabla f\rangle, \tag{3.1}$$

where Δ and ∇ denote the Laplacian and the gradient operator of the λ-hypersurface, respectively. We should notice that the $\mathcal{L}$ operator was introduced by Colding and Minicozzi in [11] for self-shrinkers.

By a direct calculation, for a constant vector $a \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned}
\mathcal{L}\langle X, a\rangle &= \Delta\langle X, a\rangle - \langle X, \nabla\langle X, a\rangle\rangle \\
&= \sum_i \langle X, a\rangle_{,ii} - \sum_i \langle X, a\rangle_{,i}\langle X, e_i\rangle \\
&= \langle HN, a\rangle - \sum_i \langle e_i, a\rangle\langle X, e_i\rangle \\
&= \langle HN, a\rangle - \langle X, a\rangle + \langle X, N\rangle\langle N, a\rangle \\
&= \lambda\langle N, a\rangle - \langle X, a\rangle,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\langle N, a\rangle &= \sum_i \langle N, a\rangle_{,ii} - \sum_i \langle N, a\rangle_{,i}\langle X, e_i\rangle \\
&= \langle -H_{,i}e_i - SN, a\rangle + \sum_i \langle X, e_i\rangle\langle \sum_j h_{ij}e_j, a\rangle \\
&= \langle X, N\rangle_{,i}\langle e_i, a\rangle - \langle SN, a\rangle + \sum_i \langle X, e_i\rangle\langle \sum_j h_{ij}e_j, a\rangle \\
&= -S\langle N, a\rangle,
\end{aligned}$$

where $S = \sum_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form.

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|X|^2) &= \langle \Delta X, X\rangle + \sum_i \langle X_{,i}, X_{,i}\rangle - \sum_i \langle X, e_i\rangle\langle X, e_i\rangle \\ &= n - |X|^2 + \lambda\langle X, N\rangle.\end{aligned}$$

Hence, we have the following

Lemma 3.1. *If $X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface, then we have*

$$\mathcal{L}\langle X, a\rangle = \lambda\langle N, a\rangle - \langle X, a\rangle, \tag{3.2}$$

$$\mathcal{L}\langle N, a\rangle = -S\langle N, a\rangle, \tag{3.3}$$

$$\frac{1}{2}\mathcal{L}(|X|^2) = n - |X|^2 + \lambda\langle X, N\rangle. \tag{3.4}$$

The following lemma due to Colding and Minicozzi [11] is needed in order to prove our results.

Lemma 3.2. *If $X : M \to \mathbb{R}^{n+1}$ is a hypersurface, u is a C^1-function with compact support and v is a C^2-function, then*

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle \nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}}d\mu. \tag{3.5}$$

Corollary 3.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a complete hypersurface. If u, v are C^2 functions satisfying*

$$\int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)e^{-\frac{|X|^2}{2}}d\mu < +\infty, \tag{3.6}$$

then we have

$$\int_M u(\mathcal{L}v)e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle \nabla u, \nabla v\rangle e^{-\frac{|X|^2}{2}}d\mu. \tag{3.7}$$

Lemma 3.3. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with polynomial area growth, then*

$$\int_M (\langle X, a\rangle - \lambda\langle N, a\rangle)e^{-\frac{|X|^2}{2}}d\mu = 0, \tag{3.8}$$

$$\int_M \big(n - |X|^2 + \lambda\langle X, N\rangle\big)e^{-\frac{|X|^2}{2}}d\mu = 0, \tag{3.9}$$

$$\begin{aligned}&\int_M \langle X, a\rangle|X|^2 e^{-\frac{|X|^2}{2}}d\mu \\ &= \int_M \Big(2n\lambda\langle N, a\rangle + 2\lambda\langle X, a\rangle(\lambda - H) - \lambda\langle N, a\rangle|X|^2\Big)e^{-\frac{|X|^2}{2}}d\mu,\end{aligned} \tag{3.10}$$

$$\int_M \langle X, a\rangle^2 e^{-\frac{|X|^2}{2}}d\mu = \int_M \Big(|a^T|^2 + \lambda\langle N, a\rangle\langle X, a\rangle\Big)e^{-\frac{|X|^2}{2}}d\mu, \tag{3.11}$$

where $a^T = \sum_i < a, e_i > e_i$.

$$\begin{aligned}&\int_M \left(|X|^2 - n - \frac{\lambda(\lambda - H)}{2}\right)^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= \int_M \left\{ (\frac{\lambda^2}{4} - 1)(\lambda - H)^2 + 2n - H^2 + \lambda^2 \right\} e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{3.12}$$

Proof. Equations (3.8) and (3.9) just follow from the corollary 3.1 and equations (3.2), and (3.4). Since $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete λ-hypersurface with polynomial area growth, by making use of $u = |X|^2$, $v = \langle X, a\rangle$ in the lemma 3.2, we have

$$\begin{aligned}&\int_M \langle X, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= -\int_M \mathcal{L}\langle X, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda\langle N, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= -\int_M \langle X, a\rangle \mathcal{L}|X|^2 e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda\langle N, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= -\int_M 2\langle X, a\rangle \big[n + \lambda\langle X, N\rangle - |X|^2\big] e^{-\frac{|X|^2}{2}} d\mu + \int_M \lambda\langle N, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= 2\int_M \langle X, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu - 2n\int_M \langle X, a\rangle - 2\lambda\langle X, a\rangle(\lambda - H) e^{-\frac{|X|^2}{2}} d\mu \\ &\quad + \int_M \lambda\langle N, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\int_M \langle X, a\rangle |X|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= \int_M \Big(2n\lambda\langle N, a\rangle + 2\lambda\langle X, a\rangle(\lambda - H) - \lambda\langle N, a\rangle |X|^2\Big) e^{-\frac{|X|^2}{2}} d\mu.\end{aligned}$$

Taking $u = v = \langle X, a\rangle$ in the lemma 3.2, we can get (3.11). Putting $u = v = |X|^2$ in the lemma 3.2, we can have

$$\begin{aligned}&\int_M \lambda(\lambda - H)|X|^2 e^{-\frac{|X|^2}{2}} d\mu \\ &= \int_M (|X|^4 - n|X|^2 + \frac{1}{2}|X|^2\mathcal{L}|X|^2) e^{-\frac{|X|^2}{2}} d\mu \\ &= \int_M (|X|^4 - n|X|^2) e^{-\frac{|X|^2}{2}} d\mu - \int_M \frac{1}{2}\langle \nabla|x|^2, \nabla|x|^2\rangle e^{-\frac{|X|^2}{2}} d\mu \\ &= \int_M \big(|X|^4 - (n+2)|X|^2 + 2(\lambda - H)^2\big) e^{-\frac{|X|^2}{2}} d\mu,\end{aligned}$$

that is,

$$\int_M \left\{ |X|^4 - [n+\lambda(\lambda-H)]|X|^2 - 2|X|^2 + 2(\lambda-H)^2 \right\} e^{-\frac{|X|^2}{2}} d\mu = 0.$$

Thus, we have

$$\begin{aligned} 0 = & \int_M \left\{ |X|^4 - 2[n+\frac{(\lambda-H)\lambda}{2}]|X|^2 + n^2 + n\lambda(\lambda-H) \right. \\ & \left. - 2n - 2\lambda(\lambda-H) + 2(\lambda-H)^2 \right\} e^{-\frac{|X|^2}{2}} d\mu \\ = & \int_M \left\{ \left[|X|^2 - (n+\frac{\lambda(\lambda-H)}{2})\right]^2 - \frac{\lambda^2(\lambda-H)^2}{4} + 2(\lambda-H)^2 \right. \\ & \left. - 2n - 2\lambda(\lambda-H) \right\} e^{-\frac{|X|^2}{2}} d\mu \\ = & \int_M \left\{ \left(|X|^2 - n - \frac{\lambda(\lambda-H)}{2}\right)^2 - (\frac{\lambda^2}{4} - 1)(\lambda-H)^2 - 2n + H^2 - \lambda^2 \right\} e^{-\frac{|X|^2}{2}} d\mu, \end{aligned}$$

namely,

$$\begin{aligned} & \int_M \left(|X|^2 - n - \frac{\lambda(\lambda-H)}{2} \right)^2 e^{-\frac{|X|^2}{2}} d\mu \\ & = \int_M \left\{ (\frac{\lambda^2}{4} - 1)(\lambda-H)^2 + 2n - H^2 + \lambda^2 \right\} e^{-\frac{|X|^2}{2}} d\mu. \end{aligned}$$

□

4. A classification of compact λ-hypersurfaces

In this section, we will give a classification of compact λ-hypersurfaces. First of all, we give some lemmas.

Lemma 4.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional λ-hypersurface. Then, the following holds.*

$$\mathcal{L}H = H + S(\lambda - H), \tag{4.1}$$

$$\frac{1}{2}\mathcal{L}S = \sum_{i,j,k} h_{ijk}^2 + (1-S)S + \lambda f_3, \tag{4.2}$$

$$\mathcal{L}\sqrt{S} = \frac{1}{\sqrt{S}}\left(\sum_{i,j,k} h_{ijk}^2 - |\nabla\sqrt{S}|^2\right) + \sqrt{S}(1-S) + \frac{1}{\sqrt{S}}\lambda f_3, \tag{4.3}$$

$$\mathcal{L}\log(H-\lambda) = 1 - S + \frac{\lambda}{H-\lambda} - |\nabla \log(H-\lambda)|^2, \quad \text{if } H - \lambda > 0, \tag{4.4}$$

where $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$.

Proof. Since $\langle X, N\rangle + H = \lambda$, one has

$$H_{,i} = \sum_j h_{ij}\langle X, e_j\rangle, \tag{4.5}$$

$$H_{,ik} = \sum_j h_{ijk}\langle X, e_j\rangle + h_{ik} + \sum_j h_{ij}h_{jk}(\lambda - H).$$

Hence,

$$\Delta H = \sum_i H_{,ii} = \sum_i H_{,i}\langle X, e_i\rangle + H + S(\lambda - H) \tag{4.6}$$

and

$$\mathcal{L}H = \Delta H - \sum_i \langle X, e_i\rangle H_{,i} = H + S(\lambda - H).$$

By a direct calculation, we have from (2.3)

$$\begin{aligned}\mathcal{L}h_{ij} &= \Delta h_{ij} - \sum_k \langle X, e_k\rangle h_{ijk}\\ &= (1-S)h_{ij} + \lambda \sum_k h_{ik}h_{kj}.\end{aligned}$$

Then it follows that

$$\begin{aligned}\frac{1}{2}\mathcal{L}S &= \frac{1}{2}\Big(\Delta \sum_{i,j} h_{ij}^2 - \sum_k \langle X, e_k\rangle \big(\sum_{i,j} h_{ij}^2\big)_{,k}\Big)\\ &= \sum_{i,j,k} h_{ijk}^2 + (1-S)S + \lambda f_3.\end{aligned}$$

Since

$$\mathcal{L}S = 2|\nabla\sqrt{S}|^2 + 2\sqrt{S}\mathcal{L}\sqrt{S}, \tag{4.7}$$

we have

$$\begin{aligned}\mathcal{L}\sqrt{S} &= \frac{1}{2\sqrt{S}}\mathcal{L}S - \frac{|\nabla\sqrt{S}|^2}{\sqrt{S}}\\ &= \frac{1}{\sqrt{S}}\Big(\sum_{i,j,k} h_{ijk}^2 - |\nabla\sqrt{S}|^2\Big) + \sqrt{S}(1-S) + \frac{1}{\sqrt{S}}\lambda f_3.\end{aligned}$$

$$\begin{aligned}\mathcal{L}\log(H-\lambda) &= \Delta\log(H-\lambda) - \sum_i \langle X, e_i\rangle(\log(H-\lambda))_{,i}\\ &= \frac{1}{H-\lambda}\mathcal{L}H - |\nabla\log(H-\lambda)|^2\\ &= 1 - S + \frac{\lambda}{H-\lambda} - |\nabla\log(H-\lambda)|^2.\end{aligned}$$

We complete the proof of the lemma. ∎

Theorem 4.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional compact λ-hypersurface in $\mathbb{R}^{n+1}$. If $H-\lambda \geq 0$ and $\lambda(f_3(H-\lambda)-S)\geq 0$, then $X : M \to \mathbb{R}^{n+1}$ is isometric to a round sphere $S^n(r)$ with $\lambda = \frac{n}{r} - r$.*

Proof. Since

$$\mathcal{L}H = H + S(\lambda - H)$$

and

$$H - \lambda \geq 0,$$

we have

$$\mathcal{L}H - H \leq 0.$$

If $\lambda \leq 0$, we conclude from the maximum principle that either $H \equiv \lambda$ or $H - \lambda > 0$. If $H \equiv \lambda$, (4.6) gives that $H = \lambda = 0$ and M is a self-shrinker, it is impossible since M is compact; If $\lambda > 0$, we have $f_3(H-\lambda) - S \geq 0$. In this case, if $H - \lambda = 0$ at some point $p \in M$, then $S = 0$ and $H = \lambda = 0$ at p, that is $\lambda \equiv 0$ and M is self-shrinker, it is also impossible since M is compact. Hence for any λ, we have $H - \lambda > 0$.
From the lemma 4.1, we can get

$$\begin{aligned}\mathcal{L}\frac{1}{(H-\lambda)^2} &= \Delta\frac{1}{(H-\lambda)^2} - \sum_i \langle X, e_i\rangle \big(\frac{1}{(H-\lambda)^2}\big)_{,i} \\ &= \frac{6}{(H-\lambda)^4}|\nabla(H-\lambda)|^2 - \frac{2}{(H-\lambda)^3}[H - S(H-\lambda)]\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}\frac{S}{(H-\lambda)^2} &= \Delta\frac{S}{(H-\lambda)^2} - \sum_i \langle X, e_i\rangle \big(\frac{S}{(H-\lambda)^2}\big)_{,i} \\ &= \frac{1}{(H-\lambda)^2}\mathcal{L}S + 2\langle \nabla S, \nabla(\frac{1}{(H-\lambda)^2})\rangle + S\mathcal{L}(\frac{1}{(H-\lambda)^2}) \\ &= \frac{2}{(H-\lambda)^2}\bigg(\sum_{i,j,k} h_{ijk}^2 + (1-S)S + \lambda f_3\bigg) + 2\langle \nabla S, \nabla(\frac{1}{(H-\lambda)^2})\rangle \\ &\quad + S\bigg(\frac{6}{(H-\lambda)^4}|\nabla(H-\lambda)|^2 - \frac{2}{(H-\lambda)^3}[H - S(H-\lambda)]\bigg).\end{aligned}$$

By multiplying $Se^{-\frac{|X|^2}{2}}$ in the above equation and using

$$\int_M S\mathcal{L}\frac{S}{(H-\lambda)^2}e^{-\frac{|X|^2}{2}}d\mu = -\int_M \langle \nabla S, \nabla(\frac{S}{(H-\lambda)^2})\rangle e^{-\frac{|X|^2}{2}}d\mu,$$

one has

$$
\begin{aligned}
&2\int_M \frac{S}{(H-\lambda)^4}\sum_{i,j,k}|h_{ijk}(H-\lambda)-h_{ij}H_{,k}|^2 e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M |\nabla(\frac{S}{(H-\lambda)^2})|^2(H-\lambda)^2 e^{-\frac{|X|^2}{2}}d\mu\\
&+2\int_M \frac{S}{(H-\lambda)^2}\lambda\left(f_3-\frac{S}{H-\lambda}\right)e^{-\frac{|X|^2}{2}}d\mu=0.
\end{aligned}
\tag{4.8}
$$

Then it follows from $\lambda(f_3(H-\lambda)-S)\geq 0$ that

$$
\lambda(f_3-\frac{S}{H-\lambda})=0,\quad \frac{S}{(H-\lambda)^2}=\text{constant},\quad h_{ijk}(H-\lambda)=h_{ij}H_{,k}. \tag{4.9}
$$

We next consider two cases.

