

Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion *

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Abstract

We study a stochastic differential equation in the sense of rough path theory driven by fractional Brownian rough path with Hurst parameter H ($1/3 < H \leq 1/2$) under the ellipticity assumption at the starting point. In such a case, the law of the solution at a fixed time has a kernel, i.e., a density function with respect to Lebesgue measure. In this paper we prove a short time off-diagonal asymptotic expansion of the kernel under mild additional assumptions. Our main tool is Watanabe's distributional Malliavin calculus.

1 Introduction

For the usual d -dimensional Brownian motion (w_t) and sufficiently regular vector fields V_i ($0 \leq i \leq d$) on $\mathbb{R}^n$, consider the following stochastic differential equation (SDE) of Stratonovich type:

$$dy_t = \sum_{i=1}^d V_i(y_t) \circ dw_t^i + V_0(y_t)dt \quad \text{with} \quad y_0 = a \in \mathbb{R}^n.$$

If the vector fields satisfy the hypoellipticity condition at the starting point a , then the law of y_t has a heat kernel i.e., a density function $p_t(a, a')$ with respect to Lebesgue measure da' for any $t > 0$.

In probability theory, the short time asymptotic (off-diagonal) problem of $p_t(a, a')$ has extensively been studied and is now a classical topic. See for instance [2, 9, 10, 11, 12, 13, 14, 26, 38, 39, 40, 41, 42, 43, 44, 48, 51, 52, 53, 54, 55, 56] and references therein. (There are also analytic approaches, of course. But, we do not discuss them in this paper.) Among many probabilistic methods, Malliavin calculus is known to be quite powerful. Bismut [14] was first to prove short time kernel asymptotics via Malliavin calculus. Among such

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proofs, we focus on Watanabe's theory of generalized Wiener functionals and asymptotic theorems for them [56, 29, 52].

Recently, the theory of "SDE" for fractional Brownian motion (fBm) was developed. As a result, an analogous asymptotic problem is gathering attention. When Hurst parameter H is larger than $1/2$, the SDE above is in the sense of Young integration. When $1/4 < H \leq 1/2$, it should be understood as a differential equation in the rough path sense driven by fractional Brownian rough path. In his previous paper [33], the author studied both on-diagonal and off-diagonal short time asymptotic expansion of $p_t(a, a')$ when $H > 1/2$. The method is Watanabe's asymptotic theory of generalized Wiener functionals (i.e., Watanabe distributions) in [56]. In [33] the coefficient vector fields are assumed to satisfy the ellipticity condition at a and some additional mild conditions are also assumed. Those conditions are almost parallel to the ones in [56]. Simply put, [33] is a "fractional version" of [56] in the framework of Young integration.

The aim of this paper is to prove a similar off-diagonal asymptotic expansion when $1/3 < H \leq 1/2$. Although the basic strategy of proof is similar to the case $H > 1/2$ in [33], the proof gets much more technically difficult since we work on the rough path space. We will carry it out by combining various recently proven results for Gaussian rough paths. A number of paper have been published on Malliavin calculus for Gaussian rough paths by now. See [1, 5, 7, 8, 15, 16, 17, 18, 20, 27, 28, 30, 34, 35] for instance. However, this type of short time kernel asymptotics seems new.

The organization of this paper is as follows: In Section 2 we give a precise formulation of our problem and the statement of our main result (Theorem 2.2). In Section 3 we prove moment estimates for Taylor expansion of Lyons-Itô map. The expansion in the deterministic sense is already known, but we need " L^p -version" (or " $\mathbf{D}_\infty$ -version") of the expansion in this paper. These estimates play a crucial role in the proof of the main theorem. In Section 4 we present two propositions (Propositions 4.1 and 4.2) on regularity in the sense of Malliavin calculus of the solution of RDE driven by fractional Brownian rough path. Thanks to these propositions, we can use Watanabe's asymptotic theory in the proof of the main theorem in Section 5, following the argument in [56, 33]. A difference from [56] is that we can work and, in particular, localize around the energy minimizing path in the domain (not in the range) of Lyons-Itô map since the map is continuous in rough path theory.

We do not give a heuristic sketch of our argument for brevity. Since formal computations are basically the same as in the Young case, the reader who wants to know it may consult the corresponding part of the author's previous paper [33].

Remark 1.1 *The first version of this paper contains detailed proofs of Theorem 3.4, Proposition 4.1, Proposition 4.2, and Lemma 5.2. It can be found on arXiv preprint server (arXiv:1403.3181).*

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2 Setting and main results

2.1 Setting

In this subsection, we introduce a stochastic process that will play a main role in this paper. From now on we denote by $w = (w_t)_{t \geq 0} = (w_t^1, \dots, w_t^d)_{t \geq 0}$ the d -dimensional fractional Brownian motion (fBm) with Hurst parameter H . Throughout this paper we assume $1/3 < H \leq 1/2$. It is a unique d -dimensional, mean-zero, continuous Gaussian process with covariance

$$\mathbb{E}[w_s^i w_t^j] = \frac{\delta_{ij}}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad (s, t \geq 0).$$

Note that, for any $c > 0$, $(w_{ct})_{t \geq 0}$ and $(c^H w_t)_{t \geq 0}$ have the same law. This property is called self-similarity or scale invariance. When $H = 1/2$, it is the usual Brownian motion. It is well-known that w admits a canonical rough path lift $\mathbf{w}$, which is called fractional Brownian rough path.

Let $V_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C_b^∞ , that is, V_i is a bounded smooth function with bounded derivatives of all order ($0 \leq i \leq d$). We consider the following rough differential equation (RDE);

$$dy_t = \sum_{i=1}^d V_i(y_t) dw_t^i + V_0(y_t) dt \quad \text{with} \quad y_0 = a \in \mathbb{R}^n. \quad (2.1)$$

This RDE is driven by the Young pairing $(\mathbf{w}, \boldsymbol{\lambda})$, where $\lambda_t = t$. The unique solution is denoted by $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2)$ and we set $y_t := a + \mathbf{y}_{0,t}^1$ as usual. We will sometimes write $y_t = y_t(a) = y_t(a, \mathbf{w})$ etc. to make explicit the dependence on a and $\mathbf{w}$.

A matrix notation is often convenient. So we set $b = V_0$ and $\sigma = [V_1, \dots, V_d]$, which is $n \times d$ matrix-valued, and often rewrite RDE (2.1) as follows;

$$dy_t = \sigma(y_t) dw_t + b(y_t) dt \quad \text{with} \quad y_0 = a \in \mathbb{R}^n.$$

2.2 Assumptions

In this subsection we introduce assumptions of the main theorems. First, we assume the ellipticity of the coefficients of (2.1) at the starting point $a \in \mathbb{R}^n$.

(A1): The set of vectors $\{V_1(a), \dots, V_d(a)\}$ linearly spans $\mathbb{R}^n$.

It is known that, under Assumption **(A1)**, the law of the solution y_t has a density $p_t(a, a')$ with respect to the Lebesgue measure on $\mathbb{R}^n$ for any $t > 0$ (see [27]). Hence, for any Borel subset $U \subset \mathbb{R}^n$, $\mathbb{P}(y_t(a) \in U) = \int_U p_t(a, a') da'$.

Let $\mathcal{H} = \mathcal{H}^H$ be the Cameron-Martin space of fBm (w_t) . Note that any $\gamma \in \mathcal{H}$ is continuous and of finite q -variation for some $q \in [1, 2)$. For $\gamma \in \mathcal{H}$, we denote by

$\phi_t^0 = \phi_t^0(\gamma)$ be the solution of the following Young ODE;

$$d\phi_t^0 = \sum_{i=1}^d V_i(\phi_t^0) d\gamma_t^i \quad \text{with} \quad \phi_0^0 = a \in \mathbb{R}^n. \quad (2.2)$$

Set, for $a' \neq a$,

$$K_a^{a'} = \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}.$$

We only consider the case where $K_a^{a'}$ is not empty. For example, if we assume **(A1)** for all a , then this set $K_a^{a'}$ is not empty. From goodness of the rate function in Schilder-type large deviation for fractional Brownian rough path (see [24]), it follows that $\inf\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\}$. Now we introduce the following assumption;

(A2): $\bar{\gamma} \in K_a^{a'}$ which minimizes $\mathcal{H}$ -norm exists uniquely.

In what follows, $\bar{\gamma}$ denotes the minimizer in Assumption **(A2)**. We also assume that $\|\cdot\|_{\mathcal{H}}^2/2$ is not so degenerate at $\bar{\gamma}$ in the following sense.

(A3): At $\bar{\gamma}$, the Hessian of the functional $K_a^{a'} \ni \gamma \mapsto \|\gamma\|_{\mathcal{H}}^2/2$ is strictly positive in the quadratic form sense. More precisely, if $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ is a smooth curve in $K_a^{a'}$ such that $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$, then $(d/du)^2|_{u=0} \|f(u)\|_{\mathcal{H}}^2/2 > 0$.

Later we will give a more analytical condition **(A3)'**, which is equivalent to **(A3)** under **(A2)**. In [56], Watanabe used **(A3)'** in his proof of off-diagonal kernel asymptotics. We will also use **(A3)'**. In order to state **(A3)'**, however, we have to introduce a lot of notations. So, we presented **(A3)** here for ease of presentation.

Remark 2.1 Assume **(A1)**. If the end point a' is sufficiently close to the starting point a , then **(A2)** and **(A3)** are satisfied. (This is shown in the author's previous paper [33] when $1/2 < H < 1$. The same proof works in our case ($1/3 < H \leq 1/2$), too. The key is the implicit function theorem.)

2.3 Index sets

In this subsection we introduce several index sets for the exponent of the small parameter $\varepsilon > 0$, which will be used in the asymptotic expansion. Unfortunately, index sets in this paper are not the set of (a constant multiple of) natural numbers and are rather complicated. (However, all these index sets are discrete subsets of $(\mathbb{Z} + H^{-1}\mathbb{Z}) \cap [0, \infty)$ with the minimum 0.)

Set

$$\Lambda_1 = \{n_1 + \frac{n_2}{H} \mid n_1, n_2 \in \mathbb{N}\},$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. We denote by $0 = \kappa_0 < \kappa_1 < \kappa_2 < \dots$ all the elements of Λ_1 in increasing order. For a while, consider the case $1/3 < H < 1/2$. Several smallest elements are explicitly given as follows;

$$\kappa_1 = 1, \quad \kappa_2 = 2, \quad \kappa_3 = \frac{1}{H}, \quad \kappa_4 = 3, \quad \kappa_5 = 1 + \frac{1}{H}, \quad \kappa_6 = 4, \dots$$

As usual, using the scale invariance (i.e., self-similarity) of fBm, we will consider the scaled version of (2.1). (See the scaled and shifted RDE (4.2) below). From its explicit form, one can easily guess why Λ_1 appears.

We also set

$$\Lambda_2 = \{\kappa - 1 \mid \kappa \in \Lambda_1 \setminus \{0\}\} = \left\{0, 1, \frac{1}{H} - 1, 2, \frac{1}{H}, 3, \dots\right\}$$

and

$$\Lambda'_2 = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1\}\} = \left\{0, \frac{1}{H} - 2, 1, \frac{1}{H} - 1, 2, \dots\right\}.$$

Next we set

$$\Lambda_3 = \{a_1 + a_2 + \dots + a_m \mid m \in \mathbb{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda_2\}.$$

In the sequel, $\{0 = \nu_0 < \nu_1 < \nu_2 < \dots\}$ stands for all the elements of Λ_3 in increasing order. Similarly,

$$\Lambda'_3 = \{a_1 + a_2 + \dots + a_m \mid m \in \mathbb{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda'_2\}.$$

In the sequel, $\{0 = \rho_0 < \rho_1 < \rho_2 < \dots\}$ stands for all the elements of Λ'_3 in increasing order. Finally,

$$\Lambda_4 = \Lambda_3 + \Lambda'_3 = \{\nu + \rho \mid \nu \in \Lambda_3, \rho \in \Lambda'_3\}.$$

We denote by $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ all the elements of Λ_4 in increasing order.

When $H = 1/2$, all these index sets Λ_i, Λ'_j above are just $\mathbb{N}$.

2.4 Statement of the main result

Now we state our main theorem, which is basically analogous to the corresponding one in Watanabe [56]. However, when $H \neq 1/2$ and the drift term exists, there are some differences. First, the exponents of t are not (a constant multiple of) natural numbers. Second, cancellation of "odd terms" as in p. 20 and p. 34, [56] does not occur in general. (These phenomena were already observed in [33] in the Young integration setting i.e., the case $H > 1/2$.)

Theorem 2.2 *Assume $a \neq a'$ and (A1)–(A3). Then, we have the following asymptotic expansion as $t \searrow 0$;*

$$p(t, a, a') \sim \exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2t^{2H}}\right) \frac{1}{t^{nH}} \{\alpha_0 + \alpha_{\lambda_1} t^{\lambda_1 H} + \alpha_{\lambda_2} t^{\lambda_2 H} + \dots\}$$

for certain real constants α_{λ_j} ($j = 0, 1, 2, \dots$). Here, $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ are all the elements of Λ_4 in increasing order. Moreover, α_0 is positive.

Remark 2.3 (i) *In theory, the constants in the asymptotic expansion in Theorem 2.2 (and in the on-diagonal case in Theorem 4.4 below) are computable. But, actual computation is quite cumbersome and we do not carry it out in this paper. We just mention here that the first constants α_0 in Theorem 2.2 and c_0 in Theorem 4.4 are non-zero.*

(ii) *It might be interesting to consider the case $1/4 < H \leq 1/3$. In that case, since the third level rough path theory is needed, calculations may become much harder.*

(iii) *Our assumptions (A1)–(A3) are quite similar to the corresponding ones in [56]. Therefore, if we set $H = 1/2$ in Theorem 2.2 above recovers most of (but not all of) the main result in Watanabe [56]. Hence, our result could also be regarded as a rough path proof of [56]. (In this case, however, the index set in Theorem 2.2 is not $\Lambda_4 = \mathbb{N}$, but is actually $2\mathbb{N}$, due to cancellation of the odd terms.) Compared to the main theorem in [56], Theorem 2.2 with $H = 1/2$ does not include the following two cases;*

(a): *In this paper the ellipticity assumption (A1) is assumed. In [56], however, something like "step 2-hypoellipticity" case was also studied. (We simply did not try this case.)*

(b): *In this paper the coefficient vector fields are of C_b^∞ . However, the condition on vector fields in [56] is as follows: "For all $m = 1, 2, \dots$ and $0 \leq i \leq d$, $\|\nabla^m V_i\|$ is bounded." (V_i itself is allowed to have linear growth.) Since Bailleul [3] recently solved RDEs with such coefficients, it might be possible to extend our theorem to include such a case by just combining existing methods.*

(iv) *In a very recent survey [6], many results on various kinds of short time asymptotic problems for RDEs (or Young ODE) driven by fBm are reviewed. For instance, Varadhan's estimate, which is short time asymptotics of $\log p(t, a, a')$, was shown in [7] under the uniform ellipticity condition on the coefficient vector fields when $H > 1/4$.*

3 Moment estimate for Taylor expansion of Lyons-Itô map

Let $p \in [2, 3)$ be the roughness constant and let $q \in [1, 2)$ be such that $1/p + 1/q > 1$. We denote by $G\Omega_p(\mathbb{R}^d)$ the geometric rough path space with p -variation topology. In this paper, the time interval is always $[0, 1]$. For the definition and basic properties of geometric rough paths, see Lyons and Qian [47], or Lyons, Caruana, and Lévy [46].

Assume that $\sigma : \mathbb{R}^n \rightarrow \text{Mat}(n, d)$ and $b : [0, 1] \times \mathbb{R}^n \rightarrow \text{Mat}(n, e)$ are C_b^∞ . For $\varepsilon \in [0, 1]$, $\mathbf{x} \in G\Omega_p(\mathbb{R}^d)$ and $h \in C_0^{q\text{-var}}([0, 1], \mathbb{R}^e)$, we consider the following RDE driven by the Young pairing $(\varepsilon \mathbf{x}, \mathbf{h}) \in G\Omega_p(\mathbb{R}^{d+e})$;

$$dy_t^\varepsilon = \sigma(y_t^\varepsilon) \varepsilon dx_t + b(\varepsilon, y_t^\varepsilon) dh_t \quad \text{with} \quad y_0^\varepsilon = a \in \mathbb{R}^n. \quad (3.1)$$

It was shown in Inahama [31] (or Inahama-Kawabi [36]) that the first level path of the solution admits a Taylor-like expansion in the deterministic sense as $\varepsilon \searrow 0$. Roughly speaking, the aim of this section is to prove that the expansion holds still true in L^r -sense for any $r \in [1, \infty)$, when $\mathbf{x}$ is the natural lift of fBm with $H \in (1/3, 1/2]$ or a similar Gaussian process.

We remark that the following RDE is a special case of (3.1) above:

$$dy_t^\varepsilon = \sigma(y_t^\varepsilon)(\varepsilon dx_t + dk_t) + \hat{b}(\varepsilon, y_t^\varepsilon)d\lambda_t \quad \text{with} \quad y_0^\varepsilon = a \in \mathbb{R}^n. \quad (3.2)$$

Here, σ and $\mathbf{x}$ are as above, $\hat{b} : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\lambda_t = t$, $k \in C_0^{q-var}([0, 1], \mathbb{R}^d)$. We can easily check this by setting $e = d + 1$, $h = (k, \lambda)$, and $b = [\sigma|\hat{b}]$ (an $n \times (d + 1)$ block matrix). This type of RDE appears when we make a Young translation of a given RDE driven by a scaled Gaussian rough path.

3.1 Notations

In this paper we work in Lyons' original framework of rough path theory. We borrow most of notations and terminologies from [47, 46]. Before we start detailed discussions, however, we need to set some additional notations.