Case 1: $\lambda=0$

In this case, we know M is isometric to $S^n(\sqrt{n})$ from Huisken's result [19].

Case 2: $\lambda\neq 0$

In this case, one gets

$$
f_3-\frac{S}{H-\lambda}=0,\quad h_{ijk}(H-\lambda)=h_{ij}H_{,k}.
$$

If H is constant, then $h_{ijk}=0$, thus M is $S^n(r)$ by the result of Lawson [22].
If H is not constant, then there exists a neighborhood U such that $|\nabla H|\neq 0$ on U. We can choose $e_1,\cdots,e_n$ such that $e_1=\frac{\nabla H}{|\nabla H|}$. It follows from $h_{ijk}=h_{ikj}$ that $h_{ij}H_{,k}=h_{ik}H_{,j}$ and

$$
\begin{aligned}
0&=\sum_{i,j,k}|h_{ij}H_{,k}-h_{ik}H_{,j}|^2\\
&=2S|\nabla H|^2-2\sum_i h_{1i}^2|\nabla H|^2\\
&=2|\nabla H|^2(S-\sum_i h_{1i}^2),
\end{aligned}
$$

that is,

$$
\sum_{i=1}^n h_{1i}^2=S=h_{11}^2+2\sum_{j\neq 1}^n h_{1j}^2+\sum_{k,l\geq 2}h_{kl}^2.
$$

Therefore, $S=h_{11}^2=H^2$ on U. On the other hand, we see from $\frac{S}{(H-\lambda)^2}=$ constant that H is constant on U. It is a contradiction. The proof of the theorem 4.1 is completed.

□

Remark 4.1. *The assumption $\lambda(f_3(H-\lambda)-S)\geq 0$ in the theorem 4.1 is satisfied for self-shrinkers of the mean curvature flow, automatically. When $\lambda>0$, this condition is needed in order to prove $H>\lambda$ since the maximum principle does not work for this case. We think that the assumption is essential. In particular, for case of complete and non-compact λ-hypersurfaces, this condition is essential in section 8. In fact, $\Gamma\times\mathbb{R}^{n-1}$ are counterexamples since $H-\lambda>0$, where Γ are compact* *embedded λ-curves other than the circle (see Remark 2.2). It is a very interesting problem to construct counterexamples for compact case.*

5. The first variation of $\mathcal{F}$-functional

In this section, we will give another variational characterization of λ-hypersurfaces. Let $X(s): M \to \mathbb{R}^{n+1}$ be immersions with $X(0) = X$. The variation vector field $\frac{\partial}{\partial s}X(s)|_{s=0}$ is the normal variation vector field fN.
For $X_0 \in \mathbb{R}^{n+1}$ and a real number t_0, the $\mathcal{F}$-functional is defined by

$$\begin{aligned}
&\mathcal{F}_{X_s,t_s}(s) = \mathcal{F}_{X_s,t_s}(X(s))\\
&= (4\pi t_s)^{-\frac{n}{2}}\int_M e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s + \lambda(4\pi t_0)^{-\frac{n}{2}}(\frac{t_0}{t_s})^{\frac{1}{2}}\int_M \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,
\end{aligned}$$

where X_s and t_s denote the variations of X_0 and t_0. Let

$$\frac{\partial t_s}{\partial s} = h(s), \quad \frac{\partial X_s}{\partial s} = y(s), \quad \frac{\partial X(s)}{\partial s} = f(s)N(s),$$

one calls that $X: M \to \mathbb{R}^{n+1}$ is *a critical point of* $\mathcal{F}_{X_s,t_s}(s)$ if it is critical with respect to all normal variations and all variations in X_0 and t_0.

Lemma 5.1. *Let $X(s)$ be a variation of X with normal variation vector field $\frac{\partial X(s)}{\partial s}|_{s=0} = fN$. If X_s and t_s are variations of X_0 and t_0 with $\frac{\partial X_s}{\partial s}|_{s=0} = y$ and $\frac{\partial t_s}{\partial s}|_{s=0} = h$, then the first variation formula of $\mathcal{F}_{X_s,t_s}(s)$ is given by*

$$\begin{aligned}
&\mathcal{F}'_{X_0,t_0}(0)\\
&= (4\pi t_0)^{-\frac{n}{2}}\int_M \Big(\lambda - (H + \langle\frac{X-X_0}{t_0}, N\rangle)\Big) f e^{-\frac{|X-X_0|^2}{2}}d\mu\\
&\quad + (4\pi t_0)^{-\frac{n}{2}}\int_M \Big(\langle\frac{X-X_0}{t_0}, y\rangle - \lambda\langle N, y\rangle\Big) e^{-\frac{|X-X_0|^2}{2}}d\mu\\
&\quad + (4\pi t_0)^{-\frac{n}{2}}\int_M \Big(\frac{|X-X_0|^2}{t_0} - n - \lambda\langle X-X_0, N\rangle\Big)\frac{h}{2t_0} e^{-\frac{|X-X_0|^2}{2}}d\mu.
\end{aligned} \tag{5.1}$$

Proof. Defining

$$\mathbb{A}(s) = \int_M e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s, \quad \mathbb{V}(s) = \int_M \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu, \tag{5.2}$$

then

$$\begin{aligned}
\mathcal{F}'_{X_s,t_s}(s) &= (4\pi t_s)^{-\frac{n}{2}}\mathbb{A}'(s) + \lambda(4\pi t_0)^{-\frac{n}{2}}(\frac{t_0}{t_s})^{\frac{1}{2}}\mathbb{V}'(s)\\
&\quad - (4\pi t_s)^{-\frac{n}{2}}\frac{n}{2t_s}h\mathbb{A}(s) - \lambda(4\pi t_0)^{-\frac{n}{2}}(\frac{t_0}{t_s})^{\frac{1}{2}}\frac{h}{2t_s}\mathbb{V}(s).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{A}'(s) = \int_M \Big\{&-\langle\frac{X(s)-X_s}{t_s}, \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}\rangle + \frac{|X(s)-X_s|^2}{2t_s^2}h\\
&- H_s\langle\frac{\partial X(s)}{\partial s}, N(s)\rangle\Big\} e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s,
\end{aligned}$$

$$\mathbb{V}'(s) = \int_M \langle \frac{\partial X(s)}{\partial s} - \frac{\partial X_s}{\partial s}, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu,$$

we have

$$\begin{aligned}
&\mathcal{F}'_{X_s,t_s}(s)\\
&= (4\pi t_s)^{-\frac{n}{2}} \int_M -(H_s + \langle \frac{X(s)-X_s}{t_s}, N(s)\rangle) f e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s\\
&\quad + (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M \lambda f \langle N(s), N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu\\
&\quad + (4\pi t_s)^{-\frac{n}{2}} \int_M \langle \frac{X(s)-X_s}{t_s}, y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s\\
&\quad + (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M \lambda \langle -y, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu\\
&\quad + (4\pi t_s)^{-\frac{n}{2}} \int_M (-\frac{n}{2t_s} + \frac{|X(s)-X_s|^2}{2t_s^2}) h e^{-\frac{|X(s)-X_s|^2}{2t_s}} d\mu_s\\
&\quad + (4\pi t_0)^{-\frac{n}{2}} \sqrt{\frac{t_0}{t_s}} \int_M -\frac{h\lambda}{2t_s} \langle X(s)-X_s, N\rangle e^{-\frac{|X-X_0|^2}{2t_0}} d\mu.
\end{aligned}$$

If $s=0$, then $X(0)=X$, $X_s=X_0$, $t_s=t_0$ and

$$\begin{aligned}
&\mathcal{F}'_{X_0,t_0}(0)\\
&= (4\pi t_0)^{-\frac{n}{2}} \int_M \left(\lambda - (H + \langle \frac{X-X_0}{t_0}, N\rangle)\right) f e^{-\frac{|X-X_0|^2}{2}} d\mu\\
&\quad + (4\pi t_0)^{-\frac{n}{2}} \int_M \left(\langle \frac{X-X_0}{t_0}, y\rangle - \lambda\langle N, y\rangle\right) e^{-\frac{|X-X_0|^2}{2}} d\mu\\
&\quad + (4\pi t_0)^{-\frac{n}{2}} \int_M \left(\frac{|X-X_0|^2}{t_0} - n - \lambda\langle X-X_0, N\rangle\right) \frac{h}{2t_0} e^{-\frac{|X-X_0|^2}{2}} d\mu.
\end{aligned}$$

□

From the lemma 5.1, we know that if $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$, then

$$H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda.$$

We next prove that if $H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda$, then $X : M \to \mathbb{R}^{n+1}$ must be a critical point of $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$. For simplicity, we only consider the case of $X_0 = 0$ and $t_0 = 1$. In this case, $H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda$ becomes

$$H + \langle X, N\rangle = \lambda. \tag{5.3}$$

Furthermore, we know that (M, X_0, t_0) is the critical point of the $\mathcal{F}$-functional if and only if M is the critical point of $\mathcal{F}$-functional with respect to fixed X_0 and t_0.

Theorem 5.1. *$X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{X_s,t_s}(s)$ if and only if*

$$H + \langle \frac{X-X_0}{t_0}, N\rangle = \lambda.$$

Proof. We only prove the result for $X_0 = 0$ and $t_0 = 1$. In this case, the first variation formula (5.1) becomes

$$\begin{aligned}\mathcal{F}'_{0,1}(0) &= (4\pi)^{-\frac{n}{2}}\int_M \Big(\lambda - (H + \langle X, N\rangle)\Big) f e^{-\frac{|X|^2}{2}} d\mu \\ &+ (4\pi)^{-\frac{n}{2}}\int_M \Big(\langle X, y\rangle - \lambda\langle N, y\rangle\Big) e^{-\frac{|X|^2}{2}} d\mu \\ &+ (4\pi)^{-\frac{n}{2}}\int_M \Big(|X|^2 - n - \lambda\langle X, N\rangle\Big)\frac{h}{2} e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{5.4}$$

If $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{0,1}$, then $X : M \to \mathbb{R}^{n+1}$ should satisfy $H + \langle X, N\rangle = \lambda$. Conversely, if $H + \langle X, N\rangle = \lambda$ is satisfied, then we know that $X : M \to \mathbb{R}^{n+1}$ is a λ-hypersurface. Therefore, the last two terms in (5.4) vanish for any h and any y from (3.8) and (3.9) of the lemma 3.3. Therefore $X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{0,1}$. □

Corollary 5.1. *$X : M \to \mathbb{R}^{n+1}$ is a critical point of $\mathcal{F}_{X_s,t_s}(s)$ if and only if M is the critical point of $\mathcal{F}$-functional with respect to fixed X_0 and t_0.*

6. The second variation of $\mathcal{F}$-functional

In this section, we shall give the second variation formula of $\mathcal{F}$-functional.

Theorem 6.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a critical point of the functional $\mathcal{F}(s) = \mathcal{F}_{X_s,t_s}(s)$. The second variation formula of $\mathcal{F}(s)$ for $X_0 = 0$ and $t_0 = 1$ is given by*

$$\begin{aligned}&(4\pi)^{\frac{n}{2}}\mathcal{F}''(0) \\ =& -\int_M fLfe^{-\frac{|X|^2}{2}} d\mu + \int_M \big(-|y|^2 + \langle X, y\rangle^2\big) e^{-\frac{|X|^2}{2}} d\mu \\ &+ \int_M \Big\{2\langle N, y\rangle + (n + 1 - |X|^2)\lambda h - 2hH - 2\lambda\langle X, y\rangle\Big\} f e^{-\frac{|X|^2}{2}} d\mu \\ &+ \int_M \Big\{\lambda\langle N, y\rangle - (n + 2)\langle X, y\rangle + \langle X, y\rangle|X|^2\Big\} h e^{-\frac{|X|^2}{2}} d\mu \\ &+ \int_M \Big\{\frac{n^2 + 2n}{4} - \frac{n + 2}{2}|X|^2 + \frac{|X|^4}{4} + \frac{3\lambda}{4}(\lambda - H)\Big\} h^2 e^{-\frac{|X|^2}{2}} d\mu,\end{aligned}$$

where the operator L is defined by

$$L = \mathcal{L} + S + 1 - \lambda^2.$$

Proof.

$$
\begin{aligned}
&\mathcal{F}^{''}(s)\\
=&(4\pi t_s)^{-\frac{n}{2}}\int_M -(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)f^{'}e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M \lambda f^{'}\langle N(s),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M \frac{nh}{2t_s}(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle f e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M -\frac{h}{2t_s}\lambda\langle N(s),N\rangle f e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M (H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)\times\\
&\qquad\qquad(\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle+H_sf)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M -\Big(\frac{dH_s}{ds}+\langle\frac{\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}}{t_s},N(s)\rangle-\langle\frac{X(s)-X_s}{t_s^2},N(s)\rangle h\\
&\qquad\qquad+\langle\frac{X(s)-X_s}{t_s},\frac{dN(s)}{ds}\rangle\Big)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M \lambda f\langle\frac{dN(s)}{ds},N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M \langle\frac{X(s)-X_s}{t_s},y^{'}\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M -\lambda\langle N,y^{'}\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}(-\frac{nh}{2t_s})\int_M \langle\frac{X(s)-X_s}{t_s},y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}(-\frac{h}{2t_s})\int_M -\lambda\langle N,y\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M (\langle\frac{\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}}{t_s},y\rangle-\langle\frac{X(s)-X_s}{t_s^2},y\rangle h)e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M \langle\frac{X(s)-X_s}{t_s},y\rangle\Big(-\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle\ -H_sf\Big)e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M (-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})h^{'}e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M -\frac{h^{'}\lambda}{2t_s}\langle X(s)-X_s,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+(4\pi t_s)^{-\frac{n}{2}}(-\frac{nh}{2t_s})\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}(-\frac{h}{2t_s})\int_M-\frac{h}{2t_s}\lambda\langle X(s)-X_s,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M(\frac{nh}{2t_s^2}-\frac{|X(s)-X_s|^2}{t_s^3}h+\frac{\langle X(s)-X_s,\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle}{t_s^2})\times\\
&\qquad he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_0)^{-\frac{n}{2}}\sqrt{\frac{t_0}{t_s}}\int_M(\frac{h}{2t_s^2}\langle X(s)-X_s,N\rangle\lambda h\\
&\qquad\qquad-\frac{1}{2t_s}\langle\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s},N\rangle\lambda h)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})h(-H_sf\\
&\qquad-\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle)e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M-(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)f\frac{|X(s)-X_s|^2}{2t_s^2}h\\
&\qquad\times e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s},y\rangle\frac{|X(s)-X_s|^2}{2t_s^2}he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})h\frac{|X(s)-X_s|^2}{2t_s^2}he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s.
\end{aligned}
$$

Since $X:M\to\mathbb{R}^{n+1}$ is a critical point, we get

$$H+\langle\frac{X-X_0}{t_0},N\rangle=\lambda,$$

$$\int_M(n+\lambda\langle X-X_0,N\rangle-\frac{|X-X_0|^2}{t_0})e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=0,$$

$$\int_M(\lambda\langle N,a\rangle-\langle\frac{X-X_0}{t_0},a\rangle)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=0.$$

On the other hand,

$$H^{'}=\Delta f+Sf,\quad N^{'}=-\nabla f.$$

Using of the above equations and letting $s = 0$, we obtain

$$
\begin{aligned}
&(4\pi t_0)^{\frac{n}{2}}\mathcal{F}''(0)\\
=&\int_M -fLfe^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+\int_M \Big(\frac{2}{t_0}\langle N, y\rangle + \frac{2h}{t_0}\langle\frac{X-X_0}{t_0}, N\rangle + \frac{n-1}{t_0}\lambda h\\
&\qquad - \frac{|X-X_0|^2}{t_0^2}\lambda h - 2\lambda\langle\frac{X-X_0}{t_0}, y\rangle\Big)fe^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+\int_M \Big(-\frac{n+2}{t_0}\langle\frac{X-X_0}{t_0}, y\rangle + \frac{\lambda}{t_0}\langle N, y\rangle\\
&\qquad + \langle\frac{X-X_0}{t_0}, y\rangle\frac{|X-X_0|^2}{t_0^2}\Big)he^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+\int_M \Big(\frac{n^2}{4t_0^2} + \frac{n}{2t_0^2} - \frac{n+2}{2t_0^3}|X-X_0|^2 + \frac{|X-X_0|^4}{4t_0^4}\\
&\qquad + \frac{3\lambda}{4t_0}\langle\frac{X-X_0}{t_0}, N\rangle\Big)h^2e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+\int_M \Big(-\frac{1}{t_0}\langle y, y\rangle + \langle\frac{X-X_0}{t_0}, y\rangle^2\Big)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu,
\end{aligned}
$$

where the operator L is defined by $L = \Delta + S + \frac{1}{t_0} - \langle\frac{X-X_0}{t_0}, \nabla\rangle - \lambda^2$. When $t_0 = 1$, $X_0 = 0$, then $L = \mathcal{L} + S + 1 - \lambda^2$.

$$
\begin{aligned}
&(4\pi)^{\frac{n}{2}}\mathcal{F}''(0)\\
=&\int_M -fLfe^{-\frac{|X-|^2}{2}}d\mu\\
&+\int_M \big(2\langle N, y\rangle + 2\lambda h + (n-1)\lambda h - 2hH\\
&\qquad - |X|^2\lambda h - 2\lambda\langle X, y\rangle\big)fe^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M \big(\lambda\langle N, y\rangle - (n+2)\langle X, y\rangle + \langle X, y\rangle|X|^2\big)he^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M \Big(\frac{n^2+2n}{4} - \frac{n+2}{2}|X|^2 + \frac{|X|^4}{4} + \frac{3\lambda}{4}\langle X, N\rangle\Big)h^2e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M -(|y|^2 - \langle X, y\rangle^2)e^{-\frac{|X|^2}{2}}d\mu
\end{aligned}
$$

$$
\begin{aligned}
&= \int_M -fLfe^{-\frac{|X|^2}{2}}d\mu \\
&+ \int_M [2\langle N, y\rangle + (n+1-|X|^2)\lambda h - 2hH - 2\lambda\langle X, y\rangle]fe^{-\frac{|X-|^2}{2}}d\mu \\
&+ \int_M (\lambda\langle N, y\rangle - (n+2)\langle X, y\rangle + \langle X, y\rangle|X|^2)he^{-\frac{|X|^2}{2}}d\mu \\
&+ \int_M (\frac{n^2+2n}{4} - \frac{n+2}{2}|X|^2 + \frac{|X|^4}{4} + \frac{3\lambda}{4}(\lambda - H))h^2e^{-\frac{|X|^2}{2}}d\mu \\
&+ \int_M (-|y|^2 + \langle X, y\rangle^2)e^{-\frac{|X|^2}{2}}d\mu.
\end{aligned}
$$

□

Definition 6.1. *One calls that a critical point $X : M \to \mathbb{R}^{n+1}$ of the $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$ is $\mathcal{F}$-stable if, for every normal variation fN, there exist variations of X_0 and t_0 such that ${\mathcal{F}''}_{X_0,t_0}(0) \geq 0$;*
One calls that a critical point $X : M \to \mathbb{R}^{n+1}$ of the $\mathcal{F}$-functional $\mathcal{F}_{X_s,t_s}(s)$ is $\mathcal{F}$-unstable if there exist a normal variation fN such that for all variations of X_0 and t_0, ${\mathcal{F}''}_{X_0,t_0}(0) < 0$.

Theorem 6.2. *If $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-stable; If $\sqrt{n} < r \leq \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-unstable.*

Proof. For the sphere $S^n(r)$, we have

$$X = -rN, \quad H = \frac{n}{r}, \quad S = \frac{H^2}{n} = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n}{r} - r$$

and

$$Lf = \mathcal{L}f + (S + 1 - \lambda^2)f = \Delta f + (\frac{n}{r^2} + 1 - \lambda^2)f. \tag{6.1}$$

Since we know that eigenvalues μ_k of Δ on the sphere $S^n(r)$ are given by

$$\mu_k = \frac{k^2 + (n-1)k}{r^2}, \tag{6.2}$$

and constant functions are eigenfunctions corresponding to eigenvalue $\mu_0 = 0$. For any constant vector $z \in \mathbb{R}^{n+1}$, we get

$$-\Delta\langle z, N\rangle = \Delta\langle z, \frac{X}{r}\rangle = \langle z, \frac{1}{r}HN\rangle = \frac{n}{r^2}\langle z, N\rangle, \tag{6.3}$$

that is, $\langle z, N\rangle$ is an eigenfunction of Δ corresponding to the first eigenvalue $\mu_1 = \frac{n}{r^2}$. Hence, for any normal variation with the variation vector field fN, we can choose a real number $a \in \mathbb{R}$ and a constant vector $z \in \mathbb{R}^{n+1}$ such that

$$f = f_0 + a + \langle z, N\rangle, \tag{6.4}$$

and f_0 is in the space spanned by all eigenfunctions corresponding to eigenvalues μ_k $(k \geq 2)$ of Δ on $S^n(r)$. Using the lemma 3.3, we get

$$
\begin{aligned}
&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0) \\
&= \int_{S^n(r)} -(f_0 + a + \langle z, N\rangle) L(f_0 + a + \langle z, N\rangle) d\mu \\
&\quad + \int_{S^n(r)} [2\langle N, y\rangle + (n+1-r^2)\lambda h - 2\frac{n}{r}h + 2\lambda\langle rN, y\rangle](f_0 + a + \langle z, N\rangle) d\mu \\
&\quad + \int_{S^n(r)} (-r)\langle N, y\rangle(r^2 - n - 1) h d\mu \\
&\quad + \int_{S^n(r)} (\frac{n^2+2n}{4} - \frac{n+2}{2}r^2 + \frac{r^4}{4} + \frac{3}{4}r^2 - \frac{3}{4}n) h^2 d\mu \\
&\quad + \int_{S^n(r)} (-|y|^2 + \langle X, y\rangle^2) d\mu \\
&\geq \int_{S^n(r)} \left\{ (\frac{n+2}{r^2} - 1 + \lambda^2) f_0^2 - (\frac{n}{r^2} + 1 - \lambda^2) a^2 + (\lambda^2 - 1)\langle z, N\rangle^2 \right\} d\mu \\
&\quad + \int_{S^n(r)} \left\{ 2(1+\lambda r)\langle N, y\rangle\langle N, z\rangle + [(n+1-r^2)\lambda - 2\frac{n}{r}] a h \right\} d\mu \\
&\quad + \int_{S^n(r)} \frac{1}{4}[r^4 - (2n+1)r^2 + n(n-1)] h^2 d\mu \\
&\quad + \int_{S^n(r)} (-|y|^2 + \langle X, y\rangle^2) d\mu.
\end{aligned}
\tag{6.5}
$$

From the lemma 3.3, we have

$$
\int_{S^n(r)} (-|y|^2 + \langle X, y\rangle^2) d\mu = -\int_{S^n(r)} (1+\lambda r)\langle N, y\rangle^2 d\mu. \tag{6.6}
$$

Putting (6.6) and $\lambda = \frac{n}{r} - r$ into (6.5), we obtain

$$
\begin{aligned}
&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0) \\
&\geq \int_{S^n(r)} \frac{1}{r^2} \left\{ (r^2 - n - \frac{1}{2})^2 + \frac{7}{4} \right\} f_0^2 d\mu \\
&\quad + \int_{S^n(r)} [r^4 - (2n+1)r^2 + n(n-1)](\frac{a}{r} + \frac{h}{2})^2 d\mu \\
&\quad + \int_{S^n(r)} \frac{1}{r^2}[r^4 - (2n+1)r^2 + n^2]\langle z, N\rangle^2 d\mu \\
&\quad + \int_{S^n(r)} 2(1+n-r^2)\langle N, y\rangle\langle N, z\rangle d\mu \\
&\quad + \int_{S^n(r)} -(1+n-r^2)\langle N, y\rangle^2 d\mu.
\end{aligned}
\tag{6.7}
$$

If we choose $h = -\frac{2a}{r}$, then we have

$$
\begin{aligned}
&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0)\\
\geq &\int_{S^n(r)} \frac{1}{r^2}\Big\{(r^2-n-\frac{1}{2})^2+\frac{7}{4}\Big\} f_0^2 d\mu\\
&+\int_{S^n(r)} (\lambda^2-1)\langle z, N\rangle^2 d\mu\\
&+\int_{S^n(r)} 2(1+\lambda r)\langle N, y\rangle\langle N, z\rangle d\mu\\
&+\int_{S^n(r)} -(1+\lambda r)\langle N, y\rangle^2 d\mu.
\end{aligned}
\tag{6.8}
$$

Let $y = kz$, then we have

$$
\begin{aligned}
&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0)\\
\geq &\int_{S^n(r)} \frac{1}{r^2}\Big\{(r^2-n-\frac{1}{2})^2+\frac{7}{4}\Big\} f_0^2 d\mu\\
&+\int_{S^n(r)} \Big\{\lambda^2-1+2(1+\lambda r)k-(1+\lambda r)k^2\Big\}\langle z, N\rangle^2 d\mu\\
= &\int_{S^n(r)} \frac{1}{r^2}\Big\{(r^2-n-\frac{1}{2})^2+\frac{7}{4}\Big\} f_0^2 d\mu\\
&+\int_{S^n(r)} \Big\{\lambda^2+\lambda r-(1+\lambda r)(1-k)^2\Big\}\langle z, N\rangle^2 d\mu.
\end{aligned}
\tag{6.9}
$$

We next consider three cases:

Case 1: $r \leq \sqrt{n}$

In this case, $\lambda \geq 0$. Taking $k = 1$, then we get

$$\mathcal{F}''(0) \geq 0.$$

Case 2: $r \geq \frac{1+\sqrt{1+4n}}{2}$.

In this case, $\lambda \leq -1$. Taking $k = 2$, we can get

$$\mathcal{F}''(0) \geq 0.$$

Case 3: $\sqrt{n+1} < r < \frac{1+\sqrt{1+4n}}{2}$.