We denote by $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2)$ a generic element in $G\Omega_p(\mathbb{R}^d)$ and we write $x_t := \mathbf{x}_{0,t}^1$ as usual. Conversely, for $x \in C_0^{\alpha-var}([0, 1], \mathbb{R}^d)$ with $\alpha \in [1, 2)$, we denote the natural lift of x (i.e., the smooth rough path lying above x) by the corresponding boldface letter $\mathbf{x}$.

Note that, for $x \in C_0^{\alpha-var}([0, 1], \mathbb{R}^d)$ and $y \in C_0^{\alpha-var}([0, 1], \mathbb{R}^e)$, $(\mathbf{x}, \mathbf{y}) \in G\Omega_p(\mathbb{R}^{d+e})$ stands for the natural lift of (x, y) , not for the pair $(\mathbf{x}, \mathbf{y}) \in G\Omega_p(\mathbb{R}^d) \times G\Omega_p(\mathbb{R}^e)$. In a similar way, for $\mathbf{x} \in G\Omega_p(\mathbb{R}^d)$ and $h \in C_0^{q-var}(\mathbb{R}^e)$ with $1/p + 1/q > 1$, $(\mathbf{x}, \mathbf{h}) \in G\Omega_p(\mathbb{R}^{d+e})$ stands for the Young pairing. These notations may be somewhat misleading. But, they make many operations intuitively clear and easy to understand when we treat rough paths over a direct sum of many vector spaces.

For a control function ω in the sense of p. 16, [47], we write $\bar{\omega} := \omega(0, 1)$. For any $\mathbf{x} \in G\Omega_p(\mathbb{R}^d)$,

$$\omega_{\mathbf{x}}(s, t) := \|\mathbf{x}^1\|_{p-var, [s, t]}^p + \|\mathbf{x}^2\|_{p/2-var, [s, t]}^{p/2} \quad (0 \leq s \leq t \leq 1) \quad (3.3)$$

defines a control function. Here, the norm on the right hand side denoted the p/j -variation ($j = 1, 2$) restricted on the subinterval $[s, t]$. (This control function is equivalent to the one defined by Carnot-Carathéodory metric.) Similarly, we set $\omega_\lambda(s, t) := \|\lambda\|_{q-var, [s, t]}^q$ for $\lambda \in C_0^{q-var}([0, 1], \mathbb{R}^e)$.

For $\alpha > 0$ and $\mathbf{x} \in G\Omega_p(\mathbb{R}^d)$, set $\tau_0(\alpha) = 0$ and

$$\tau_{i+1}(\alpha) = \inf\{t \in (\tau_i(\alpha), 1] \mid \omega_{\mathbf{x}}(\tau_i(\alpha), t) \geq \alpha\} \wedge 1 \quad (i = 1, 2, \dots)$$

Define

$$N_\alpha(\mathbf{x}) = \sup\{i \in \mathbb{N} \mid \tau_i(\alpha) < 1\}. \quad (3.4)$$

Superadditivity of $\omega_{\mathbf{x}}$ yields $\alpha N_\alpha(\mathbf{x}) \leq \bar{\omega}_{\mathbf{x}}$. This quantity (3.4) was first studied by Cass, Litterer, and Lyons [19].

Let $x(k)$ be continuous paths which takes values in $\mathbb{R}^{d_k}$ for $k = 1, \dots, m$. Then, we write

$$\mathcal{I}[x(1), \dots, x(m)]_{s, t} = \int_{s < t_1 < \dots < t_m < t} dx(1)_{t_1} \otimes \dots \otimes dx(m)_{t_m} \quad (3.5)$$

whenever the iterated integral on the right hand side makes sense. For example, if $(\mathbf{x}, \mathbf{y})$ is a smooth rough path lying above (x, y) , then its second level path is given by $(\mathbf{x}^2, \mathcal{I}[x, y], \mathcal{I}[y, x], \mathbf{y}^2)$. Slightly abusing notations, we denote by $\mathcal{I}[x, y]$ the " (x, y) -component" of the second level path of any $\mathbf{z} = "(\mathbf{x}, \mathbf{y})" \in G\Omega_p(\mathbb{R}^d \oplus \mathbb{R}^e)$, when no confusion may occur.

For brevity we will often write $\mathcal{V} = \mathbb{R}^d$, $\hat{\mathcal{V}} = \mathbb{R}^e$, and $\mathcal{W} = \mathbb{R}^n$ in this section.

3.2 ODEs for ordinary Taylor terms

Ordinary terms in the Taylor expansion are known to satisfy a very simple ODE. In this section we recall them, following [31], etc. We will first calculate in 1-variational setting (i.e., the Riemann-Stieltjes sense). After that we will continuously extend these objects to the rough path setting.

The ODE that corresponds to (3.1) is the following;

$$dy_t^\varepsilon = \sigma(y_t^\varepsilon)\varepsilon dx_t + b(\varepsilon, y_t^\varepsilon)dh_t \quad \text{with} \quad y_0^\varepsilon = a. \quad (3.6)$$

Here, $(x, h) \in C_0^{1-var}([0, 1], \mathbb{R}^{d+e})$. By setting $\varepsilon = 0$, we can easily see that the 0th term $\phi^0 = \phi^0(h)$ satisfies the following ODE;

$$d\phi_t^0 = b(0, \phi_t^0)dh_t \quad \text{with} \quad \phi_0^0 = a. \quad (3.7)$$

ODEs for ϕ^1 and ϕ^2 are given as follows;

$$d\phi_t^1 - \nabla b(0, \phi_t^0)\langle \phi_t^1, dh_t \rangle = \sigma(\phi_t^0)dx_t + \partial_\varepsilon b(0, \phi_t^0)dh_t \quad \text{with} \quad \phi_0^1 = 0, \quad (3.8)$$

and

$$\begin{aligned} d\phi_t^2 - \nabla b(0, \phi_t^0)\langle \phi_t^2, dh_t \rangle &= \nabla \sigma(\phi_t^0)\langle \phi_t^1, dx_t \rangle + \frac{1}{2}\nabla^2 b(0, \phi_t^0)\langle \phi_t^1, \phi_t^1, dh_t \rangle \\ &\quad + \partial_\varepsilon \nabla b(0, \phi_t^0)\langle \phi_t^1, dh_t \rangle + \frac{1}{2}\partial_\varepsilon^2 b(0, \phi_t^0)dh_t \quad \text{with} \quad \phi_0^2 = 0. \end{aligned} \quad (3.9)$$

ODEs for $\phi^k = \phi^k(x, h)$ ($k = 2, 3, 4, \dots$) are given as follows. A heuristic explanation for how to derive these ODEs was given in [31]. We write $\partial_\varepsilon b$ for the partial derivative in ε and ∇b for the (partial) gradient in y for fixed ε .

$$d\phi_t^k - \nabla b(0, \phi_t^0)\langle \phi_t^k, dh_t \rangle = dA_t^k + dB_t^k \quad \text{with} \quad \phi_0^k = 0, \quad (3.10)$$

where

$$dA_t^k[x, h, \phi^0, \dots, \phi^{k-1}] = \sum_{j=1}^{k-1} \sum_{i_1+\dots+i_j=k-1} \frac{1}{j!} \nabla^j \sigma(\phi_t^0)\langle \phi_t^{i_1}, \dots, \phi_t^{i_j}, dx_t \rangle \quad (3.11)$$

and

$$\begin{aligned}
dB_t^k[x, h, \phi^0, \dots, \phi^{k-1}] &= \sum_{j=2}^k \sum_{i_1 + \dots + i_j = k} \frac{1}{j!} \nabla^j b(0, \phi_t^0) \langle \phi_t^{i_1}, \dots, \phi_t^{i_j}, dh_t \rangle \\
&+ \sum_{m=1}^{k-1} \sum_{j=1}^{k-m} \sum_{i_1 + \dots + i_j = k-m} \frac{1}{m!j!} \partial_\varepsilon^m \nabla^j b(0, \phi_t^0) \langle \phi_t^{i_1}, \dots, \phi_t^{i_j}, dh_t \rangle \\
&+ \frac{1}{k!} \partial_\varepsilon^k b(0, \phi_t^0) dh_t.
\end{aligned} \tag{3.12}$$

Note that in the definition of A^k , the summation is taken over all positive $i_1, \dots, i_j$ such that $i_1 + \dots + i_j = k - 1$. A similar remark goes for the summations in the definition of B^k . (As usual we set $A_0^k = B_0^k = 0$.)

Let us recall that we can obtain ϕ^k by the variation of constants formula since the right hand side of (3.8)–(3.10) is known. Set $K_t = K_t[h] = \int_0^t \nabla b(0, \phi_t^0) \langle \cdot, dh_t \rangle$ and consider the following $\text{Mat}(n, n)$ -valued ODE;

$$dM_t = (dK_t) \cdot M_t \quad \text{with} \quad M_0 = \text{Id}_n. \tag{3.13}$$

It is easy to see that M_t^{-1} exists and satisfies a similar ODE. Using this, we can easily check that ϕ^k has the following expression;

$$\phi_t^k = M_t \int_0^t M_s^{-1} dZ_s^k = Z_t^k - M_t \int_0^t dM_s^{-1} \cdot Z_s^k. \tag{3.14}$$

Here, Z_t^k (with $Z_0^k = 0$) is a shorthand for the right hand side of (3.8)–(3.10). Finally, we set

$$r_\varepsilon^{k+1} = y^\varepsilon - (\phi^0 + \varepsilon \phi^1 + \dots + \varepsilon^k \phi^k). \tag{3.15}$$

It is obvious that for each $\varepsilon \in [0, 1]$ and $k \in \mathbb{N}$

$$(x, h) \mapsto (x, h, y^\varepsilon, \phi^0, \dots, \phi^k, r_\varepsilon^{k+1}) \tag{3.16}$$

is continuous from $C_0^{1-var}([0, 1], \mathcal{V} \oplus \hat{\mathcal{V}})$ to $C^{1-var}([0, 1], \mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus k+3})$. It is known that this map extends to a continuous map with respect to the rough path topology in the following sense (after the initial values are suitably adjusted, precisely speaking. Note that $y_0^\varepsilon = a = \phi_0^0$.)

Proposition 3.1 *Let $2 \leq p < 3$ and $1 \leq q < 2$ such that $1/p + 1/q > 1$. Then, for each $\varepsilon \in [0, 1]$ and $k \in \mathbb{N}$, the map (3.16) naturally extends to the following locally Lipschitz continuous map;*

$$G\Omega_p(\mathcal{V}) \times C_0^{q-var}(\hat{\mathcal{V}}) \ni (\mathbf{x}, h) \mapsto (\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^k, \mathbf{r}_\varepsilon^{k+1}) \in G\Omega_p(\mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus k+3}).$$

Proof. This was already shown in [31] for arbitrary $p \geq 2$. Here, we only give a sketch of proof for later use.

First, $(\mathbf{x}, \mathbf{h})$ is just Young pairing of $\mathbf{x}$ and h . Since $(\mathbf{y}^\varepsilon, \phi^0)$ is a unique solution of an RDE driven by $(\mathbf{x}, \mathbf{h})$, we obtain $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0)$. Next, assume that we have $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1})$. Then, $A_k + B_k$ on the right hand side of (3.10) can be interpreted as a rough path integral, we obtain $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1}, A_k + B_k)$. For $\tilde{M}_t := \text{Id}_{\mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus k+1}} \oplus M_t$, we can use a rough path version of variation of constant method to obtain $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^k)$. (Observe (3.14) above.) Finally, since $\mathbf{r}_\varepsilon^{k+1}$ is a linear combination of $y^\varepsilon, \phi^0, \dots, \phi^k$, we obtain $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^k, \mathbf{r}_\varepsilon^{k+1})$. $\blacksquare$

By the following proposition, this expansion can be called a Taylor(-like) expansion of Lyons-Itô map.

Proposition 3.2 *Keep the same notations and assumptions as in Proposition 3.1 above. Then, the following (i) and (ii) hold.*

(i) *For any any $\rho > 0$ and $k = 1, 2, \dots$, there exists a positive constants $C = C(\rho, k)$ which satisfies that*

$$\|(\phi^k)^1\|_{p\text{-var}} \leq C(1 + \overline{\omega}_{\mathbf{x}}^{1/p})^k.$$

for any $\mathbf{x} \in G\Omega_p(\mathcal{V})$ and any $h \in C_0^{q\text{-var}}([0, 1], \hat{\mathcal{V}})$ with $\|h\|_{q\text{-var}} \leq \rho$.

(ii) *For any $\rho_1, \rho_2 > 0$ and $k = 1, 2, \dots$, there exists a positive constants $\tilde{C} = \tilde{C}(\rho_1, \rho_2, k)$, which is independent of ε and satisfies that*

$$\|(\mathbf{r}_\varepsilon^{k+1})^1\|_{p\text{-var}} \leq \tilde{C}(\varepsilon + \varepsilon \overline{\omega}_{\mathbf{x}}^{1/p})^{k+1}$$

for any $\mathbf{x} \in G\Omega_p(\mathcal{V})$ with $\overline{\omega}_{\varepsilon \mathbf{x}}^{1/p} = \varepsilon \overline{\omega}_{\mathbf{x}}^{1/p} \leq \rho_1$ and any $h \in C_0^{q\text{-var}}([0, 1], \hat{\mathcal{V}})$ with $\|h\|_{q\text{-var}} \leq \rho_2$.

Proof. This was already shown in [31] for arbitrary $p \geq 2$. In that paper, estimates not only for the first level path, but also for the higher level paths are given. $\blacksquare$

Remark 3.3 *In a very recent preprint [4], Bailleul gave a simplified proof of Propositions 3.1 and 3.2 for any $p \geq 2$ in the framework of Gubinelli's controlled path theory.*

3.3 Main results in this section

In this subsection we state the main result of this section, that is, moment estimates for Taylor expansion of Lyons-Itô map. We will prove this theorem rigorously in subsequent subsections. Note that η_k may depend on $k, \mathbf{x}, h, p, q$, but not on ε .

Theorem 3.4 Let $2 \leq p < 3$ and $1 \leq q < 2$ such that $1/p + 1/q > 1$ and let $h \in C_0^{q-var}([0, 1], \mathbb{R}^e)$. Assume that $\mathbf{x}$ is a $G\Omega_p(\mathbb{R}^d)$ -valued random variable such that (a) $\overline{\omega_{\mathbf{x}}} = \omega_{\mathbf{x}}(0, 1) \in \cap_{1 \leq r < \infty} L^r$ and (b) $\exp(N_\alpha(\mathbf{x})) \in \cap_{1 \leq r < \infty} L^r$ for any $\alpha > 0$.

Then, for any $\mathbf{x}$, h , and $k \in \mathbb{N}$, there exist control functions $\eta_k = \eta_{k, \mathbf{x}, h}$ such that the following (i)–(iii) hold:

- (i) η_k are non-decreasing in k , i.e., $\eta_{k, \mathbf{x}, h}(s, t) \leq \eta_{k+1, \mathbf{x}, h}(s, t)$ for all $k, \mathbf{x}, h, (s, t)$.
- (ii) $\overline{\eta_{k, \mathbf{x}, h}} \in \cap_{1 \leq r < \infty} L^r$ for all k, h .
- (iii) For all $\varepsilon \in (0, 1]$, $k \in \mathbb{N}$, $h, \mathbf{x}$, and $0 \leq s \leq t \leq 1$, $j = 1, 2$, we have

$$\left| (\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^k, \varepsilon^{-(k+1)} \mathbf{r}_\varepsilon^{k+1})_{s,t}^j \right| \leq \eta_{k, \mathbf{x}, h}(s, t)^{j/p}.$$

In particular, for all $k \in \mathbb{N}$ and h , $\|(\phi^k)^1\|_{p-var} \in \cap_{1 \leq r < \infty} L^r$ and $\|(\mathbf{r}_\varepsilon^{k+1})^1\|_{p-var} = O(\varepsilon^{k+1})$ in L^r for any $1 < r < \infty$.

Remark 3.5 (1) Examples of Gaussian processes whose rough path lifts satisfy the integrability assumptions

$$\omega_{\mathbf{x}}(0, 1) \in \cap_{1 \leq r < \infty} L^r \quad \text{and} \quad \exp(N_\alpha(\mathbf{x})) \in \cap_{1 \leq r < \infty} L^r \quad (\forall \alpha > 0)$$

can be found in Friz and Oberhauser [22] (a Fernique-type theorem) and Cass, Litterer, and Lyons [19] (Integrability of N_α). FBM with Hurst parameter $H \in (1/4, 1/2]$ is a typical example.

(2) The estimate above is actually uniform in h when it varies in a bounded subset in q -variation space. (But, the uniform version is not needed in this paper.)

3.4 Proof of Theorem 3.4 for $k = 0$

The rest of this section is devoted to showing Theorem 3.4. Without loss of generality we may assume that the initial value $a = 0$. In this proof $c_1, c_2, \dots$ stands for unimportant positive constants, which is independent of $\varepsilon \in (0, 1]$ and $\mathbf{x}$, but may depend on $p, q, \sigma, b, \|h\|_{q-var}$, etc. We say that a geometric p -rough path $\mathbf{x}$ is controlled by a control function ω if $|\mathbf{x}_{s,t}^j| \leq \omega(s, t)^{j/p}$ for any $s \leq t$ and $j = 1, 2$.

The expansion of the Itô map in the deterministic case is already given in [31, 36] by mathematical induction. We will closely look at it and check the integrability holds or not. In this subsection, we will obtain the moment estimates of $\mathbf{r}_\varepsilon^1$. Surprisingly, for those who understand the proof for the deterministic sense, the most difficult part is this initial step of the induction. However, that problem is somewhat similar to the moment estimates of Jacobian process driven by Gaussian rough paths, which was solved by Cass, Litterer, and Lyons [19]. In the sequel we will check that their method also applies to this kind of problem as they conjectured in [19].