In this case, $-1 < \lambda < 0$, $1 + \lambda r < 0$, we can take k such that $(1-k)^2 \geq \frac{\lambda(\lambda+r)}{1+\lambda r}$, then we have

$$\mathcal{F}''(0) \geq 0.$$

Thus, if $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-stable;

If $\sqrt{n} < r \leq \sqrt{n+1}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is $\mathcal{F}$-unstable. In fact, in this case, $-1 < \lambda < 0$, $1 + \lambda r \geq 0$. We can choose f such that $f_0 = 0$, then we have

$$
\begin{aligned}
(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{F}''(0) = & \int_{S^n(r)} (\lambda^2 - 1)\langle z, N\rangle^2 d\mu \\
& + \int_{S^n(r)} 2(1 + \lambda r)\langle N, y\rangle\langle N, z\rangle d\mu \\
& + \int_{S^n(r)} -(1 + \lambda r)\langle N, y\rangle^2 d\mu \\
= & (\lambda^2 + \lambda r) \int_{S^n(r)} \langle z, N\rangle^2 d\mu \\
& - (1 + \lambda r) \int_{S^n(r)} (\langle z, N\rangle - \langle y, N\rangle)^2 d\mu \\
< & \ 0.
\end{aligned}
\tag{6.10}
$$

This completes the proof of the theorem 6.2. $\square$

According to our theorem 6.2, we would like to propose the following:

Problem 6.1. Is it possible to prove that spheres $S^n(r)$ with $r \leq \sqrt{n}$ or $r > \sqrt{n+1}$ are the only $\mathcal{F}$-stable compact λ-hypersurfaces?

Remark 6.1. *Colding and Minicozzi* [10] *have proved that the sphere $S^n(\sqrt{n})$ is the only $\mathcal{F}$-stable compact self-shrinkers. In order to prove this result, the property that the mean curvature H is an eigenfunction of L-operator plays a very important role. But for λ-hypersurfaces, the mean curvature H is not an eigenfunction of L-operator in general.*

7. The weak stability of the weighted area functional for weighted volume-preserving variations

Define

$$
\mathcal{T}(s) = (4\pi t_s)^{-\frac{n}{2}} \int_M e^{-\frac{|X(s) - X_s|^2}{2t_s}} d\mu_s. \tag{7.1}
$$

We compute the first and the second variation formulas of the general $\mathcal{T}$-functional for weighted volume-preserving variations. By a direct calculation, we have

$$\begin{aligned}
&\mathcal{T}'(s)\\
&=(4\pi t_s)^{-\frac{n}{2}}\int_M -(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s},y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{T}''(s)\\
&=(4\pi t_s)^{-\frac{n}{2}}\int_M -(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)f'e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M\frac{nh}{2t_s}(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)\times\\
&\qquad\qquad(\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle+H_sf)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M-\Big(\frac{dH_s}{ds}+\langle\frac{\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}}{t_s},N(s)\rangle-\langle\frac{X(s)-X_s}{t_s^2},N(s)\rangle h\\
&\qquad\qquad+\langle\frac{X(s)-X_s}{t_s},\frac{dN(s)}{ds}\rangle\Big)fe^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s},y'\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}(-\frac{nh}{2t_s})\int_M\langle\frac{X(s)-X_s}{t_s},y\rangle e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M(\langle\frac{\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}}{t_s},y\rangle-\langle\frac{X(s)-X_s}{t_s^2},y\rangle h)e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s},y\rangle\Big(-\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle\\
&\qquad\qquad-H_sf\Big)e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})h'e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\quad+(4\pi t_s)^{-\frac{n}{2}}(-\frac{nh}{2t_s})\int_M(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2})he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s
\end{aligned}$$

$$
\begin{aligned}
&+(4\pi t_s)^{-\frac{n}{2}}\int_M\Big(\frac{nh}{2t_s^2}-\frac{|X(s)-X_s|^2}{t_s^3}h+\frac{\langle X(s)-X_s,\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle}{t_s^2}\Big)\times\\
&\qquad he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&\qquad\qquad -\frac{1}{2t_s}\langle\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s},N\rangle\lambda h)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M\Big(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2}\Big)h(-H_sf\\
&\qquad -\langle\frac{X(s)-X_s}{t_s},\frac{\partial X(s)}{\partial s}-\frac{\partial X_s}{\partial s}\rangle)e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M-(H_s+\langle\frac{X(s)-X_s}{t_s},N(s)\rangle)f\frac{|X(s)-X_s|^2}{2t_s^2}h\\
&\qquad\times e^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M\langle\frac{X(s)-X_s}{t_s},y\rangle\frac{|X(s)-X_s|^2}{2t_s^2}he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s\\
&+(4\pi t_s)^{-\frac{n}{2}}\int_M\Big(-\frac{n}{2t_s}+\frac{|X(s)-X_s|^2}{2t_s^2}\Big)h\frac{|X(s)-X_s|^2}{2t_s^2}he^{-\frac{|X(s)-X_s|^2}{2t_s}}d\mu_s.
\end{aligned}
$$

Lemma 7.1.

$$\int_M f^{'}(0)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=0.$$

Proof. Since $V(t)=\int_M\langle X(t)-X_0,N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=V(0)$ for any t, we have

$$\int_M f(t)\langle N(t),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu=0.$$

Hence, we get

$$
\begin{aligned}
0&=\frac{d}{dt}|_{t=0}\int_M f(t)\langle N(t),N\rangle e^{-\frac{|X-X_0|^2}{2t_0}}d\mu\\
&=\int_M f^{'}(0)e^{-\frac{|X-X_0|^2}{2t_0}}d\mu.
\end{aligned}
$$

□

Since M is a critical point of $\mathcal{T}(s)$, we have

$$H+\langle\frac{X-X_0}{t_0},N\rangle=\lambda.$$

On the other hand, we have

$$H^{'}=\Delta f+Sf,\quad N^{'}=-\nabla f. \tag{7.2}$$

Then for $t_0 = 1$ and $X_0 = 0$, the second variation formula becomes

$$
\begin{aligned}
&(4\pi)^{\frac{n}{2}}\mathcal{T}''(0)\\
=&\int_M \langle X, y'\rangle e^{-\frac{|X-|^2}{2}}d\mu + \int_M (\frac{|X|^2}{2} - \frac{n}{2})h' e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M -f\big(\mathcal{L}f + (S+1-\lambda^2)f\big)e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M \Big(2\langle N, y\rangle + (n-|X|^2)\lambda h + 2\langle X, N\rangle h\\
&\qquad + 2\langle N, y\rangle - 2\lambda\langle X, y\rangle\Big) f e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M (-(n+2)\langle X, y\rangle + \langle X, y\rangle |X|^2) h e^{-\frac{|X-|^2}{2}}d\mu\\
&+\int_M (\frac{n^2+2n}{4} - \frac{n+2}{2}|X|^2 + \frac{|X|^4}{4})h^2 e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M (-|y|^2 + \langle X, y\rangle^2) e^{-\frac{|X|^2}{2}}d\mu.
\end{aligned}
$$

Theorem 7.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a critical point of the functional $\mathcal{T}(s)$ for the weighted volume-preserving variations with fixed $X_0 = 0$ and $t_0 = 1$. The second variation formula of $\mathcal{T}(s)$ is given by*

$$
(4\pi)^{\frac{n}{2}}\mathcal{T}''(0) = \int_M -f\big(\mathcal{L}f + (S+1-\lambda^2)f\big)e^{-\frac{|X|^2}{2}}d\mu. \tag{7.3}
$$

Definition 7.1. *A critical point $X : M \to \mathbb{R}^{n+1}$ of the functional $\mathcal{T}(s)$ is called weakly stable if, for any weighted volume-preserving normal variation, $\mathcal{T}''(0) \geq 0$; A critical point $X : M \to \mathbb{R}^{n+1}$ of the functional $\mathcal{T}(s)$ is called weakly unstable if there exists a weighted volume-preserving normal variation, such that $\mathcal{T}''(0) < 0$.*

Theorem 7.2. *If $r \leq \frac{-1+\sqrt{1+4n}}{2}$ or $r \geq \frac{1+\sqrt{1+4n}}{2}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly stable; If $\frac{-1+\sqrt{1+4n}}{2} < r < \frac{1+\sqrt{1+4n}}{2}$, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly unstable.*

Proof. For the sphere $S^n(r)$, we have

$$
X = -rN, \quad H = \frac{n}{r}, \quad S = \frac{n}{r^2}, \quad \lambda = H - r = \frac{n}{r} - r
$$

and

$$
Lf = \mathcal{L}f + (S+1-\lambda^2)f = \Delta f + (\frac{n}{r^2} + 1 - \lambda^2)f. \tag{7.4}
$$

Since we know that eigenvalues μ_k of Δ on the sphere $S^n(r)$ are given by

$$
\mu_k = \frac{k^2 + (n-1)k}{r^2}, \tag{7.5}
$$

and constant functions are eigenfunctions corresponding to eigenvalue $\mu_0 = 0$. For any constant vector $z \in \mathbb{R}^{n+1}$, we get

$$-\Delta\langle z, N\rangle = \frac{n}{r^2}\langle z, N\rangle, \tag{7.6}$$

that is, $\langle z, N\rangle$ is an eigenfunction of Δ corresponding to the first eigenvalue $\mu_1 = \frac{n}{r^2}$. Hence, for any weighted volume-preserving normal variation with the variation vector field fN satisfying

$$\int_{S^n(r)} f e^{-\frac{r^2}{2}} d\mu = 0,$$

we can choose a constant vector $z \in \mathbb{R}^{n+1}$ such that

$$f = f_0 + \langle z, N\rangle, \tag{7.7}$$

and f_0 is in the space spanned by all eigenfunctions corresponding to eigenvalues μ_k $(k \geq 2)$ of Δ on $S^n(r)$. By making use of the theorem 7.1, we have

$$\begin{aligned}&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{T}''(0)\\ =&\int_{S^n(r)} -(f_0 + \langle z, N\rangle) L(f_0 + \langle z, N\rangle) d\mu\\ \geq&\int_{S^n(r)} \left\{(\frac{n+2}{r^2} - 1 + \lambda^2) f_0^2 + (\lambda^2 - 1)\langle z, N\rangle^2\right\} d\mu.\end{aligned} \tag{7.8}$$

According to $\lambda = \frac{n}{r} - r$, we obtain

$$\begin{aligned}&(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{T}''(0)\\ \geq&\int_{S^n(r)} \frac{1}{r^2}\{(r^2 - n - \frac{1}{2})^2 + \frac{7}{4}\} f_0^2 d\mu + \int_{S^n(r)} (\frac{n}{r} - r - 1)(\frac{n}{r} - r + 1)\langle z, N\rangle^2 d\mu \geq 0\end{aligned}$$

if

$$r \leq \frac{-1 + \sqrt{4n+1}}{2} \quad \text{or} \quad r \geq \frac{1 + \sqrt{4n+1}}{2}.$$

Thus, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly stable.
If

$$\frac{-1 + \sqrt{4n+1}}{2} < r < \frac{1 + \sqrt{4n+1}}{2},$$

choosing $f =< z, N >$, we have

$$\int_{S^n(r)} f e^{-\frac{r^2}{2}} d\mu = 0.$$

Hence, there exists a weighted volume-preserving normal variation with the variation vector filed fN such that

$$(4\pi)^{\frac{n}{2}} e^{\frac{r^2}{2}} \mathcal{T}''(0) = \int_{S^n(r)} (\frac{n}{r} - r - 1)(\frac{n}{r} - r + 1)\langle z, N\rangle^2 d\mu < 0.$$

Thus, the n-dimensional round sphere $X : S^n(r) \to \mathbb{R}^{n+1}$ is weakly unstable. It finishes the proof. $\square$

Remark 7.1. *From the theorem 6.2 and theorem 7.2, we know the $\mathcal{F}$-stability and the weak stability are different. The $\mathcal{F}$-stability is a weaker notation than the weak stability.*

Remark 7.2. *Is it possible to prove that spheres $S^n(r)$ with $r \le \frac{-1+\sqrt{1+4n}}{2}$ or $r \ge \frac{1+\sqrt{1+4n}}{2}$ are the only weak stable compact λ-hypersurfaces?*

8. Complete and non-compact λ-hypersurfaces

In this section, we will give a classification of complete and non-compact λ-hypersurfaces.

Theorem 8.1. *$S^k(r) \times \mathbb{R}^{n-k}$, $0 \le k \le n$, are the only complete embedded λ-hypersurfaces with polynomial area growth in $\mathbb{R}^{n+1}$ if $H - \lambda \ge 0$ and $\lambda(f_3(H-\lambda) - S) \ge 0$.*

Remark 8.1. *The assumption $\lambda(f_3(H-\lambda) - S) \ge 0$ in the theorem 8.1 is satisfied for self-shrinkers of the mean curvature flow, automatically and the assumption is essential. In fact, $\Gamma \times \mathbb{R}^{n-1}$ are counterexamples, which satisfy $H - \lambda > 0$, where Γ are compact embedded λ-curves other than the circle (see Remark 2.2).*

At first, we prepare the following lemmas and propositions.

Lemma 8.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional immersed hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. At any point $p \in M$, we have*

$$|\nabla\sqrt{S}|^2 \le \sum_{i,k} h_{iik}^2 \le \sum_{i,j,k} h_{ijk}^2, \tag{8.1}$$

$$\frac{n+3}{n+1}|\nabla\sqrt{S}|^2 \le \sum_{i,j,k} h_{ijk}^2 + \frac{2n}{n+1}|\nabla H|^2. \tag{8.2}$$

Its proof is standard. See Schoen, Simon and Yau [28] and Colding and Minicozzi [11].

Proposition 8.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with $H - \lambda > 0$ and $\lambda(f_3 - \frac{S}{H-\lambda}) \ge 0$. If η is a function with compact support, then*

$$\int_M \eta^2(S + |\nabla \log(H-\lambda)|^2) e^{-\frac{|X|^2}{2}} d\mu \le c(n,\lambda) \int_M (|\nabla\eta|^2 + \eta^2) e^{-\frac{|X|^2}{2}} d\mu, \tag{8.3}$$

where $c(n,\lambda)$ is constant depending on n and λ.

Proof. Since $H - \lambda > 0$, $\log(H-\lambda)$ is well-defined. Suppose η is a function with compact support, the lemma 4.1 and the corollary 3.1 give

$$\begin{aligned}
&\int_M \langle \nabla\eta^2, \nabla \log(H-\lambda)\rangle e^{-\frac{|X|^2}{2}} d\mu \\
&= -\int_M \eta^2(\mathcal{L} \log(H-\lambda)) e^{-\frac{|X|^2}{2}} d\mu \\
&= \int_M \eta^2 \Big(S - 1 - \frac{\lambda}{H-\lambda} + |\nabla \log(H-\lambda)|^2\Big) e^{-\frac{|X|^2}{2}} d\mu.
\end{aligned} \tag{8.4}$$

Combining this with inequality:

$$\langle \nabla \eta^2, \nabla \log(H-\lambda)\rangle \leq \varepsilon |\nabla \eta|^2 + \frac{1}{\varepsilon}\eta^2 |\nabla \log(H-\lambda)|^2 \tag{8.5}$$

gives that

$$\begin{aligned} &\int_M (\eta^2 S + \eta^2(1-\frac{1}{\varepsilon})|\nabla \log(H-\lambda)|^2) e^{-\frac{|X|^2}{2}} d\mu \\ \leq &\int_M (\varepsilon|\nabla\eta|^2 + \eta^2 + \frac{\lambda}{H-\lambda}\eta^2) e^{-\frac{|X|^2}{2}} d\mu, \end{aligned} \tag{8.6}$$

for $\varepsilon > 0$. Since

$$\frac{\lambda}{H-\lambda} \leq \frac{\lambda f_3}{S} \leq |\lambda|\sqrt{S} \leq |\lambda|(\frac{S}{2\delta} + \frac{\delta}{2}) \tag{8.7}$$

for $\delta > 0$, we have from (8.6) and (8.7)

$$\begin{aligned} &\int_M \left\{(1-\frac{|\lambda|}{2\delta})\eta^2 S + \eta^2(1-\frac{1}{\varepsilon})|\nabla \log(H-\lambda)|^2\right\} e^{-\frac{|X|^2}{2}} d\mu \\ \leq &\int_M \left(\varepsilon|\nabla\eta|^2 + \big(1+\frac{|\lambda|}{2}\delta\big)\eta^2\right) e^{-\frac{|X|^2}{2}} d\mu. \end{aligned} \tag{8.8}$$

By choosing ε, δ and constant $c(n,\lambda)$, we get

$$\int_M \eta^2 (S + |\nabla \log(H-\lambda)|^2) e^{-\frac{|X|^2}{2}} d\mu \leq c(n,\lambda) \int_M (|\nabla\eta|^2 + \eta^2) e^{-\frac{|X|^2}{2}} d\mu. \tag{8.9}$$

□

Proposition 8.2. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete λ-hypersurface with $H-\lambda > 0$ and $\lambda(f_3 - \frac{S}{H-\lambda}) \geq 0$. If M has polynomial area growth, then*

$$\begin{aligned} &\int_M \langle \nabla S, \nabla \log(H-\lambda)\rangle e^{-\frac{|X|^2}{2}} d\mu \\ =& -\int_M S\mathcal{L}\log(H-\lambda) e^{-\frac{|X|^2}{2}} d\mu \\ =& \int_M S\left(S - 1 - \frac{\lambda}{H-\lambda} + |\nabla \log(H-\lambda)|^2\right) e^{-\frac{|X|^2}{2}} d\mu, \end{aligned} \tag{8.10}$$

and

$$\begin{aligned} &\int_M |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu \\ =& -\int_M \sqrt{S}\mathcal{L}\sqrt{S} e^{-\frac{|X|^2}{2}} d\mu \\ \leq& \int_M (S^2 - S - \lambda f_3) e^{-\frac{|X|^2}{2}} d\mu. \end{aligned} \tag{8.11}$$

Proof. Taking $\eta = \phi$ in (8.4), we have

$$\begin{aligned}&\int_M \langle \nabla \phi^2, \nabla \log(H-\lambda)\rangle e^{-\frac{|X|^2}{2}} d\mu\\ &= -\int_M \phi^2 (\mathcal{L}\log(H-\lambda)) e^{-\frac{|X|^2}{2}} d\mu\\ &= \int_M \phi^2 \Big(S - 1 - \frac{\lambda}{H-\lambda} + |\nabla \log(H-\lambda)|^2\Big) e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{8.12}$$

Since

$$\langle \nabla \phi^2, \nabla \log(H-\lambda)\rangle \leq |\nabla\phi|^2 + \phi^2 |\nabla \log(H-\lambda)|^2, \tag{8.13}$$

we derive

$$\int_M \phi^2 S e^{-\frac{|X|^2}{2}} d\mu \leq \int_M (|\nabla\phi|^2 + \phi^2 + \frac{\lambda}{H-\lambda}\phi^2) e^{-\frac{|X|^2}{2}} d\mu. \tag{8.14}$$

Let $\phi = \eta\sqrt{S}$, where $\eta \geq 0$ has a compact support, for $\alpha > 0$, we have

$$\begin{aligned}&\int_M \eta^2 S^2 e^{-\frac{|X|^2}{2}} d\mu\\ &\leq \int_M \big\{\eta^2 |\nabla\sqrt{S}|^2 + 2\eta\sqrt{S}|\nabla\eta||\nabla\sqrt{S}|\\ &\quad + S|\nabla\eta|^2 + (1 + \frac{\lambda}{H-\lambda})\eta^2 S\big\} e^{-\frac{|X|^2}{2}} d\mu\\ &\leq \int_M (1+\alpha)\eta^2 |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu\\ &\quad + \int_M S\big\{(1+\frac{1}{\alpha})|\nabla\eta|^2 + (1+\frac{\lambda}{H-\lambda})\eta^2\big\} e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{8.15}$$

The lemma 4.1 and lemma 8.1 give the following inequality

$$\begin{aligned}\mathcal{L}S &= 2\sum_{i,j,k} h_{ijk}^2 + 2(1-S)S + 2\lambda f_3\\ &\geq \frac{2(n+3)}{n+1}|\nabla\sqrt{S}|^2 - \frac{4n}{n+1}|\nabla H|^2 + 2S - 2S^2 + 2\lambda f_3,\end{aligned} \tag{8.16}$$

Integrating this with $\frac{1}{2}\eta^2$ and using the lemma 3.2, we obtain

$$\begin{aligned}&-2\int_M \eta\sqrt{S}\langle \nabla\eta, \nabla\sqrt{S}\rangle e^{-\frac{|X|^2}{2}} d\mu\\ &\geq \int_M \Big\{\eta^2 \frac{(n+3)}{n+1}|\nabla\sqrt{S}|^2 - \frac{2n}{n+1}\eta^2|\nabla H|^2 + S\eta^2 - S^2\eta^2 + \lambda f_3 \eta^2\Big) e^{-\frac{|X|^2}{2}} d\mu.\end{aligned}$$

Since $2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon}$ for $\epsilon > 0$, we infer

$$\begin{aligned}&\int_M \Big\{\eta^2 S^2 + \frac{2n}{n+1}\eta^2 |\nabla H|^2 + \frac{1}{\epsilon} S|\nabla\eta|^2\Big\} e^{-\frac{|X|^2}{2}} d\mu\\ &\geq \int_M \Big\{(\frac{n+3}{n+1} - \epsilon)\eta^2 |\nabla\sqrt{S}|^2 + S\eta^2 + \lambda f_3 \eta^2\Big\} e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{8.17}$$

From (8.15), (8.17) and $\lambda\frac{S}{H-\lambda}\leq \lambda f_3$, by taking α and ϵ such that $\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}>0$, we have

$$\begin{aligned}
&\int_M \eta^2 S^2 e^{-\frac{|X|^2}{2}}d\mu\\
\leq &\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\int_M \eta^2 S^2 e^{-\frac{|X|^2}{2}}d\mu+\frac{2n}{n+1}\cdot\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\int_M \eta^2|\nabla H|^2 e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\left[\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}(\frac{1}{\epsilon}|\nabla\eta|^2-\eta^2)+(1+\frac{1}{\alpha})|\nabla\eta|^2+(1+\frac{\lambda}{H-\lambda})\eta^2\right]Se^{-\frac{|X|^2}{2}}d\mu\\
&+\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\int_M(-\lambda f_3\eta^2)e^{-\frac{|X|^2}{2}}d\mu\\
\leq &\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\int_M \eta^2 S^2 e^{-\frac{|X|^2}{2}}d\mu+\frac{2n}{n+1}\cdot\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\int_M \eta^2|\nabla H|^2 e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\Big\{[\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\times\frac{1}{\epsilon}+1+\frac{1}{\alpha}]|\nabla\eta|^2+(1-\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon})\eta^2\\
&\qquad+\frac{\lambda}{H-\lambda}(1-\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon})\eta^2\Big\}Se^{-\frac{|X|^2}{2}}d\mu.
\end{aligned}$$

Using

$$\lambda\frac{S}{H-\lambda}\leq\lambda f_3\leq|\lambda|S\sqrt{S}\leq\frac{1}{2\delta}|\lambda|S^2+\frac{\delta}{2}|\lambda|S,$$

for $\delta>0$, we obtain, by taking α and ϵ such that $1-\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}>0$

$$\begin{aligned}
&\left(1-\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\right)\left(1-\frac{|\lambda|}{2\delta}\right)\int_M\eta^2S^2e^{-\frac{|X|^2}{2}}d\mu\\
\leq&\frac{2n}{n+1}\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\int_M\eta^2|\nabla H|^2e^{-\frac{|X|^2}{2}}d\mu\\
&+\int_M\Big\{\left(\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\frac{1}{\epsilon}+1+\frac{1}{\alpha}\right)|\nabla\eta|^2+\left(1-\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\right)\eta^2\\
&+\left(1-\frac{1+\alpha}{\frac{n+3}{n+1}-\epsilon}\right)\eta^2\frac{\delta}{2}|\lambda|\Big\}Se^{-\frac{|X|^2}{2}}d\mu.
\end{aligned}\tag{8.18}$$

Assuming $|\eta|\leq1$ and $|\nabla\eta|\leq1$, choosing δ such that $\frac{|\lambda|}{2\delta}<1$, we have

$$\int_M\eta^2S^2e^{-\frac{|X|^2}{2}}d\mu\leq C(n,\lambda)\int_M(|\nabla H|^2+S)e^{-\frac{|X|^2}{2}}d\mu\tag{8.19}$$

for some constant $C(n,\lambda)$ depending on n and λ. Since $|\nabla H|\leq\sqrt{S}|X|$ holds from (4.5), one has from (8.19)

$$\int_M\eta^2S^2e^{-\frac{|X|^2}{2}}d\mu\leq C(n,\lambda)\int_M S(1+|X|^2)e^{-\frac{|X|^2}{2}}d\mu.\tag{8.20}$$

Since $H-\lambda>0$ and $\lambda f_3 \geq \lambda \frac{S}{H-\lambda}$, let η_j be one on B_j and cut off linearly to zero from ∂B_j to ∂B_{j+1}, where $B_j = X(M)\cap B_j(0)$ with $B_j(0)$ is the Euclidean ball of radius j centered at the origin. Applying the proposition 8.1 with $\eta=\eta_j|X|$, letting $j\to\infty$, the dominated convergence theorem and the polynomial area growth give that $\int_M S(1+|X|^2)e^{-\frac{|X|^2}{2}}d\mu<+\infty$. Thus (8.20) and the dominated convergence theorem give that

$$\int_M S^2 e^{-\frac{|X|^2}{2}}d\mu<+\infty.$$

Hence, from (8.17), we also have

$$\int_M |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}}d\mu<+\infty.$$

We next prove $\int_M \sum\limits_{i,j,k} h_{ijk}^2 e^{-\frac{|X|^2}{2}}d\mu<+\infty$. From (4.2), one has

$$\begin{aligned}
&\int_M \eta^2\sum_{i,j,k} h_{ijk}^2 e^{-\frac{|X|^2}{2}}d\mu\\
&=\int_M \eta^2(S^2-S)e^{-\frac{|X|^2}{2}}d\mu-\int_M \lambda f_3\eta^2 e^{-\frac{|X|^2}{2}}d\mu\\
&\quad-\int_M 2\eta\sqrt{S}\langle\nabla\eta,\nabla\sqrt{S}\rangle e^{-\frac{|X|^2}{2}}d\mu\\
&\leq C_0(n,\lambda)\int_M(\eta^2S^2+\eta^2S+|\nabla\eta|^2|\nabla\sqrt{S}|^2)e^{-\frac{|X|^2}{2}}d\mu\\
&<+\infty,
\end{aligned}\tag{8.21}$$

where $C_0(n,\lambda)$ is constant depending on n and λ. The dominated convergence theorem gives that

$$\int_M\sum_{i,j,k} h_{ijk}^2 e^{-\frac{|X|^2}{2}}d\mu<+\infty.\tag{8.22}$$

This shows that

$$\int_M (S+S^2+|\nabla\sqrt{S}|^2+\sum_{i,j,k} h_{ijk}^2)e^{-\frac{|X|^2}{2}}d\mu<+\infty.\tag{8.23}$$

From (8.23), we have

$$\int_M (S^2+|\nabla\sqrt{S}|^2)e^{-\frac{|X|^2}{2}}d\mu<+\infty,$$

that is, $\sqrt{S}$ is in the weighted $W^{1,2}$ space. Applying the proposition 8.1 with $\eta=\eta_j\sqrt{S}$, letting $j\to\infty$, using the dominated convergence theorem, one has

$$\int_M S|\nabla\log(H-\lambda)|^2 e^{-\frac{|X|^2}{2}}d\mu<+\infty.\tag{8.24}$$

It follows that

$$
\begin{aligned}
&\int_M |\nabla S||\nabla \log(H-\lambda)|e^{-\frac{|X|^2}{2}}d\mu \\
&\leq \int_M (|\nabla\sqrt{S}|^2 + S|\nabla \log(H-\lambda)|^2)e^{-\frac{|X|^2}{2}}d\mu < +\infty.
\end{aligned}
\tag{8.25}
$$

(4.4) gives that

$$
\begin{aligned}
&\int_M S|\mathcal{L}\log(H-\lambda)|e^{-\frac{|X|^2}{2}}d\mu \\
=&\int_M S\left|1-S+\frac{\lambda}{H-\lambda}-|\nabla\log(H-\lambda)|^2\right|e^{-\frac{|X|^2}{2}}d\mu \\
\leq& C_1(n,\lambda)\int_M \left\{S^2+S+S|\nabla\log(H-\lambda)|^2\right\}e^{-\frac{|X|^2}{2}}d\mu \\
<& +\infty,
\end{aligned}
\tag{8.26}
$$

where $C_1(n,\lambda)$ is constant. Thus, we obtain

$$
\int_M \left\{S|\nabla\log(H-\lambda)|+|\nabla S||\nabla\log(H-\lambda)|+S\mathcal{L}\log(H-\lambda)|\right\}e^{-\frac{|X|^2}{2}}d\mu < +\infty.
\tag{8.27}
$$