Now we prove Theorem 3.4 for $k = 0$. Set $\omega_h(s, t) = \|h\|_{q-var, [s, t]}^q$ and $\omega_{\mathbf{x}, h}(s, t) = \omega_{\mathbf{x}}(s, t) + \omega_h(s, t)$. Then, the Young pairing $(\mathbf{x}, \mathbf{h}) \in G\Omega_p(\mathcal{V} \oplus \hat{\mathcal{V}})$ is controlled by $c_1(1 + \overline{\omega_{\mathbf{x}, h}})^{c_2} \omega_{\mathbf{x}, h}$, that is,

$$|(\mathbf{x}, \mathbf{h})_{s,t}^i| \leq \{c_1(1 + \overline{\omega_{\mathbf{x}, h}})^{c_2} \omega_{\mathbf{x}, h}(s, t)\}^{i/p}$$

for all i and (s, t) .

Next we consider $(\mathbf{y}^\varepsilon, \phi^0)$ which is a solution of a $\mathcal{W}^{\oplus 2}$ -valued RDE driven by $(\mathbf{x}, \mathbf{h})$. Since the $C_b^{[p]+1}$ -norm of the coefficients of the RDE is bounded in ε , $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0) \in G\Omega_p(\mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus 2})$ is controlled by $c_3(1 + \overline{\omega_{\mathbf{x}, h}})^{c_4}\omega_{\mathbf{x}, h}$.

It is easy to see from (3.6) and (3.7) that $r_{\varepsilon, t}^1$ satisfies the following equation in the 1-variational setting;

$$\frac{1}{\varepsilon} dr_{\varepsilon, t}^1 = \sigma(y_t^\varepsilon) dx_t + \frac{1}{\varepsilon} \{b(\varepsilon, y_t^\varepsilon) - b(0, \phi_t^0)\} dh_t \quad \text{with} \quad r_{\varepsilon, 0}^1 = 0. \quad (3.17)$$

The first term on the right hand side can be interpreted as a rough path integration of a $C_b^{[p]+1}$ one-form along $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0)$. Hence, $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \int \sigma(\mathbf{y}^\varepsilon) d\mathbf{x})$ is controlled by $c_5(1 + \overline{\omega_{\mathbf{x}, h}})^{c_6}\omega_{\mathbf{x}, h}$, namely,

$$\left| (\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \int \sigma(\mathbf{y}^\varepsilon) d\mathbf{x})_{s, t}^i \right| \leq \left\{ c_5(1 + \overline{\omega_{\mathbf{x}, h}})^{c_6}\omega_{\mathbf{x}, h}(s, t) \right\}^{i/p} \quad (3.18)$$

for all i and (s, t) . With (3.18) in hand we have only to obtain a nice estimate of q -variation norm of the second term on the right hand side of (3.17).

Let us estimate the first level path of $\mathbf{r}_\varepsilon^1$, that is the difference of the first level paths of $\mathbf{y}^\varepsilon$ and ϕ^0 . $\mathbf{y}^\varepsilon$ and ϕ^0 are the solutions of the RDEs (whose coefficient are the $n \times (d+e)$ -matrix $[\sigma, b(\varepsilon, \cdot)]$ and $[\sigma, b(0, \cdot)]$, resp.) driven by $(\varepsilon\mathbf{x}, \mathbf{h})$ and $(\mathbf{0}, \mathbf{h})$, resp. A useful estimate of difference of two solutions of RDEs can be found in Theorem 10.26, pp. 233–236, [25]. ($\mathbf{0}$ denotes a rough path such that $\mathbf{x}^j \equiv 0$ for $j = 1, 2$.) There are positive constants c_7, c_8 such that $C_b^{[p]}$ -norm of $[\sigma, b(\varepsilon, \cdot)]$ and $[\sigma, b(\varepsilon, \cdot)] - [\sigma, b(0, \cdot)]$ are dominated by c_7 and $c_8\varepsilon$, respectively. Set $\omega' = c_9\omega_{\mathbf{x}, h}(s, t)$. If we take c_9 sufficiently large, we have $\|(\varepsilon\mathbf{x}, \mathbf{h})\|_{p-\omega'} \leq 1$ for all $\varepsilon \in [0, 1]$ (see Chapter 8, [25] the definition and $\|\cdot\|_{p-\omega'}$ and details) and $(\varepsilon\mathbf{x}, \mathbf{h})$ satisfies the assumption (iii), Theorem 10.26, [25]. Note also that $|(\varepsilon\mathbf{x}, \mathbf{h})_{s, t}^j - (\mathbf{0}, \mathbf{h})_{s, t}^j| \leq \varepsilon c_{10}(1 + \omega'(S, T))^{c_{11}}\omega'(s, t)^{j/p}$ for any S, T, s, t with $S \leq s \leq t \leq T$.

Then, (a trivial modification of) Theorem 10.26, [25] implies that, on any subinterval $[S, T] \subset [0, 1]$, there exists a constant $c_{12} > 0$ such that

$$\begin{aligned} & |(\mathbf{y}^\varepsilon)_{s, t}^1 - (\phi^0)_{s, t}^1| \\ & \leq c_{12} [c_7 |(\mathbf{y}^\varepsilon)_{0, S}^1 - (\phi^0)_{0, S}^1| + c_8\varepsilon + \varepsilon c_{10}(1 + \omega'(S, T))^{c_{11}}] \omega'(s, t)^{1/p} \exp(c_{12}c_7^p \omega'(S, T)) \\ & \leq c_{13} [|(\mathbf{y}^\varepsilon)_{0, S}^1 - (\phi^0)_{0, S}^1| + \varepsilon(1 + \omega_{\mathbf{x}, h}(S, T))^{c_{13}}] \omega_{\mathbf{x}, h}(s, t)^{1/p} \exp(c_{13}\omega_{\mathbf{x}, h}(S, T)) \end{aligned} \quad (3.19)$$

for any $S \leq s \leq t \leq T$.

Let $\tau_i = \tau_i(\alpha)$ be as in the definition of $N_\alpha(\mathbf{x})$ in (3.4). We choose $\alpha > 0$ so small that $c_{13}(1 + 2\alpha)^{c_{13}}(2\alpha)^{1/p} \leq 1$ holds. Consider each subinterval $I_i := [\tau_{i-1}, \tau_i]$ ($i = 1, 2, \dots, N_\alpha(\mathbf{x})$). Let $\{\tau_{i-1} = \sigma_0^{(i)} < \sigma_1^{(i)} < \dots < \sigma_{K_i}^{(i)} = \tau_i\}$ be a partition of I_i such that $\omega_h(\sigma_{j-1}^{(i)}, \sigma_j^{(i)}) = \alpha$ for $1 \leq j \leq K_i - 1$ and $\omega_h(\sigma_{K_i-1}^{(i)}, \sigma_{K_i}^{(i)}) \leq \alpha$. It is easy to see that $K_i - 1 \leq \omega_h(\tau_{i-1}, \tau_i)/\alpha$. Let $\{0 = t_0 < t_1 < \dots < t_J = 1\}$ be all $\sigma_j^{(i)}$'s in increasing order.

The total number J of the subintervals is now at most

$$J = \sum_{i=1}^{N_\alpha(\mathbf{x})+1} K_i \leq \sum_{i=1}^{N_\alpha(\mathbf{x})+1} \left(1 + \frac{\omega_h(\tau_{i-1}, \tau_i)}{\alpha}\right) \leq N_\alpha(\mathbf{x}) + 1 + \frac{\|h\|_{q-var}^q}{\alpha}.$$

On each subinterval $\hat{I}_i := [t_{i-1}, t_i]$, $\omega_{\mathbf{x},h}(t_{i-1}, t_i) \leq 2\alpha$. Hence we have from (3.19) that

$$|(\mathbf{y}^\varepsilon)_{s,t}^1 - (\phi^0)_{s,t}^1| \leq [|(\mathbf{y}^\varepsilon)_{0,t_{i-1}}^1 - (\phi^0)_{0,t_{i-1}}^1 | + \varepsilon] \exp(c_{13}\omega_{\mathbf{x},h}(t_{i-1}, t_i))$$

for any $t_{i-1} \leq s \leq t \leq t_i$. By mathematical induction, we have

$$\begin{aligned} |(\mathbf{y}^\varepsilon)_{0,t_{i-1}}^1 - (\phi^0)_{0,t_{i-1}}^1| &\leq \varepsilon \prod_{k=1}^{i-1} \{1 + \exp(c_{13}\omega_{\mathbf{x},h}(t_{k-1}, t_k))\} \\ &\leq \varepsilon 2^{i-1} \exp\left(c_{13} \sum_{k=1}^{i-1} \omega_{\mathbf{x},h}(t_{k-1}, t_k)\right). \end{aligned}$$

Putting this back into (3.19), we have on each interval $\hat{I}_i$,

$$\begin{aligned} |(\mathbf{y}^\varepsilon)_{s,t}^1 - (\phi^0)_{s,t}^1| &\leq \varepsilon \left\{ c_{13} 2^{i-1} \exp\left(c_{13} \sum_{k=1}^{i-1} \omega_{\mathbf{x},h}(t_{k-1}, t_k)\right) + (2\alpha)^{-1/p} \right\} \\ &\quad \times \omega_{\mathbf{x},h}(s, t)^{1/p} \exp(c_{13}\omega_{\mathbf{x},h}(t_{i-1}, t_i)) \\ &\leq \varepsilon c_{14} \exp\left(J \log 2 + c_{13} \sum_{k=1}^J \omega_{\mathbf{x},h}(t_{k-1}, t_k)\right) \omega_{\mathbf{x},h}(s, t)^{1/p} \\ &\leq \varepsilon c_{14} \exp\left[\left(N_\alpha(\mathbf{x}) + 1 + \frac{\|h\|_{q-var}^q}{\alpha}\right) \log 2 \right. \\ &\quad \left. + c_{13}\{\alpha(N_\alpha(\mathbf{x}) + 1) + \|h\|_{q-var}^q\} \right] \omega_{\mathbf{x},h}(s, t)^{1/p} \\ &\leq \varepsilon c_{15} \exp(c_{16}N_\alpha(\mathbf{x})) \omega_{\mathbf{x},h}(s, t)^{1/p}. \end{aligned}$$

Here, the positive constants c_i ($14 \leq i \leq 16$) depend on α , too. Since there are J subintervals, we have on the whole interval that

$$\begin{aligned} |(\mathbf{y}^\varepsilon)_{s,t}^1 - (\phi^0)_{s,t}^1| &\leq J^{1-1/p} \varepsilon c_{15} \exp(c_{16}N_\alpha(\mathbf{x})) \omega_{\mathbf{x},h}(s, t)^{1/p} \\ &\leq \varepsilon c_{17} (N_\alpha(\mathbf{x}) + 1)^{1-1/p} \exp(c_{16}N_\alpha(\mathbf{x})) \omega_{\mathbf{x},h}(s, t)^{1/p} \\ &\leq \varepsilon c_{18} \exp(c_{18}N_\alpha(\mathbf{x})) \omega_{\mathbf{x},h}(s, t)^{1/p} \end{aligned} \tag{3.20}$$

for any $0 \leq s \leq t \leq 1$. This is the most difficult part in this subsection. For brevity we set a control function ξ_1 by $\xi_1(s, t)^{1/p} = c_{18} \exp(c_{18}N_\alpha(\mathbf{x})) \omega_{\mathbf{x},h}(s, t)^{1/p}$. Obviously, $\overline{\xi_1} \in \cap_{1 < r < \infty} L^r$ by assumption.

We see from (3.18) and (3.20) that

$$\begin{aligned}
& \left| \{b(\varepsilon, y_t^\varepsilon) - b(0, \phi_t^0)\} - \{b(\varepsilon, y_s^\varepsilon) - b(0, \phi_s^0)\} \right| \\
& \leq \|\nabla b\|_\infty |(\mathbf{y}^\varepsilon)_{s,t}^1 - (\phi^0)_{s,t}^1| + \left(\varepsilon \|\partial_\varepsilon \nabla b\|_\infty + 2 \|\nabla^2 b\|_\infty \|y^\varepsilon - \phi^0\|_\infty \right) |(\phi^0)_{s,t}^1| \\
& \leq \varepsilon \left[\|\nabla b\|_\infty \xi_1(s, t)^{1/p} + \left(\|\partial_\varepsilon \nabla b\|_\infty + 2 \|\nabla^2 b\|_\infty \bar{\xi}_1 \right) \left\{ c_5 (1 + \overline{\omega_{\mathbf{x},h}})^{c_6} \omega_{\mathbf{x},h}(s, t) \right\}^{1/p} \right] \\
& \leq \varepsilon 2^{(p-1)/p} \left[\|\nabla b\|_\infty^p \xi_1(s, t) + \left(\|\partial_\varepsilon \nabla b\|_\infty + 2 \|\nabla^2 b\|_\infty \bar{\xi}_1 \right)^p c_5 (1 + \overline{\omega_{\mathbf{x},h}})^{c_6} \omega_{\mathbf{x},h}(s, t) \right]^{1/p}.
\end{aligned}$$

We denote the right hand side by $\varepsilon \xi_2(s, t)^{1/p}$. Then, ξ_2 is a control function such that $\bar{\xi}_2 \in \cap_{1 < r < \infty} L^r$.

From a basic property of Young integration and the above estimate, we have

$$\frac{1}{\varepsilon} \left| \int_s^t \{b(\varepsilon, y_t^\varepsilon) - b(0, \phi_t^0)\} dh_t \right| \leq c_{19} (1 + \bar{\xi}_2 + \overline{\omega_h})^{c_{19}} \{\xi_2(s, t) + \omega_h(s, t)\}^{1/q}. \quad (3.21)$$

In particular, the Young integral on the left hand side above is of finite q -variation.

From (3.17), (3.18), (3.21) and a basic property of Young pairing, we have

$$\left| (\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \varepsilon^{-1} \mathbf{r}_\varepsilon^1)_{s,t}^i \right| \leq \xi_3(s, t)^{i/p}$$

for some control function ξ_3 such that $\bar{\xi}_3 \in \cap_{1 < r < \infty} L^r$. This ξ_3 can be written as a simple combination of control functions which appear on the right hand sides on (3.18) and (3.21) and is independent of ε . Thus, we have shown Theorem 3.4 for $k = 0$

3.5 Proof of Theorem 3.4 for general $k \geq 1$

In this subsection we prove Theorem 3.4 for k , assuming that it holds for the cases up to $k - 1$. In the proof of the deterministic case in [31, 36], it is explained how to obtain an estimate of $\mathbf{r}_\varepsilon^{k+1}$, which can be expressed as a rough path integral along $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1})$.

Our strategy is quite simple. We carefully look at the proof in [31, 36] once again and make sure that every operation is "of at most polynomial order." Therefore, for those who already know the proof for the deterministic case, this subsection is not very difficult. Since the full proof is quite lengthy, we only give a sketch of proof here.

Let us calculate r_ε^{k+1} . From (3.6)–(3.10), we have

$$\begin{aligned}
dr_{\varepsilon,t}^{k+1} - \nabla b(0, \phi_t^0) \langle r_{\varepsilon,t}^{k+1}, dh_t \rangle &= \left[\sigma(y_t^\varepsilon) \varepsilon dx_t - \sum_{l=1}^k \varepsilon^l dA_t^l \right] \\
&+ \left[b(\varepsilon, y_t^\varepsilon) dh_t - b(0, \phi_t^0) dh_t - \nabla b(0, \phi_t^0) \langle y_t^\varepsilon - \phi_t^0, dh_t \rangle - \sum_{l=1}^k \varepsilon^l dB_t^l \right] \\
&=: dI_t^{k+1} + dJ_t^{k+1}, \quad \text{with } r_{\varepsilon,0}^{k+1} = 0.
\end{aligned} \quad (3.22)$$

Here, I^{k+1} and J^{k+1} stand for sums of the integrals with respect to x and h , respectively. Observe the right hand side of (3.22). There are only $x, h, y^\varepsilon, \phi^0, \dots, \phi^{k-1}$ (and no ϕ^k). See (3.11) and (3.12). Therefore, the right hand side can be regarded as a rough path integral along $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1}, \varepsilon^{-k} \mathbf{r}_\varepsilon^k)$. As a result we obtain

$$(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1}, \varepsilon^{-k} \mathbf{r}_\varepsilon^k, \mathbf{I}^{k+1} + \mathbf{J}^{k+1}) \in G\Omega_p(\mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus k+3}).$$

We will prove that the rough path above is controlled by a nice control function with moments of all order. Note that J^{k+1} is a path of finite q -variation and hence the above rough path is a Young translation of $(\mathbf{x}, \mathbf{h}, \dots, \varepsilon^{-k} \mathbf{r}_\varepsilon^k, \mathbf{I}^{k+1})$ by J^{k+1} .

From Taylor expansion and the way the rough path integral is defined, we can see that the above rough path satisfies essentially the same estimate as in Theorem 3.4, (iii) as follows.

Lemma 3.6 *Keep the same notations and assumptions as in Theorem 3.4. Assume that Theorem 3.4 holds for the cases $1, 2, \dots, k-1$. Then, there exists a control function $\xi = \xi_{\mathbf{x}, h}$ such that $\eta_{k-1, \mathbf{x}, h}(s, t) \leq \xi_{\mathbf{x}, h}(s, t)$, $\bar{\xi} \in \cap_{1 \leq r < \infty} L^r$, and*

$$|(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1}, \varepsilon^{-k} \mathbf{r}_\varepsilon^k, \varepsilon^{-(k+1)}(\mathbf{I}^{k+1} + \mathbf{J}^{k+1}))_{s,t}^j| \leq \xi_{\mathbf{x}, h}(s, t)^{j/p}. \quad (3.23)$$

for all $0 \leq s \leq t \leq 1$ and $j = 1, 2$. (Note that ξ may not depend on ε .)