By applying the corollary 3.1 to S and $\log(H-\lambda)$, we get

$$
\begin{aligned}
&\int_M \langle\nabla S, \nabla\log(H-\lambda)\rangle e^{-\frac{|X|^2}{2}}d\mu \\
=&-\int_M S\mathcal{L}\log(H-\lambda)e^{-\frac{|X|^2}{2}}d\mu \\
=&\int_M S\left(S-1-\frac{\lambda}{H-\lambda}+|\nabla\log(H-\lambda)|^2\right)e^{-\frac{|X|^2}{2}}d\mu.
\end{aligned}
\tag{8.28}
$$

On one hand, (4.3) gives

$$
\begin{aligned}
&\int_M \sqrt{S}|\mathcal{L}\sqrt{S}|e^{-\frac{|X|^2}{2}}d\mu \\
=&\int_M \left|\sum_{i,j,k}h_{ijk}^2-|\nabla\sqrt{S}|^2+S(1-S)+\lambda f_3\right|e^{-\frac{|X|^2}{2}}d\mu \\
\leq& C_2(n,\lambda)\int_M\left(\sum_{i,j,k}h_{ijk}^2+|\nabla\sqrt{S}|^2+S+S^2\right)e^{-\frac{|X|^2}{2}}d\mu \\
<& +\infty.
\end{aligned}
\tag{8.29}
$$

Hence

$$
\int_M\left(\sqrt{S}|\nabla\sqrt{S}|+|\nabla\sqrt{S}|^2+\sqrt{S}|\mathcal{L}\sqrt{S}|\right)e^{-\frac{|X|^2}{2}}d\mu < +\infty.
\tag{8.30}
$$

On the other hand, we have from (4.3) and the lemma 8.1

$$
\mathcal{L}\sqrt{S} \geq \sqrt{S}-\sqrt{S}S+\frac{\lambda f_3}{\sqrt{S}}.
\tag{8.31}
$$

Then we can apply the corollary 3.1 to $\sqrt{S}$ and $\sqrt{S}$ and obtain

$$\begin{aligned}&\int_M |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu\\ &= -\int_M \sqrt{S}\mathcal{L}\sqrt{S} e^{-\frac{|X|^2}{2}} d\mu\\ &\leq \int_M (S^2 - S - \lambda f_3) e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{8.32}$$

□

Proof of Theorem 8.1. Since $H - \lambda \geq 0$ and $\mathcal{L}H - H \leq 0$, if $\lambda \leq 0$, we have from the maximum principle that either $H - \lambda \equiv 0$ or $H - \lambda > 0$, if $H - \lambda \equiv 0$, (4.5) and (4.6) give that $\lambda = 0 = H$, then M is a self-shrinker of the mean curvature flow. According to the results of Colding and Minicozzi [11], M is $\mathbb{R}^n$. If $\lambda > 0$ and $H - \lambda = 0$ at some point $p \in M$, then we see from $\lambda(f_3(H - \lambda) - S) \geq 0$ that $S = 0$ and $H = 0$ at p, then $\lambda \equiv 0$, according to the results of Colding and Minicozzi [11], we know that M is $\mathbb{R}^n$. Hence, for any λ, we have either M is $\mathbb{R}^n$ or $H - \lambda > 0$. Next, we assume that $H - \lambda > 0$. From the proposition 8.2, we have

$$\begin{aligned}&\int_M \langle \nabla S, \nabla \log(H - \lambda)\rangle e^{-\frac{|X|^2}{2}} d\mu\\ &= -\int_M S\mathcal{L}\log(H - \lambda) e^{-\frac{|X|^2}{2}} d\mu\\ &= \int_M S\Big(S - 1 - \frac{\lambda}{H - \lambda} + |\nabla \log(H - \lambda)|^2\Big) e^{-\frac{|X|^2}{2}} d\mu,\end{aligned} \tag{8.33}$$

and

$$\begin{aligned}&\int_M |\nabla\sqrt{S}|^2 e^{-\frac{|X|^2}{2}} d\mu\\ &= -\int_M \sqrt{S}\mathcal{L}\sqrt{S} e^{-\frac{|X|^2}{2}} d\mu\\ &\leq \int_M (S^2 - S - \lambda f_3) e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{8.34}$$

Substituting (8.34) into (8.33) and using $\lambda f_3 \geq \lambda \frac{S}{H-\lambda}$, one has

$$\begin{aligned}0 &\geq \int_M \Big\{ |\nabla\sqrt{S}|^2 - 2\sqrt{S}\langle \nabla\sqrt{S}, \nabla\log(H - \lambda)\rangle + S|\nabla \log(H - \lambda)|^2\\ &\qquad + \lambda f_3 - \lambda \frac{S}{H - \lambda}\Big\} e^{-\frac{|X|^2}{2}} d\mu\\ &\geq \int_M \big|\nabla\sqrt{S} - \sqrt{S}\nabla\log(H - \lambda)\big|^2 e^{-\frac{|X|^2}{2}} d\mu.\end{aligned} \tag{8.35}$$

Hence we conclude that $\nabla\sqrt{S} = \sqrt{S}\nabla\log(H - \lambda)$. Thus, we obtain

$$\sqrt{S} = \beta(H - \lambda) \tag{8.36}$$

for a constant $\beta > 0$. Since all inequalities in above equations become equalities, we obtain

$$\sum_{i,j,k} h_{ijk}^2 = |\nabla \sqrt{S}|^2, \quad \lambda f_3 = \lambda \frac{S}{H-\lambda}. \tag{8.37}$$

From the lemma 8.1 and (8.37), we know

(1) There is a constant C_k such that $h_{iik} = C_k \lambda_i$ for every i and k.

(2) If $i \neq j$, then $h_{ijk} = 0$, that is, $h_{ijk} = 0$ unless $i = j = k$ since $h_{ijk} = h_{ikj}$.

If $\lambda_i \neq 0$ and $j \neq i$, then $0 = h_{iij} = C_j \lambda_i$. It follows that $C_j = 0$. If the rank of matrix (h_{ij}) is at least two at p, then $C_j = 0$ for $j \in \{1, 2, \cdots, n\}$. Hence, we have $h_{ijk}(p) = 0$.
We next consider two cases.

Case 1: The rank of matrix (h_{ij}) is greater than one at p.

In this case, we will prove that the rank of (h_{ij}) is at least two everywhere. In fact, for $q \in M$, let $\lambda_1(q)$ and $\lambda_2(q)$ be the two eigenvalues of $(h_{ij})(q)$ that are largest in absolute value and define the set

$$\Omega = \{q \in M | \lambda_1(q) = \lambda_1(p), \lambda_2(q) = \lambda_2(p)\}. \tag{8.38}$$

Then $p \in \Omega$, since λ_i's are continuous, so Ω is closed. Given any point $q \in \Omega$, it follows that the rank of (h_{ij}) is at least two at q. Hence there is an open set U, $q \in U$, where the rank of (h_{ij}) is at least two. On U, we have $h_{ijk} = 0$ and the eigenvalues of (h_{ij}) are constant on U. Thus, $U \subset \Omega$, Ω is open. Since M is connected, we have $\Omega = M$ and $h_{ijk} \equiv 0$ on M. We know that $M = S^k(r) \times \mathbb{R}^{n-k}$, where $k > 1$.

Case 2: The rank of matrix (h_{ij}) is one.

From Case 1, we know that the rank of (h_{ij}) is one everywhere. Hence $S = H^2$. On the other hand, $S = \beta^2 (H - \lambda)^2$, hence $H^2 = \beta^2 (H - \lambda)^2$. If $\lambda = 0$, then M is a self-shrinker of the mean curvatue flow. If $\lambda \neq 0$, then we have H is constant. M is $S^1(r) \times \mathbb{R}^{n-1}$ from the proposition 2.2. This completes the proof of Theorem 8.1.

□

9. Properness and polynomial area growth for λ-hypersurfaces

For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, the well-known theorem of Bishop and Gromov says that geodesic balls have at most polynomial area growth:

$$\text{Area}(B_r(x_0)) \leq C r^n.$$

For n-dimensional complete and non-compact gradient shrinking Ricci soliton, Cao and Zhou [5] have proved geodesic balls have at most polynomial area growth. For self-shrinkers, Ding and Xin [12] proved that any complete non-compact properly immersed self-shrinker in the Euclidean space has polynomial area growth. X. Cheng and Zhou [10] showed that any complete immersed self-shrinker with polynomial area growth in the Euclidean space is proper. Hence any complete immersed self-shrinker is proper if and only if it has polynomial area growth.

It is our purposes in this section to study the area growth for λ-hypersurfaces. First of all, we study the equivalence of properness and polynomial area growth for λ-hypersurfaces. If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional hypersurface in $\mathbb{R}^{n+1}$, we say M has polynomial area growth if there exist constant C and d such that for all $r \geq 1$,

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq Cr^d, \tag{9.1}$$

where $B_r(0)$ is a round ball in $\mathbb{R}^{n+1}$ with radius r and centered at the origin.

Theorem 9.1. *Let $X : M \to \mathbb{R}^{n+1}$ be a complete and non-compact properly immersed λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Then, there is a positive constant C such that for $r \geq 1$,*

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq Cr^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}}, \tag{9.2}$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$.

Proof. Since $X : M \to \mathbb{R}^{n+1}$ is a complete and non-compact properly immersed λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$, we have

$$\langle X, N\rangle + H = \lambda.$$

Defining $f = \frac{|X|^2}{4}$, we have

$$f - |\nabla f|^2 = \frac{|X|^2}{4} - \frac{|X^T|^2}{4} = \frac{|X^\perp|^2}{4} = \frac{1}{4}(\lambda - H)^2, \tag{9.3}$$

$$\begin{aligned} \Delta f &= \frac{1}{2}(n + H\langle N, X\rangle) \\ &= \frac{1}{2}(n + \lambda\langle N, X\rangle - \langle N, X\rangle^2) \\ &= \frac{1}{2}n + \frac{\lambda^2}{4} - \frac{H^2}{4} - f + |\nabla f|^2. \end{aligned} \tag{9.4}$$

Hence, we obtain

$$|\nabla(f - \beta)|^2 \leq (f - \beta), \tag{9.5}$$

$$\Delta(f - \beta) - |\nabla(f - \beta)|^2 + (f - \beta) \leq (\frac{n}{2} + \frac{\lambda^2}{4} - \beta - \frac{\inf H^2}{4}). \tag{9.6}$$

Since the immersion X is proper, we know that $\overline{f} = f - \beta$ is proper. Applying the theorem 2.1 of X. Cheng and Zhou [10] to $\overline{f} = f - \beta$ with $k = (\frac{n}{2} + \frac{\lambda^2}{4} - \beta - \frac{\inf H^2}{4})$, we obtain

$$\text{Area}(B_r(0) \cap X(M)) = \int_{B_r(0)\cap X(M)} d\mu \leq Cr^{n+\frac{\lambda^2}{2}-2\beta-\frac{\inf H^2}{2}}, \tag{9.7}$$

where $\beta = \frac{1}{4}\inf(\lambda - H)^2$ and C is a constant. □

Remark 9.1. *The estimate in our theorem 9.1 is the best possible because the cylinders $S^k(r_0)\times\mathbb{R}^{n-k}$ satisfy the equality.*

Remark 9.2. *By making use of the same assertions as in X. Cheng and Zhou* [10] *for self-shrinkers, we can prove the weighted area of a complete and non-compact properly immersed λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is bounded.*

By making use of to the same assertions as in X. Cheng and Zhou [10] for self-shrinkers, we can prove the following theorem. We will leave it for readers.

Theorem 9.2. *If $X: M\to\mathbb{R}^{n+1}$ is an n-dimensional complete immersed λ-hypersurface with polynomial area growth, then $X: M\to\mathbb{R}^{n+1}$ is proper.*

10. A lower bound growth of the area for λ-hypersurfaces

For n-dimensional complete and non-compact Riemannian manifolds with nonnegative Ricci curvature, the well-known theorem of Calabi and Yau says that geodesic balls have at least linear area growth:

$$\mathrm{Area}(B_r(x_0))\geq Cr.$$

Cao and Zhu [6] have proved that n-dimensional complete and non-compact gradient shrinking Ricci soliton must have infinite volume. Furthermore, Munteanu and Wang [26] have proved that areas of geodesic balls for n-dimensional complete and non-compact gradient shrinking Ricci soliton has at least linear growth. For self-shrinkers, Li and Wei [24] proved that any complete and non-compact proper self-shrinker has at least linear area growth.
In this section, we study the lower bound growth of the area for λ-hypersurfaces. The following lemmas play a very important role in order to prove our results.