Let M be as in (3.13). Then, M and M^{-1} are deterministic, depends only on h , and are of finite q -variation. We see from (3.22) that at least formally

$$r_{\varepsilon, t}^{k+1} = M_t \int_0^t M_s^{-1} d[I_s^{k+1} + J_s^{k+1}] = [I_t^{k+1} + J_t^{k+1}] - M_t \int_0^t dM_s^{-1} \cdot [I_s^{k+1} + J_s^{k+1}].$$

Note that the last expression takes the form of Young translation.

To be more precise, set $\tilde{M}_t := \text{Id}_{\mathcal{V} \oplus \hat{\mathcal{V}} \oplus \mathcal{W}^{\oplus k+2}} \oplus M_t$ and apply (a rough path version of) variation of constant method as in (3.14) to the rough path in (3.23) in Lemma 3.6 above. Then, we obtain $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1}, \varepsilon^{-k} \mathbf{r}_\varepsilon^k, \varepsilon^{-(k+1)} \mathbf{r}_\varepsilon^{k+1})$. We can easily see that this rough path satisfies the same inequality as in (3.23) (if ξ is suitably replaced).

Note that $\phi^k = \varepsilon^{-k} r_\varepsilon^k - \varepsilon \{ \varepsilon^{-(k+1)} r_\varepsilon^{k+1} \}$. By applying a simple linear map to the above rough path, we can obtain $(\mathbf{x}, \mathbf{h}, \mathbf{y}^\varepsilon, \phi^0, \dots, \phi^{k-1}, \phi^k, \varepsilon^{-(k+1)} \mathbf{r}_\varepsilon^{k+1})$. Since the operator norm of this ε -dependent linear map is bounded in ε , this rough path also satisfies the same inequality as in (3.23) (if ξ is suitably replaced by another control function, which we call $\eta_{k, \mathbf{x}, h}$). This is the sketch of proof of Theorem 3.4. ■

3.6 Remark for fractional order case

In this subsection we consider the case where the coefficients of RDEs are of fractional order in ε and present analogous results to Proposition 3.1, Proposition 3.2, and Theorem 3.4. The contents of this subsection will be used in later sections.

In this subsection we assume that $1/3 < 1/p < H \leq 1/2$. Let $\sigma : \mathbb{R}^n \rightarrow \text{Mat}(n, d)$ and $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C_b^∞ . Let $\mathbf{x} \in G\Omega_p(\mathbb{R}^d)$ and $h \in C_0^{q-var}(\mathbb{R}^d)$ with $1/p + 1/q > 1$ and we set $\lambda_t = t$. We consider the following RDE driven by the Young pairing $(\varepsilon \mathbf{x}, \mathbf{h}, \boldsymbol{\lambda})$;

$$\begin{aligned} d\tilde{y}_t^\varepsilon &= \sigma(\tilde{y}_t^\varepsilon)(\varepsilon dx_t + dh_t) + \varepsilon^{1/H} b(\tilde{y}_t^\varepsilon) dt \\ &= \sigma(\tilde{y}_t^\varepsilon) \varepsilon dx_t + [\sigma(\tilde{y}_t^\varepsilon) dh_t + \varepsilon^{1/H} b(\tilde{y}_t^\varepsilon) d\lambda_t] \quad \text{with} \quad \tilde{y}_0^\varepsilon = a \in \mathbb{R}^n. \end{aligned} \quad (3.24)$$

This is a variant of RDE (3.2). Strictly speaking, unless $H = 1/2$ the results in previous subsections cannot be used for RDE (3.24). With minor modifications, however, similar results hold in this case, too. We will explain it below. (Proofs are essentially the same and will be omitted).

Let us fix some notations for fractional order expansions. For

$$\Lambda_1 = \{n_1 + \frac{n_2}{H} \mid n_1, n_2 \in \mathbb{N}\},$$

let $0 = \kappa_0 < \kappa_1 < \kappa_2 < \dots$ be all elements of Λ_1 in increasing order. More concretely, leading terms are as follows if $H \in (1/3, 1/2)$;

$$(\kappa_0, \kappa_1, \kappa_2, \dots) = (0, 1, 2, \frac{1}{H}, 3, 1 + \frac{1}{H}, 4, 2 + \frac{1}{H}, 5 \wedge \frac{2}{H}, \dots). \quad (3.25)$$

If $H = 1/2$, then $\Lambda_1 = \mathbb{N}$.

Instead of (3.15) the Taylor expansion of Lyons-Itô map takes the following form;

$$r_\varepsilon^{\kappa_{k+1}} = \tilde{y}^\varepsilon - (\phi^0 + \varepsilon^{\kappa_1} \phi^{\kappa_1} + \dots + \varepsilon^{\kappa_k} \phi^{\kappa_k}). \quad (3.26)$$

In this case, ϕ^{κ_k} is the term of "order κ_k " and is explicitly given in essentially the same way as in (3.10), (3.11), and (3.12). For the reader's convenience, we will give explicit formal expressions of ϕ^{κ_k} for $k = 0, 1, 2, 3$ when $1/3 < H < 1/2$.

$$d\phi_t^0 = \sigma(\phi_t^0) dh_t \quad \text{with} \quad \phi_0^0 = a, \quad (3.27)$$

$$d\phi_t^1 - \nabla \sigma(\phi_t^0) \langle \phi_t^1, dh_t \rangle = \sigma(\phi_t^0) dx_t \quad \text{with} \quad \phi_0^1 = 0, \quad (3.28)$$

$$\begin{aligned} d\phi_t^2 - \nabla \sigma(\phi_t^0) \langle \phi_t^2, dh_t \rangle &= \nabla \sigma(\phi_t^0) \langle \phi_t^1, dx_t \rangle \\ &\quad + \frac{1}{2} \nabla^2 \sigma(\phi_t^0) \langle \phi_t^1, \phi_t^1, dh_t \rangle \quad \text{with} \quad \phi_0^2 = 0, \end{aligned} \quad (3.29)$$

$$d\phi_t^{1/H} - \nabla \sigma(\phi_t^0) \langle \phi_t^{1/H}, dh_t \rangle = b(\phi_t^0) dt \quad \text{with} \quad \phi_0^{1/H} = 0. \quad (3.30)$$

Proposition 3.1 holds still true with a slight modification. Namely, if $1/p + 1/q > 1$, the map

$$\begin{aligned} G\Omega_p(\mathcal{V}) \times C_0^{q-var}([0, 1], \mathcal{V}) &\ni (\mathbf{x}, h) \\ \mapsto (\mathbf{x}, \mathbf{h}, \boldsymbol{\lambda}, \tilde{\mathbf{y}}^\varepsilon, \phi^0, \phi^{\kappa_1}, \dots, \phi^{\kappa_k}, \mathbf{r}_\varepsilon^{\kappa_{k+1}}) &\in G\Omega_p(\mathcal{V}^{\oplus 2} \oplus \mathbb{R} \oplus \mathcal{W}^{\oplus k+3}). \end{aligned}$$

is locally Lipschitz continuous for any k .

The deterministic estimates for terms in the expansion (Proposition 3.2) can easily be modified as follows (This proposition was already used in [32]);

Proposition 3.7 Assume $1/3 < 1/p < H < 1/2$ and $1/p + 1/q > 1$. Consider RDE (3.24) and keep the same notations as above. Then, the following (i) and (ii) hold.

(i) For any $\rho > 0$ and $k = 1, 2, \dots$, there exists a positive constants $C = C(\rho, k)$ which satisfies that

$$\|(\phi^{\kappa_k})^1\|_{p-var} \leq C(1 + \overline{\omega}_{\mathbf{x}}^{1/p})^{\kappa_k}.$$

for any $\mathbf{x} \in G\Omega_p(\mathcal{V})$ and $h \in C_0^{q-var}([0, 1], \mathcal{V})$ with $\|h\|_{q-var} \leq \rho$.

(ii) For any $\rho_1, \rho_2 > 0$ and $k = 1, 2, \dots$, there exists a positive constants $\tilde{C} = \tilde{C}(\rho_1, \rho_2, k)$, which is independent of ε and satisfies that

$$\|(\mathbf{r}_{\varepsilon}^{\kappa_{k+1}})^1\|_{p-var} \leq \tilde{C}(\varepsilon + \varepsilon \overline{\omega}_{\mathbf{x}}^{1/p})^{\kappa_{k+1}}$$

for any $\mathbf{x} \in G\Omega_p(\mathcal{V})$ with $\overline{\omega}_{\varepsilon \mathbf{x}}^{1/p} = \varepsilon \overline{\omega}_{\mathbf{x}}^{1/p} \leq \rho_1$ and any $h \in C_0^{q-var}([0, 1], \mathcal{V})$ with $\|h\|_{q-var} \leq \rho_2$.

The moment estimates for terms in the expansion (Theorem 3.4) can be modified in the following way. This can be shown in essentially the same way as in Theorem 3.4.

Theorem 3.8 We consider RDE (3.24). Assume $1/3 < 1/p < H < 1/2$ and $1/p + 1/q > 1$ and let $h \in C_0^{q-var}([0, 1], \mathcal{V})$. Assume that $\mathbf{x}$ be a $G\Omega_p(\mathcal{V})$ -valued random variable such that (i) $\overline{\omega}_{\mathbf{x}} = \omega_{\mathbf{x}}(0, 1) \in \cap_{1 \leq r < \infty} L^r$ and (ii) $\exp(N_{\alpha}(\mathbf{x})) \in \cap_{1 \leq r < \infty} L^r$ for any $\alpha > 0$.

Then, for any $\mathbf{x}$, h and $k \in \mathbb{N}$, there exist control functions $\eta_k = \eta_{k, \mathbf{x}, h}$ such that the following (i)–(iii) hold:

- (i) η_k are non-decreasing in k , i.e., $\eta_{k, \mathbf{x}, h}(s, t) \leq \eta_{k+1, \mathbf{x}, h}(s, t)$ for all $k, \mathbf{x}, h, (s, t)$.
- (ii) $\overline{\eta_{k, \mathbf{x}, h}} \in \cap_{1 \leq r < \infty} L^r$ for all k, h .
- (iii) For all $\varepsilon \in (0, 1]$, $k \in \mathbb{N}$, $h, \mathbf{x}$, and $0 \leq s \leq t \leq 1$, $j = 1, 2$, we have

$$\left| (\mathbf{x}, \mathbf{h}, \mathbf{y}^{\varepsilon}, \phi^0, \phi^{\kappa_1}, \dots, \phi^{\kappa_k}, \varepsilon^{-\kappa_{k+1}} \mathbf{r}_{\varepsilon}^{\kappa_{k+1}})^j_{s,t} \right| \leq \eta_{k, \mathbf{x}, h}(s, t)^{j/p}.$$

In particular, for all $k \in \mathbb{N}$ and h , $\|(\phi^{\kappa_k})^1\|_{p-var} \in \cap_{1 \leq r < \infty} L^r$ and $\|(\mathbf{r}_{\varepsilon}^{\kappa_{k+1}})^1\|_{p-var} = O(\varepsilon^{\kappa_{k+1}})$ in L^r for any $1 \leq r < \infty$.

Remark 3.9 (i) This section (Section 3) may look a little bit lengthy. But, we will only use Proposition 3.7 and Theorem 3.8 in later sections.

(ii) The author guesses that the results in this section naturally extends to the case of $[p] \geq 3$. But, computation may be hard and it has not been confirmed yet.

4 Malliavin Calculus for solution of RDE driven by fBM

In this section we study the solution of a (scaled) RDE driven by fractional Brownian motion with $H \in (1/3, 1/2]$ via Malliavin calculus. It was already done by Hairer and Pillai [27] (and Cass, Hairer, Litterer, and Tindel [18]). In this section we basically follow

their arguments, but in our case we need to check dependency on the small parameter $\varepsilon \in (0, 1]$.

To keep our argument concise, we do not explain much about Malliavin calculus here. The reader should refer to well-known textbooks such as Nualart [49] and Shigekawa [50]. In this paper we use Watanabe distribution theory and asymptotic theorems for them, which can be found in [56] or Section V-9, [29]. (The results in [56, 29] are formulated on the classical Wiener space, but they are still true on an abstract Wiener space.) One thing different from is [56, 29] that the index sets of asymptotic expansions may not be $\mathbb{N} = \{0, 1, 2, \dots\}$ in this paper. So, we need to slightly modify these asymptotic theorems. However, we skip details here since a summary was already given in the author's previous work [33].

In this paper, we use the following notations. D stands for the $\mathcal{H}$ -derivative. Sobolev space of the integral index $r \in (1, \infty)$ and the differential index $s \in \mathbb{R}$ is denoted by $\mathbf{D}_{r,s}$. As in [56, 29], we set $\mathbf{D}_\infty = \bigcap_{k=1}^\infty \bigcap_{1 < r < \infty} \mathbf{D}_{r,k}$, $\mathbf{D}_{-\infty} = \bigcup_{k=1}^\infty \bigcup_{1 < r < \infty} \mathbf{D}_{r,-k}$. Moreover, we also use $\tilde{\mathbf{D}}_\infty = \bigcap_{k=1}^\infty \bigcup_{1 < r < \infty} \mathbf{D}_{r,k}$ and $\tilde{\mathbf{D}}_{-\infty} = \bigcup_{k=1}^\infty \bigcap_{1 < r < \infty} \mathbf{D}_{r,-k}$ in Watanabe distribution theory. The Sobolev space of vector-valued Wiener functionals is denoted by $\mathbf{D}_{r,s}(\mathcal{K})$, etc., where $\mathcal{K}$ is a real separable Hilbert space.

Let $1/3 < H \leq 1/2$ and choose p so that $1/3 < 1/p < H$. The d -dimensional fBm $(w_t)_{0 \leq t \leq 1}$ with Hurst parameter H admits a natural rough path lift $\mathbf{w}$ as a random rough path that takes values in $G\Omega_p(\mathbb{R}^d)$. We denote by $\mathcal{H} = \mathcal{H}^H$ the Cameron-Martin space associated with d -dimensional fBm with $H \in (1/3, 1/2]$. Throughout this section $\gamma \in \mathcal{H}$ is arbitrary, but fixed. By Friz-Victoir [23], there is a continuous embedding

$$\mathcal{H}^H \hookrightarrow W^{1/q,2}([0, 1], \mathbb{R}^d) \hookrightarrow C_0^{q-var}([0, 1], \mathbb{R}^d) \quad (4.1)$$

for any $q \in ((H+1/2)^{-1}, 2)$. (In a very recent preprint [21], the above embedding is shown to still hold for $q = (H+1/2)^{-1}$.) The Banach space in the middle is the fractional Sobolev (i.e., Besov) space with the differential index $1/q$ and the integral index 2. Note that if p and q are sufficiently close to $1/H$ and $(H+1/2)^{-1}$, respectively, then $1/p + 1/q > 1$, which makes Young integration/translation/paring possible.

Let us make a remark on Hölder regularity of the above RDE. It is well-known that $\mathbf{w}$ is actually an α -Hölder geometric rough path a.s., where we set $\alpha := 1/p$. At first, it is not so obvious whether $\tau_\gamma(\varepsilon \mathbf{w})$ is an α -Hölder geometric rough path, even though $\mathcal{H}^H \hookrightarrow C_0^{\alpha-hld}(\mathbb{R}^d)$. It was shown to be true in Friz and Victoir [23] and Exercise 9.37, p. 211, [25]. Using (4.1) with $1/q = \alpha + 1/2$, they showed that

$$\begin{aligned} \|\tau_\gamma(\mathbf{x})^1\|_{\alpha-hld} &\leq \text{const.} \times (\|\mathbf{x}^1\|_{\alpha-hld} + \|\gamma\|_{\mathcal{H}}), \\ \|\tau_\gamma(\mathbf{x})^2\|_{2\alpha-hld} &\leq \text{const.} \times (\|\mathbf{x}^2\|_{2\alpha-hld} + \|\mathbf{x}^1\|_{\alpha-hld} \|\gamma\|_{\mathcal{H}} + \|\gamma\|_{\mathcal{H}}^2) \end{aligned}$$

for any $\gamma \in \mathcal{H}$ and $\mathbf{x} \in G\Omega_\alpha^H(\mathbb{R}^d)$. These imply that the driving signal $(\tau_\gamma(\varepsilon \mathbf{w}), \varepsilon^{1/H} \lambda)$ of RDE (4.2) is actually a α -Hölder geometric rough path a.s. Consequently, so is $\tilde{\mathbf{y}}^\varepsilon$.

As before $\sigma : \mathbb{R}^n \rightarrow \text{Mat}(n, d)$ and $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C_b^∞ . For notational convenience, we will sometimes denote by $V_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the i th column vector field of σ ($1 \leq i \leq d$), i.e., $\sigma = [V_1; \dots; V_d]$. In a similar way we will write $V_0 = b$.