Lemma 10.1. *Let $X: M\to\mathbb{R}^{n+1}$ be an n-dimensional complete noncompact proper λ-hypersurface, then there exist constants $C_1(n,\lambda)$ and $c(n,\lambda)$ such that for all $t\geq C_1(n,\lambda)$,*

$$\mathrm{Area}(B_{t+1}(0)\cap X(M))-\mathrm{Area}(B_t(0)\cap X(M))\leq c(n,\lambda)\frac{\mathrm{Area}(B_t(0)\cap X(M))}{t}\tag{10.1}$$

and

$$\mathrm{Area}(B_{t+1}(0)\cap X(M))\leq 2\mathrm{Area}(B_t(0)\cap X(M)).\tag{10.2}$$

Proof. Since $X: M\to\mathbb{R}^{n+1}$ is a complete λ-hypersurface, one has

$$\frac{1}{2}\Delta|X|^2=n+H\langle N,X\rangle=n+H\lambda-H^2.\tag{10.3}$$

Integrating (10.3) over $B_r(0) \cap X(M)$, we obtain

$$
\begin{aligned}
& n\text{Area}(B_r(0) \cap X(M)) + \int_{B_r(0)\cap X(M)} H\lambda d\mu - \int_{B_r(0)\cap X(M)} H^2 d\mu \\
&= \frac{1}{2}\int_{B_r(0)\cap X(M)} \triangle |X|^2 d\mu \\
&= \frac{1}{2}\int_{\partial(B_r(0)\cap X(M))} \nabla |X|^2 \cdot \frac{\nabla\rho}{|\nabla\rho|} d\sigma \\
&= \int_{\partial(B_r(0)\cap X(M))} |X^T| d\sigma \\
&= \int_{\partial(B_r(0)\cap X(M))} \frac{|X|^2 - (\lambda - H)^2}{|X^T|} d\sigma \\
&= r\big(\text{Area}(B_r(0) \cap X(M))\big)' - \int_{\partial(B_r(0)\cap X(M))} \frac{(\lambda - H)^2}{|X^T|} d\sigma,
\end{aligned} \tag{10.4}
$$

where $\rho(x) := |X(x)|$, $\nabla\rho = \frac{X^T}{|X|}$. Here we used, from the co-area formula,

$$
\big(\text{Area}(B_r(0) \cap X(M))\big)' = r\int_{\partial(B_r(0)\cap X(M))} \frac{1}{|X^T|} d\sigma. \tag{10.5}
$$

Hence, we obtain

$$
\begin{aligned}
& (n + \frac{\lambda^2}{4})\text{Area}(B_r(0) \cap X(M)) - r\big(\text{Area}(B_r(0) \cap X(M))\big)' \\
&= \int_{B_r(0)\cap X(M)} (H - \frac{\lambda}{2})^2 d\mu - \int_{\partial(B_r(0)\cap X(M))} \frac{(\lambda - H)^2}{|X^T|} d\sigma,
\end{aligned} \tag{10.6}
$$

From (10.5), $(H-\lambda)^2 = \langle N, X\rangle^2 \leq |X|^2 = r^2$ on $\partial(B_r(0) \cap X(M))$ and (10.6), we conclude

$$
\int_{B_r(0)\cap X(M)} (H - \frac{\lambda}{2})^2 d\mu \leq (n + \frac{\lambda^2}{4})\text{Area}(B_r(0) \cap X(M)). \tag{10.7}
$$

Furthermore, we have

$$
\begin{aligned}
\int_{B_r(0)\cap X(M)} (H - \lambda)^2 d\mu &\leq \int_{B_r(0)\cap X(M)} 2\big[(H - \frac{\lambda}{2})^2 + \frac{\lambda^2}{4}\big] d\mu \\
&\leq (2n + \lambda^2)\text{Area}(B_r(0) \cap X(M)).
\end{aligned} \tag{10.8}
$$

(10.6) implies that

$$
\begin{aligned}
& \big(r^{-n-\frac{\lambda^2}{4}}\text{Area}(B_r(0) \cap X(M))\big)' \\
&= r^{-n-1-\frac{\lambda^2}{4}}\Big(r\big(\text{Area}(B_r(0) \cap X(M))\big)' - (n + \frac{\lambda^2}{4})\text{Area}(B_r(0) \cap X(M))\Big) \\
&= r^{-n-1-\frac{\lambda^2}{4}}\int_{\partial(B_r(0)\cap X(M))} \frac{(H - \lambda)^2}{|X^T|} d\sigma - r^{-n-1-\frac{\lambda^2}{4}}\int_{B_r(0)\cap X(M)} (H - \frac{\lambda}{2})^2 d\mu.
\end{aligned} \tag{10.9}
$$

Integrating (10.9) from r_2 to r_1 ($r_1 > r_2$), one has

$$
\begin{aligned}
&r_1^{-n-\frac{\lambda^2}{4}}\mathrm{Area}(B_{r_1}(0)\cap X(M)) - r_2^{-n-\frac{\lambda^2}{4}}\mathrm{Area}(B_{r_2}(0)\cap X(M))\\
&= r_1^{-n-2-\frac{\lambda^2}{4}}\int_{B_{r_1}(0)\cap X(M)}(H-\lambda)^2 d\mu - r_2^{-n-2-\frac{\lambda^2}{4}}\int_{B_{r_2}(0)\cap X(M)}(H-\lambda)^2 d\mu\\
&\quad + (n+2+\frac{\lambda^2}{4})\int_{r_2}^{r_1} s^{-n-3-\frac{\lambda^2}{4}}(\int_{B_s(0)\cap X(M)}(H-\lambda)^2 d\mu)ds\\
&\quad - \int_{r_2}^{r_1} s^{-n-1-\frac{\lambda^2}{4}}(\int_{B_s(0)\cap X(M)}(H-\frac{\lambda}{2})^2 d\mu)ds\\
&\le \left(r_1^{-n-2-\frac{\lambda^2}{4}} + r_2^{-n-2-\frac{\lambda^2}{4}}\right)\int_{B_{r_1}(0)\cap X(M)}(H-\lambda)^2 d\mu.
\end{aligned}
\tag{10.10}
$$

Here we used

$$
\left(\int_{B_r(0)\cap X(M)}(H-\lambda)^2 d\mu\right)' = r\int_{\partial(B_r(0)\cap X(M))}\frac{(H-\lambda)^2}{|X^T|}d\sigma
$$

and $\mathrm{Area}(B_r(0)\cap X(M))$ is non-decreasing in r from (10.5). Combining (10.10) with (10.8), we have

$$
\begin{aligned}
&\frac{\mathrm{Area}(B_{r_1}(0)\cap X(M))}{r_1^{n+\frac{\lambda^2}{4}}} - \frac{\mathrm{Area}(B_{r_2}(0)\cap X(M))}{r_2^{n+\frac{\lambda^2}{4}}}\\
&\le (2n+\lambda^2)\Big(\frac{1}{r_1^{n+2+\frac{\lambda^2}{4}}} + \frac{1}{r_2^{n+2+\frac{\lambda^2}{4}}}\Big)\mathrm{Area}(B_{r_1}(0)\cap X(M)).
\end{aligned}
\tag{10.11}
$$

Putting $r_1 = t+1$, $r_2 = t > 0$, we get

$$
\begin{aligned}
&\left(1 - \frac{2(2n+\lambda^2)(t+1)^{n+\frac{\lambda^2}{4}}}{t^{n+2+\frac{\lambda^2}{4}}}\right)\mathrm{Area}(B_{t+1}(0)\cap X(M))\\
&\le \mathrm{Area}(B_t(0)\cap X(M))(\frac{t+1}{t})^{n+\frac{\lambda^2}{4}}.
\end{aligned}
\tag{10.12}
$$

For t sufficiently large, one has, from (10.12),

$$
\begin{aligned}
&\mathrm{Area}(B_{t+1}(0)\cap X(M)) - \mathrm{Area}(B_t(0)\cap X(M))\\
&\le \mathrm{Area}(B_t(0)\cap X(M))\left((1+\frac{1}{t})^n - 1 + \frac{C(t+1)^{2n+\lambda^2 4}}{t^{2n+2+\lambda^2}}\right),
\end{aligned}
\tag{10.13}
$$

where C is constant only depended on n, λ. Therefore, there exists some constant $C_1(n,\lambda)$ such that for all $t \ge C_1(n,\lambda)$,

$$
\begin{aligned}
&\mathrm{Area}(B_{t+1}(0)\cap X(M)) - \mathrm{Area}(B_t(0)\cap X(M))\\
&\le c(n,\lambda)\frac{\mathrm{Area}(B_t(0)\cap X(M))}{t},
\end{aligned}
\tag{10.14}
$$

$$
\mathrm{Area}(B_{t+1}(0)\cap X(M)) \le 2\mathrm{Area}(B_t(0)\cap X(M)),
\tag{10.15}
$$

where $c(n,\lambda)$ depends only on n and λ. This completes the proof of the lemma 10.1.

□

Using Logarithmic Sobolev inequality for hypersurfaces in Euclidean space due to Ecker [14] and conformal theory, we can show

Lemma 10.2. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface with measure $d\mu$. Then the following inequality*

$$\begin{aligned}&\int_M f^2(\ln f^2)e^{-\frac{|X|^2}{2}}d\mu - \int_M f^2e^{-\frac{|X|^2}{2}}d\mu\ln(\int_M f^2e^{-\frac{|X|^2}{2}}2^{\frac{n}{2}}d\mu)\\ &\leq \int_M |\nabla f|^2e^{-\frac{|X|^2}{2}}d\mu + \frac{1}{4}\int_M |H+\langle X,N\rangle|^2f^2e^{-\frac{|X|^2}{2}}d\mu\\ &+C(n)\int_M f^2e^{-\frac{|X|^2}{2}}d\mu\end{aligned} \tag{10.16}$$

holds for any nonnegative function f for which all integrals are well-defined and finite, where $C(n)$ is a positive constant depending on n.

Corollary 10.1. *For an n-dimensional λ-hypersurface $X : M \to \mathbb{R}^{n+1}$, we have the following inequality*

$$\int_M f^2(\ln f)e^{-\frac{|X|^2}{2}}d\mu \leq \frac{1}{2}\int_M |\nabla f|^2e^{-\frac{|X|^2}{2}}d\mu + (\frac{1}{2}C(n)+\frac{1}{8}\lambda^2)2^{-\frac{n}{2}} \tag{10.17}$$

for any nonnegative function f which satisfies

$$\int_M f^2e^{-\frac{|X|^2}{2}}2^{\frac{n}{2}}d\mu = 1. \tag{10.18}$$

Corollary 10.2. *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional λ-hypersurface, then the following inequality*

$$\begin{aligned}&\int_M u^2(\ln u^2)d\mu - \int_M u^2d\mu\ln(\int_M u^2d\mu)\\ &\leq 2\int_M |\nabla u|^2d\mu + (\frac{1}{4}\lambda^2+\frac{n}{2}\ln 2 + C(n))\int_M u^2d\mu\end{aligned} \tag{10.19}$$

holds for any nonnegative function f which satisfies

$$f = ue^{\frac{|X|^2}{4}}. \tag{10.20}$$

Lemma 10.3. ([24]) *Let $X : M \to \mathbb{R}^{n+1}$ be a complete properly immersed hypersurface. For any $x_0 \in M$, $r \leq 1$, if $|H| \leq \frac{C}{r}$ in $B_r(X(x_0)) \cap X(M)$ for some constant $C > 0$. Then*

$$Area(B_r(X(x_0)) \cap X(M)) \geq \kappa r^n, \tag{10.21}$$

where $\kappa = \omega_n e^{-C}$.

Lemma 10.4. *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete and non-compact proper λ-hypersurface. then it has infinite area.*

Proof. Let

$$\begin{aligned}\Omega(k_1,k_2) &= \{x \in M : 2^{k_1-\frac{1}{2}} \leq \rho(x) \leq 2^{k_2-\frac{1}{2}}\},\\ A(k_1,k_2) &= \mathrm{Area}(X(\Omega(k_1,k_2))),\end{aligned}$$

where $\rho(x) = |X(x)|$. Since $X : M \to \mathbb{R}^{n+1}$ is a complete and non-compact proper immersion, $X(M)$ can not be contained in a compact Euclidean ball. Then, for k large enough, $\Omega(k, k+1)$ contains at least 2^{2k-1} disjoint balls

$$B_r(x_i) = \{x \in M : \rho_{x_i}(x) < 2^{-\frac{1}{2}} r\}, \quad x_i \in M, \ r = 2^{-k}$$

where $\rho_{x_i}(x) = |X(x) - X(x_i)|$. Since, in $\Omega(k, k+1)$,

$$|H| \leq |H - \lambda| + |\lambda| = |\langle X, N\rangle| + |\lambda| \leq |X| + |\lambda| \leq 2^k\sqrt{2} + |\lambda| \leq \frac{\sqrt{2} + |\lambda|}{r}, \tag{10.22}$$

by using of the lemma 10.3, we get

$$A(k, k+1) \geq \kappa_1 2^{2k-1-kn}, \tag{10.23}$$

with $\kappa_1 = \omega_n e^{-(\sqrt{2}+|\lambda|)2^{-\frac{1}{2}}} 2^{-\frac{n}{2}}$.

Claim: If $Area(X(M)) < \infty$, then, for every $\varepsilon > 0$, there exists a large constant $k_0 > 0$ such that,

$$A(k_1, k_2) \leq \varepsilon \quad \text{and} \quad A(k_1, k_2) \leq 2^{4n} A(k_1 + 2, k_2 - 2), \quad \text{if} \quad k_2 > k_1 > k_0. \tag{10.24}$$

In fact, we may choose $K > 0$ sufficiently large such that $k_1 \approx \frac{K}{2}$, $k_2 \approx \frac{3K}{2}$. Assume (10.24) does not hold, that is,

$$A(k_1, k_2) \geq 2^{4n} A(k_1 + 2, k_2 - 2).$$

If

$$A(k_1 + 2, k_2 - 2) \leq 2^{4n} A(k_1 + 4, k_2 - 4),$$

then we complete the proof of the claim. Otherwise, we can repeat the procedure for j times, we have

$$A(k_1, k_2) \geq 2^{4nj} A(k_1 + 2j, k_2 - 2j).$$

When $j \approx \frac{K}{4}$, we have from (10.23)

$$\text{Area}(X(M)) \geq A(k_1, k_2) \geq 2^{nK} A(K, K+1) \geq \kappa_1 2^{2K-1}.$$

Thus, (10.24) must hold for some $k_2 > k_1$ because Area$(M) < \infty$. Hence for any $\varepsilon > 0$, we can choose k_1 and $k_2 \approx 3k_1$ such that (10.24) holds.
We define a smooth cut-off function $\psi(t)$ by

$$\psi(t) = \begin{cases} 1, & 2^{k_1+\frac{3}{2}} \leq t \leq 2^{k_2-\frac{5}{2}}, \\ 0, & \text{outside } [2^{k_1-\frac{1}{2}}, 2^{k_2-\frac{1}{2}}]. \end{cases} \qquad 0 \leq \psi(t) \leq 1, \qquad |\psi'(t)| \leq 1. \tag{10.25}$$