We consider the following RDE for $\varepsilon \in (0, 1]$ and $a \in \mathbb{R}^n$;

$$d\tilde{y}_t^\varepsilon = \sigma(\tilde{y}_t^\varepsilon)(\varepsilon dw_t + d\gamma_t) + \varepsilon^{1/H} b(\tilde{y}_t^\varepsilon) dt \quad \text{with} \quad \tilde{y}_0^\varepsilon = a \in \mathbb{R}^n. \quad (4.2)$$

We write $\tilde{y}_t^\varepsilon = a + (\tilde{\mathbf{y}}^\varepsilon)_{0,t}^1$ and study this process. When $\gamma = 0$, we write $\tilde{\mathbf{y}}^\varepsilon = \mathbf{y}^\varepsilon$. When $\gamma = 0$ and $\varepsilon = 1$, we write $\tilde{\mathbf{y}}^\varepsilon = \mathbf{y}$. If Φ denotes the Lyons-Itô map that corresponds to $[\sigma, b]$ and a , then $\tilde{y}^\varepsilon = \Phi((\tau_\gamma(\varepsilon \mathbf{w}), \varepsilon^{1/H} \lambda))$. Here, (i) $\tau_\gamma(\varepsilon \mathbf{w})$ denotes the Young translation of $\varepsilon \mathbf{w}$ by γ and (ii) $(\tau_\gamma(\varepsilon \mathbf{w}), \varepsilon^{1/H} \lambda)$ denotes the Young pairing of $\tau_\gamma(\varepsilon \mathbf{w})$ and the one-dimensional path $\varepsilon^{1/H} \lambda_t = \varepsilon^{1/H} t$. Using V_i 's we can rewrite RDE (4.2) as follows:

$$d\tilde{y}_t^\varepsilon = \sum_{i=1}^d V_i(\tilde{y}_t^\varepsilon)(\varepsilon dw_t^i + d\gamma_t^i) + \varepsilon^{1/H} V_0(\tilde{y}_t^\varepsilon) dt \quad \text{with} \quad \tilde{y}_0^\varepsilon = a \in \mathbb{R}^n. \quad (4.3)$$

Note that $(y_t^\varepsilon)_{0 \leq t \leq 1}$ and $(y_{\varepsilon^{1/H}t}^\varepsilon)_{0 \leq t \leq 1}$ have the same law. (See Inahama [32] for a proof).

In Hairer and Pillai [27], they proved the following: (i) $y_t \in \mathbf{D}_\infty(\mathbb{R}^n)$ for any $t > 0$, i.e., $D^m y_t$ exists and in $\cap_{1 < r < \infty} L^r$ for any $m = 0, 1, 2, \dots$ (ii) Under Hörmander's hypoellipticity condition on vector fields $\{V_1, \dots, V_d, V_0\}$ at the starting point a , Malliavin covariance matrix of y_t is non-degenerate in the sense of Malliavin for any $t > 0$, i.e.,

$$\det \left[\{ \langle Dy_t^{(i)}, Dy_t^{(j)} \rangle_{\mathcal{H}} \}_{i,j=1}^n \right]^{-1} \in \cap_{1 < r < \infty} L^r,$$

where $y_t^{(i)}$ denoted the i th component of y_t .

It is almost obvious that $\tilde{y}_1^\varepsilon$ also satisfies (i) and (ii) above for each fixed ε . In this paper, however, we need to check dependency on $\varepsilon \in (0, 1]$ as it varies. The precise statements are given in the following two propositions. We will prove them later by slightly modifying the proofs in [27, 18, 35].

Proposition 4.1 *Assume σ and b are C_b^∞ and let $\gamma \in \mathcal{H}$ be arbitrary but fixed. Then, for any $m = 0, 1, 2, \dots$ and $r \in (1, \infty)$, there exists a positive constant $c = c_{m,r}$ such that*

$$\mathbb{E} [\|D^m \tilde{y}_1^\varepsilon\|_{\mathcal{H}^{\otimes m}}^r]^{1/r} \leq c \varepsilon^m.$$

Proof. In [35] the author proved $\mathbf{D}_\infty$ -property of solutions of RDEs driven by Gaussian rough path $\mathbf{w}$ including fBm with $H > 1/4$. The proof is so flexible that we can replace $\mathbf{w}$ by $\tau_\gamma(\varepsilon \mathbf{w}) = \varepsilon \mathbf{w} + \gamma$. If we keep track of ε -dependency in that argument, then we can easily see that $D^m \tilde{y}^\varepsilon$ is $O(\varepsilon^m)$ as $\varepsilon \searrow 0$ for any $m \in \mathbb{N}$. In that proof, the uniform estimate of Jacobian process and its inverse plays a crucial role. $\blacksquare$

Proposition 4.2 *In addition to the assumption of Proposition 4.1, we assume the ellipticity assumption (A1). Then, $(\tilde{y}_1^\varepsilon - a)/\varepsilon$ is uniformly non-degenerate in the sense of Malliavin, that is,*

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E} \left[\det \left[\left\{ D \left(\frac{\tilde{y}_1^{\varepsilon, (i)} - a}{\varepsilon} \right), D \left(\frac{\tilde{y}_1^{\varepsilon, (j)} - a}{\varepsilon} \right) \right\}_{\mathcal{H}} \}_{i,j=1}^n \right]^{-r} \right] < \infty$$

for any $r \in (1, \infty)$.

Proof. Note that the special case " $\gamma = 0$ and $b \equiv 0$ and uniformly elliptic coefficients" was already shown in [5, 7], etc. Since this proposition can be shown in a similar way, we omit the proof. (However, we note that uniform non-degeneracy of $(\tilde{y}_1^\varepsilon - a)/\varepsilon$ for the shifted RDE becomes quite complicated under a Hörmander-type condition instead of **(A1)**.)

■

Consider the asymptotic expansion of $\tilde{y}^\varepsilon$ as in (3.26). We have already seen that this expansion holds true both in the deterministic sense and the L^r -sense. Moreover, evaluated at time $t = 1$, it also holds true in $\mathbf{D}_\infty$ -sense.

Proposition 4.3 *We keep the same assumptions as in Proposition 4.1. Then, we have the following asymptotic expansion as $\varepsilon \searrow 0$:*

$$\tilde{y}_1^\varepsilon \sim \phi_1^0 + \varepsilon^{\kappa_1} \phi_1^{\kappa_1} + \cdots + \varepsilon^{\kappa_k} \phi_1^{\kappa_k} + \cdots \quad \text{in } \mathbf{D}_\infty(\mathbb{R}^n).$$

This means that for each k , (i) $\phi_1^{\kappa_k} \in \mathbf{D}_\infty(\mathbb{R}^n)$ and (ii) $\mathbf{D}_{r,s}$ -norm of $r_{\varepsilon,1}^{\kappa_{k+1}}$ is $O(\varepsilon^{\kappa_{k+1}})$ for any $r \in (1, \infty)$ and $s \geq 0$.

Proof. By the way it is constructed, $\phi_1^{\kappa_k}$ is an element of inhomogeneous Wiener chaos of order at most $[\kappa_k]$. Hence, $\phi_1^{\kappa_k} \in \mathbf{D}_\infty(\mathbb{R}^n)$ and $D^{[\kappa_k]+1} \phi_1^{\kappa_k} = 0$. Next we estimate Sobolev norms of the remainder terms. We see from the stronger form of Meyer's equivalence that, for any integer $s \geq [\kappa_k] + 1$ and any $r \in (1, \infty)$, there exists $C = C_{r,s}$ such that

$$\|r_{\varepsilon,1}^{\kappa_{k+1}}\|_{\mathbf{D}_{r,s}} \leq C(\|r_{\varepsilon,1}^{\kappa_{k+1}}\|_{L^r} + \|D^s r_{\varepsilon,1}^{\kappa_{k+1}}\|_{L^r}) = C(\|r_{\varepsilon,1}^{\kappa_{k+1}}\|_{L^r} + \|D^s \tilde{y}_1^\varepsilon\|_{L^r})$$

holds. By Theorem 3.8 and Proposition 4.1, the right hand side is $O(\varepsilon^{\kappa_{k+1}}) + O(\varepsilon^s) = O(\varepsilon^{\kappa_{k+1}})$. Thus, we have the desired estimate for such (r, s) . Since $\mathbf{D}_{r,s}$ -norm is increasing in s , the proof is done. ■

Now we state and prove on-diagonal short time asymptotics of $p_t(a, a) = \mathbb{E}[\delta_a(y_t)]$. Compared to the off-diagonal case, this is not so difficult. From Propositions 4.2, and 4.3, and Watanabe's asymptotic theory for generalized Wiener functionals (i.e., Watanabe distributions), we can obtain the following theorem.

Theorem 4.4 *Assume the ellipticity assumption **(A1)**. Then, the diagonal of the kernel $p(t, a, a)$ admits the following asymptotics as $t \searrow 0$:*

$$p(t, a, a) \sim \frac{1}{t^{nH}} (c_0 + c_{\nu_1} t^{\nu_1 H} + c_{\nu_2} t^{\nu_2 H} + \cdots)$$

for certain real constants $c_0, c_{\nu_1}, c_{\nu_2}, \dots$. Here, $\{0 = \nu_0 < \nu_1 < \nu_2 < \cdots\}$ are all the elements of Λ_3 in increasing order. Moreover, c_0 is positive.

Proof. In this proof, $\gamma = 0$. From the scaling property, we see that

$$p(\varepsilon^{1/H}, a, a) = \mathbb{E}[\delta_a(y_1^\varepsilon(a))] = \mathbb{E}[\delta_0(\varepsilon \frac{y_1^\varepsilon(a) - a}{\varepsilon})] = \varepsilon^{-n} \mathbb{E}[\delta_0(\frac{y_1^\varepsilon(a) - a}{\varepsilon})].$$

By Proposition 4.2, $(y_1^\varepsilon(a) - a)/\varepsilon$ is uniformly non-degenerate. It admits asymptotic expansion in $\mathbf{D}_\infty(\mathbf{R}^n)$ as in Proposition 4.3 with the index set for the exponents being Λ_2 . Then, by (a slight generalization of) Theorem 9.4, p. 387, Ikeda and Watanabe [29], the following asymptotic expansion holds in $\tilde{\mathbf{D}}_{-\infty}$ as $\varepsilon \searrow 0$;

$$\delta_0\left(\frac{y_1^\varepsilon(a) - a}{\varepsilon}\right) \sim \phi_0 + \varepsilon^{\nu_1} \phi_{\nu_1} + \varepsilon^{\nu_2} \phi_{\nu_2} + \cdots \quad \text{as } \varepsilon \searrow 0.$$

Formally, this is a composition of Taylor expansion of $\delta_0(\cdot)$ and the asymptotic expansion of $(y_1^\varepsilon(a) - a)/\varepsilon$. Hence, the new index set is $\mathbb{N}\langle\Lambda_2\rangle = \Lambda_3$. By taking the generalized expectation and setting $c_{\nu_k} = \mathbb{E}[\phi_{\nu_k}]$, we have

$$p(\varepsilon^{1/H}, a, a) \sim \varepsilon^{-n} (c_0 + c_{\nu_1} \varepsilon^{\nu_1} + c_{\nu_2} \varepsilon^{\nu_2} + \cdots) \quad \text{as } \varepsilon \searrow 0.$$

Putting $\varepsilon = t^H$, we prove the asymptotic expansion. It is straight forward to see that

$$c_0 = \mathbb{E}[\delta_0(\sum_{j=1}^d V_j(a) w_1^j)] = (2\pi)^{-n/2} \{\det(\sigma(a)\sigma(a)^*)\}^{-1/2} > 0,$$

which completes the proof. $\blacksquare$

5 Off-diagonal short time asymptotics

In this section, following Watanabe [56], we prove the short time asymptotics of kernel function $p_t(a, a')$ when $a \neq a'$ and $1/3 < H \leq 1/2$. Unlike in [56], we can localize around the energy minimizing path in the geometric rough path space in this paper, since Lyons-Itô map is continuous in this setting. (The case $H > 1/2$ was done in [33]. The result in this section can be regarded as a rough path version of that in [33].)

5.1 Localization around energy minimizing path

Let $G\Omega_{\alpha, m}^B(\mathbb{R}^d)$ be the geometric rough path space with (α, m) -Besov norm for $\alpha \in (1/3, 1/2]$ and $m > 1$ with $\alpha - 1/m > 1/3$. Explicitly, the norms are given by

$$\|\mathbf{x}^i\|_{i\alpha, m/i-B} := \left(\iint_{0 \leq s < t \leq 1} \frac{|\mathbf{x}_{s,t}^i|^{m/i}}{|t-s|^{1+m\alpha}} ds dt \right)^{i/m} \quad (i = 1, 2).$$

We have the following continuous embeddings

$$G\Omega_\beta^H(\mathbb{R}^d) \hookrightarrow G\Omega_{\alpha', m}^B(\mathbb{R}^d) \hookrightarrow G\Omega_\alpha^H(\mathbb{R}^d) \hookrightarrow G\Omega_p(\mathbb{R}^d) \quad (5.1)$$

if $1/3 < 1/p = \alpha < \alpha' - 1/m < \alpha' < \beta \leq 1/2$ (see Appendix A2, Friz and Victoir [25]).

Next, we introduce a measure. Let $\mu = \mu^H$ be the law of the fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2]$. This is a probability measure on $\mathcal{W} = \overline{\mathcal{H}}$, which is the closure of $\mathcal{H} = \mathcal{H}^H$ in $C_0^{p-var}([0, 1], \mathbb{R}^d)$. Then, the triple $(\mathcal{W}, \mathcal{H}, \mu)$ is an abstract Wiener space.

For any $\beta \in (1/3, H)$, fBm (w_t) admits a natural lift a.s. via dyadic piecewise linear approximation and the lift $\mathbf{w}$ is a random variable taking values in $G\Omega_\beta^H(\mathbb{R}^d)$. Note that the lift of Cameron-Martin space $\mathcal{H}$ is contained in $G\Omega_\beta^H(\mathbb{R}^d)$. Moreover, as $\varepsilon \searrow 0$, Schilder-type large deviation holds for the laws of $\varepsilon \mathbf{w}$, which will be denoted by $\nu_\varepsilon = \nu_\varepsilon^H$. (See Friz and Victoir [24]). Because of Besov-Hölder embedding mentioned above, these properties also hold with respect to (α', m) -Besov topology if $\alpha' < H$. As usual, the good rate function $\mathcal{I}$ is given as follows: $\mathcal{I}(\mathbf{x}) = \|h\|_{\mathcal{H}}^2/2$ if $\mathbf{x}$ is the lift of some $h \in \mathcal{H}$ and $\mathcal{I}(\mathbf{x}) = \infty$ if otherwise.

Let us clarify the conditions on various indices here. From now on, these will be assumed unless otherwise stated. First, for given $H \in (1/3, 1/2]$, we choose $p := 1/\alpha \in (1/H, 3)$ and $q \in ((H+1/2)^{-1}, 2)$ so that $1/p + 1/q > 1$ holds. Then, we choose $\alpha' \in (\alpha, H)$ and $m \in \mathbb{N}$ such that $(\alpha' - \alpha) \vee (H - \alpha') > 1/(4m)$ and consider $G\Omega_{\alpha', 4m}^B(\mathbb{R}^d)$. (Heuristically, m is a very large integer.)

Since m is an integer, $w \mapsto \|\mathbf{w}^i - \mathbf{h}^i\|_{i\alpha, 4m/i-B}^{4m/i}$ is $\mathbf{D}_\infty$ in the sense of Malliavin calculus for $i = 1, 2$, where $\mathbf{h}$ is a fixed element. Actually, it is an element of an inhomogeneous Wiener chaos. Due to this fact, the localization is allowed even in the framework of Watanabe distribution theory. This is the reason why we use this Besov-type norm on the geometric rough path space.

The Young translation τ_γ works on $G\Omega_{\alpha', 4m}^B(\mathbb{R}^d)$ for any $\gamma \in \mathcal{H}$. The proof is just a slight modification of the Hölder case.

Lemma 5.1 *Let H, α', m be as above. Then, for any $\gamma \in \mathcal{H}$, the Young translation τ_γ is a continuous map from $G\Omega_{\alpha', 4m}^B(\mathbb{R}^d)$ to itself.*

Proof. Generally, we have the following basic result for Young integrals. Let $p', q' > 0$ with $1/p' + 1/q' > 1$. Then, there is a constant $C > 0$ which depends only on p', q' such that

$$\left| \int_s^t (x_u - x_s) \otimes dy_u \right| \leq C \|x\|_{p'-var;[s,t]} \cdot \|y\|_{q'-var;[s,t]}.$$

for any $[s, t] \subset [0, 1]$.

Now we prove the lemma. We have

$$\begin{aligned} \tau_\gamma(\mathbf{x})_{s,t}^1 &= \mathbf{x}_{s,t}^1 + \gamma_{s,t}^1 \\ \tau_\gamma(\mathbf{x})_{s,t}^2 &= \mathbf{x}_{s,t}^2 + \gamma_{s,t}^2 + \int_s^t \mathbf{x}_{s,u}^1 \otimes d\gamma_u + \int_s^t \gamma_{s,u}^1 \otimes dx_u \end{aligned} \quad (5.2)$$

Here, the second, the third, and the fourth terms on the right hand side of (5.2) are Young integrals. As usual we set $x_t = \mathbf{x}_{0,t}^1$. By Besov-Hölder embedding theorem, x is $\alpha' - 1/(4m)$ Hölder continuous. Moreover, there is a constant c such that

$$\|\gamma\|_{q-var;[s,t]} \leq c\|\gamma\|_{W^{1/q,2}} \cdot (t-s)^{\frac{1}{q}-\frac{1}{2}} \leq c\|\gamma\|_{\mathcal{H}} \cdot (t-s)^{\frac{1}{q}-\frac{1}{2}} \quad (\gamma \in \mathcal{H}, \quad \frac{1}{q} < H + \frac{1}{2}).$$

(See p. 211, [25]. The constant $c > 0$ may vary from line to line.) Therefore, γ^2 is of finite $2(1/q - 1/2)$ Hölder norm. The third and the fourth terms are of finite $(1/q - 1/2) + (\alpha' - 1/(4m))$ Hölder norm. Since $H - \alpha' < 1/(4m)$ and we may choose q so that $1/q - 1/2$ can be arbitrarily close to H , these three terms are actually of finite $(2\alpha' + \delta)$ Hölder norm for some $\delta > 0$ and hence are of finite $(2\alpha', 2m)$ Besov norm. Thus, we have shown that τ_γ maps $G\Omega_{\alpha',4m}^B(\mathbb{R}^d)$ to itself.

We can show continuity of τ_γ by estimating the difference $|\tau_\gamma(\mathbf{x})_{s,t}^i - \tau_\gamma(\tilde{\mathbf{x}})_{s,t}^i|$ for $i = 1, 2$ in essentially the same way. So, we omit details. ■

For $\gamma \in \mathcal{H} \subset C_0^{q-var}([0,1], \mathbb{R}^d)$, let $\phi^0 = \phi^0(\gamma)$ be a unique solution of (3.27) in the q -variational Young sense, which starts at $a \in \mathbb{R}^n$. Set, for $a \neq a'$,

$$K_a^{a'} = \{\gamma \in \mathcal{H} \mid \phi_1^0(\gamma) = a'\}.$$

This is a closed set in $\mathcal{H}$. We only consider the case that $K_a^{a'}$ is not empty. For example, if **(A1)** is satisfied for any a , then $K_a^{a'}$ is not empty for any a' . From the Schilder-type large deviation theory, we see that $\inf\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\} = \min\{\|\gamma\|_{\mathcal{H}} \mid \gamma \in K_a^{a'}\}$.

We continue to assume **(A1)**. Now we introduce another assumption;

(A2): $\bar{\gamma} \in K_a^{a'}$ which minimizes $\mathcal{H}$ -norm exists uniquely.

In the sequel, $\bar{\gamma}$ denotes the minimizer in Assumption **(A2)** and we use the results of the previous section for this $\bar{\gamma}$.

Note that (i) the mapping $\gamma \in \mathcal{H} \hookrightarrow C_0^{q-var}([0,1], \mathbb{R}^d) \mapsto \phi_1^0(\gamma) \in \mathbb{R}^n$ is Fréchet differentiable and (ii) its Jacobian is a surjective linear mapping from $\mathcal{H}$ to $\mathbb{R}^n$ at any γ , because there exists a positive constant $c = c(\gamma)$ such that

$$\left(\langle D\phi_1^{0,i}(\gamma), D\phi_1^{0,j}(\gamma) \rangle_{\mathcal{H}^*} \right)_{1 \leq i,j \leq n} \geq c \cdot \text{Id}_n. \quad (5.3)$$

This can be shown in the same way as in the proof of non-degeneracy of y_t under ellipticity assumption. (Actually, it is easier since γ is non-random and fixed here.)

Therefore, by the Lagrange multiplier method, there exists $\bar{\nu} = (\bar{\nu}_1, \dots, \bar{\nu}_n) \in \mathbb{R}^n$ uniquely such that the map

$$\mathcal{H} \times \mathbb{R}^n \ni (\gamma, \nu) \mapsto \frac{1}{2}\|\gamma\|_{\mathcal{H}}^2 - \langle \nu, \phi_1^0(\gamma) - a' \rangle_{\mathbb{R}^n} \in \mathbb{R} \quad (5.4)$$

attains an extremum at $(\bar{\gamma}, \bar{\nu})$. By differentiating in the direction of $k \in \mathcal{H}$, we have

$$\langle \bar{\gamma}, k \rangle_{\mathcal{H}} = \langle \bar{\nu}, D_k \phi_1^0(\bar{\gamma}) \rangle_{\mathbb{R}^n} = \langle \bar{\nu}, \hat{J}(\bar{\gamma})_1 \int_0^1 \hat{J}(\bar{\gamma})_t^{-1} \sigma(\phi_t^0(\bar{\gamma})) dk_t \rangle_{\mathbb{R}^n}. \quad (5.5)$$

Here, $\hat{J}(\bar{\gamma})^{\pm 1}$ are of finite q -variation and $\hat{J}(\bar{\gamma})$ satisfies the following ODE in Young sense;

$$dJ_t = \nabla \sigma(\phi_t^0(\bar{\gamma})) \langle J_t, d\bar{\gamma}_t \rangle \quad \text{with } J_0 = \text{Id}_n.$$

Since the integral on the right hand side is of (5.5) Young integral, $\langle \bar{\gamma}, \cdot \rangle_{\mathcal{H}}$ naturally extends to a continuous linear functional on $C_0^{p-var}([0, 1], \mathbb{R}^d)$.

Next, set $\hat{\nu}_\varepsilon = \nu_\varepsilon \otimes \delta_{\varepsilon^{1/H}\lambda}$, where λ is a one-dimensional path defined by $\lambda_t = t$ and $\otimes$ stands for the product of probability measures. This measure is supported on $G\Omega_{\alpha', 4m}^B(\mathbb{R}^d) \times \mathbb{R}\langle \lambda \rangle$. The Young pairing map $G\Omega_{\alpha', 4m}^B(\mathbb{R}^d) \times \mathbb{R}\langle \lambda \rangle \rightarrow G\Omega_{\alpha', 4m}^B(\mathbb{R}^{d+1})$ is continuous. The law of $\hat{\nu}_\varepsilon$ induced by this map is the law of $(\varepsilon \mathbf{w}, \varepsilon^{1/H}\lambda)$, the Young pairing of $\varepsilon \mathbf{w}$ and $\varepsilon^{1/H}\lambda$.

Define $\hat{\mathcal{I}}(\mathbf{x}; l) = \|h\|_{\mathcal{H}}^2/2$ if $\mathbf{x}$ is the lift of some $h \in \mathcal{H}$ and $l_t \equiv 0$ and define $\hat{\mathcal{I}}(w, l) = \infty$ if otherwise. Here, l is a one-dimensional path. We can easily show that $\{\hat{\nu}_\varepsilon\}_{\varepsilon > 0}$ also satisfies a large deviation principle as $\varepsilon \searrow 0$ with a good rate function $\hat{\mathcal{I}}$. We will use this in Lemma 5.2 below to show that we may localize on a neighborhood of the minimizer $\bar{\gamma}$ in order to obtain the asymptotic expansion.

Now we introduce a cut-off function. Let $\psi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\psi(u) = 1$ if $|u| \leq 1/2$ and $\psi(u) = 0$ if $|u| \geq 1$. For each $\eta > 0$ and $\varepsilon > 0$, we set

$$\chi_\eta(\varepsilon, w) = \prod_{i=1}^2 \psi\left(\frac{1}{\eta^{4m}} \|\tau_{-\bar{\gamma}}(\varepsilon \mathbf{w})^i\|_{i\alpha', 4m/i-B}^{4m/i}\right).$$

Here, $\tau_{-\bar{\gamma}}$ is the Young translation by $-\bar{\gamma}$. It is a continuous map from $G\Omega_\beta^H(\mathbb{R}^d)$ to itself. So, the right hand side is defined for almost all $w \in \mathcal{W}$. Shifting by $\bar{\gamma}/\varepsilon$, we have

$$\chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}) = \prod_{i=1}^2 \psi\left(\frac{\varepsilon^{4m}}{\eta^{4m}} \|\mathbf{w}^i\|_{i\alpha', 4m/i-B}^{4m/i}\right).$$

This is a $\mathbf{D}_\infty$ -functional. Moreover, from Taylor expansion for ψ , the following asymptotics holds; for any $\eta > 0$ and any $M \in \mathbb{N}$,

$$\chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}) = 1 + O(\varepsilon^M) \quad \text{in } \mathbf{D}_\infty \text{ as } \varepsilon \searrow 0. \quad (5.6)$$

Since $\|\mathbf{w}^i\|_{i\alpha', 4m/i-B}^{4m/i}$ is an element of an inhomogeneous Wiener chaos of order $4m$, so is its Cameron-Martin shift $\|\tau_{-\bar{\gamma}}(\varepsilon \mathbf{w})^i\|_{i\alpha', 4m/i-B}^{4m/i}$. For any $r \in (0, \infty)$, L^r -norm of this Wiener functional is bounded in ε . Hence, so is its $\mathbf{D}_{r,k}$ -norm for any r, k .

The following lemma states that only rough paths sufficiently close to the lift of the minimizer $\bar{\gamma}$ contribute to the asymptotics.

Lemma 5.2 *Assume (A1) and (A2). Then, for any $\eta > 0$, there exists $c = c_\eta > 0$ such that*

$$0 \leq \mathbb{E}[(1 - \chi_\eta(\varepsilon, w)) \cdot \delta_{a'}(y_1^\varepsilon)] = O\left(\exp\left\{-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2 + c}{2\varepsilon^2}\right\}\right) \quad \text{as } \varepsilon \searrow 0.$$

Proof. The proof of this lemma is a bit lengthy and quite similar to the proof for the corresponding lemma in [33] or [56], except that we work on the geometric rough path space. So, we only give a sketch of proof here.

Set $g(u) = u \vee 0$ for $u \in \mathbb{R}$. Then, in the sense of distributional derivative, $g'' = \delta_0$. Take a bounded continuous function $C : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $C(u_1, \dots, u_n) = g(u_1 - a'_1)g(u_2 - a'_2) \cdots g(u_n - a'_n)$ if $|u - a'| \leq 2\eta'$. Take $\eta' > 0$ arbitrarily small.

Then, we have

$$\begin{aligned} 0 \leq \mathbb{E}[(1 - \chi_\eta(\varepsilon, w)) \cdot \delta_{a'}(y_1^\varepsilon)] &= \mathbb{E}\left[\{1 - \chi_\eta(\varepsilon, w)\} \psi\left(\frac{|y_1^\varepsilon - a'|^2}{\eta'^2}\right) \cdot \delta_{a'}(y_1^\varepsilon)\right] \\ &= \mathbb{E}\left[\{1 - \chi_\eta(\varepsilon, w)\} \psi\left(\frac{|y_1^\varepsilon - a'|^2}{\eta'^2}\right) (\partial_1^2 \cdots \partial_n^2 C)(y_1^\varepsilon)\right] \quad (5.7) \end{aligned}$$

The idea is that by using the integration by parts formula for generalized expectations as in [56, 29], we reduce the problem to the upper bound of the large deviation principle for $\{\hat{\nu}_\varepsilon\}_{\varepsilon>0}$. (The reason is as follows. Thanks to the formula, we can remove the partial differentiations in $(\partial_1^2 \cdots \partial_n^2 C)(y_1^\varepsilon)$ at a certain price. Then, we have only to treat $C(y_1^\varepsilon)$ which is just a bounded function.) ■

5.2 Integrability lemmas

In this subsection, we prove a few lemmas for integrability of Wiener functionals of exponential type which will be used in the proof of the short time asymptotic expansion.

Throughout this subsection we assume **(A2)**. Let $\bar{\gamma}$ be as in **(A2)** and let ϕ^{κ_j} and $r_\varepsilon^{\kappa_j+} = r_\varepsilon^{\kappa_{j+1}}$ ($j = 0, 1, 2, \dots$) be as in (3.26) with $\gamma = \bar{\gamma}$. First we consider

$$\frac{r_\varepsilon^{\kappa_3}}{\varepsilon^2} = \frac{r_\varepsilon^{2+}}{\varepsilon^2} = \frac{1}{\varepsilon^2}(\tilde{y}^\varepsilon - \phi^0 - \varepsilon\phi^1 - \varepsilon^2\phi^2) = \varepsilon^{\kappa_3-2}\phi^{\kappa_3} + \varepsilon^{\kappa_4-2}\phi^{\kappa_4} + \dots$$

Recall that $\kappa_3 = 1/H$ if $H \in (1/3, 1/2)$ and $\kappa_3 = 3$ if $H = 1/2$. When evaluated at time $t = 1$, this quantity has a kind of exponential integrability in the following sense. (Now that $\bar{\gamma}$ is fixed, $r_\varepsilon^{2+}(\mathbf{x})$, $\phi^2(\mathbf{x})$, etc. are function of $\mathbf{x}$ alone. We will often write r_ε^{2+} , ϕ^2 , etc. for simplicity.)

Lemma 5.3 Assume **(A2)**. For any $M > 0$, there exists $\eta > 0$ such that

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E}[\exp(M\langle \bar{\nu}, r_{\varepsilon,1}^{2+} \rangle / \varepsilon^2) I_{U_\eta}(\varepsilon \mathbf{w})] < \infty.$$

Here, we set $U_\eta = \cap_{i=1,2} \{\|\mathbf{x}^i\|_{i\alpha', 4m/i-B}^{1/i} < \eta\}$ as before.

Proof. Let $\omega_{\mathbf{x}}$ be as in (3.3). Note that U_1 is bounded with respect to p -variation norm. So we may use Proposition 3.2 to see that, for some positive constants c_1, c_2 ,

$$\|r_{\varepsilon}^{2+}\|_{p-var} \leq c_1(\varepsilon + \bar{\omega}_{\varepsilon\mathbf{x}}^{1/p})^{\kappa_3} \leq c_2(\varepsilon + \|(\varepsilon\mathbf{x})^1\|_{\alpha', 4m-B} + \|(\varepsilon\mathbf{x})^2\|_{2\alpha', 2m-B}^{1/2})^{\kappa_3} \quad (\varepsilon\mathbf{x} \in U_1).$$

(In this paragraph we used Besov-Hölder-variation embedding theorem on geometric rough path spaces. See Proposition A.9, p. 578, [25] for instance.) Hence, if $\varepsilon\mathbf{x} \in U_{\eta}$ for $0 < \eta \leq 1$, then

$$\frac{\|r_{\varepsilon}^{2+}\|_{p-var}}{\varepsilon^2} \leq c_2(1 + \|\mathbf{x}^1\|_{\alpha', 4m-B} + \|\mathbf{x}^2\|_{2\alpha', 2m-B}^{1/2})^2(\varepsilon + 2\eta)^{\kappa_3-2}. \quad (5.8)$$

Recall that Fernique's theorem holds for fractional Brownian rough path $\mathbf{w}$ with respect to β -Hölder topology and hence with respect to $(\alpha', 4m)$ -Besov topology. It states that for some $\rho > 0$ we have

$$\mathbb{E}\left[\exp\left(\rho(1 + \|\mathbf{w}^1\|_{\alpha', 4m-B} + \|\mathbf{w}^2\|_{2\alpha', 2m-B}^{1/2})^2\right)\right] < \infty.$$

(See Friz and Oberhauser [22] for a proof.)

For given M , take $0 < \eta \leq 1$ so that $M|\bar{\nu}|c_2(3\eta)^{\kappa_3-2} \leq \rho$. Then, we have

$$\sup_{0 < \varepsilon \leq \eta} \mathbb{E}\left[\exp(M\langle \bar{\nu}, r_{\varepsilon,1}^{2+} \rangle / \varepsilon^2) I_{U_{\eta}}(\varepsilon\mathbf{w})\right] < \infty.$$

Note that, if $\varepsilon\mathbf{w} \in U_{\eta}$ and $\eta \leq \varepsilon \leq 1$, then $\|r_{\varepsilon}^{2+}\|_{p-var}/\varepsilon^2$ is bounded. (The bound may depend on η .) This completes the proof. ■

Next we consider

$$\frac{r_{\varepsilon}^2}{\varepsilon} = \frac{r_{\varepsilon}^{1+}}{\varepsilon} = \frac{1}{\varepsilon}(\tilde{y}^{\varepsilon} - \phi^0 - \varepsilon\phi^1) = \varepsilon\phi^2 + \varepsilon^{\kappa_3-1}\phi^{\kappa_3} + \dots.$$

Lemma 5.4 *Assume (A2). For any $M > 0$, there exists $\eta > 0$ such that*

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E}\left[\exp(M\|r_{\varepsilon}^2\|_{p-var}^2/\varepsilon^2) I_{U_{\eta}}(\varepsilon\mathbf{w})\right] < \infty.$$

Proof. We can prove the lemma in the same way as in Lemma 5.3 above. So we only give a sketch of proof.

In this case we have the following inequality instead of (5.8):

$$\frac{\|r_{\varepsilon}^2\|_{p-var}^2}{\varepsilon^2} \leq c_2(1 + \|\mathbf{x}^1\|_{\alpha', 4m-B} + \|\mathbf{x}^2\|_{2\alpha', 2m-B}^{1/2})^2(\varepsilon + 2\eta)^2 \quad (\varepsilon\mathbf{x} \in U_{\eta}).$$

The rest is similar. So we omit details. ■

From now on we assume **(A1)** and **(A2)**. In addition, we introduce the following assumption;

$$\textbf{(A3)'}: \quad \mathbb{E}[\exp(\langle \bar{\nu}, \phi_1^2(\mathbf{w}) \rangle) \mid \phi_1^1 = 0] < \infty.$$

Note that $\phi_T^1(w) = \hat{J}_T \int_0^T \hat{J}_t^{-1} \sigma(\phi_t^0) dw_t$. Here $\phi_t^0 = \phi_t^0(\bar{\gamma})$, $\hat{J}_t = \hat{J}(\bar{\gamma})_t$. Note that the right hand side is Young integral and, consequently, is continuous in $w \in \mathcal{W}$. We regard its j th component $\phi_1^{1,j} \in \mathcal{W}^* \subset \mathcal{H}^*$ as an element of $\mathcal{H}$ by Riesz isometry, we write $\sharp \phi_1^{1,j} \in \mathcal{H} \subset \mathcal{W}$. We have an orthogonal decomposition $\mathcal{H} = \ker \phi_1^1 \oplus (\ker \phi_1^1)^\perp$. We denote by π the orthogonal projection from $\mathcal{H}$ onto $\ker \phi_1^1$. Note that $(\ker \phi_1^1)^\perp$ is an n -dimensional linear subspace spanned by $\{\sharp \phi_1^{1,1}, \dots, \sharp \phi_1^{1,n}\}$. Since $\dim(\ker \phi_1^1)^\perp < \infty$, the abstract Wiener space splits into two; $\mathcal{W} = \overline{\ker \phi_1^1}^{\|\cdot\|_{p-var}} \oplus (\ker \phi_1^1)^\perp$. The projection π naturally extends to the one from $\mathcal{W}$ onto $\overline{\ker \phi_1^1}^{\|\cdot\|_{p-var}}$, which is again denoted by the same symbol. There exist Gaussian measures μ_1 and μ_2 such that $(\overline{\ker \phi_1^1}^{\|\cdot\|_{p-var}}, \ker \phi_1^1, \mu_1)$ and $((\ker \phi_1^1)^\perp, (\ker \phi_1^1)^\perp, \mu_2)$ are abstract Wiener spaces. Naturally, $\mu_1 = \pi_* \mu$, $\mu_2 = \pi_*^\perp \mu$ and $\mu = \mu_1 \times \mu_2$ (the product measure). One may think μ_1 is the definition of the conditional measure $\mathbb{P}[\cdot \mid \phi_1^1 = 0]$ in **(A3)'** above.