Letting

$$f(x) = e^{L + \frac{|X|^2}{4}} \psi(\rho(x)), \tag{10.26}$$

we choose L satisfying

$$1 = \int_M f^2 e^{-\frac{|X|^2}{2}} 2^{\frac{n}{2}} d\mu = e^{2L} \int_{\Omega(k_1, k_2)} \psi^2(\rho(x)) 2^{\frac{n}{2}} d\mu. \tag{10.27}$$

We obtain from the corollary 10.1 and $t\ln t \geq -\frac{1}{e}$ for $0 \leq t \leq 1$

$$
\begin{aligned}
(\frac{1}{2}C(n)+\frac{1}{8}\lambda^2)2^{-\frac{n}{2}} &\geq \int_{\Omega(k_1,k_2)} e^{2L}\psi^2(L+\frac{|X|^2}{4}+\ln\psi)d\mu \\
&\quad -\frac{1}{2}\int_{\Omega(k_1,k_2)} e^{2L}|\psi'\nabla\rho+\psi\frac{X^T}{2}|^2 d\mu \\
&\geq \int_{\Omega(k_1,k_2)} e^{2L}\psi^2(L+\frac{|X|^2}{4}+\ln\psi)d\mu \\
&\quad -\int_{\Omega(k_1,k_2)} e^{2L}|\psi'|^2 d\mu - \frac{1}{4}\int_{\Omega(k_1,k_2)} e^{2L}\psi^2|X|^2 d\mu \\
&= 2^{-\frac{n}{2}}L + \int_{\Omega(k_1,k_2)} e^{2L}\psi^2\ln\psi d\mu - \int_{\Omega(k_1,k_2)} e^{2L}|\psi'|^2 d\mu \\
&\geq 2^{-\frac{n}{2}}L - (\frac{1}{2e}+1)e^{2L}A(k_1,k_2).
\end{aligned} \tag{10.28}
$$

Therefore, it follows from (10.24) that

$$
\begin{aligned}
(\frac{1}{2}C(n)+\frac{1}{8}\lambda^2)2^{-\frac{n}{2}} &\geq 2^{-\frac{n}{2}}L - (\frac{1}{2e}+1)e^{2L}2^{4n}A(k_1+2,k_2-2) \\
&\geq 2^{-\frac{n}{2}}L - (\frac{1}{2e}+1)e^{2L}2^{4n}\int_{\Omega(k_1,k_2)}\psi^2(\rho(x))d\mu \\
&= 2^{-\frac{n}{2}}L - (\frac{1}{2e}+1)2^{4n}2^{-\frac{n}{2}}.
\end{aligned} \tag{10.29}
$$

On the other hand, we have, from (10.24) and definition of $f(x)$,

$$1 \leq e^{2L}\varepsilon 2^{\frac{n}{2}}. \tag{10.30}$$

Letting $\varepsilon > 0$ sufficiently small, then L can be arbitrary large, which contradicts (10.29). Hence, M has infinite area. □

Theorem 10.1. *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete proper λ-hypersurface. Then, for any $p \in M$, there exists a constant $C > 0$ such that*

$$\mathit{Area}(B_r(X(x_0)) \cap X(M)) \geq Cr,$$

for all $r > 1$.

Proof. We can choose $r_0 > 0$ such that $\text{Area}(B_r(0) \cap X(M)) > 0$ for $r \geq r_0$. It is sufficient to prove there exists a constant $C > 0$ such that

$$\text{Area}(B_r(0) \cap X(M)) \geq Cr \tag{10.31}$$

holds for all $r \geq r_0$. In fact, if (10.31) holds, then for any $x_0 \in M$ and $r > |X(x_0)|$,

$$B_r(X(x_0)) \supset B_{r-|X(x_0)|}(0), \tag{10.32}$$

and

$$\text{Area}(B_r(X(x_0)) \cap X(M)) \geq \text{Area}(B_{r-|X(x_0)|}(0) \cap X(M)) \geq \frac{C}{2}r, \tag{10.33}$$

for $r \geq 2|X(x_0)|$.

We next prove (10.31) by contradiction. Assume for any $\varepsilon > 0$, there exists $r \geq r_0$ such that

$$\text{Area}(B_r(0) \cap X(M)) \leq \varepsilon r. \tag{10.34}$$

Without loss of generality, we assume $r \in \mathbb{N}$ and consider a set:

$$D := \{k \in \mathbb{N} : \text{Area}(B_t(0) \cap X(M)) \leq 2\varepsilon t \text{ for any integer } t \text{ satisfying } r \leq t \leq k\}.$$

Next, we will show that $k \in D$ for any integer k satisfying $k \geq r$. For $t \geq r_0$, we define a function u by

$$u(x) = \begin{cases} t+2-\rho(x), & \text{in } B_{t+2}(0) \cap X(M) \setminus B_{t+1}(0) \cap X(M), \\ 1, & \text{in } B_{t+1}(0) \cap X(M) \setminus B_t(0) \cap X(M), \\ \rho(x)-(t-1), & \text{in } B_t(0) \cap X(M) \setminus B_{t-1}(0) \cap X(M), \\ 0, & \text{otherwise.} \end{cases} \tag{10.35}$$

Using the corollary 10.2, $|\nabla \rho| \leq 1$ and $t \ln t \geq -\frac{1}{e}$ for $0 \leq t \leq 1$, we have

$$\begin{aligned} &-\Big(\int_M u^2 d\mu\Big)\ln\big\{\big(\text{Area}(B_{t+2}(0)\cap X(M)) - \text{Area}(B_{t-1}(0)\cap X(M))\big)2^{\frac{n}{2}}\big\} \\ &\leq C_0\Big(\text{Area}(B_{t+2}(0)\cap X(M)) - \text{Area}(B_{t-1}(0)\cap X(M))\Big), \end{aligned} \tag{10.36}$$

where $C_0 = 2 + \frac{1}{e} + \frac{\lambda^2}{4} + \frac{n}{2}\ln 2 + C(n)$, $C(n)$ is the constant of the corollary 10.2. For all $t \geq C_1(n,\lambda)+1$, we have from the lemma 10.1

$$\begin{aligned} &\text{Area}(B_{t+2}(0)\cap X(M)) - \text{Area}(B_{t-1}(0)\cap X(M)) \\ &\leq c(n,\lambda)\bigg(\frac{\text{Area}(B_{t+1}(0)\cap X(M))}{t+1} \\ &\quad + \frac{\text{Area}(B_t(0)\cap X(M))}{t} + \frac{\text{Area}(B_{t-1}(0)\cap X(M))}{t-1}\bigg) \\ &\leq c(n,\lambda)\bigg(\frac{2}{t+1} + \frac{1}{t} + \frac{1}{t}(1+\frac{1}{C_1(n,\lambda)})\bigg)\text{Area}(B_t(0)\cap X(M)) \\ &\leq C_2(n,\lambda)\frac{\text{Area}(B_t(0)\cap X(M))}{t}, \end{aligned} \tag{10.37}$$

where $C_2(n,\lambda)$ is constant depended only on n and λ. Note that we can assume $r \geq C_1(n,\lambda)+1$ for the r satisfying (10.34). In fact, if for any given $\varepsilon > 0$, all the r which satisfies (10.34) is bounded above by $C_1(n,\lambda)+1$, then $\text{Area}(B_r(0)\cap X(M)) \geq Cr$ holds for any $r > C_1(n,\lambda)+1$. Thus, we know that M has at least linear area growth. Hence, for any $k \in D$ and any t satisfying $r \leq t \leq k$, we have

$$\text{Area}(B_{t+2}(0)\cap X(M)) - \text{Area}(B_{t-1}(0)\cap X(M)) \leq 2C_2(n,\lambda)\varepsilon. \tag{10.38}$$

Since

$$\int_M u^2 d\mu \geq \text{Area}(B_{t+1}(0)\cap X(M)) - \text{Area}(B_t(0)\cap X(M)), \tag{10.39}$$

holds, if we choose ε such that $2C_2(n,\lambda)\varepsilon 2^{\frac{n}{2}}<1$, from (10.36), we obtain

$$\begin{aligned}&(\mathrm{Area}(B_{t+1}(0)\cap X(M))-\mathrm{Area}(B_t(0)\cap X(M)))\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}\\ &\leq C_0\bigg(\mathrm{Area}(B_{t+2}(0)\cap X(M))-\mathrm{Area}(B_{t-1}(0)\cap X(M))\bigg).\end{aligned}\tag{10.40}$$

Iterating from $t=r$ to $t=k$ and taking summation on t, we infer, from the lemma 10.1

$$\begin{aligned}&(\mathrm{Area}(B_{k+1}(0)\cap X(M))-\mathrm{Area}(B_r(0)\cap X(M)))\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}\\ &\leq 3C_0\mathrm{Area}(B_{k+2}(0)\cap X(M))\leq 6C_0\mathrm{Area}(B_{k+1}(0)\cap X(M)).\end{aligned}\tag{10.41}$$

Hence, we get

$$\begin{aligned}&\mathrm{Area}(B_{k+1}(0)\cap X(M))\\ &\leq\frac{\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}}{\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}-6C_0}\mathrm{Area}(B_r(0)\cap X(M))\\ &\leq\frac{\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}}{\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}-6C_0}\varepsilon r.\end{aligned}\tag{10.42}$$

We can choose ε small enough such that

$$\frac{\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}}{\ln(2^{\frac{n}{2}+1}C_2(n,\lambda)\varepsilon)^{-1}-6C_0}\leq 2.\tag{10.43}$$

Therefore, it follows from (10.42) that

$$\mathrm{Area}(B_{k+1}(0)\cap X(M))\leq 2\varepsilon r,\tag{10.44}$$

for any $k\in D$. Since $k+1\geq r$, we have, from (10.44) and the definition of D, that $k+1\in D$. Thus, by induction, we know that D contains all of integers $k\geq r$ and

$$\mathrm{Area}(B_k(0)\cap X(M))\leq 2\varepsilon r,\tag{10.45}$$

for any integer $k\geq r$. This implies that M has finite volume, which contradicts with the lemma 10.4. Hence, there exist constants C and r_0 such that $\mathrm{Area}(B_r(0)\cap X(M))\geq Cr$ for $r>r_0$. It completes the proof of the theorem 10.1.

□

Remark 10.1. *The estimate in our theorem is the best possible because the cylinders $S^{n-1}(r_0)\times\mathbb{R}$ satisfy the equality.*

References

[1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geom., **23** (1986), 175-196.

[2] S. Angenent, *Shrinking doughnuts*, In nonlinear diffusion equations and their equilibrium states, Birkhaüser, Boston-Basel-Berlin, **7**, 21-38, 1992.

[3] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z., **185** (1984), 339-353.

[4] H. -D. Cao and H. Li, *A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension*, Calc. Var. Partial Differential Equations, **46** (2013), 879-889.

[5] H. -D. Cao and D. Zhou, *On complete gradient shrinking Ricci solitons*, J. Diff. Geom. , **85** (2010), 175-185.
[6] H. -D. Cao and X.-P. Zhu, *A complete proof of the Poincare and geometrization conjectures, application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math. , **10** (2006), 165-492.
[7] J. -E. Chang, *One dimensional solutions of the λ-self shrinkers*, arXiv:1410.1782.
[8] Q. -M. Cheng and Y. Peng, *Complete self-shrinkers of the mean curvature flow*, Calc. Var. Partial Differential Equations, **52** (2015), 497-506.
[9] Q. -M. Cheng and G. Wei,*The weighted volume-preserving mean curvature flow*, preprint.
[10] X. Cheng and D. Zhou, *Volume estimate about shrinkers*, Proc. Amer. Math. Soc., **141** (2013), 687-696.
[11] T. H. Colding and W. P. Minicozzi II, *Generic mean curvature flow I; Generic singularities*, Ann. of Math., **175** (2012), 755-833.
[12] Q. Ding and Y. L. Xin, *Volume growth, eigenvalue and compactness for self-shrinkers*, Asia J. Math., **17** (2013), 443-456.
[13] Q. Ding and Y. L. Xin, *The rigidity theorems of self shrinkers*, Trans. Amer. Math. Soc., **366** (2014), 5067-5085.
[14] K. Ecker, *Logarithmic Sobolev inequalities on submanifolds of Euclidean space*, J. Reine Angew. Math., **522** (2000), 105-118.
[15] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math., **130** (1989), 453-471.
[16] P. Guan and J. Li, *A mean curvature type flow in space forms*, arXiv:1309.5099.
[17] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom., **22** (1984), 237-266.
[18] G. Huisken, *The volume preserving mean curvature flow*, J. reine angew. Math., **382** (1987), 35-48.
[19] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom., **31** (1990), 285-299.
[20] G. Huisken, *Local and global behaviour of hypersurfaces moving by mean curvature*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), 175C191, Proc. Sympos. Pure Math., 54, Part 1, Amer. Math. Soc., Providence, RI, (1993).
[21] S. Kleene and N. M. Møller, *Self-shrinkers with a rotation symmetry*, Trans. Amer. Math. Soc., **366** (2014), 3943-3963.
[22] H. B. Lawson, *Local rigidity theorem for minimal surfaces*, Ann. of Math., **89** (1969), 187-197.
[23] Nam Q. Le and N. Sesum, *Blow-up rate of the mean curvature during the mean curvature flow and a gap theorem for self-shrinkers*, Comm. Anal. Geom., **19** (2011), 1-27.
[24] H. Li and Y. Wei, *Lower volume growth estimates for self-shrinkers of mean curvature flow*, preprint.
[25] H. Li and Y. Wei, *Classification and rigidity of self-shrinkers in the mean curvature flow*, J. Math. Soc. Japan, **66** (2014), 709-734.
[26] O. Munteanu and J. Wang, *Analysis of the weighted Laplacian and applications to Ricci solitons*, Comm. Anal. Geom., **20** (2012), 55-94.
[27] F. Morgan, *Geometric measure theory. A beginner's guide*, fourth ed., Elsevier/Academic Press, Amsterdam, 2009.
[28] R. M. Schoen, L. M. Simon and S. T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math., **134** (1975), 275-288.

Qing-Ming Cheng, Department of Applied Mathematics, Faculty of Sciences , Fukuoka University, 814-0180, Fukuoka, Japan, cheng@fukuoka-u.ac.jp

Guoxin Wei, School of Mathematical Sciences, South China Normal University, 510631, Guangzhou, China, weiguoxin@tsinghua.org.cn