Therefore, **(A3)'** is equivalent to the following;

$$\mathbb{E}[\exp(\langle \bar{\nu}, \phi_1^2(\pi \mathbf{w}) \rangle)] < \infty. \quad (5.9)$$

Precisely, $\pi \mathbf{w} := \mathcal{L}(\pi w) = \lim_{m \rightarrow \infty} \mathcal{L}((\pi w)(m))$. Here, $\mathcal{L}$ stands for the rough path lift map. Now we will see that $\pi \mathbf{w}$ is well-defined and has nice properties.

Note that $\phi_1^1(k) = D_k \phi_1^0(\bar{\gamma})$ and recall (5.3), (5.5). Hence, $\{\phi_1^{1,1}, \dots, \phi_1^{1,n}\}$ are of rank n in $\mathcal{H}^*$. Let C be the positive symmetric matrix in (5.3) and set $K = (K_{ij}) = C^{-1}$, $M = (M_{ij}) = C^{-1/2}$, which are again positive symmetric. Then we have

$$\pi w = w - \sum_{j=1}^n \langle w, \sum_{l'=1}^n M_{jl} \sharp \phi_1^{1,l'} \rangle \sum_{l'=1}^n M_{jl'} \sharp \phi_1^{1,l'} = w - \sum_{l,l'=1}^n K_{ll'} \phi_1^{1,l}(w) \cdot \sharp \phi_1^{1,l'}. \quad (5.10)$$

This projection also works in q -variational setting. Note that the second term on the right hand side is $\mathcal{H}$ -valued. Therefore, the lift of πw is actually a Young translation of $\mathbf{w}$ by $\sum_{l,l'} K_{ll'} \phi_1^{1,l}(w) \cdot \sharp \phi_1^{1,l'}$. It also holds that $\pi \mathbf{w} = \lim_{m \rightarrow \infty} \mathcal{L}[\pi(w(m))]$.

For $k, k' \in C_0^{q-var}([0, 1], \mathbb{R}^d)$, we set

$$\begin{aligned} \mathcal{A}(k, k') &= \frac{1}{2} \hat{J}_1 \int_0^1 \hat{J}_t^{-1} \{ \nabla \sigma(\phi_t^0) \langle \phi_t^1(k'), dk_t \rangle + \nabla \sigma(\phi_t^0) \langle \phi_t^1(k), dk_t' \rangle \} \\ &\quad + \frac{1}{2} \hat{J}_1 \int_0^1 \hat{J}_t^{-1} \nabla^2 \sigma(\phi_t^0) \langle \phi_t^1(k), \phi_t^1(k'), d\bar{\gamma}_t \rangle \end{aligned} \quad (5.11)$$

and $\hat{\mathcal{A}}(k, k') = \langle \bar{\nu}, \mathcal{A}(\pi k, \pi k') \rangle$. Here, $\hat{J} = \hat{J}(\bar{\gamma})$ and $\phi^0 = \phi^0(\bar{\gamma})$ for brevity. Then, $\hat{\mathcal{A}}$ is a symmetric bounded bilinear mapping form on $\mathcal{H} \times \mathcal{H}$. Notice that

$$\mathcal{A}(k, k) = \phi_1^2(k) - \frac{1}{2} \delta^{H, 1/2} \cdot \hat{J}_1 \int_0^1 \hat{J}_t^{-1} b(\phi_t^0) dt, \quad (5.12)$$

where $\delta^{H,1/2} = 1$ if $H = 1/2$ and $\delta^{H,1/2} = 0$ if otherwise. Therefore, $\hat{\mathcal{A}}(k, k) = \phi_1^2(k) + (\text{const})$.

Now we will see that (i) $\hat{\mathcal{A}}$ is actually Hilbert-Schmidt and (ii) $\phi_1^2(\pi \mathbf{w}) \in \mathcal{C}_2 \oplus \mathcal{C}_0$ whose $\mathcal{C}_2$ -component corresponds to $\hat{\mathcal{A}}$, that is, $\phi_1^2(\pi \mathbf{w}) = \Xi_{\hat{\mathcal{A}}}(w) + (\text{const})$. Here, $\mathcal{C}_j$ denotes the j th homogeneous Wiener chaos of order j and $\Xi_{\mathcal{B}}$ denotes the element in $\mathcal{C}_2$ which unitarily corresponds to a symmetric Hilbert-Schmidt bilinear form $\mathcal{B}$.

For $m \in \mathbb{N}$, set $\hat{\mathcal{A}}_m(k, k') = \langle \bar{\nu}, \mathcal{A}((\pi k)(m), (\pi k')(m)) \rangle$. The corresponding bounded self-adjoint operator on $\mathcal{H}$ is denoted by $\hat{A}_m$. This bilinear form extends to a bounded bilinear form on $\mathcal{W} \times \mathcal{W}$. Hence, by Goodman's theorem (see Theorem 4.6, p. 83, [37]), it is of trace class (and consequently Hilbert-Schmidt). $\hat{\mathcal{A}}_m(w, w) = \Xi_{\hat{\mathcal{A}}_m}(w) + \text{Trace}(\hat{A}_m)$. As a result, $\phi_1^2((\pi w)(m)) = \Xi_{\hat{\mathcal{A}}_m}(w) + s_m$, where the constant s_m may depend on m .

By a straight forward rough path calculation as in Section 5, Inahama [32], we can prove that $\phi_1^2((\pi w)(m))$ converges to $\phi_1^2(\pi \mathbf{w})$ in $L^2(\mu)$. (In Inahama [32], the convergence $\phi_1^2(w(m)) \rightarrow \phi_1^2(\mathbf{w})$ as $m \rightarrow \infty$ is shown. We can modify that proof, since the effect of the projection π appears as Young translation as we have already seen.) Hence, both $\Xi_{\hat{\mathcal{A}}_m}$ and s_m converge in $\mathcal{C}_2$ and $\mathcal{C}_0$, respectively. By the unitary correspondence, there exists a symmetric Hilbert-Schmidt bilinear form $\mathcal{B}$ such that $\hat{\mathcal{A}}_m \rightarrow \mathcal{B}$ as $m \rightarrow \infty$ in Hilbert-Schmidt norm. From a basic property of Young integral, we see that $\hat{\mathcal{A}}_m(k, k') \rightarrow \hat{\mathcal{A}}(k, k')$ as $m \rightarrow \infty$ for each fixed $k, k' \in \mathcal{H}$. Thus we have shown (i) and (ii) above.

Exponentially integrability of quadratic Wiener functionals is well-known. (5.9) is equivalent to $\mathbb{E}[\exp(\Xi_{\hat{\mathcal{A}}})] < \infty$, which in turn is equivalent to $\sup \text{Spec}(\hat{A}) < 1/2$. Since the inequality is strict, there exists $\rho > 1$ such that $\sup \text{Spec}(\rho \hat{A}) < 1/2$, which is equivalent to $\mathbb{E}[\exp(\rho \Xi_{\hat{\mathcal{A}}})] < \infty$. Summing it up, we have seen that **(A3)'** is equivalent to the following;

$$\mathbb{E}[\exp(\rho \langle \bar{\nu}, \phi_1^2(\pi \mathbf{w}) \rangle)] < \infty \quad \text{for some } \rho > 1. \quad (5.13)$$

Let us check here that **(A3)** and **(A3)'** are equivalent under **(A1)**, **(A2)**.

Proposition 5.5 *Under **(A1)** and **(A2)**, the two conditions **(A3)** and **(A3)'** are equivalent.*

Proof. As is explained above, **(A3)'** is equivalent to $\sup \text{Spec}(\hat{A}) < 1/2$. Keep in mind that the only accumulation point of $\text{Spec}(\hat{A})$ is 0, since $\hat{A}$ is Hilbert-Schmidt. Let $(-\varepsilon_0, \varepsilon_0) \ni u \mapsto f(u) \in K_a^{a'}$ be a smooth curve in $K_a^{a'}$ such that $f(0) = \bar{\gamma}$ and $f'(0) \neq 0$ as in **(A3)**. Then, a straight forward calculation shows that

$$\begin{aligned} \frac{d^2}{du^2} \Big|_{u=0} \frac{\|f(u)\|_{\mathcal{H}}^2}{2} &= \frac{d^2}{du^2} \Big|_{u=0} \left(\frac{\|f(u)\|_{\mathcal{H}}^2}{2} - \langle \bar{\nu}, \phi_1^0(f_u) - a' \rangle \right) \\ &= \|f'(0)\|_{\mathcal{H}}^2 + \langle f''(0), \bar{\gamma} \rangle_{\mathcal{H}} - \langle \bar{\nu}, D\phi_1^0(\bar{\gamma}) \langle f''(0) \rangle \rangle - \langle \bar{\nu}, D^2\phi_1^0(\bar{\gamma}) \langle f'(0), f'(0) \rangle \rangle \\ &= \|f'(0)\|_{\mathcal{H}}^2 - \langle \bar{\nu}, D^2\phi_1^0(\bar{\gamma}) \langle \pi f'(0), \pi f'(0) \rangle \rangle \\ &= \|f'(0)\|_{\mathcal{H}}^2 - 2 \langle \bar{\nu}, \psi \langle \pi f'(0), \pi f'(0) \rangle \rangle, \end{aligned} \quad (5.14)$$

where we used (5.4)–(5.5) and the fact that $f'(0)$ is tangent to the submanifold $K_a^{a'}$. Since $f'(0)$ can be any non-zero vector h such that $\pi h = h$, we see from (5.14) that **(A3)** is equivalent to

$$\langle \bar{\nu}, \psi \langle \pi f'(0), \pi f'(0) \rangle \rangle < \frac{1}{2} \|h\|_{\mathcal{H}}^2 \quad (h \in \mathcal{H} \setminus \{0\}),$$

which in turn is equivalent to $\sup \text{Spec}(\hat{A}) < 1/2$. $\blacksquare$

The following is a key technical lemma. Roughly speaking, it states that restricted on a sufficiently small subset, $\exp(\langle \bar{\nu}, r_{\varepsilon,1}^2 \rangle / \varepsilon^2) \in \cup_{1 < q < \infty} L^q$ uniformly in ε .

Lemma 5.6 *Assume **(A1)**, **(A2)** and **(A3)**. Then, there exists $\rho_1 > 1$ and $\eta > 0$ such that*

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E} \left[\exp(\rho_1 \langle \bar{\nu}, r_{\varepsilon,1}^2 \rangle / \varepsilon^2) I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1|/\varepsilon| \leq \eta_1\}} \right] < \infty$$

for any $\eta_1 > 0$.

Proof. By Lemma 5.3 and the relation $r_{1,\varepsilon}^2/\varepsilon^2 = \phi_1^2 + r_{\varepsilon,1}^{2+}/\varepsilon^2$, it is sufficient to show that

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E} \left[\exp(\rho_1 \langle \bar{\nu}, \phi_1^2 \rangle) I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1|/\varepsilon| \leq \eta_1\}} \right] < \infty. \quad (5.15)$$

Then, from (5.10) and (5.12) we have

$$\begin{aligned} \phi_1^2(\mathbf{w}) &= \lim_{m \rightarrow \infty} \phi_1^2(w(m)) = \lim_{m \rightarrow \infty} \mathcal{A}(w(m), w(m)) - (\text{const}) \\ &= \phi_1^2(\pi \mathbf{w}) + 2 \sum_{j,j'} \phi_1^{1,j}(w) K_{jj'} \cdot \mathcal{A}\langle w, \# \phi_1^{1,j'} \rangle \\ &\quad + \sum_{j,j',k,k'} \phi_1^{1,j}(w) \phi_1^{1,k}(w) K_{jj'} K_{kk'} \cdot \mathcal{A}\langle \# \phi_1^{1,j'}, \# \phi_1^{1,k'} \rangle =: Z_1 + Z_2 + Z_3. \end{aligned} \quad (5.16)$$

Note that $\mathcal{A}\langle w, \# \phi_1^{1,j'} \rangle$ and $\mathcal{A}\langle \# \phi_1^{1,j'}, \# \phi_1^{1,k'} \rangle$ are well-defined as Young integrals.

Exponential integrability of the first term Z_1 on the right hand side of (5.16) is given in (5.13). So, we estimate the second term Z_2 . Since $\varepsilon \phi_1^1(w) = r_{\varepsilon,1}^2(\mathbf{w}) - r_{\varepsilon,1}^1(\mathbf{w})$ and $|\mathcal{A}\langle w, \# \phi_1^{1,j'} \rangle| \lesssim \|w\|_{p\text{-var}}$, we have

$$\begin{aligned} |\phi_1^{1,j}(w) \mathcal{A}\langle w, \# \phi_1^{1,j'} \rangle| &\leq c_1 \left\{ \left| \frac{r_{\varepsilon,1}^2(\mathbf{w})}{\varepsilon} \right| + \left| \frac{r_{\varepsilon,1}^1(\mathbf{w})}{\varepsilon} \right| \right\} \|w\|_{p\text{-var}} \\ &\leq c_1 \left\{ \left| \frac{c' r_{\varepsilon,1}^2(\mathbf{w})}{\varepsilon} \right|^2 + \frac{\|w\|_{p\text{-var}}^2}{4c'^2} \right\} + c_1 \left| \frac{r_{\varepsilon,1}^1(\mathbf{w})}{\varepsilon} \right| \|w\|_{p\text{-var}} \end{aligned}$$

for any $c' > 0$.

Set $c_2 = 2c_1 n^2 \sup_{j,j'} |K_{j,j'}|$ and let $M > 0$. Then, by Hölder's inequality,

$$\begin{aligned} \mathbb{E}[e^{M|Z_2|} I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1/\varepsilon| \leq \eta_1\}}] &\leq \mathbb{E}[\exp(3Mc_2 c'^2 |r_{\varepsilon,1}^2/\varepsilon|^2) I_{U_\eta}(\varepsilon \mathbf{w})]^{1/3} \\ &\quad \times \mathbb{E}[e^{3Mc_2 \|w\|_{p-var}^2/(4c')}]^{1/3} \mathbb{E}[e^{3Mc_2 \eta_1 \|w\|_{p-var}}]^{1/3}. \end{aligned}$$

For any $M > 0$ and $\eta_1 > 0$, the third factor is integrable. If c' is chosen sufficiently large, then the second factor is also integrable by Fernique's theorem. By Lemma 5.4, there exists $\eta > 0$ such that $\sup_\varepsilon$ of the first factor is finite and, hence,

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E}[e^{M|Z_2|} I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1/\varepsilon| \leq \eta_1\}}] < \infty. \quad (5.17)$$

Since $\phi_1^{1,j}(w)\phi_1^{1,k}(w) = \varepsilon^{-1}\{r_{\varepsilon,1}^2(\mathbf{w})^j - r_{\varepsilon,1}^1(\mathbf{w})^j\}\phi_1^{1,k}(w)$, we can deal with Z_3 in the same way. For any $M > 0$ and $\eta_1 > 0$, there exists $\eta > 0$ such that

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E}[e^{M|Z_3|} I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1/\varepsilon| \leq \eta_1\}}] < \infty. \quad (5.18)$$

Let $\rho > 1$ be as in (5.13). Set $\rho_1 = (1+\rho)/2 > 1$, $s = 2\rho/(1+\rho) > 1$, and $1/s + 1/s' = 1$. Then, from Hölder's inequality and (5.13), (5.16)–(5.18), we can easily see that

$$\begin{aligned} &\mathbb{E}[\exp(\rho_1 \langle \bar{\nu}, \phi_1^2 \rangle) I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1/\varepsilon| \leq \eta_1\}}] \\ &\leq \mathbb{E}[\exp(\rho \langle \bar{\nu}, \phi_1^2 \circ \pi \rangle)]^{1/s} \prod_{i=1}^2 \mathbb{E}[e^{2q'\rho_1 |\bar{\nu}| |Z_i|} I_{U_\eta}(\varepsilon \mathbf{w}) I_{\{|r_{\varepsilon,1}^1/\varepsilon| \leq \eta_1\}}]^{1/(2s')}. \end{aligned}$$

From this, (5.15) is immediate. This completes the proof. $\blacksquare$

5.3 Proof of off-diagonal short time asymptotics

In this subsection we prove Theorem 2.2, namely, off-diagonal short time asymptotics of the density of the solution $(y_t) = (y_t(a))$ of RDE (2.1) driven by fractional Brownian rough path $\mathbf{w}$ with $1/3 < H \leq 1/2$ under Assumptions **(A1)**–**(A3)**.

First, let us calculate the kernel $p(t, a, a')$. Take $\eta > 0$ as in Lemma 5.6. Then, we see

$$\begin{aligned} p(\varepsilon^{1/H}, a, a') &= \mathbb{E}[\delta_{a'}(y_1^\varepsilon)] \\ &= \mathbb{E}[\delta_{a'}(y_1^\varepsilon) \chi_\eta(\varepsilon, w)] + \mathbb{E}[\delta_{a'}(y_1^\varepsilon) \{1 - \chi_\eta(\varepsilon, w)\}] =: I_1 + I_2. \end{aligned}$$

As we have shown in Lemma 5.2, the second term I_2 on the right hand side does not contribute to the asymptotic expansion. So, we have only to calculate the first term I_1 . By Cameron-Martin formula,

$$I_1 = \mathbb{E}\left[\exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2} - \frac{1}{\varepsilon} \langle \bar{\gamma}, w \rangle\right) \delta_{a'}(\tilde{y}_1^\varepsilon) \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon})\right].$$

Recall that $\langle \bar{\gamma}, w \rangle = \langle \bar{\nu}, \phi_1^1(w) \rangle$ for all w . Hence, we have

$$\begin{aligned}
I_1 &= \exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\right) \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\langle \bar{\nu}, \phi_1^1 \rangle\right) \delta_{a'}(a' + \varepsilon\phi_1^1 + r_{\varepsilon,1}^2) \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon})\right] \\
&= \frac{1}{\varepsilon^n} \exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\right) \mathbb{E}\left[\exp\left(-\frac{1}{\varepsilon}\langle \bar{\nu}, \phi_1^1 \rangle\right) \delta_0(\phi_1^1 + \varepsilon^{-1}r_{\varepsilon,1}^2) \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon})\right] \\
&= \frac{1}{\varepsilon^n} \exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\right) \mathbb{E}\left[\exp\left(\langle \bar{\nu}, r_{\varepsilon,1}^2 \rangle / \varepsilon^2\right) \delta_0(\phi_1^1 + \varepsilon^{-1}r_{\varepsilon,1}^2) \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon})\right] \\
&= \frac{1}{\varepsilon^n} \exp\left(-\frac{\|\bar{\gamma}\|_{\mathcal{H}}^2}{2\varepsilon^2}\right) \mathbb{E}\left[F(\varepsilon, w) \delta_0\left(\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right)\right],
\end{aligned}$$

where

$$F(\varepsilon, w) = \exp(\varepsilon^{-2}\langle \bar{\nu}, r_{\varepsilon,1}^2 \rangle) \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}) \psi\left(\frac{1}{\eta_1^2} \left|\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right|^2\right) \quad (5.19)$$

for any positive constant η_1 . Here, ψ is the cut-off function introduced in Subsection 5.1. It is easy to see that (i) $\chi_\eta(\varepsilon, w + \bar{\gamma}/\varepsilon)$ and its derivatives vanish outside $\{w \mid \varepsilon w \in U_\eta\}$ and (ii) $\psi\left(\frac{1}{\eta_1^2} \left|\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right|^2\right)$ and its derivatives vanish outside $\{|r_{\varepsilon,1}^1/\varepsilon| \leq \eta_1\}$. Hence, by Lemma 5.6, $F(\varepsilon, w) \in \tilde{\mathbf{D}}_\infty$ and $F(\varepsilon, w) = O(1)$ with respect to that topology. Since $\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)$ admits an asymptotic expansion in $\tilde{\mathbf{D}}_{-\infty}$, the problem reduces to whether $F(\varepsilon, w)$ admits an asymptotic expansion in $\tilde{\mathbf{D}}_\infty$.

Lemma 5.7 *Assume (A1)–(A3). For any $M \in \mathbb{N}$, we have*

$$\mathbb{E}\left[F(\varepsilon, w) \delta_0\left(\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right)\right] = \mathbb{E}\left[F(\varepsilon, w) \psi(|\phi_1^1/\eta_1|^2) \delta_0\left(\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right)\right] + O(\varepsilon^M)$$

as $\varepsilon \searrow 0$.

Proof. By using Taylor expansion for ψ , we see that, for given M , there exist $m \in \mathbb{N}$ and $G_j(\varepsilon, w) \in \mathbf{D}_\infty$ ($1 \leq j \leq m$) such that

$$\psi\left(\frac{1}{\eta_1^2} \left|\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right|^2\right) = \psi\left(\left|\frac{\phi_1^1}{\eta_1}\right|^2\right) + \sum_{j=1}^m \psi^{(j)}\left(\left|\frac{\phi_1^1}{\eta_1}\right|^2\right) G_j(\varepsilon, w) + O(\varepsilon^M) \quad (5.20)$$

in $\mathbf{D}_\infty$ as $\varepsilon \searrow 0$. $G_j(\varepsilon, w) = O(1)$, but its explicit form is not important. Note that $\psi^{(j)}(|\phi_1^1/\eta_1|^2)T(\phi_1^1) = 0$ if $j \geq 1$ and $\text{supp}(T) \subset \{a \in \mathbb{R}^n \mid |a| < \eta_1/2\}$.

By Proposition 4.2 and Watanabe's asymptotic theory in [56, 29], $\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)$ admits an asymptotic expansion in $\tilde{\mathbf{D}}_{-\infty}$ as follows. As before, we set $\{0 = \nu_0 < \nu_1 < \nu_2 < \dots\}$ to be all the elements of Λ_3 in increasing order. For given M , let $l \in \mathbb{N}$ be the smallest integer such that $M \leq \nu_{l+1}$. Then, for some $\Phi_{\nu_j} \in \tilde{\mathbf{D}}_{-\infty}$ ($1 \leq j \leq l$), it holds that

$$\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon) = \delta_0(\phi_1^1) + \varepsilon^{\nu_1} \Phi_{\nu_1} + \dots + \varepsilon^{\nu_l} \Phi_{\nu_l} + O(\varepsilon^{\nu_{l+1}}) \quad (5.21)$$

in $\tilde{\mathbf{D}}_{-\infty}$ as $\varepsilon \searrow 0$. Here, Φ_{ν_j} is a finite linear combination of terms of the form

$$\partial^\beta \delta_0(\phi_1^1) \times \{\text{a polynomial of the components of } \phi_1^{\kappa_i} \text{'s}\},$$

where β stands for a multi-index. Hence, $\psi^{(j')}(|\phi_1^1/\eta_1|^2)\Phi_{\nu_j}$ vanish for all j, j' .

Now, using (5.20) and (5.21), we prove the lemma.

$$\begin{aligned} & \mathbb{E}[F(\varepsilon, w)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)] \\ &= \mathbb{E}[F(\varepsilon, w)\psi\left(\frac{1}{\eta_1^2}\left|\frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon}\right|^2\right)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)] \\ &= \mathbb{E}[F(\varepsilon, w)\psi(|\phi_1^1/\eta_1|^2)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)] \\ &\quad + \mathbb{E}[F(\varepsilon, w)\left(\sum_{j=1}^m \psi^{(j)}\left(\left|\frac{\phi_1^1}{\eta_1}\right|^2\right)G_j(\varepsilon, w)\right)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)] + O(\varepsilon^M) \\ &= \mathbb{E}[F(\varepsilon, w)\psi(|\phi_1^1/\eta_1|^2)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)] \\ &\quad + \mathbb{E}[F(\varepsilon, w)\left(\sum_{j=1}^m \psi^{(j)}\left(\left|\frac{\phi_1^1}{\eta_1}\right|^2\right)G_j(\varepsilon, w)\right)(\delta_0(\phi_1^1) + \cdots + \varepsilon^{\nu_l}\Phi_{\nu_l})] + O(\varepsilon^M) \\ &= \mathbb{E}[F(\varepsilon, w)\psi(|\phi_1^1/\eta_1|^2)\delta_0((\tilde{y}_1^\varepsilon - a')/\varepsilon)] + O(\varepsilon^M). \end{aligned}$$

Thus, we have shown the lemma. $\blacksquare$

Set $\Lambda'_2 = \{\kappa - 2 \mid \kappa \in \Lambda_1 \setminus \{0, 1\}\}$. If $H \neq 1/2$, then $\Lambda'_2 = \{0 < H^{-1} - 2 < 1 < \cdots\}$. Next we set $\Lambda'_3 = \{a_1 + a_2 + \cdots + a_m \mid m \in \mathbf{N}_+ \text{ and } a_1, \dots, a_m \in \Lambda'_2\}$. In the following lemma, $\{0 = \xi_0 < \xi_1 < \xi_2 < \cdots\}$ stands for all the elements of Λ'_3 in increasing order.

Note that the following lemma does not claim $F_{k+1}(\varepsilon, w) = O(\varepsilon^{\xi_{k+1}})$, but it claims $F_{k+1}(\varepsilon, w)T(\phi_1^1) = O(\varepsilon^{\xi_{k+1}})$ if $T \in \mathcal{S}'(\mathbb{R}^n)$ is for example of the form $\partial^\beta \delta_0$.

Lemma 5.8 *Assume (A1)–(A3) and let $F(\varepsilon, w) \in \tilde{\mathbf{D}}_\infty$ as in (5.19). Then, for every $k = 1, 2, 3, \dots$,*

$$\begin{aligned} & F(\varepsilon, w)\psi(|\phi_1^1(w)/\eta_1|^2) \\ &= \exp(\langle \bar{\nu}, \phi_1^2(\mathbf{w}) \rangle)\psi(|\phi_1^1(w)/\eta_1|^2)^2 \{1 + \varepsilon^{\xi_1}K_{\xi_1}(w) + \cdots + \varepsilon^{\xi_k}K_{\xi_k}(w)\} + F_{k+1}(\varepsilon, w), \end{aligned}$$

where $F_{k+1}(\varepsilon, w) \in \tilde{\mathbf{D}}_\infty$ satisfies that

$$F_{k+1}(\varepsilon, w)T(\phi_1^1) = O(\varepsilon^{\xi_{k+1}}) \quad \text{in } \mathbf{D}_{-\infty} \text{ as } \varepsilon \searrow 0$$

for any $T \in \mathcal{S}'(\mathbb{R}^n)$ with $\text{supp}(T) \subset \{a \in \mathbb{R}^n \mid |a| \leq \eta_1/2\}$. Moreover, $K_{\xi_j} \in \mathbf{D}_\infty$ ($j = 1, 2, \dots$) are determined by the following formal expansion ($\kappa_3 = H^{-1}$ if $H \neq 1/2$);

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{\langle \bar{\nu}, r_{\varepsilon,1}^{\kappa_3}/\varepsilon^2 \rangle^m}{m!} &= \sum_{m=0}^{\infty} \frac{1}{m!} \left\{ \varepsilon^{\kappa_3-2} \langle \bar{\nu}, \phi_1^{\kappa_3} \rangle + \varepsilon^{\kappa_4-2} \langle \bar{\nu}, \phi_1^{\kappa_3} \rangle + \cdots \right\}^m \\ &= 1 + \varepsilon^{\xi_1}K_{\xi_1} + \varepsilon^{\xi_2}K_{\xi_2} + \cdots. \end{aligned}$$

Proof. Let $\rho_1 > 1$ be as in Lemma 5.6. First we show that, for any $\eta_1 > 0$,

$$\mathbb{E}[\exp(\rho_1 \langle \bar{\nu}, \phi_1^2 \rangle) I_{\{|\phi_1^1| \leq \eta_1\}}] < \infty. \quad (5.22)$$

We can choose a subsequence $\{\varepsilon_k\}$ such that, as $k \rightarrow \infty$, $\varepsilon_k \searrow 0$ and $R_1^{1, \varepsilon_k} / \varepsilon_k \rightarrow \phi_1^1$ a.s. To prove (5.22), we apply Fatou's lemma to (5.15) with η_1 replaced by $2\eta_1$.

$$\begin{aligned} \infty &> \liminf_{k \rightarrow \infty} \mathbb{E}[\exp(\rho_1 \langle \bar{\nu}, \phi_1^2 \rangle) I_{U_\eta}(\varepsilon_k \mathbf{w}) I_{\{|r_{\varepsilon_k, 1}^1 / \varepsilon_k| \leq 2\eta_1\}}] \\ &\geq \mathbb{E}[\exp(\rho_1 \langle \bar{\nu}, \phi_1^2 \rangle) \liminf_{k \rightarrow \infty} I_{\{|r_{\varepsilon_k, 1}^1 / \varepsilon_k| \leq 2\eta_1\}}] \geq \mathbb{E}[\exp(\rho_1 \langle \bar{\nu}, \phi_1^2 \rangle) I_{\{|\phi_1^1| \leq \eta_1\}}]. \end{aligned}$$

From (5.22), it is easy to check that $\exp(\langle \bar{\nu}, \phi_1^2 \rangle) \psi(|\phi_1^1 / \eta_1|^2) \in \tilde{\mathbf{D}}_\infty$.

Now we expand $\exp(\langle \bar{\nu}, r_{\varepsilon, 1}^2 \rangle / \varepsilon^2) = \exp(\langle \bar{\nu}, \phi_1^2 \rangle) \exp(\langle \bar{\nu}, r_{\varepsilon, 1}^{\kappa_3} \rangle / \varepsilon^2)$ in ε . Set $Q_{l+1} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$Q_{l+1}(u) = e^u - \left(1 + u + \frac{u^2}{2!} + \cdots + \frac{u^l}{l!}\right) = u^{l+1} \int_0^1 \frac{(1-\theta)^l}{l!} e^{\theta u} d\theta \quad (u \in \mathbb{R}).$$

We will prove that, for sufficiently large $l \in \mathbb{N}$, as $\varepsilon \searrow 0$,

$$e^{\langle \bar{\nu}, \phi_1^2 \rangle} Q_{l+1}(\langle \bar{\nu}, r_{\varepsilon, 1}^{\kappa_3} \rangle / \varepsilon^2) \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}) \psi(|\phi_1^1 / \eta_1|^2) = O(\varepsilon^{\xi_{k+1}}) \quad \text{in } \tilde{\mathbf{D}}_\infty. \quad (5.23)$$

Note that $\chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}) = O(1)$ in $\mathbf{D}_\infty$ as $\varepsilon \searrow 0$ by (5.6). By Proposition 4.3, $r_{\varepsilon, 1}^{\kappa_3} / \varepsilon^2 = O(\varepsilon^{\kappa_3-2})$ in $\mathbf{D}_\infty$. So, if $l+1 \geq \xi_{k+1} / (\kappa_3 - 2)$, then $(\langle \bar{\nu}, r_{\varepsilon, 1}^{\kappa_3} \rangle / \varepsilon^2)^{l+1} = O(\varepsilon^{\xi_{k+1}})$ in $\mathbf{D}_\infty$. Therefore, in order to verify (5.23), it is sufficient to show that, as $\varepsilon \searrow 0$,

$$\int_0^1 (1-\theta)^l e^{\langle \bar{\nu}, \phi_1^2 + \theta r_{\varepsilon, 1}^{\kappa_3} \rangle / \varepsilon^2} d\theta \cdot \chi_\eta(\varepsilon, w + \frac{\bar{\gamma}}{\varepsilon}) \psi(|\phi_1^1 / \eta_1|^2) = O(1) \quad \text{in } \tilde{\mathbf{D}}_\infty. \quad (5.24)$$

To verify the integrability of this Wiener functional, note that $e^{\theta u} \leq 1 + e^u$ for all $u \in \mathbb{R}$ and $0 \leq \theta \leq 1$. This implies that the first factor on the left hand side of (5.24) is dominated by $e^{\langle \bar{\nu}, \phi_1^2 \rangle} + e^{\langle \bar{\nu}, r_{\varepsilon, 1}^2 \rangle / \varepsilon^2}$. From Lemma 5.6 and (5.22), we see that the left hand side of (5.24) is $O(1)$ in any L^r ($1 < r < \infty$). In the same way, the Malliavin derivatives of the left hand side of (5.24) are $O(1)$ in any L^r .

It is easy to see that, as $\varepsilon \searrow 0$,

$$\sum_{k=0}^l \frac{\{\langle \bar{\nu}, r_{\varepsilon, 1}^{\kappa_3} \rangle / \varepsilon^2\}^k}{k!} = 1 + \varepsilon^{\xi_1} K_{\xi_1} + \cdots + \varepsilon^{\xi_k} K_{\xi_k} + O(\varepsilon^{\xi_{k+1}}) \quad \text{in } \mathbf{D}_\infty. \quad (5.25)$$

From this and (5.6), we see that

$$\begin{aligned} &F(\varepsilon, w) \psi(|\phi_1^1(w) / \eta_1|^2) \\ &= \exp(\langle \bar{\nu}, \phi_1^2(\mathbf{w}) \rangle) \psi(|\phi_1^1(w) / \eta_1|^2) \psi\left(\frac{1}{\eta_1^2} \left| \frac{\tilde{y}_1^\varepsilon - a'}{\varepsilon} \right|^2\right) \{1 + \varepsilon^{\xi_1} K_{\xi_1}(w) + \cdots + \varepsilon^{\xi_k} K_{\xi_k}(w)\} \\ &\quad + O(\varepsilon^{\xi_{k+1}}) \quad \text{in } \tilde{\mathbf{D}}_\infty. \end{aligned}$$

Using (5.20), we finish the proof. $\blacksquare$

Proof of the main theorem (Theorem 2.2) Now we prove our main theorem in this paper. We set

$$\Lambda_4 = \Lambda_3 + \Lambda'_3 = \{\nu + \xi \mid \nu \in \Lambda_3, \xi \in \Lambda'_3\}.$$

We denote by $\{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ all the elements of Λ_4 in increasing order. It is no mystery why this index set appears in the short time expansion of the kernel because, very formally speaking, the problem reduces to finding asymptotic behavior of $\mathbb{E}[\exp(\langle \bar{\nu}, r_{\varepsilon,1}^2 \rangle / \varepsilon^2) \cdot \delta_0(r_{\varepsilon,1}^1 / \varepsilon)]$, as we have seen. Now, by (5.19), Lemma 5.7 Lemma 5.8, and (5.21), we can easily prove the asymptotic expansion in Theorem 2.2. It is easy to see that $\alpha_0 = \mathbb{E}[e^{\langle \bar{\nu}, \phi_1^2 \rangle} \delta(\phi_1^1)] > 0$. ■

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